

The Burgers equation driven by a stochastic measure

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Abstract The class of one-dimensional equations driven by a stochastic measure μ is studied. For μ only σ -additivity in probability is assumed. This class of equations includes the Burgers equation and the heat equation. The existence and uniqueness of the solution are proved, and the averaging principle for the equation is studied.

Keywords Stochastic Burgers equation, stochastic heat equation, stochastic measure, mild solution, averaging principle

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1 Introduction

In this paper, we consider the stochastic equation, which can formally be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x, u(t, x)) + \frac{\partial g}{\partial x}(t, x, u(t, x)) + \sigma(t, x) \frac{\partial \mu}{\partial x}, \quad u(0, x) = u_0(x), \quad (1)$$

where $(t, x) \in [0, T] \times \mathbb{R}$, and μ is a stochastic measure defined on $\mathcal{B}(\mathbb{R})$ (Borel σ -algebra in \mathbb{R}). For μ we assume only σ -additivity in probability, assumptions for f , g , σ and u_0 are given in Section 3. We consider the solution to the formal equation (1) in the mild form (see (5) below).

If $f = 0$ and $g(x, t, v) = v^2/2$ then (1) is the Burgers equation, for $g = \text{const}$ we get the heat equation.

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In the paper, we prove the existence and uniqueness of the solution and obtain its L^2 -continuity. In addition, we obtain that the averaging principle holds for equation (1).

Equation (1) driven by the Wiener process was studied in [7] for $x \in \mathbb{R}$, in [6] and [8] for $x \in [0, 1]$. Equation driven by Lévy process was considered in [9]. The main reason to study equation (1) is that it is a generalization of the Burgers equation which is very important in fluid mechanics. The stochastic Burgers equation was studied, for example, in [4], [11], [14], [16, Chapter 18], [26], [27]. All these equations are driven by Gaussian processes or Lévy processes. In our paper, we consider a more general integrator. Our proofs are based on methods and results of [7].

Stochastic equations driven by stochastic measures were studied, for example, in [1] (general parabolic equation), [2] (wave equation), [17] (heat equation). The averaging principle for such equations was considered in [3], [18], [13]. The detailed theory of stochastic measures is presented in [20]. Note that in all these publications, the functions in the equations are assumed to be bounded.

The rest of the paper is organized as follows. In Section 2 we have compiled some basic facts about stochastic measures. The precise formulation of the problem and our assumptions are given in Section 3, some regularity properties of the stochastic integral are studied in Section 4, and one auxiliary equation is considered in Section 5. The existence and uniqueness of the solution to equation (1) are proved in Section 6. The averaging principle for our equation is obtained in Section 7.

2 Preliminaries

In this section, we give basic information concerning stochastic measures in a general setting. In equation (1), μ is defined on Borel subsets of \mathbb{R} .

Let $L_0 = L_0(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and \mathcal{B} be a σ -algebra of subsets of X .

Definition 1. A σ -additive mapping $\mu : \mathcal{B} \rightarrow L_0$ is called *stochastic measure* (SM).

We do not assume the moment existence or martingale properties for SM. In other words, μ is L_0 -valued measure.

Important examples of SMs are orthogonal stochastic measures, α -stable random measures defined on a σ -algebra for $\alpha \in (0, 1) \cup (1, 2]$ (see [21, Chapter 3]). Theorem 8.3.1 of [10] states the conditions under which the increments of a real-valued process with independent increments generate an SM.

Many examples of the SMs on the Borel subsets of $[0, T]$ may be given by the Wiener-type integral

$$\mu(A) = \int_{[0, T]} \mathbf{1}_A(t) dX_t. \quad (2)$$

We note the following cases of processes X_t in (2) that generate SM.

1. X_t is any square integrable martingale.
2. $X_t = W_t^H$ is the fractional Brownian motion with Hurst index $H > 1/2$, see Theorem 1.1 [15].

3. $X_t = S_t^k$ is the sub-fractional Brownian motion for $k = H - 1/2, 1/2 < H < 1$, see Theorem 3.2 (ii) and Remark 3.3 c) in [23].
4. $X_t = Z_H^k(t)$ is the Hermite process, $1/2 < H < 1, k \geq 1$, see [12], [24, Section 3.1.3]. $Z_H^2(t)$ is known as the Rosenblatt process, see [22, Section 3].

The following analogue of the Nikodym theorem is satisfied for SMs.

Theorem 1 ([5, Theorem 8.6]). *Let μ_n are SMs on \mathcal{B} , $n \geq 1$, and*

$$\forall A \in \mathcal{B} \exists \mu(A) = \text{p-} \lim_{n \rightarrow \infty} \mu_n(A).$$

Then μ is an SM on \mathcal{B} .

Applying Theorem 1, we can construct the following example of the SM on all Borel subsets of \mathbb{R} :

$$\mu(A) = \int_{\mathbb{R}} \mathbf{1}_A(t) \xi(t) dW_t := \lim_{T \rightarrow \infty} \int_{[-T, T]} \mathbf{1}_A(t) \xi(t) dW_t, \tag{3}$$

where W_t is the Wiener process, $\xi(t)$ is an adapted process such that $\int_{\mathbb{R}} \mathbb{E} \xi(t)^2 dt < \infty$, the limit is taken in $L^2(\Omega)$.

For deterministic measurable functions $g : \mathbf{X} \rightarrow \mathbb{R}$, an integral of the form $\int_{\mathbf{X}} g d\mu$ is studied in [20, Chapter 1] (see also [10, Chapter 7]). In particular, every bounded measurable g is integrable w.r.t. any SM μ .

To estimate the stochastic integral, we will use the following statement.

Lemma 1 (Lemma 3.1 [17], Lemma 2.3 [20]). *Let $f_l : \mathbf{X} \rightarrow \mathbb{R}, l \geq 1$, be measurable functions such that $\bar{f}(x) = \sum_{l=1}^{\infty} |f_l(x)|$ is integrable w.r.t. μ . Then*

$$\sum_{l=1}^{\infty} \left(\int_{\mathbf{X}} f_l d\mu \right)^2 < \infty \quad \text{a.s.}$$

We consider the Besov spaces $B_{22}^\alpha([c, d])$. Recall that the norm in this classical space for $0 < \alpha < 1$ may be introduced by

$$\|g\|_{B_{22}^\alpha([c, d])} = \|g\|_{L_2([c, d])} + \left(\int_0^{d-c} (w_2(g, r))^2 r^{-2\alpha-1} dr \right)^{1/2}, \tag{4}$$

where

$$w_2(g, r) = \sup_{0 \leq h \leq r} \left(\int_c^{d-h} |g(y+h) - g(y)|^2 dy \right)^{1/2}.$$

By C and $C(\omega)$ we will denote positive constants and positive random constants, respectively, whose values may change.

