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On the Feynman–Kac semigroup for some Markov processes

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Abstract For a (non-symmetric) strong Markov process X, consider the Feynman–Kac semigroup

$$T_t^A f(x) := \mathbb{E}^x\big[e^{A_t} f(X_t)\big], \quad x \in \mathbb{R}^n,\ t > 0,$$

where A is a continuous additive functional of X associated with some signed measure. Under the assumption that X admits a transition probability density that possesses upper and lower bounds of certain type, we show that the kernel corresponding to T_t^A possesses the density $p_t^A(x, y)$ with respect to the Lebesgue measure and construct upper and lower bounds for $p_t^A(x, y)$. Some examples are provided.

Keywords Transition probability density, continuous additive functional, Kato class, Feynman–Kac semigroup

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1 Introduction

Let $(X_t)_{t\geq 0}$ be a Markov process with the state space $\mathbb{R}^n$. For a Borel measurable function $V : \mathbb{R}^n \to \mathbb{R}$, we can define the functional A_t of X by

$$A_t := \int_0^t V(X_s)ds, \quad t > 0. \tag{1.1}$$

Suppose that $\lim_{t\to 0}\sup_x \mathbb{E}^x|A_t| = 0$. Then, by the Khasminski lemma there exist constants $C, b > 0$, such that

$$\sup_x \mathbb{E}^x e^{|A_t|} \le Ce^{bt}; \tag{1.2}$$

see, for example, [11, Lemma 3] or [12, Lemma 3.3.7]. Estimate (1.2) allows us to define the operator

$$T_t^A f(x) := \mathbb{E}^x\big[e^{A_t} f(X_t)\big], \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.3}$$

where the function f is bounded and Borel measurable. The family of operators $(T_t^A)_{t\ge 0}$ forms a semigroup, called the *Feynman–Kac semigroup*.

Feynman–Kac semigroup is well studied in the case of a Brownian motion (see [23, 24, 12, 3]); in particular, in [3] more general functionals are treated. The case of a general Markov process is much more complicated; see, however, [12, Chap. 3.3.2] and [24]. The essential condition on the process, stated in the papers cited, is that the Markov process X is symmetric and possesses a transition probability density $p_t(x, y)$.

In this paper, we construct and investigate the Feynman–Kac semigroups for a wider class of Markov processes. First, we construct the Feynman–Kac semigroup for a (non-symmetric) Markov process, admitting a transition density. We also treat a more general class of functionals A_t, that is, in our setting the functional A_t is not necessarily of the form (1.1), but is constructed by means of some measure ϖ, which is in the *Kato class* with respect to the transition probability density of X (cf. (2.3)). The approach used in [8] allows us to show the existence of the kernel $p_t^A(x, y)$ of the semigroup $(T_t^A)_{t\ge 0}$ and to give its representation. The method from [8] relies on the construction of the *Markov bridge density*, which in turn employs the regularity properties of the transition probability density of the initial process X rather than its symmetry.

In such a way, this prepares the base for the main result of the paper, which is devoted to the investigation of the Feynman–Kac semigroup for the particular class of processes constructed in [18]. In [20, 19], we develop the approach that allows us to relate to a pseudo-differential operator of certain type a Markov process possessing a transition probability density $p_t(x, y)$ and construct for this density two-sided estimates. In particular, such estimates provide an easily checkable condition when a measure ϖ belongs to the Kato class with respect to $p_t(x, y)$. This allows us to describe the respective continuous additive functional A_t and to show (1.2). Starting with the class of processes investigated in [18], we construct (see Theorem 3) the upper and lower estimates for the Feynman–Kac density $p_t^A(x, y)$. In particular, we show that the structure of such estimates is "inherited" from the structure of the estimates on $p_t(x, y)$. In some cases when the upper bound on $p_t(x, y)$ can be written in a rather compact way, we can describe explicitly the Kato class of measures. For example, this is the case if $p_t(x, y)$ is comparable for small t with the density of a symmetric stable process; see also [4, Cor. 12] for refined results. In Proposition 4 we show that if the initial transition probability density possesses an upper bound of a rather simple (polynomial) form, this form is inherited by the Feynman–Kac density $p_t^A(x, y)$.

Up to the author's knowledge, in general, the results on two-sided estimates of $p_t^A(x, y)$ are yet unavailable. For X being an α-stable-like process, the estimates of the kernel $p_t^A(x, y)$ are obtained in [22]; see also [10] and the references therein for more recent results in this direction, including two-sided estimates on $p_t^A(x, y)$ in the case when the functional A is not necessarily continuous. The approach used in [22, 10] to construct the Feynman–Kac semigroup is based on the Dirichlet form technique. See also [5] for yet another approach to investigate Feynman–Kac semigroups.

The paper is organized as follows. In Section 2, we give the basic notions and introduce the main results. Proofs are given in Sections 3 and 4. In Section 5, we illustrate our results with examples.

Notation

For functions f, g, by $f \asymp g$ we mean that there exist some constants $c_1, c_2 > 0$ such that $c_1 f(x) \le g(x) \le c_2 f(x)$ for all $x \in \mathbb{R}^n$. By $x \cdot y$ and $\|x\|$ we denote, respectively, the scalar product and the norm in $\mathbb{R}^n$, and $\mathbb{S}^n$ denotes the unit sphere in $\mathbb{R}^n$. By $B_b(\mathbb{R}^n)$ we denote the family of bounded Borel functions on $\mathbb{R}^n$. By $C_\infty^k(\mathbb{R}^n)$ we denote the space of k-times differentiable functions, with derivatives vanishing at infinity. By c_i, c and C we denote arbitrary positive constants. The symbols $*$, $\square$, and $\lozenge$ denote, respectively, the convolutions

$$(f * g)(x, y) := \int_{\mathbb{R}^n} f(x - z) g(z - y) dz,$$

$$(f \square g)(x, y) := \int_{\mathbb{R}^n} f(x - z) g(z - y) \varpi(dz),$$

and

$$(f \lozenge g)_t(x, y) := \int_0^t \int_{\mathbb{R}^n} f_{t-s}(x, z) g_s(z, y) \varpi(dz) ds,$$

where ϖ is a (signed) measure.

2 Settings and the main results

Let X be a Markov process with the state space $\mathbb{R}^n$. We call X a *Feller process* if the corresponding operator

$$T_t f(x) := \mathbb{E}^x f(X_t) \tag{2.1}$$

maps the space $C_\infty(\mathbb{R}^n)$ of continuous functions vanishing at infinity into itself. Assume that X possesses a transition probability density $p_t(x, y)$ which satisfies the following assumption.

P1. For fixed $x \in \mathbb{R}^n$, the mapping $y \mapsto p_s(x, y)$ is continuous for all $s \in (0, t]$, and the mapping $s \mapsto p_s(x, y)$ is continuous for all $x, y \in \mathbb{R}^n$.

Recall some notions on the Kato class of measures and related continuous additive functionals.

We say that a functional φ_t of a Markov process X_t is a W-functional (see [13, §6.11]) if φ_t is a positive continuous additive functional, almost surely homogeneous, and such that $\sup_x \mathbb{E}^x \varphi_t < \infty$. By additivity we mean that φ_t satisfies the following equality:

$$\varphi_{t+s} = \varphi_t + \varphi_s \circ \theta_t, \tag{2.2}$$

where θ_t is the shift operator, that is, $X_s \circ \theta_t = X_{t+s}$. The function $v_t(x) := \mathbb{E}^x \varphi_t$ is called the characteristic of φ_t and determines φ_t in the unique way; see [13, Thm. 6.3].