3 The problem

We consider (1) in the mild form

$$\begin{aligned}
 u(t, x) &= \int_{\mathbb{R}} p(t, x - y) u_0(y) dy \\
 &+ \int_0^t \int_{\mathbb{R}} p(t - s, x - y) f(s, y, u(s, y)) dy ds \\
 &- \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t - s, x - y) g(s, y, u(s, y)) dy ds \\
 &+ \int_{\mathbb{R}} \int_0^t p(t - s, x - y) \sigma(s, y) ds d\mu(y),
 \end{aligned} \tag{5}$$

where equality holds a.s. for each $t \in [0, T]$ for almost all $x \in \mathbb{R}$, $u \in \mathbb{C}([0, T], \mathbb{L}^2(\mathbb{R}))$. Here

$$p(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

is the heat kernel, and μ is an SM on $\mathcal{B}(\mathbb{R})$.

We will refer to the following assumptions on elements of (5).

Assumption A1. $u_0(y) = u_0(y, \omega) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is measurable and $u_0(y) = u_0(y, \omega) \in \mathbb{L}^2(\mathbb{R})$ for each fixed ω .

Assumption A2. $f(s, y, r) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Borel function satisfying the linear growth and Lipschitz-type condition:

$$\begin{aligned}
 |f(s, y, r)| &\leq a_1(y) + K|r|, \\
 |f(s, y, r_1) - f(s, y, r_2)| &\leq (a_2(y) + L|r_1| + L|r_2|)|r_1 - r_2|
 \end{aligned}$$

for all $s \in [0, T]$, $y, r_i \in \mathbb{R}$ and for some constants K, L and nonnegative functions $a_i \in \mathbb{L}^2(\mathbb{R})$.

Assumption A3. The function g is of the form

$$g(s, y, r) = g_1(s, y, r) + g_2(s, r),$$

where

$$g_1(s, y, r) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad g_2(s, r) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$$

are Borel functions satisfying the following linear and quadratic growth conditions:

$$\begin{aligned}
 |g_1(s, y, r)| &\leq b_1(y) + b_2(y)|r|, \\
 |g_2(s, r)| &\leq K|r|^2,
 \end{aligned}$$

for all $s \in [0, T]$, $y, r \in \mathbb{R}$, where K is a constant, and

$$b_1 \in \mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R}), \quad b_2 \in \mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^\infty(\mathbb{R})$$

are nonnegative functions. Moreover, g satisfies the following Lipschitz-type condition:

$$|g(s, y, r_1) - g(s, y, r_2)| \leq (b_3(y) + L|r_1| + L|r_2|)|r_1 - r_2|$$

for all $s \in [0, T]$, $y, r_i \in \mathbb{R}$ and for some constant L and nonnegative function $b_3 \in \mathbb{L}^2(\mathbb{R})$.

Assumption A4. $\sigma(s, y) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, and

$$|\sigma(s, y)| \leq C_\sigma, \quad |\sigma(s, y_1) - \sigma(s, y_2)| \leq L_\sigma |y_1 - y_2|^{\beta(\sigma)}$$

for $1/2 < \beta(\sigma) < 1$ and some constants C_σ, L_σ .

Assumption A5. $|y|^\tau$ is integrable w.r.t. μ on \mathbb{R} for some $\tau > 1/2$.

For example, Assumption A5 holds for the SM defined in (3) and given τ if

$$\int_{\mathbb{R}} t^{2\tau} \mathbf{E} \xi(t)^2 dt < \infty.$$

If A4 and A5 hold, then, by Theorem of [1] (or Theorem 3.1 [20]), the random function

$$\vartheta(t, x) = \int_{\mathbb{R}} \int_0^t p(t-s, x-y) \sigma(s, y) ds d\mu(y)$$

has a version $\tilde{\vartheta}(t, x)$ such that for any fixed

$$\delta \in (0, T), \quad M > 0, \quad 0 < \gamma_1 < 1/2, \quad 0 < \gamma_2 < 1/4$$

there exists $L_{\tilde{\vartheta}}(\omega)$ such that

$$|\tilde{\vartheta}(t_1, x_1) - \tilde{\vartheta}(t_2, x_2)| \leq L_{\tilde{\vartheta}}(\omega) (|t_1 - t_2|^{\gamma_2} + |x_1 - x_2|^{\gamma_1}), \quad (6)$$

$$t_i \in [\delta, T], \quad |x_i| \leq M, \quad i = 1, 2.$$

In the sequel, we will use that

$$\left| \frac{\partial p}{\partial y}(t-s, x-y) \right| \leq \frac{C_\lambda}{t-s} e^{-\frac{\lambda(x-y)^2}{t-s}} \quad (7)$$

for any $\lambda \in (0, 1/4)$ and some constant C_λ .

4 Regularity property of the stochastic integral

In this section, we obtain some properties of $\tilde{\vartheta}$.

For any $j \in \mathbb{R}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(j)} = (d_{(k-1)n}^{(j)}, d_{kn}^{(j)}], \quad 1 \leq k \leq 2^n.$$

The following estimate is a key tool for the estimates of the stochastic integral.

Lemma 2 (Lemma 3.2 [17]). *Let μ be defined on $\mathcal{B}(\mathbb{R})$, Z be an arbitrary set, and $q(z, x) : Z \times [j, j+1] \rightarrow \mathbb{R}$ be a function such that for some $1/2 < \alpha < 1$ and for each $z \in Z$ $q(z, \cdot) \in B_{22}^\alpha([j, j+1])$. Then the random function*

$$\eta(z) = \int_{[j, j+1]} q(z, x) d\mu(x), \quad z \in Z,$$

has a version $\tilde{\eta}(z)$ such that for some constant C (independent of z, j, ω) and each $\omega \in \Omega$,

$$|\tilde{\eta}(z)| \leq |q(z, j)\mu([j, j+1])| + C \|q(z)\|_{B_{22}^\alpha([j, j+1])} \left\{ \sum_{n \geq 1} 2^{n(1-2\alpha)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}^{1/2}. \quad (8)$$

Further, we study the properties of stochastic integral ϑ .