A positive Borel measure ϖ is said to belong to the Kato class S_K with respect to $p_t(x, y)$ if

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} p_s(x, y)\varpi(dy)ds = 0. \tag{2.3}$$

By [13, Thm. 6.6], the condition $\varpi \in S_K$ implies that the function

$$\chi_t(x) := \int_0^t \int_{\mathbb{R}^n} p_s(x, y)\varpi(dy)ds \tag{2.4}$$

for which the mapping $x \mapsto \chi_t(x)$ is measurable for all $t \geq 0$, is the characteristic of some W-functional φ_t.

Let $\varpi = \varpi^+ - \varpi^-$ be a signed measure such that $\varpi^\pm \in S_K$ with respect to $p_t(x, y)$. Then

$$\chi_t^\pm := \int_0^t \int_{\mathbb{R}^n} p_s(x, y)\varpi^\pm(dy)ds \tag{2.5}$$

are the characteristics of some W-functionals $A_t^\pm$, respectively, that is, there exist $A_t^\pm$ such that $\chi_t^\pm(x) = \mathbb{E}^x A_t^\pm$. Since for such functionals we have

$$\lim_{t\to 0} \sup_x \mathbb{E}^x A_t^\pm = 0,$$

then estimate (1.2) holds true, and thus the Feynman–Kac semigroup $(T_t^A)_{t\geq 0}$ for $A_t := A_t^+ - A_t^-$ is correctly defined.

To show that the semigroup $(T_t^A)_{t\geq 0}$ can be written as

$$T_t^A f(x) = \int_{\mathbb{R}^n} f(y) p_t^A(x, y)dy, \quad f \in B_b\big(\mathbb{R}^n\big),$$

and to find the representation of the density $p_t^A(x, y)$ in terms of the probability density of the initial process, recall some notions on Markov bridge measures.

Denote by $(\mathcal{F}_t)_{t\geq 0}$ the admissible filtration related to X. A *Markov bridge* $X_t^{x,y}$ of X_t is a Markov processes conditioned by $X_0 = x$ and $X_t = y$. In the proof of [8, Thm. 1], it is shown that under P1 there exists the corresponding Markov bridge measure $\mathbb{P}_{x,y}^t$ on $\mathcal{F}_{t-}$ for (t, x, y) such that $p_t(x, y) > 0$. We denote by $\mathbb{E}_{x,y}^t$ the expectation with respect to $\mathbb{P}_{x,y}^t$.

The next proposition is essentially contained in [8, Thm. 1], but we reformulate the result in the way convenient for our purposes.

Proposition 1. *Let X be a Feller process, admitting the transition probability density $p_t(x, y)$, for which assumption* P1 *holds. Let $\varpi = \varpi^+ - \varpi^-$ be a signed Borel measure, $\varpi^\pm \in S_K$, and $A_t = A_t^+ - A_t^-$, where $A^\pm$ are continuous additive functionals with characteristics* (2.5)*, respectively. Then*

$$T_t^A f(x) = \int_{\{y:\, p_t(x,y)>0\}} f(y) p_t^A(x, y)dy \quad \textit{for any } f \in B_b\big(\mathbb{R}^n\big),$$

where

$$p_t^A(x,y) = p_t(x,y)\mathbb{E}_{x,y}^t e^{A_t}, \quad x, y \in \mathbb{R}^n, t > 0. \tag{2.6}$$

Remark 1. When X is a Brownian motion, the statement of Proposition 1 is known, see [23] and also [3]. The construction from [3, 23] can be extended to the case of a *symmetric* Markov process, see [24]. On the contrary, the construction presented in [8] relies on P1 and does not require the symmetry of the initial process.

Proposition 1 implicitly gives the representation of the function $p_t^A(x,y)$. However, when one wants to get quantitative information about $p_t^A(x,y)$, like the upper bound on $p_t^A(x,y)$, estimation of the expectation $\mathbb{E}_{x,y}^t e^{A_t}$ in (2.6) appears to be non-trivial. Instead, for some class of Feller processes, we can use another approach, which enables us to get explicitly an upper estimate of $p_t^A(x,y)$. Namely, in [18] we formulated the assumptions under which one can construct a Feller process possessing the transition probability density $p_t(x,y)$ satisfying assumption P1 and admitting upper and lower bounds of certain form. In order to make the presentation self-contained, we quote this result below.

Let

$$\mathcal{L}f(x) := a(x)\cdot\nabla f(x) + \int_{\mathbb{R}^n} \big(f(x+u) - f(x) - u\cdot\nabla f(x)\mathbb{1}_{\{\|u\|\le 1\}}\big)m(x,u)\mu(du), \tag{2.7}$$

where $f \in C_\infty^2(\mathbb{R}^n)$, and μ is a Lévy measure, that is, a Borel measure such that

$$\int_{\mathbb{R}^n} \big(\|u\|^2 \wedge 1\big)\mu(du) < \infty.$$

Assume that μ satisfies the following assumption.

A1. There exists $\beta > 1$ such that

$$\sup_{\ell\in\mathbb{S}^n} q^U(r\ell) \le \beta \inf_{\ell\in\mathbb{S}^n} q^L(r\ell) \quad \text{for all } r > 0 \text{ large enough,}$$

where

$$q^U(\xi) := \int_{\mathbb{R}^n} \big[(\xi\cdot u)^2 \wedge 1\big]\mu(du), \qquad q^L(\xi) := \int_{|u\cdot\xi|\le 1} (\xi\cdot u)^2\mu(du). \tag{2.8}$$

Assume that the functions $a(x)$ and $m(x,u)$ in (2.7) satisfy the assumptions A2–A4 given below.

A2. The functions $m(x,u)$ and $a(x)$ are measurable, and satisfy with some constants $b_1, b_2, b_3 > 0$, the inequalities

$$b_1 \le m(x,u) \le b_2, \qquad \big|a(x)\big| \le b_3, \quad x, u \in \mathbb{R}^n.$$

A3. There exist constants $\gamma \in (0,1]$ and $b_4 > 0$ such that

$$\big|m(x,u) - m(y,u)\big| + \big\|a(x) - a(y)\big\| \le b_4\big(\|x-y\|^\gamma \wedge 1\big), \quad u, x, y \in \mathbb{R}^n. \tag{2.9}$$

A4. In the case $\beta > 2$, we assume that $a(x) = 0$ and the kernel $m(x, u)\mu(du)$ is symmetric with respect to u for all $x \in \mathbb{R}^n$.

Denote by $f_{\rm low}$ and $f_{\rm up}$ the functions of the form

$$f_{\rm low}(x) := a_1\big(1 - a_2\|x\|\big)_+, \qquad f_{\rm up}(x) := a_3 e^{-a_4\|x\|}, \quad x \in \mathbb{R}^n, \tag{2.10}$$

where $a_i > 0$, $1 \le i \le 4$, are some constants.

Finally, define $q^*(r) := \sup_{\ell\in\mathbb{S}^n} q^U(r\ell)$, $r > 0$. It was shown in [17] (see also [20]) that condition A1 implies that

$$q^*(r) \ge r^{2/\beta}, \quad r \ge 1.$$

Note also that the continuity of q^U in ξ implies the continuity of q^* in r. Therefore, we can define its generalized inverse

$$\rho_t := \inf\big\{r : q^*(r) = 1/t\big\}, \quad t \in (0, 1]. \tag{2.11}$$

Theorem 2 ([18]). *Under assumptions* A1–A4*, the operator* $(\mathcal{L}, C^2_\infty(\mathbb{R}^n))$ *extends to the generator of a Feller process, admitting a transition probability density* $p_t(x, y)$*. This density is continuous in* $(t, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$*, and there exist constants* $a_i > 0$*,* $1 \le i \le 4$*, and a family of sub-probability measures* $\{Q_t, t \ge 0\}$ *such that*

$$\rho_t^n f_{\rm low}\big((x-y)\rho_t\big) \le p_t(x, y) \le \rho_t^n \big(f_{\rm up}(\rho_t\cdot) * Q_t\big)(x-y), \quad t \in (0, 1],\ x, y \in \mathbb{R}^n, \tag{2.12}$$

where $f_{\rm low}$ *and* $f_{\rm up}$ *are functions of the form* (2.10) *with constants* a_i*, and* ρ_t *is defined in* (2.11).

The constructed process is a *Lévy type* process. In the "constant coefficient case," that is, where $a(x) \equiv$ const and $m(x, u) =$ const, (2.7) is just the representation of the generator of a Lévy process; in other words, a Lévy type process is the process with "locally independent increments." It is known (cf. the Courrège–Waldenfels theorem, see [16, Thm. 4.5.21]) that if the class $C_c^\infty(\mathbb{R}^n)$ of infinitely differentiable compactly supported functions belongs to the domain $D(A)$ of the generator A of a Feller process, then on this set $C_c^\infty(\mathbb{R}^n)$ the operator A coincides with $\mathcal{L}+$"Gaussian component." Thus, the class of processes satisfying the conditions of Theorem 2 is rather wide.

Let us show that, under the conditions of Theorem 2, we have

$$p_t(x, y) > 0 \quad \text{for all } t > 0,\ x, y \in \mathbb{R}^n.$$

We find the minimal N such that the distance from x to y can be covered by N balls of the radius smaller than $(2a_2\rho_{t/N})^{-1}$ (where $a_2 > 0$ is the constant appearing in $f_{\rm low}$ in (2.12)), that is, the minimal N for which

$$\frac{\|x-y\|}{N} \le \frac{1}{a_2\rho_{t/N}}. \tag{2.13}$$

Observe that $q^*(r) \le c_1 r^2$, $r \ge 1$, implying $c_2 t^{-1/2} \le \rho_t$ for all t small enough. Hence, (2.13) holds with $N \ge \frac{(a_2c_2\|x-y\|)^2}{t}$. Therefore, putting $y_0 = x$ and $y_N = y$, we get

$$\begin{aligned} p_t(x,y) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big(\prod_{i=1}^N p_{t/N}(y_{i-1}, y_i) \Big) dy_1 \dots dy_N \\ &\ge \int_{B(y_0,(2a_2\rho_{t/N})^{-1})} \cdots \int_{B(y_{N-1},(2a_2\rho_{t/N})^{-1})} \prod_{i=1}^N p_{t/N}(y_{i-1}, y_i) dy_i \\ &\ge c_0 \rho_{t/N}^{Nn}, \end{aligned}$$

where in the last line we used that

$$p_{t/N}(y_{i-1}, y_i) \ge 2^{-1} a_1 \rho_{t/N}^n \quad \text{for all } y_i \in B\big(y_{i-1}, (2a_2\rho_{t/N})^{-1}\big).$$

Thus, the transition probability density $p_t(x, y)$ is strictly positive.

Finally, for a signed Borel measure ϖ, define

$$h(r) := \sup_x |\varpi| \big\{ y : \|x - y\| \le r \big\}, \tag{2.14}$$

where $|\varpi| := \varpi^+ + \varpi^-$ is the total variation of ϖ. Denote by $\hat{h}$ the Laplace transform of h.

The following theorem is the main result of the paper. Let $t_0 \in (0, 1]$ be small enough.

Theorem 3. *Let X be the Feller process constructed in Theorem 2. Take a signed Borel measure ϖ such that its volume function* (2.14) *satisfies*

$$\int_0^t \rho_s^{n+1} \hat{h}(\rho_s) ds \le C t^\zeta, \quad t \in [0, 1], \tag{2.15}$$

with some constants $C, \zeta > 0$, where ρ_t is given by (2.11). *Then*

a) *There exists a continuous functional A_t such that*

$$\mathbb{E}^x A_t = \int_0^t \int_{\mathbb{R}^n} p_s(x, y) \varpi(dy) ds;$$

b) *The semigroup $(T_t^A)_{t\ge 0}$ is well defined, and its kernel possesses a density $p_t^A(x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^n$;*

c) *There exist constants $a_i > 0$, $1 \le i \le 4$, and a family of sub-probability measures $\{\mathcal{R}_t, t \ge 0\}$ such that for $t \in (0, t_0]$ and $x, y \in \mathbb{R}^n$,*

$$\rho_t^n f_{\text{low}}\big((x-y)\rho_t\big) \le p_t^A(x, y) \le \rho_t^n \big(f_{\text{up}}(\rho_t \cdot) * \mathcal{R}_t\big)(y - x); \tag{2.16}$$

here f_{low} and f_{up} are the function of the form (2.10) *with some constants a_i, $1 \le i \le 4$.*

Remark 2. In general, a_i in estimate (2.16) are *some* constants, that may not coincide with those in estimate (2.12). In order to simplify the notation, we assume that in Theorem 2, $a_1 = a_3 = 1$, $a_2 = a$, and $a_4 = b$.

Assumption (2.15) can be relaxed, provided that more information about the initial transition probability density is available. Put

$$\mathfrak{g}_t(x) := \frac{1}{t^{\frac{n}{\alpha}}(1 + \|x\|/t^{1/\alpha})^{d+\alpha}}, \quad t > 0, \ x \in \mathbb{R}^n. \tag{2.17}$$

Note that for $d = n$, this function is equivalent to the transition probability density of a symmetric α-stable process in $\mathbb{R}^n$ (that is, the process whose characteristic function is $e^{-t\|\xi\|^\alpha}$). Denote by $\mathcal{K}_{n,\alpha}$ the class of Borel signed measures such that

$$\lim_{t\to 0} \sup_x \int_0^t \frac{|\varpi|\{y : \|x - y\| \le s\}}{s^{n+1-\alpha}} ds = 0. \tag{2.18}$$

The following lemma shows that for $d > n - \alpha$ the Kato class of measures with respect to $\mathfrak{g}_t(x - y)$ coincides with $\mathcal{K}_{n,\alpha}$. The proof uses the idea from [4], and will be given in Appendix A.

Lemma 1. *A finite Borel signed measure ϖ belongs to S_K with respect to $\mathfrak{g}_t(x - y)$, given by (2.17) with $d > n - \alpha$, if and only if $|\varpi| \in \mathcal{K}_{n,\alpha}$.*

Corollary 1. *In particular, it follows from Lemma 1 that $\varpi \in S_K$ with respect to the transition probabiility density of a symmetric α-stable process if an only if $\varpi \in \mathcal{K}_{n,\alpha}$.*

In the proposition below, we state the "compact" upper bound for $p_t^A(x, y)$.

Proposition 4. *Let X be a Feller process satisfying the conditions of Proposition 1, and in addition assume that the transition density $p_t(x, y)$ of X is such that for all $t \in (0, 1]$, $x, y \in \mathbb{R}^n$, the inequality*

$$p_t(x, y) \le c\mathfrak{g}_t(x - y), \quad t \in (0, 1], \ x, y \in \mathbb{R}^n, \tag{2.19}$$

where the function $\mathfrak{g}_t(x)$ is defined in (2.17) with $d > n - \alpha$. Suppose that $\varpi \in \mathcal{K}_{n,\alpha}$. Then

$$p_t^A(x, y) \le C\mathfrak{g}_t(x - y), \quad t \in (0, 1], \ x, y \in \mathbb{R}^n. \tag{2.20}$$

Remark 3. a) For X being a symmetric α-stable-like process, such a result is known, see [22]. In particular, the upper bound (2.20) holds with $n = d$. In our case, X is from a wider class; in particular, we do not assume the symmetry of the initial process, and the method of constructing the Feynman–Kac semigroup is completely different.

b) In view of Lemma 1, under the assumptions of this proposition, we can take $\varpi \in \mathcal{K}_{n,\alpha}$ rather than $\varpi \in S_K$ with respect to $\mathfrak{g}_t$, which is more convenient for usage.