Lemma 3. *Let Assumptions A4 and A5 hold. Then for version $\tilde{\vartheta}$ that satisfies (6), we have $\tilde{\vartheta} \in \mathbb{C}([0, T], L^2(\mathbb{R}))$*

Proof. To prove the continuity in $L^2(\mathbb{R})$, for fixed $\omega \in \Omega$ and arbitrary $t_n \rightarrow t_0$, consider

$$\|\tilde{\vartheta}(t_n) - \tilde{\vartheta}(t_0)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\tilde{\vartheta}(t_n, x) - \tilde{\vartheta}(t_0, x)|^2 dx.$$

By (6), $\tilde{\vartheta}$ is continuous in both variables, therefore $\tilde{\vartheta}(t_n, x) \rightarrow \tilde{\vartheta}(t_0, x)$ for each x .

We will find $g \in L^2(\mathbb{R})$ such that

$$|\tilde{\vartheta}(t, x)| = \left| \int_{\mathbb{R}} \int_0^t p(t-s, x-y)\sigma(s, y) ds d\mu(y) \right| \leq Cg(x), \tag{9}$$

then the dominated convergence theorem implies our statement.

We apply Lemma 2 for

$$q(z, y) = \int_0^t p(t-s, x-y)\sigma(s, y) ds, \quad z = (t, x).$$

Our considerations will imply that the function $q(z, y)$ is continuous in y (see (14) and (15) below), thus the conditions of Lemma 2 are satisfied.

We fix $\omega \in \Omega, \theta > 1$ and estimate

$$\begin{aligned} |\tilde{\vartheta}(t, x)|^2 &= \left| \int_{\mathbb{R}} \int_0^t p(t-s, x-y)\sigma(s, y) ds d\mu(y) \right|^2 \\ &= \left| \sum_{j \in \mathbb{Z}} \int_{(j, j+1]} \int_0^t p(t-s, x-y)\sigma(s, y) ds d\mu(y) \right|^2 \\ &\leq \sum_{j \in \mathbb{Z}} (|j|+1)^{-\theta} \sum_{j \in \mathbb{Z}} (|j|+1)^\theta \left| \int_{(j, j+1]} \int_0^t p(t-s, x-y)\sigma(s, y) ds d\mu(y) \right|^2 \\ &= C \sum_{j \in \mathbb{Z}} (|j|+1)^\theta \left| \int_{(j, j+1]} q(t, x, y) d\mu(y) \right|^2 \\ &\stackrel{(8)}{\leq} C \sum_{j \in \mathbb{Z}} (|j|+1)^\theta \left(|q(t, x, j)\mu((j, j+1])|^2 \right. \\ &\quad \left. + \|q(t)\|_{B_\alpha^{2,2}((j, j+1])}^2 \left\{ \sum_{n \geq 1} 2^{-n(2\alpha-1)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\} \right). \end{aligned} \tag{10}$$

Below we will use the following simple estimates:

$$\begin{aligned} \int_0^t \frac{1}{r} e^{-\frac{b}{r}} dr &\stackrel{b/r=z}{=} \int_{b/t}^\infty \frac{1}{z} e^{-z} dz \\ &\leq \mathbf{1}_{\{b \geq t\}} \int_{b/t}^\infty e^{-z} dz + \mathbf{1}_{\{b < t\}} \left(\int_{b/t}^1 \frac{1}{z} dz + \int_1^\infty e^{-z} dz \right) \\ &\leq \mathbf{1}_{\{b \geq t\}} e^{-b/t} + \mathbf{1}_{\{b < t\}} \left(\ln \frac{t}{b} + 1 \right), \end{aligned} \tag{11}$$

$$\int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac{b}{t-s}} ds \leq \int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac{b}{t}} ds \leq C e^{-\frac{b}{t}}. \tag{12}$$

We have that

$$\begin{aligned} |q(t, x, y)| &= \left| \int_0^t p(t-s, x-y) \sigma(s, y) ds \right| \\ &\leq C_\sigma C \int_0^t \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{t-s}} ds \stackrel{(12)}{\leq} C e^{-\frac{(x-y)^2}{4t}}. \end{aligned} \tag{13}$$

To estimate $\|q(t)\|_{B_\alpha^{2,2}([j, j+1])}$, we consider

$$\begin{aligned} q(t, x, y+h) - q(t, x, y) &= \int_0^t p(t-s, x-y) (\sigma(s, y+h) - \sigma(s, y)) ds \\ &+ \int_0^t (p(t-s, x-y-h) - p(t-s, x-y)) \sigma(s, y+h) ds := I_1 + I_2. \end{aligned}$$

For I_1 , we obtain

$$\begin{aligned} |I_1| &\leq L_\sigma h^{\beta(\sigma)} \int_0^t p(t-s, x-y) ds = C h^{\beta(\sigma)} \int_0^t \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{t-s}} ds \\ &\stackrel{(12)}{\leq} C h^{\beta(\sigma)} e^{-\frac{(x-y)^2}{4t}}. \end{aligned} \tag{14}$$

For I_2 , assuming $0 \leq h \leq 1$, we get

$$\begin{aligned} |I_2| &\stackrel{A4}{\leq} C_\sigma \int_0^t |p(t-s, x-y-h) - p(t-s, x-y)| ds \\ &= C_\sigma \int_0^t \left| \int_{x-y-h}^{x-y} \frac{\partial p(t-s, v)}{\partial v} dv \right| ds \\ &\stackrel{(7)}{\leq} C_\sigma C_\lambda \int_0^t \frac{1}{t-s} \int_{x-y-h}^{x-y} e^{-\frac{\lambda v^2}{t-s}} dv ds \\ &\stackrel{(11)}{\leq} C \int_{x-y-h}^{x-y} e^{-\frac{\lambda v^2}{t}} \mathbf{1}_{\{\lambda v^2 \geq t\}} dv + C \int_{x-y-h}^{x-y} \left(\left| \ln \frac{T}{\lambda v^2} \right| + 1 \right) \mathbf{1}_{\{\lambda v^2 < t\}} dv \\ &\leq C \int_{x-y-h}^{x-y} e^{-\frac{\lambda v^2}{t}} \mathbf{1}_{\{\lambda v^2 \geq t\}} dv + C \int_{x-y-h}^{x-y} |\ln |v|| \mathbf{1}_{\{\lambda v^2 < t\}} dv \\ &\stackrel{(*)}{\leq} C \int_{x-y-h}^{x-y} e^{-\frac{\lambda v^2}{t}} \mathbf{1}_{\{\lambda v^2 \geq t\}} dv + C \mathbf{1}_{\{|x-y| < \sqrt{t/\lambda} + 1\}} \int_0^{h/2} |\ln z| dz \\ &\stackrel{(**)}{\leq} C h e^{-\frac{\tilde{\lambda}(x-y)^2}{T}} + C \mathbf{1}_{\{|x-y| < \sqrt{t/\lambda} + 1\}} (z - z \ln z) \Big|_0^{h/2} \\ &\leq C h e^{-\frac{\tilde{\lambda}(x-y)^2}{T}} + C h |\ln h| \mathbf{1}_{\{|x-y| < \sqrt{t/\lambda} + 1\}} \\ &\leq C h e^{-\frac{\tilde{\lambda}(x-y)^2}{T}} + C h^{\beta(\sigma)} \mathbf{1}_{\{|x-y| < \sqrt{T/\lambda} + 1\}} \leq C h^{\beta(\sigma)} e^{-\frac{\tilde{\lambda}(x-y)^2}{T}}. \end{aligned} \tag{15}$$