In Section 5, we provide examples that illustrate Theorem 3 and Proposition 4.

2.1 Discussion and overview

1. *On continuous additive functionals.* Loosely speaking, there are two approaches for constructing continuous additive functionals. One approach, which we described previously, relies on the Dynkin theory of W-functionals. Another approach, based on the Dirichlet form technique, establishes the one-to-one correspondence between the class of positive continuous additive functionals and the class of *smooth measures*, see [14, Lemmas 5.1.7, 5.1.8] or [15, Thm. 5.1.4] in the case when the process under consideration is symmetric; see also [21, Thm. 2.4] for the non-symmetric case. In this paper, we use Dynkin's approach as more appropriate in our situation, in particular, we do not assume that the initial Markov process X is symmetric. Our standard reference in this paper is [13].

2. *On the generator of* $(T_t^A)_{t\geq 0}$. Suppose that the Markov process X and the *positive* functional A_t are as in Proposition 1. In this case, the semigroup $(T_t^A)_{t\geq 0}$ is contractive, and thus there exists a sub-Markov process with transition sub-probability density $p_t^A(x,y)$. Formally, we can describe the generator of $(T_t^A)_{t\geq 0}$ as
$$\mathcal{L}^A = \mathcal{L} - \varpi, \tag{2.21}$$
where $\mathcal{L}$ is the generator of the semigroup associated with X, and ϖ is the measure appearing in the characteristic of A_t (cf. (2.4)), see [13, Thms. 9.5, 9.6] for the (equivalent) formulation. Nevertheless, in this framework the problem of defining the domain $D(\mathcal{L}^A)$ of $\mathcal{L}^A$ still remains open. In the general case, that is, when A can attain negative values, in order to define the generator of (non-contractive) semigroup $(T_t^A)_{t\geq 0}$, we can use the quadratic form approach, see [1, 2], and also [9].

3 Proof of Theorem 3

3.1 Proof of statements a) and b)

a) By the upper bound in (2.12) on $p_t(x,y)$ (see also Remark 2), (2.15) implies that $\varpi \in S_K$:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}\int_0^t\int_{\mathbb{R}^n} p_s(x,y)|\varpi|(dy)ds \\
&\quad\leq \sup_{x\in\mathbb{R}^n}\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_s^n f_{\mathrm{up}}\big((y-x-z)\rho_s\big)Q_s(dz)|\varpi|(dy)ds \\
&\quad\leq b\sup_{x\in\mathbb{R}^n}\int_0^t\int_{\mathbb{R}^n}\int_0^\infty \rho_s^n|\varpi|\big\{y:\ \|y-x-z\|\leq v/\rho_s\big\}e^{-bv}dv\,Q_s(dz)ds \\
&\quad\leq b\int_0^t \rho_s^{n+1}\hat{h}(b\rho_s)ds \to 0, \quad t\to 0.
\end{aligned}$$

Hence, applying [13, Thm. 6.6], we derive the existence of a continuous functional A_t with claimed characteristic.

Statement b) is already contained in Proposition 1.

3.2 Outline of the proof of c)

For the proof of Theorem 3(c), we use the Duhamel principle. First, we show that the function $p_t^A(x, y)$ satisfies the integral equation

$$p_t^A(x, y) = p_t(x, y) + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(x, z) p_s^A(z, y) \varpi(dz) ds, \tag{3.1}$$

provided that the integral on the right-hand side converges. We show that if the series

$$\pi_t(x, y) := \sum_{k=1}^{\infty} p_t^{\diamondsuit k}(x, y) \tag{3.2}$$

converges, then it satisfies Eq. (3.1). We derive an upper estimate for the convolutions $p_t^{\diamondsuit k}(x, y)$, which guarantees the absolute convergence of the series and allows to find the upper estimate for $\pi_t(x, y)$.

Second, we show that on $(0, t_0] \times \mathbb{R}^n \times \mathbb{R}^n$ the solution (3.2) to (3.1) is unique in the class of non-negative functions $\{f(t, x, y) \ge 0,\ t \in (0, t_0],\ x, y \in \mathbb{R}^n\}$ such that

$$\int_{\mathbb{R}^n} f(t, x, y) dy \le C \quad \text{for all } t \in (0, t_0],\ x \in \mathbb{R}^n. \tag{3.3}$$

We use the standard method, based on the Gronwall–Bellman inequality.

Finally, observe that the kernel $p_t^A(x, y)$ of T_t^A belongs to the class of functions satisfying (3.3). Indeed, since for A_t we have (1.2), it follows that

$$\left|T_t^A f(x)\right| \le c_1 \mathbb{E}^x e^{|A_t|} \le c_2, \quad f \in B_b\big(\mathbb{R}^n\big),\ x \in \mathbb{R}^n,\ t \in (0, t_0]. \tag{3.4}$$

Thus, $p_t^A(x, y) \equiv \pi_t(x, y)$ on $(0, t_0] \times \mathbb{R}^n \times \mathbb{R}^n$.

Before we prove that (3.2) is the solution to Eq. (3.1) on $(0, t_0] \times \mathbb{R}^n \times \mathbb{R}^n$, let us discuss a simple case when ϖ is the Lebesgue measure on $\mathbb{R}^n$. In this case $h(r) = c_n r^n$, and thus assumption (2.15) is satisfied:

$$\int_0^t \rho_s^{n+1} \hat{h}(\rho_s) ds = c_n t.$$

Therefore, the procedure of estimation of convolutions reduces to those treated in [18, Lemmas 3.1, 3.2].

Rewrite the upper bound in (2.12) as

$$p_t(x, y) \le C_1 t^{-1/2} \big(g_t^{(1)} * Q_t\big)(y - x), \tag{3.5}$$

where $C_1 > 0$ is some constant,

$$g_t^{(1)}(x) := t^{1/2} g_t(x), \tag{3.6}$$

and (cf. Remark 2)

$$g_t(x) := \rho_t^n f_{\text{up}}(\rho_t x) = \rho_t^n e^{-b\rho_t |x|}. \tag{3.7}$$

This modification is technical, but proves to be useful for estimating the convolutions $p_t^{\diamond k}(x,y)$. Let us estimate $p_t^{\diamond k}(x,y)$. Take now a sequence $(\theta_k)_{k\geq 1}$ such that $0 < \theta_{k+1} < \theta_k$, $\theta_1 = 1$, and put

$$g_t^{(k)}(x) := t^{k/2} g_t(\theta_k x), \quad k \geq 1. \tag{3.8}$$

Since ρ_t is monotone decreasing, for $0 < s < \frac{t}{2}$, we have $\rho_{t-s} \leq \rho_{t/2}$. Note that $\rho_t \asymp \rho_{t/2}$; this follows from condition A1 and the definition of ρ_t; see [20] for the detailed proof. Then, for $0 < s < t/2$,

$$\begin{aligned}\left(g_{t-s}^{(k-1)} * g_s^{(1)}\right)(x) &\leq t^{k/2} \int_{\mathbb{R}^n} g_{t-s}(\theta_{k-1}x - \theta_{k-1}y) g_s(\theta_{k-1}y) dy \\ &= t^{k/2} \theta_{k-1}^{-n} \int_{\mathbb{R}^n} g_{t-s}(\theta_{k-1}x - y) g_s(y) dy \\ &\leq t^{k/2} \theta_{k-1}^{-n} \int_{\mathbb{R}^n} \rho_{t-s}^n \rho_s^n e^{-\frac{b\rho_t \theta_k}{\theta_{k-1}}(|\theta_{k-1}x - z| + |z|) - b\rho_s(1 - \frac{\theta_k}{\theta_{k-1}})|z|} dz \\ &\leq c_1 t^{k/2} \theta_{k-1}^{-n} \rho_t^n e^{-b\rho_t \theta_k |x|} \int_{\mathbb{R}^n} \rho_t e^{-b\rho_s(1 - \frac{\theta_k}{\theta_{k-1}})|z|} dz \\ &= D_k g_t^{(k)}(x), \end{aligned} \tag{3.9}$$

where $D_k = c(\theta_{k-1} - \theta_k)^{-n}$, $c = c_1 \int_{\mathbb{R}^n} e^{-b|z|} dz$, and in the second line from below, we used the triangle inequality and monotonicity of ρ_t. In the case $t/2 \leq s \leq t$, calculation is similar.