In (*) we have used that the maximal value of $\int_{x_1}^{x_2} |\ln |v|| \mathbf{1}_{\{|v|<C\}} dv$ for fixed $|x_1 - x_2|$ is achieved when $x_2 = -x_1$.

In (***) we applied

$$C e^{-\frac{\lambda v^2}{T}} \leq e^{-\frac{\tilde{\lambda}(x-y)^2}{T}},$$

which holds for $0 < \tilde{\lambda} < \lambda$, $x - y - h \leq v \leq x - y$.

Further, we note that for $y \in [j, j + 1]$

$$e^{-\frac{(x-y)^2 \tilde{\lambda}}{T}} \leq e^{-\frac{(1-(|x-j|-1)^2) \tilde{\lambda}}{T}},$$

and (14) and (15) imply that

$$|q(t, x, y + h) - q(t, x, y)| \leq C h^{\beta(\sigma)} e^{-\frac{(1-(|x-j|-1)^2) \tilde{\lambda}}{T}}. \tag{16}$$

Now, from (4), (13), and (16), for some $1/2 < \alpha < \beta(\sigma)$ we get that

$$\|q(t)\|_{B_{\alpha}^{2,2}([j, j+1])} \leq C e^{-\frac{(1-(|x-j|-1)^2) \tilde{\lambda}}{T}}. \tag{17}$$

From (10), (13), and (17) we obtain (9), where

$$\begin{aligned} g^2(x) = & \sum_{j \in \mathbb{Z}} (|j| + 1)^{\theta} \left(e^{-\frac{2(1-(|x-j|-1)^2) \tilde{\lambda}}{T}} \mu^2((j, j + 1]) \right. \\ & \left. + e^{-\frac{2(1-(|x-j|-1)^2) \tilde{\lambda}}{T}} \left\{ \sum_{n \geq 1} 2^{-n(2\alpha-1)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\} \right). \end{aligned} \tag{18}$$

We will check that $g \in L^2(\mathbb{R})$, and get

$$\begin{aligned} \int_{\mathbb{R}} g^2(x) dx \leq & C \sum_{j \in \mathbb{Z}} (|j| + 1)^{\theta} \mu^2((j, j + 1]) \\ & + C \sum_{j \in \mathbb{Z}} (|j| + 1)^{\theta} \left\{ \sum_{n \geq 1} 2^{-n(2\alpha-1)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}. \end{aligned} \tag{19}$$

Here the sums with μ have a form $\sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} f_l d\mu \right)^2$, where

$$\begin{aligned} \{f_l(y), l \geq 1\} &= \{(|j| + 1)^{\theta/2} \mathbf{1}_{(j, j+1]}(y), j \in \mathbb{Z}\}, \\ \{f_l(y), l \geq 1\} &= \{(|j| + 1)^{\theta/2} 2^{-n(\alpha-1/2)} \mathbf{1}_{\Delta_{kn}^{(j)}}(y), \\ & j \in \mathbb{Z}, n \geq 1, 1 \leq k \leq 2^n\}. \end{aligned}$$

We have

$$\sum_{l=1}^{\infty} |f_l(y)| \leq C(|y| + 1)^{\theta/2}.$$

By A5, $\sum_{l=1}^{\infty} |f_l|$ is integrable w.r.t. μ on \mathbb{R} in both cases. From Lemma 1 it follows that for

$$\Omega_{\theta,\alpha} = \left\{ \omega \in \Omega : \sum_{l=1}^{\infty} \left(\int_{\mathbb{X}} f_l d\mu \right)^2 < +\infty \right\} \tag{20}$$

we have $\mathbf{P}(\Omega_{\theta,\alpha}) = 1$. For each $\omega \in \Omega_{\theta,\alpha}$, we have $g(\cdot, \omega) \in L^2(\mathbb{R})$, and $\tilde{\vartheta}(\cdot, \omega) \in C([0, T], L^2(\mathbb{R}))$. \square

In particular, Lemma 3 implies that for each $\omega \in \Omega_{\theta,\alpha}$ it holds

$$\sup_{t \in [0, T]} \|\tilde{\vartheta}(t)\|_{L^2(\mathbb{R})} < \infty. \tag{21}$$

We will need one more boundness result about $\tilde{\vartheta}$.

Lemma 4. *Let Assumptions A4 and A5 hold. Then, for version $\tilde{\vartheta}$ that satisfies (6), we have*

$$\sup_{t \in [0, T], x \in \mathbb{R}} |\tilde{\vartheta}(t, x)| < \infty \quad a.s.$$

Proof. In the proof of Lemma 3, for function g defined in (18), we obtained that estimate (9) holds. From (18) it follows that

$$\begin{aligned} g^2(x) &\leq C \sum_{j \in \mathbb{Z}} (|j| + 1)^\theta \mu^2((j, j + 1]) \\ &\quad + C \sum_{j \in \mathbb{Z}} (|j| + 1)^\theta \left\{ \sum_{n \geq 1} 2^{-n(2\alpha-1)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}. \end{aligned}$$

Further, we repeat the proof of Lemma 3 after (19), and for each $\omega \in \Omega_{\theta,\alpha}$ obtain that $\sup_{x \in \mathbb{R}} |g(x, \omega)| < \infty$. \square

5 Solution to the auxiliary equation

Consider the operator $\pi_N : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ such that

$$\pi_N(v) = \begin{cases} v, & \|v\|_{L^2(\mathbb{R})} \leq N, \\ N \frac{v}{\|v\|_{L^2(\mathbb{R})}}, & \|v\|_{L^2(\mathbb{R})} > N. \end{cases}$$

For the Hilbert space $L^2(\mathbb{R})$, it is easy to check that

$$\|\pi_N(v) - \pi_N(w)\|_{L^2(\mathbb{R})} \leq \|v - w\|_{L^2(\mathbb{R})}. \tag{22}$$

In this section, we study the auxiliary equation

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}} p(t, x - y) u_0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}} p(t - s, x - y) f(s, y, \pi_N(u)(s, y)) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t - s, x - y) g(s, y, \pi_N(u)(s, y)) dy ds + \tilde{\vartheta}(t, x). \end{aligned} \tag{23}$$

We consider (23) for fixed $\omega \in \Omega_{\theta, \alpha}$ (see (20)), and will prove the existence and uniqueness of the solution.

Let us consider the linear operators

$$\begin{aligned} (J_1 v)(t, x) &= \int_0^t \int_{\mathbb{R}} p(t-s, x-y)v(s, y) dy ds, \\ (J_2 w)(t, x) &= \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t-s, x-y)w(s, y) dy ds, \end{aligned}$$

where $t \in [0, T]$, $x \in \mathbb{R}$, $v, w \in L^\infty([0, T], L^2(\mathbb{R}))$.

We recall some lemmas from [7].

Lemma 5 (Lemma 3.1 [7]). *The operator J_1 is bounded from $L^2([0, T], L^2(\mathbb{R}))$ to $\mathbb{C}([0, T], L^2(\mathbb{R}))$, and the following estimate holds:*

$$\|J_1 v(t) - J_1 v(r)\|_{L^2(\mathbb{R})} \leq C|t-r|^{1/3} \left(\int_0^t \|v(s)\|_{L^2(\mathbb{R})}^2 ds \right)^{1/2}, \quad (24)$$

where $0 \leq r, t \leq T$.

Lemma 6 (Lemma 3.2 [7]). *The operator J_2 is bounded from $L^5([0, T], L^1(\mathbb{R}))$ to $\mathbb{C}([0, T], L^2(\mathbb{R}))$, and the following estimates hold:*

$$\|J_2 w(t)\|_{L^2(\mathbb{R})} \leq C \int_0^t (t-s)^{-3/4} \|w(s)\|_{L^1(\mathbb{R})} ds, \quad (25)$$

$$\|J_2 w(t) - J_2 w(r)\|_{L^2(\mathbb{R})} \leq C|t-r|^{1/21} \left(\int_0^t \|w(s)\|_{L^1(\mathbb{R})}^5 ds \right)^{1/5}, \quad (26)$$

where $0 \leq r, t \leq T$.

The following statement is the analogue of Proposition 4.1 [7].

Lemma 7. *Let A2 and A3 hold, and*

$$u_0 \in L^2(\mathbb{R}), \quad |\tilde{\vartheta}(t, x)| \leq C(\omega), \quad \|\tilde{\vartheta}(t)\|_{L^2(\mathbb{R})} \leq C(\omega)$$

for some finite constant $C(\omega)$. Then for any fixed $N > 0$ and $\omega \in \Omega_{\theta, \alpha}$ equation (23) has a unique solution $u \in \mathbb{C}([0, T], L^2(\mathbb{R}))$.

Proof. *Step 1 (existence and uniqueness of the solution).*

Denote

$$\begin{aligned} (\mathcal{A}_1 u)(t, x) &= \int_0^t \int_{\mathbb{R}} p(t-s, x-y)f(s, y, \pi_N(u)(s, y)) dy ds, \\ (\mathcal{A}_2 u)(t, x) &= - \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t-s, x-y)g(s, y, \pi_N(u)(s, y)) dy ds. \end{aligned}$$

Fix $\lambda > 0$. Let \mathcal{H} denote the Banach space of $L^2(\mathbb{R})$ -valued functions $u(t, x)$ such that $u(0, x) = u_0(x)$, with the norm

$$\|u(t)\|_{\mathcal{H}}^2 = \int_0^T e^{-\lambda t} \|u(t)\|_{L^2(\mathbb{R})}^2 dt.$$

Define the operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$(\mathcal{A}u)(t, x) = \int_{\mathbb{R}} p(t, x - y)u_0(y) dy + (\mathcal{A}_1u)(t, x) + (\mathcal{A}_2u)(t, x) + \tilde{\vartheta}(t, x). \quad (27)$$

We will prove that operator \mathcal{A} is a contraction for large λ .

Using the inequality

$$\left\| \int_0^t f(s) ds \right\| \leq \int_0^t \|f(s)\| ds,$$

we have that

$$\begin{aligned} & \|(\mathcal{A}_1u)(t) - (\mathcal{A}_1v)(t)\|_{L^2(\mathbb{R})} \\ & \leq \int_0^t \left\| \int_{\mathbb{R}} p(t-s, x-y)(f(s, y, \pi_N(u)(s, y)) - f(s, y, \pi_N(v)(s, y))) dy \right\|_{L^2(\mathbb{R})} ds \\ & \stackrel{A2}{\leq} \int_0^t \left\| \int_{\mathbb{R}} p(t-s, x-y)(a_2(y) + L|\pi_N(u)(s, y)| + L|\pi_N(v)(s, y)|) \right. \\ & \quad \times |\pi_N(u)(s, y) - \pi_N(v)(s, y)| dy \left. \right\|_{L^2(\mathbb{R})} ds \\ & \stackrel{(**)}{\leq} \int_0^t \|p(t-s, x-y)\|_{L^2(\mathbb{R})} \|(a_2(y) + L|\pi_N(u)(s, y)| + L|\pi_N(v)(s, y)|) \\ & \quad \times |\pi_N(u)(s, y) - \pi_N(v)(s, y)|\|_{L^1(\mathbb{R})} ds. \end{aligned} \quad (28)$$