By induction we can get

$$\left|p_t^{\diamond k}(x,y)\right| \leq C_k t^{\frac{k}{2}-1} \left(g_t^{(k)} * Q_t^{(k)}\right)(y - x), \quad k \geq 2, \tag{3.10}$$

where

$$C_k := c^{k-1} C_1^k \frac{\Gamma^k(1/2)}{\Gamma(k/2)} \prod_{j=2}^{k} \frac{1}{(\theta_{j-1} - \theta_j)^n},$$

and for $k \geq 2$

$$Q_t^{(k)}(dw) := \frac{1}{B(\frac{k-1}{2}, \frac{1}{2})} \int_0^1 \int_{\mathbb{R}} (1-r)^{(k-1)/2-1/2} r^{-1/2} Q_{t(1-r)}^{(k-1)}(dw - u) Q_{tr}^{(1)}(du) dr.$$

Since $\{Q_t^{(k)},\ t > 0,\ k \geq 1\}$ is the sequence of sub-probability measures and $g_t^{(k)}(x) \leq \rho_t^n t^{k/2}$, we obtain

$$\left|p_t^{\diamond k}(x,y)\right| \leq C_k t^{k-1} \rho_t^n.$$

Thus, to show the absolute convergence of the series $\sum_{k=1}^{\infty} p_t^{\diamond k}(x,y)$, we may check that $\sum_{k=1}^{\infty} C_k < \infty$. However, the behaviour of C_k as $k \to \infty$ is rather complicated. To see this, take, for example, $\theta_k = \frac{1}{2} + \frac{1}{2k}$. Then

$$C_k = c^{k-1} C_1^k \frac{\Gamma^k(1/2)}{\Gamma(k/2)} \left(2^k k!(k-1)!\right)^n,$$

and thus C_k explodes as $k \to \infty$. Therefore, this procedure of estimation of convolutions is too rough, and needs to be modified. For this, we change the estimation procedure after some finite number of steps; this allows us to control the decay of coefficients and, in such a way, to prove that $\sum_{k=1}^{\infty} p_t^{\diamondsuit k}(x, y) < \infty$.

In the next subsection, we handle the general case, in particular,

- We give the generic calculation, which allows us to estimate the convolution $(g_{t-s} \square g_s)(x)$;
- We estimate the convolutions $p_t^{\diamondsuit k}(x, y)$, $k \geq 2$;
- We change the estimation procedure after k_0 steps, where k_0 is properly chosen, and estimate $p_t^{\diamondsuit(k_0+\ell)}(x, y)$, $\ell \geq 1$.

The change of the estimation procedure could be unnecessary if we would know that $p_t(x, y)$ possesses a more regular upper bound than (2.12). In this case, we obtain a sufficient control on the coefficients C_k, $k \geq 1$. This is exactly the case under the conditions of Proposition 4.

3.3 Representation lemma, generic calculation, and estimation of convolutions

Lemma 2. *The function* $p_t^A(x, y)$ *given by* (2.6) *satisfies Eq.* (3.1).

Proof. In the case when X is a symmetric stable-like process and $\varpi \in S_K$ with respect to the transition probability density of X, the sketch of the proof is given in [22]. In the general case, the proof is the same; in order to make the presentation self-contained, we present it below. Using the equality

$$e^{A_t} = \int_0^t e^{A_t - A_s} dA_s + 1,$$

the strong Markov property of X, and the additivity of A_t (cf. (2.2)), we write

$$\begin{aligned} T_t^A f(x) &= \mathbb{E}^x \left[f(X_t) e^{A_t} \right] \\ &= \mathbb{E}^x f(X_t) + \mathbb{E}^x \left[\int_0^t \left[f(X_t) e^{A_t - A_s} \right] dA_s \right] \\ &= \mathbb{E}^x f(X_t) + \mathbb{E}^x \left[\int_0^t \mathbb{E}^{X_s} \left[f(X_{t-s}) e^{A_{t-s}} \right] dA_s \right] \\ &= \mathbb{E}^x f(X_t) + \mathbb{E}^x \int_0^t T_{t-s}^A f(X_s) dA_s. \end{aligned}$$

Observe that for $f \in B_b(\mathbb{R}^n)$, we have

$$\mathbb{E}^x \int_0^t f(X_s) dA_s = \int_0^t \int_{\mathbb{R}^n} f(y) p_s(x, y) \varpi(dy) ds. \tag{3.11}$$

Indeed, since $\chi_t = \chi_t^+ - \chi_t^-$ with $\chi_t^{\pm}$ given by (2.5) is the characteristic of A_t, Eq. (3.11) holds for a finite linear combination of indicators. Approximating $f \in B_b(\mathbb{R}^n)$ by such linear combinations and passing to the limit, we get (3.11). $\square$

For $\theta \in [0,1]$, put

$$g_{t,\theta}(x) := g_t(\theta x), \tag{3.12}$$

where $g_t(x)$ is defined in (3.7), and

$$\phi_\nu(s) := \rho_s^{n+1}\hat{h}(\nu\rho_s), \quad \nu > 0, \tag{3.13}$$

where h is the volume function (cf. (2.14)) appearing in condition (2.15). Lemma below gives the generic calculation, needed for the proof of Theorem 3.

Lemma 3. *For $\theta \in (0,1)$, we have*

$$(g_{t-s} \,\square\, g_s)(x) \le C\big[\phi_{(1-\theta)b}(t-s) + \phi_{(1-\theta)b}(s)\big]g_{t,\theta}(x), \quad x \in \mathbb{R}^n,\ 0 < s < t \le 1, \tag{3.14}$$

where $C > 0$ is some constant, independent of θ, and $b > 0$ comes from the definition of g_t, see (3.7).

Proof. Take $\theta \in (0,1)$. Since by definition the function ρ_t is decreasing, we have

$$\|x-z\|\rho_{t-s} + \|z\|\rho_s \ge \|x\|\rho_t,$$

which implies

$$\begin{aligned}
&(g_{t-s} \,\square\, g_s)(x)\\
&\quad \le e^{-\theta b\|x-y\|\rho_t}\rho_{t-s}^n\rho_s^n \int_{\mathbb{R}^n}\big[f_{\mathrm{up}}\big((z-x)\rho_{t-s}\big)f_{\mathrm{up}}\big((y-z)\rho_s\big)\big]^{(1-\theta)}|\varpi|(dz).
\end{aligned}$$

By integration by parts we derive, using that ρ_t is monotone decreasing, that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho_{t-s}^n\rho_s^n\big[f_{\mathrm{up}}\big((x-z)\rho_{t-s}\big)f_{\mathrm{up}}\big((z-y)\rho_s\big)\big]^{1-\theta}|\varpi|(dz)\\
&\quad \le \rho_{t/2}^n\int_{\mathbb{R}^n}\rho_s^n f_{\mathrm{up}}^{1-\theta}\big((z-y)\rho_s\big)|\varpi|(dz)\\
&\quad \le c_1\rho_t^n\rho_s^n\int_0^\infty|\varpi|\big\{z : e^{-b(1-\theta)\|z-y\|\rho_s} \ge e^{-v}\big\}e^{-v}dv\\
&\quad = (1-\theta)bc_1\rho_t^n\rho_s^n\int_0^\infty|\varpi|\big\{z : \|z-y\| \le v/\rho_s\big\}e^{-b(1-\theta)v}dv\\
&\quad \le (1-\theta)bc_1\rho_t^n\rho_s^n\int_0^\infty h(v/\rho_s)e^{-b(1-\theta)v}dv\\
&\quad = c_1\rho_t^n\rho_s^{n+1}\hat{h}\big(b(1-\theta)\rho_s\big)\\
&\quad = c_1\rho_t^n\phi_{b(1-\theta)}(s).
\end{aligned} \tag{3.15}$$

Similar estimate holds true for $s > \frac{t}{2}$, which finishes the proof of (3.14). □

Take a sequence $(\theta_k)_{k\ge1}$ such that

$$\theta_1 = 1, \qquad \theta_k > 0, \qquad \theta_{k-1} > \theta_k, \quad k \ge 2. \tag{3.16}$$

Let

$$k_0 := \left[\frac{n}{\alpha\zeta}\right], \tag{3.17}$$

where ζ is the parameter appearing in (2.15). Define

$$\kappa := \min\{b(\theta_{j-1}-\theta_j),\ 1 \le j \le k_0\}, \tag{3.18}$$

$$F(t) := \int_0^t \phi_\kappa(r)dr, \tag{3.19}$$

and

$$\tilde g_t^{(k)}(x) := \begin{cases} g_{t,\theta_k}(x)F^{k-1}(t), & 1 \le k \le k_0,\\ e^{-b\theta_{k_0}\rho_t\|x\|}F^{k-k_0}(t), & k > k_0, \end{cases} \tag{3.20}$$

where $g_{t,\theta}(x)$ is defined in (3.12).

Finally, define inductively the sequence of measures

$$\mathcal{R}_t^{(1)}(dw) := Q_t(dw) \quad \text{if } k = 1,$$

$$\mathcal{R}_t^{(k)}(dw) := \big(2F(t)\big)^{-1}\int_0^t\int_{\mathbb{R}^n}\big[\phi_\kappa(t-s)+\phi_\kappa(s)\big]Q_{t-s}(dw-u)\mathcal{R}_s^{(k-1)}(du)ds \tag{3.21}$$

if $k \ge 2$. Since $(Q_t)_{t\ge 0}$ is the family of sub-probability measures (see Theorem 2), we have

$$\mathcal{R}_t^{(2)}\big(\mathbb{R}^n\big) \le \big(2F(t)\big)^{-1}\int_0^t\big[\phi_\kappa(t-s)+\phi_\kappa(s)\big]Q_{t-s}\big(\mathbb{R}^n\big)Q_s\big(\mathbb{R}^n\big)ds \le 1,$$

and we can see by induction that $\mathcal{R}_t^{(k)}(\mathbb{R}^n) \le 1$, $t \in [0,1]$, for all $k \ge 2$.

Lemma 4. *For $k \ge 2$ we have*

$$\big|p_t^{\diamond k}(x,y)\big| \le \tilde C_k\big(\tilde g_t^{(k)} * \mathcal{R}_t^{(k)}\big)(y-x), \quad x,y \in \mathbb{R}^n,\ t \in (0,1], \tag{3.22}$$

where the sequence $(\tilde g_t^{(k)})_{k\ge 1}$ is given by (3.20), $\mathcal{R}_t^{(k)}$ is defined in (3.21), $k \ge 2$, and for $k > k_0$, the constants $\tilde C_k$ can be expressed as

$$\tilde C_k = C^{k-k_0}M,$$

where $M, C > 0$ are some constants.

Proof. We use induction. Rewrite the upper estimate on $p_t(x,y)$ in the form (3.5). For $k = 2$ we get, using (3.5) and (3.15), the following estimates:

$$\begin{aligned}\big|p_t^{\diamond 2}(x,y)\big| &\le C_1^2\int_0^t\int_{\mathbb{R}^{2n}}\Big[\int_{\mathbb{R}^n}\tilde g_s^{(1)}(z-x-w_1)\tilde g_{t-s}^{(1)}(y-z-w_2)|\varpi|(dz)\Big]\\ &\quad\cdot Q_{t-s}(dw_1)Q_s(dw_2)ds\\ &\le C_2\int_{\mathbb{R}^n} g_{t,\theta_2}(x-w)\Big\{\int_0^t\big[\phi_{b(\theta_1-\theta_2)}(t-s)+\phi_{b(\theta_1-\theta_2)}(s)\big]\\ &\quad\cdot\int_{\mathbb{R}^n}Q_{t-s}(dw-u)Q_s(du)ds\Big\}\end{aligned}$$

$$
\begin{aligned}
&\le C_2 \int_{\mathbb{R}^n} g_{t,\theta_2}(x-w)\Big\{\int_0^t \big[\phi_\kappa(t-s)+\phi_\kappa(s)\big] \\
&\quad \cdot \int_{\mathbb{R}^n} Q_{t-s}(dw-u)Q_s(du)ds\Big\} \\
&\le 2C_2 F(t)\big(g_{t,\theta_2} * \mathcal{R}^{(2)}\big)(y-x) \\
&= 2C_2\big(\tilde{g}_t^{(2)} * \mathcal{R}^{(2)}\big)(y-x), \qquad (3.23)
\end{aligned}
$$

where $C_1 > 0$ comes from (3.5), and in the third line from below we used that by the definition of κ and monotonicity of ϕ_ν in ν,

$$
\phi_{b(\theta_{j-1}-\theta_j)}(t) \le \varphi_\kappa(t), \qquad t \in (0,1].
$$

Suppose that (3.22) holds for some $2 \le k \le k_0$. Then

$$
\begin{aligned}
\big|p_t^{\Diamond(k+1)}(x,y)\big| &\le 2^{k-1}C_kC_1 \int_0^t \int_{\mathbb{R}^n} \big(\tilde{g}_{t-s}^{(1)} * Q_{t-s}\big)(z-x) \\
&\quad \cdot \big(\tilde{g}_s^{(k)} * \mathcal{R}_s^{(k)}\big)(y-z)dzds \\
&= 2^{k-1}C_kC_1 \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\tilde{g}_{t-s}^{(1)} \square \tilde{g}_s^{(k)}\big)(y-x-w_1-w_2) \\
&\quad \cdot Q_{t-s}(dw_1)\mathcal{R}_s^{(k)}(dw_2)ds. \qquad (3.24)
\end{aligned}
$$

By the same argument as those used in the proof of Lemma 3, we have

$$
\begin{aligned}
\big(\tilde{g}_{t-s}^{(1)} \square \tilde{g}_s^{(k)}\big)(x) &\le (g_{t-s,\theta_k} \square g_{s,\theta_k})(x)F^{k-1}(t) \\
&\le c_{k+1} g_{t,\theta_{k+1}}(x)F^{k-1}(t)\big[\phi_{b(\theta_{k-1}-\theta_k)}(t-s)+\phi_{b(\theta_{k-1}-\theta_k)}(s)\big] \\
&= c_{k+1}\big(F(t)\big)^{-1}\big[\phi_\kappa(t-s)+\phi_\kappa(s)\big]\tilde{g}_t^{(k+1)}(x).
\end{aligned}
$$

Substituting this estimate into (3.27), performing the change of variables and normalizing, we get (3.22) for $2 \le k \le k_0$.