Here in (**) we used the inequality for convolution $\|v * w\|_{L^2(\mathbb{R})} \leq \|v\|_{L^2(\mathbb{R})} \|w\|_{L^1(\mathbb{R})}$. Further, we have

$$\begin{aligned} & \|p(t-s, x-y)\|_{L^2(\mathbb{R})}^2 = C(t-s)^{-1/2}, \\ & \|(a_2(y) + L|\pi_N(u)(s, y)| + L|\pi_N(v)(s, y)|)|\pi_N(u)(s, y) - \pi_N(v)(s, y)|\|_{L^1(\mathbb{R})} \\ & \leq \|(a_2(y) + L|\pi_N(u)(s, y)| + L|\pi_N(v)(s, y)|)\|_{L^2(\mathbb{R})} \\ & \quad \times \|\pi_N(u)(s, y) - \pi_N(v)(s, y)\|_{L^2(\mathbb{R})} \\ & \stackrel{(22)}{\leq} (\|a_2\|_{L^2(\mathbb{R})} + 2LN)\|u(s) - v(s)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned} \|(\mathcal{A}_1u)(t) - (\mathcal{A}_1v)(t)\|_{L^2(\mathbb{R})}^2 & \leq C_N \left(\int_0^t (t-s)^{-1/4} \|u(s) - v(s)\|_{L^2(\mathbb{R})} ds \right)^2 \\ & \leq C_N \int_0^t (t-s)^{-3/4} \|u(s) - v(s)\|_{L^2(\mathbb{R})}^2 ds, \end{aligned} \quad (29)$$

where C_N denotes some constants that may depend on N .

For \mathcal{A}_2 , we have

$$\begin{aligned}
& \|(\mathcal{A}_2 u)(t) - (\mathcal{A}_2 v)(t)\|_{\mathbb{L}^2(\mathbb{R})}^2 \\
&= \left\| \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t-s, x-y)(g(s, y, \pi_N(u)(s, y)) \right. \\
&\quad \left. - g(s, y, \pi_N(v)(s, y))) dy ds \right\|_{\mathbb{L}^2(\mathbb{R})}^2 \\
&\stackrel{(25)}{\leq} \left(\int_0^t (t-s)^{-3/4} \|b_3(y) + L|\pi_N(u)(s, y)| \right. \\
&\quad \left. + L|\pi_N(v)(s, y)|\| |\pi_N(u)(s, y) - \pi_N(v)(s, y)| \|_{\mathbb{L}^1(\mathbb{R})} ds \right)^2 \\
&\leq C \int_0^t (t-s)^{-3/4} \|b_3(y) + L|\pi_N(u)(s, y)| \\
&\quad + L|\pi_N(v)(s, y)|\| |\pi_N(u)(s, y) - \pi_N(v)(s, y)| \|_{\mathbb{L}^1(\mathbb{R})}^2 ds \\
&\leq C \int_0^t (t-s)^{-3/4} \|b_3(y) + L|\pi_N(u)(s, y)| + L|\pi_N(v)(s, y)|\|_{\mathbb{L}^2(\mathbb{R})}^2 \\
&\quad \times \|\pi_N(u)(s, y) - \pi_N(v)(s, y)\|_{\mathbb{L}^2(\mathbb{R})}^2 ds \\
&\leq C \int_0^t (t-s)^{-3/4} (\|b_3\|_{\mathbb{L}^2(\mathbb{R})} + 2LN)^2 \|u(s, y) - v(s, y)\|_{\mathbb{L}^2(\mathbb{R})}^2 ds \\
&= C_N \int_0^t (t-s)^{-3/4} \|u(s) - v(s)\|_{\mathbb{L}^2(\mathbb{R})}^2 ds.
\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}
\|Au - Av\|_{\mathcal{H}}^2 &= \int_0^T e^{-\lambda t} \|(Au)(t) - (Av)(t)\|_{\mathbb{L}^2(\mathbb{R})}^2 dt \\
&\stackrel{(29), (30)}{\leq} C_N \int_0^T e^{-\lambda t} \int_0^t (t-s)^{-3/4} \|u(s) - v(s)\|_{\mathbb{L}^2(\mathbb{R})}^2 ds dt \\
&= C_N \int_0^T e^{-\lambda s} \|u(s) - v(s)\|_{\mathbb{L}^2(\mathbb{R})}^2 \int_s^T (t-s)^{-3/4} e^{-\lambda(t-s)} dt ds \\
&\leq C_N \int_0^T e^{-\lambda s} \|u(s) - v(s)\|_{\mathbb{L}^2(\mathbb{R})}^2 \int_0^\infty r^{-3/4} e^{-\lambda r} dr ds \\
&= C_N \|u - v\|_{\mathcal{H}}^2 \int_0^\infty r^{-3/4} e^{-\lambda r} dr,
\end{aligned}$$

and for large λ we can get

$$C_N \int_0^\infty r^{-3/4} e^{-\lambda r} dr < 1.$$

Then the operator \mathcal{A} on \mathcal{H} is a contraction and has a unique fixed point which is the solution of (23).

Step 2 (continuity of the solution in $\mathbb{L}^2(\mathbb{R})$).

We will demonstrate that $u \in \mathbb{C}([0, T], L^2(\mathbb{R}))$ provided that $\mathcal{A}u = u$. We consider each term in (27), and obtain

$$\begin{aligned} & \|(\mathcal{A}_1 u)(t) - (\mathcal{A}_1 u)(r)\|_{L^2(\mathbb{R})}^2 \\ & \stackrel{(24)}{\leq} (t-r)^{2/3} \int_0^t \|f(s, y, \pi_N(v)(s, y))\|_{L^2(\mathbb{R})}^2 ds \\ & \stackrel{A2}{\leq} (t-r)^{2/3} 2T (\|a_1\|_{L^2(\mathbb{R})}^2 + K^2 N^2), \\ & \|(\mathcal{A}_2 u)(t) - (\mathcal{A}_2 u)(r)\|_{L^2(\mathbb{R})}^2 \stackrel{(26)}{\leq} (t-r)^{5/21} \int_0^t \|g(s, y, \pi_N(v)(s, y))\|_{L^1(\mathbb{R})}^5 ds \\ & \stackrel{A3}{\leq} (t-r)^{5/21} \cdot CT (\|b_1\|_{L^1(\mathbb{R})}^5 + \|b_2 \pi_N(v)\|_{L^1(\mathbb{R})}^5 + K^5 \|\pi_N^2(v)\|_{L^1(\mathbb{R})}^5) \\ & \leq C(t-r)^{5/21} (\|b_1\|_{L^1(\mathbb{R})}^5 + \|b_2\|_{L^2(\mathbb{R})}^5 \|\pi_N(v)\|_{L^2(\mathbb{R})}^5 + K^5 \|\pi_N(v)\|_{L^2(\mathbb{R})}^5) \\ & \leq C(t-r)^{5/21} (\|b_1\|_{L^1(\mathbb{R})}^5 + \|b_2\|_{L^2(\mathbb{R})}^5 N^5 + K^5 N^5), \end{aligned}$$

therefore $(\mathcal{A}_1 u)(t)$ and $(\mathcal{A}_2 u)(t)$ are continuous. For \tilde{v} , we refer to Lemma 3, convolution $p * u_0$ is continuous by standard properties of the heat semigroup. \square

6 Solution to the main equation

We will use one more statement from [7].