Take $c_0 > 0$. Note that for some $c_1 > 0$, we have $c_0\rho_t \le \rho_{c_1 t}$, $t \in (0,1]$. Then, by (2.15),

$$
\int_0^t \rho_t^{n+1}\hat{h}(c_0\rho_t)dt \le c_2 \int_0^t \rho_{c_1 t}^{n+1}\hat{h}(\rho_{c_1 t})dt \le c_3 \int_0^{c_1 t} \rho_t^{n+1}\hat{h}(\rho_t)dt \le c_4 t^\zeta.
$$

Therefore, taking k_0 as in (3.17), we get

$$
\rho_t^n F^{k_0}(t) \le c_5 t^{-n/\alpha + k_0\zeta} \le c_6, \qquad t \in [0,1]. \qquad (3.25)
$$

In such a way, on the (k_0+1)-th step, we obtain

$$
\begin{aligned}
\big(\tilde{g}_{t-s}^{(k_0)} \square \tilde{g}_s^{(1)}\big)(x) &\le c e^{-b\theta_{k_0}\rho_t\|x\|} \int_{\mathbb{R}^n} e^{-b\rho_s(1-\theta_{k_0})\|z-x\|}|\varpi|(dz) \\
&= c e^{-b\theta_{k_0}\rho_t\|x\|} \int_0^\infty |\varpi|\big\{z : \rho_s b(1-\theta_{k_0})\|z-x\| \le r\big\}e^{-r}dr
\end{aligned}
$$

$$\leq ce^{-b\theta_{k_0}\rho_t\|x\|}\phi_{b(1-\theta_{k_0})}(s)$$
$$\leq c\tilde{g}_t^{(k_0+1)}(x)\phi_\kappa(s)F^{-1}(t)$$

(cf. (3.15)), where in the last line we used the inequality $\kappa < b(1-\theta_{k_0})$ and the monotonicity of ϕ_ν in ν. Using this estimate, we derive

$$p_t^{\diamondsuit(k_0+1)}(x,y) \leq C_{k_0}C_1 \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\tilde{g}_{t-s}^{(k_0)}\square\tilde{g}_s^{(1)}\big)(y-x-w_1-w_2)$$
$$\cdot\, Q_s(dw_1)\mathcal{R}_{t-s}^{(k_0)}(dw_2)ds$$
$$\leq 2cC_1C_{k_0}\cdot\big(\tilde{g}_t^{(k_0+1)} * \mathcal{R}_t^{(k_0+1)}\big)(y-x). \tag{3.26}$$

Then (3.22) follows by induction. Indeed, assume that (3.22) holds for $k = k_0+\ell-1$. For $\ell \geq 2$ we get

$$\big(\tilde{g}_{t-s}^{(k_0+\ell-1)}\square\,\tilde{g}_s^{(1)}\big)(x) \leq cF^{\ell-1}(t)e^{-b\theta_{k_0}\rho_t\|x\|}\phi_\kappa(s) = cF^{-1}(t)\tilde{g}_t^{(k_0+\ell)}(x)\phi_\kappa(s).$$

Therefore,

$$\big|p_t^{\diamondsuit(k_0+\ell)}(x,y)\big| \leq (2C_1c)^{\ell-1}C_1C_{k_0}\int_0^t\int_{\mathbb{R}^n}\big(\tilde{g}_{t-s}^{(k_0+\ell-1)} * \mathcal{R}_{t-s}^{(k_0+\ell-1)}\big)(z-x)$$
$$\cdot\,\big(\tilde{g}_s^{(1)} * Q_s\big)(y-z)dzds$$
$$= C_{k_0}(2C_1c)^\ell\big(\tilde{g}_t^{(k_0+\ell)} * \mathcal{R}_t^{(k_0+\ell)}\big)(y-x). \tag{3.27}$$

□

Remark 4. As we observed in the proof, the estimation procedure depends on condition H1, which guarantees the existence of the number k_0 such that (3.25) holds. In general, without H1 we cannot guarantee the existence of such a number, which is crucial in our approach. For example, suppose that $\rho_s \asymp s^{-1}$ for small s, and take the measure ϖ such that

$$h(r) \asymp \frac{1}{\ln^2 r}, \quad r \in (0,1].$$

By the Tauberian theorem, we have $\hat{h}(\lambda) \asymp [\lambda \ln^2\lambda]^{-1}$ for large λ. Therefore, $\phi_\nu(t) \sim |\ln t|^{-1}$ as $t \to 0$, and thus the integral $F(t)$ diverges. Nevertheless, assumption H1 can be dropped, if the function $p_t(x,y)$ possesses a more precise upper bound. We discuss this question later in Section 4.

3.4 Proof of statement c)

From (3.27) we get for all $x, y \in \mathbb{R}^n$,

$$\big|p_t^{\diamondsuit(k_0+\ell)}(x,y)\big| \leq M\big(CF(t)\big)^\ell, \quad \ell \geq 1, \tag{3.28}$$

where $M = C_{k_0}$ and $C = 2C_1c$. Without loss of generality, assume that $C \geq 1$. Since $F(t) \to 0$ as $t \to 0$, there exists $t_0 > 0$, such that

$$CF(t) < 1/2, \quad t \in (0, t_0]. \tag{3.29}$$

Thus, for $t \in (0, t_0]$, the series (3.2) converges absolutely and is the solution to (3.1).

Let us show that the integral equation (3.1) possesses a unique solution in the class of functions $\{f(t,x,y)\ge 0,\ t\in(0,t_0],\ x,y\in\mathbb{R}^n\}$, such that

$$\int_{\mathbb{R}^n} f(t,x,y)dy \le c, \quad t\in(0,t_0],\ x\in\mathbb{R}^n. \tag{3.30}$$

Then the series (3.2) is a unique representation of the Feynman–Kac kernel $p_t^A(x,y)$ for $t\in(0,t_0]$, $x,y\in\mathbb{R}^n$.

Suppose that there are two solutions $p_t^{(1),A}(x,y)$ and $p_t^{(2),A}(x,y)$ to (3.1). Put $\tilde p_t^A(x,y) := |p_t^{(1),A}(x,y) - p_t^{(2),A}(x,y)|$ and $v_t(x) := \int_{\mathbb{R}^n} \tilde p_t^A(x,y)dy$. Then, by (3.1) we have

$$v_t(x) \le \int_0^t \int_{\mathbb{R}^n} p_{t-s}(x,z)v_s(z)\varpi(dz)ds. \tag{3.31}$$

By induction we get

$$v_t(x) \le \int_0^t \int_{\mathbb{R}^n} p_{t-s}^{\diamondsuit k}(x,z)v_s(z)\varpi(dz)ds. \tag{3.32}$$

Note that there exists $c>0$ such that $p_t^{\diamondsuit(k_0+1)}(x,y)\le c$ for all $t\in(0,t_0]$, $x,y\in\mathbb{R}^n$ (cf. (3.26)). In such a way, by the finiteness of measure ϖ, we get

$$v_t(x) \le c_1 \int_0^t \int_{\mathbb{R}^n} v_s(z)\varpi(dz)ds \le c_2 \int_0^t \tilde v_s ds, \tag{3.33}$$

where $\tilde v_s := \sup_{z\in\mathbb{R}^n} v_s(z)$. Taking $\sup_{x\in\mathbb{R}^n}$ in the left-hand side of (3.33), we derive

$$\tilde v_t \le c_2 \int_0^t \tilde v_s ds, \quad t\in(0,t_0]. \tag{3.34}$$

Applying the Gronwall–Bellman lemma, we derive $\tilde v_t \equiv 0$ for all $t\in(0,t_0]$. Thus, the solution to (3.1) is unique in the class of functions

$$\left\{f(t,x,y)\ge 0,\ t\in(0,t_0],\ x,y\in\mathbb{R}^n\right\}$$

satisfying (3.30).