Lemma 8 (Lemma 4.2 [7]). *Let $\zeta = \{\zeta(t, x), t \in [0, T], x \in \mathbb{R}\}$ be a continuous and bounded function belonging to $\mathbb{C}([0, T], L^2(\mathbb{R}))$. Let $v \in \mathbb{C}([0, T], L^2(\mathbb{R}))$ be a solution of the integral equation*

$$\begin{aligned} v(t, x) &= \int_{\mathbb{R}} p(t, x-y) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} p(t-s, x-y) f(s, y, v(s, y) + \zeta(s, y)) dy ds \\ &- \int_0^t \int_{\mathbb{R}} \frac{\partial p}{\partial y}(t-s, x-y) g(s, y, v(s, y) + \zeta(s, y)) dy ds, \end{aligned} \tag{31}$$

where $u_0(y) \in L^2(\mathbb{R})$, and f and g satisfy the assumptions A2 and A3.

Then we have

$$\|v(t)\|_{L^2(\mathbb{R})}^2 \leq (\|u_0\|_{L^2(\mathbb{R})}^2 + C_1(1 + R_1(\zeta))) \exp\{C_2(1 + R_2(\zeta))\}, \tag{32}$$

where the constants C_1 and C_2 depend only on T and on the functions a_i, b_i and the constants K and L appearing in the hypothesis A2 and A3, and

$$R_1(\zeta) = \sup_{s \in [0, T]} (\|\zeta(s)\|_{L^2(\mathbb{R})}^2 + \|\zeta(s)\|_{L^4(\mathbb{R})}^4 + \|\zeta(s)\|_{L^\infty(\mathbb{R})}^2),$$

$$R_2(\zeta) = \sup_{s \in [0, T]} \|\zeta(s)\|_{L^\infty(\mathbb{R})}^2.$$

The main result of the paper is the following.

Theorem 2. *Let Assumptions A1–A5 hold. Then equation (5) has a unique solution which is continuous with values in $L^2(\mathbb{R})$.*

Proof. We consider equation (5) for each fixed $\omega \in \Omega_{\theta, \alpha}$, take the version \tilde{v} that satisfies (6).

We apply Lemma 8 to

$$\zeta(t, x) = \int_{\mathbb{R}} \int_0^t p(t-s, x-y) \sigma(s, y) ds d\mu(y) = \tilde{v}(t, x),$$

assumptions on ζ are fulfilled by Lemmas 3 and 4.

For given ζ , set

$$N = N_1 + \sup_{t \in [0, T]} \|\zeta(t)\|_{L^2(\mathbb{R})} + 1, \tag{33}$$

where N_1^2 is equal to the right-hand side of (32). For given N , take u which is a solution of (23).

Denote

$$t_N = \sup\{t : \|u(t)\|_{L^2(\mathbb{R})} \leq N\},$$

for $t \leq t_N$ we have $\pi_N(u) = u$ and $v = u - \zeta$ satisfies (31).

By Lemma 8, $\|v(t)\|_{L^2(\mathbb{R})} \leq N_1$, and

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|v(t)\|_{L^2(\mathbb{R})} + \|\zeta(t)\|_{L^2(\mathbb{R})} \leq N - 1, \quad t \leq t_N.$$

Therefore, $t_N = T$, and we never get $\|u(t)\|_{L^2(\mathbb{R})} > N$. The solution of (23) will satisfy (5).

Conversely, in (5) we have that $\|u_0\|_{L^2(\mathbb{R})} \leq N - 1$, up to the moment $t_N = T$ equation (5) coincides with (23) and has a unique solution. \square

7 Averaging principle

In this section, we consider equation (1), where functions f and g do not depend on the time variable t , and study the averaging of the stochastic term.

For $\varepsilon > 0$, consider equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} &= \frac{\partial^2 u_\varepsilon}{\partial x^2} + f(x, u_\varepsilon(t, x)) + \frac{\partial g}{\partial x}(x, u_\varepsilon(t, x)) + \sigma(t/\varepsilon, x) \frac{\partial \mu}{\partial x}, \\ u_\varepsilon(0, x) &= u_0(x). \end{aligned} \tag{34}$$

Assume that for each $y \in \mathbb{R}$ there exists the limit

$$\bar{\sigma}(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma(s, y) ds. \tag{35}$$

It is easy to see that $\bar{\sigma}(y)$ satisfies Assumption A4 with the same constants C_σ, L_σ .

We will study convergence $u_\varepsilon(t, x) \rightarrow \bar{u}(t, x)$, $\varepsilon \rightarrow 0$, where \bar{u} is the solution of the averaged equation

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= \frac{\partial^2 \bar{u}}{\partial x^2} + f(x, \bar{u}(t, x)) + \frac{\partial g}{\partial x}(x, \bar{u}(t, x)) + \bar{\sigma}(x) \frac{\partial \mu}{\partial x}, \\ \bar{u}(0, x) &= u_0(x). \end{aligned} \tag{36}$$

The mild forms of (34) and (36) are, respectively,

$$\begin{aligned}
 u_\varepsilon(t, x) &= \int_0^t p(t, x - y)u_0(y) dy \\
 &+ \int_0^t \int_{\mathbb{R}} p(t - s, x - y) f(y, u_\varepsilon(s, y)) dy ds \\
 &- \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} p(t - s, x - y) g(y, u_\varepsilon(s, y)) dy ds \\
 &+ \int_{\mathbb{R}} \int_0^t p(t - s, x - y) \sigma(s/\varepsilon, y) ds d\mu(y).
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{u}(t, x) &= \int_{\mathbb{R}} p(t, x - y)u_0(y) dy \\
 &+ \int_0^t \int_{\mathbb{R}} p(t - s, x - y) f(y, \bar{u}(s, y)) dy ds \\
 &- \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} p(t - s, x - y) g(y, \bar{u}(s, y)) dy ds \\
 &+ \int_{\mathbb{R}} \int_0^t p(t - s, x - y) \bar{\sigma}(y) ds d\mu(y).
 \end{aligned}$$

We also impose the following additional condition that is standard in the averaging principle.

Assumption A6. *The function $G_\sigma(r, y) = \int_0^r (\sigma(s, y) - \bar{\sigma}(y)) ds, r \in \mathbb{R}_+, y \in \mathbb{R}$ is bounded.*

This holds, for example, if $\sigma(s, y)$ is bounded and periodic in s for each fixed y , and the set of values of minimal periods is bounded. Obviously, A6 implies (35).

Theorem 3. *Assume that Assumptions A1–A6 hold. Then there exists a versions of u_ε and \bar{u} such that*

$$\sup_{t \in [0, T]} \|u_\varepsilon(t) - \bar{u}(t)\|_{L^2(\mathbb{R})} \rightarrow 0 \quad a.s. \tag{37}$$

Proof. In Step 1 of the proof of Theorem 1 [19] (or Theorem 7.1 [20]), for the stochastic integrals in

$$\begin{aligned}
 \xi_\varepsilon(t, x) &= \int_{\mathbb{R}} \int_0^t p(t - s, x - y) \sigma(s/\varepsilon, y) ds d\mu(y) \\
 &- \int_{\mathbb{R}} \int_0^t p(t - s, x - y) \bar{\sigma}(y) ds d\mu(y)
 \end{aligned}$$

and some $\gamma_1 > 0$ it was proved that there exists a version such that

$$|\xi_\varepsilon(t, x)| \leq C(\omega)\varepsilon^{\gamma_1} \tag{38}$$

for all $\omega \in \Omega_1$, $\mathbf{P}(\Omega_1) = 1$ (here Assumption A6 was used).

As in the proof of Lemma 3, for function g defined in (18), we obtain that

$$|\xi_\varepsilon(t, x)| \leq Cg(x),$$

where $g \in L^2(\mathbb{R})$ for each $\omega \in \Omega_{\theta, \alpha}$, $\mathbf{P}(\Omega_{\theta, \alpha}) = 1$, and g is independent of ε .

By (38), $|\xi_{\varepsilon_n}(t, x)| \rightarrow 0$, $\varepsilon_n \rightarrow 0$. The dominated convergence theorem imply that for each $t \in [0, T]$ and each $\omega \in \Omega_{\theta, \alpha} \cap \Omega_1$ it holds

$$(\|\xi_{\varepsilon_n}(t)\|_{L^2(\mathbb{R})})^2 = \int_{\mathbb{R}} |\xi_{\varepsilon_n}(t, x)|^2 dx \rightarrow 0, \quad \varepsilon_n \rightarrow 0,$$

therefore

$$\|\xi_\varepsilon(t)\|_{L^2(\mathbb{R})} \rightarrow 0, \quad \varepsilon \rightarrow 0. \tag{39}$$

In the proof of Theorem 2, it was obtained that

$$\|u(t)\|_{L^2(\mathbb{R})} \leq N, \tag{40}$$

where N is defined in (33).

We can see that also

$$\|u_\varepsilon(t)\|_{L^2(\mathbb{R})} \leq N \tag{41}$$

for all $\varepsilon > 0$ for the same N . To explain this, note that in (33) N_1 depends only on T , on the functions a_i, b_i and the constants K and L appearing in the assumptions A2 and A3. For $\zeta(t, s) = \vartheta(t, s)$, in the proof of Lemma 3 we obtained that

$$\sup_{t \in [0, T]} \|\vartheta(t)\|_{L^2(\mathbb{R})}^2 \leq C \int_{\mathbb{R}} g^2(x) dx,$$

where the right-hand side may be estimated by (19). The constants in (19) may depend on C_σ and L_σ from assumption A4, but are independent of ε .

Further, we obtain

$$\begin{aligned} & \|u_\varepsilon(t) - \bar{u}(t)\|_{L^2(\mathbb{R})} \\ & \leq \left\| \int_0^t \int_{\mathbb{R}} p(t-s, x-y)(f(y, u_\varepsilon(s, y)) - f(y, \bar{u}(s, y))) dy ds \right\|_{L^2(\mathbb{R})} \\ & \quad + \left\| \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} p(t-s, x-y)(g(y, u_\varepsilon(s, y)) - g(y, \bar{u}(s, y))) dy ds \right\|_{L^2(\mathbb{R})} \\ & \quad + \|\xi_\varepsilon(t)\|_{L^2(\mathbb{R})} := J_1 + J_2 + \|\xi_\varepsilon(t)\|_{L^2(\mathbb{R})}. \end{aligned}$$

For J_1 , as in (28)–(29), taking into account (40) and (41), we get

$$J_1^2 \leq C \int_0^t (t-s)^{-1/4} \|u_\varepsilon(s) - \bar{u}(s)\|_{L^2(\mathbb{R})}^2 ds.$$

For J_2 , as in (30), we obtain

$$J_2^2 \leq C \int_0^t (t-s)^{-3/4} \|u_\varepsilon(s) - \bar{u}(s)\|_{L^2(\mathbb{R})}^2 ds.$$

Therefore,

$$\|u_\varepsilon(t) - \bar{u}(t)\|_{L^2(\mathbb{R})}^2 \leq C \int_0^t (t-s)^{-3/4} \|u_\varepsilon(s) - \bar{u}(s)\|_{L^2(\mathbb{R})}^2 ds + 3 \|\xi_\varepsilon(t)\|_{L^2(\mathbb{R})}^2.$$

We use the generalized Gronwall's inequality (see, for example, Corollary 1 in [25]) and get

$$\begin{aligned} \|u_\varepsilon(t) - \bar{u}(t)\|_{L^2(\mathbb{R})}^2 &\leq C(\omega) \|\xi_\varepsilon(t)\|_{L^2(\mathbb{R})}^2 \\ &\quad + C(\omega) \int_0^t \sum_{n=1}^{\infty} \frac{\Gamma^n(1/4)}{\Gamma(n/4)} (t-s)^{n/4-1} \|\xi_\varepsilon(s)\|_{L^2(\mathbb{R})}^2 ds. \end{aligned}$$

By (21), $\|\xi_\varepsilon(s)\|_{L^2(\mathbb{R})} \leq C$. It is easy to check that the function

$$h(s) = \sum_{n=1}^{\infty} \frac{\Gamma^n(1/4)}{\Gamma(n/4)} (t-s)^{n/4-1}$$

is integrable. Applying (39) and the dominated convergence theorem, we obtain (37). \square

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