Estimating series (3.2) from above, we get an upper bound in (2.16) with f_{up} of the form (2.10) and

$$\mathcal{R}_t(dw) = c_0 \sum_{k\ge 1} c^k \mathcal{R}_t^{(k)}(dw),$$

with some $c\in(0,1)$ and the normalizing constant $c_0>0$ chosen so that $\mathcal{R}_t(\mathbb{R}^n)\le 1$ for all $t\in(0,t_0]$.

For the lower bound, observe that by (3.20) we have

$$\left|p_t^{\diamondsuit k}(x,y)\right| \le C(k_0)\rho_t^n F(t), \quad 2\le k\le k_0. \tag{3.35}$$

By (3.28) and (3.29) we get

$$\sum_{\ell\ge 1} p_t^{\diamondsuit(k_0+\ell)}(x,y) \le 2MCF(t), \quad t\in(0,t_0],$$

which, together with (3.35) and the observation that ρ_t is decreasing, yields the estimate

$$\Big|\sum_{k=2}^{\infty} p_t^{\Diamond k}(x,y)\Big| \le C_0 F(t)\rho_t^n, \quad t\in(0,t_0], \tag{3.36}$$

where $C_0>0$ is some constant. Therefore, choosing t_0 small enough, we have by the lower bound in (2.12) the inequalities

$$\begin{aligned} p_t^A(x,y) &\ge \rho_t^n f_{\text{low}}\big((y-x)\rho_t\big) - C_0 F(t)\rho_t^n \\ &\ge c\rho_t^n f_{\text{low}}\big((y-x)\rho_t\big), \quad t\in(0,t_0]. \end{aligned} \tag{3.37}$$

□

4 Proof of Proposition 4

Since the proof of the proposition follows with minor changes from the proof of the upper estimate in [22, Thm. 3.3], we only sketch the argument. For $(t,x,y)\in(0,t_0]\times\mathbb{R}^n\times\mathbb{R}^n$, put

$$I_0(t,x,y) := \mathfrak{g}_t(x-y), \quad I_k(t,x,y) = \int_0^t\int_{\mathbb{R}^n} \mathfrak{g}_{t-s}(x-z) I_{k-1}(s,z,y)\varpi(dz)ds.$$

By the same argument as in [22], we can get

$$\big|I_k(t,x,y)\big| \le c^k \mathfrak{g}_t(y-x), \quad k\ge 1,\ t\in(0,t_0],$$

where $c\in(0,1)$ is some constant. Thus, for $k\ge1$, we have

$$\big|p_t^{\Diamond k}(x,y)\big| \le c^k \mathfrak{g}_t(x-y), \quad x,y\in\mathbb{R}^n,\ t\in(0,t_0]. \tag{4.1}$$

This proves the convergence of the series (3.2) and the upper estimate (2.20). □

Remark 5. Let us briefly discuss the crucial difference between the proofs of Theorem 3 and Proposition 4. We changed the procedure of estimation of $p_t^{\Diamond k}(x,y)$ after a certain step, which was possible due to (2.15). In the case when we have a single-kernel estimate for $p_t(x,y)$, for example, (2.19), we can drop condition (2.15). In fact, it is enough to require that $\varpi\in S_K$ with respect to $\mathfrak{g}_t(y-x)$. This happens because in the case of the single-kernel estimate of type (2.19), it is possible to show that the convolutions $p_t^{\Diamond k}(x,y)$ satisfy the upper bound (4.1) with $c\in(0,1)$, which implies the convergence of the series (3.2).

5 Examples

As one might observe, the scope of applicability of Theorem 3 heavily relies on the properties of the initial process X. To assure the existence of such a process, we applied Theorem 2. Below we give the examplesin which condition A1 is satisfied. Since conditions A2–A4 are easy to check, we may assume that the functions $a(x)$ and $m(x,u)$ are appropriate. We confine ourselves to the case when the measure μ in the generator of X is "discretized α-stable; up to the author's knowledge, in this case the corresponding Feynman–Kac semigroup was not investigated. Examples below illustrate that our approach is applicable also in the situation when the "Lévy-type measure" $m(x,u)\mu(du)$ related to the initial process X is not absolutely continuous with respect to the Lebesgue measure.

Example 1. a) Consider a "discretized version" of an α-stable Lévy measure in $\mathbb{R}^n$. Let $m_{k,\upsilon}(dy)$ be the uniform distribution on a sphere $\mathbb{S}_{k,\upsilon}$ centered at 0 with radius $2^{-k\upsilon}$, $\upsilon > 0$, $k \in \mathbb{Z}$. Consider the Lévy measure

$$\mu(dy) = \sum_{k=-\infty}^{\infty} 2^{k\gamma} m_{k,\upsilon}(dy), \tag{5.1}$$

where $0 < \gamma < 2\upsilon$. In [17], it is shown that for such a Lévy measure condition A1 is satisfied, and

$$\rho_t \asymp t^{-1/\alpha}, \quad t \in (0,1], \tag{5.2}$$

where $\alpha = \gamma/\upsilon$.

Take some functions $a(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and a non-negative bounded function $m(\cdot,\cdot)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying assumptions A2–A4. By Theorem 2 the operator of the form (2.7) with μ, $a(x)$, and $m(x,u)$ as before can be extended to the generator of a Feller process X that admits the transition density $p_t(x,y)$ satisfying (2.12).

Let ϖ be a finite Borel measure, and let h be its volume function, see (2.14). Let us show that if the inequality

$$\int_0^t \frac{h(v)}{v^{n+1-\alpha}} dv \le c_1 t^{\zeta}, \quad t \in (0,1], \tag{5.3}$$

for some $\zeta > 0$, then we have (2.15). Using (5.2), changing variables, and applying the Fubini theorem, we derive

$$\begin{aligned} \int_0^t \rho_s^{n+1} \hat{h}(\rho_s) ds &\le \int_0^t s^{-\frac{n+1}{\alpha}} \hat{h}\big(c_2 s^{-1/\alpha}\big) ds \\ &= \alpha \int_0^\infty \bigg[\int_0^{t^{1/\alpha} v} \frac{h(\tau)}{\tau^{n+1-\alpha}} d\tau\bigg] v^{n-\alpha} e^{-c_2 v} dv. \end{aligned}$$

Denote by $I(t)$ the right-hand side in this expression. Applying (5.3), we get

$$I(t) \le c_1 \int_0^\infty \big(t^{1/\alpha} v\big)^{\zeta} v^{n-\alpha} e^{-c_2 v} dv \le c_3 t^{\zeta/\alpha}.$$

In particular, if $h(v) \le c v^d$, $d > n - \alpha$, then (5.3) holds.

Thus, by Theorem 3, the Feynman–Kac semigroup $(T_t^A)_{t\ge 0}$ is well defined, and the kernel $p_t^A(x,y)$ satisfies (2.16) with some constants a_i, $1 \le i \le 4$, and some family of sub-probability measures $(\mathcal{R}^{(k)})_{t\ge 0}$.

b) Consider now the one-dimensional situation. In this case, the Lévy measure μ from (5.1) is just

$$\mu(dy) = \sum_{n=-\infty}^{\infty} 2^{n\gamma} \big(\delta_{2^{-n\upsilon}}(dy) + \delta_{-2^{-n\upsilon}}(dy)\big). \tag{5.4}$$

Let X be a Lévy process with characteristic exponent

$$\psi(\xi) := \int_{\mathbb{R}^n} \big(1 - \cos(\xi u)\big)\mu(du).$$

In [20] we show that if $1 < \alpha < 2$, then the transition probability density $p_t(x, y)$ of X, $X_0 = x$, is continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}$ and admits the following upper bound:

$$p_t(x, y) \le ct^{-1/\alpha}\big(1 + |y - x|/t^{1/\alpha}\big)^{-\alpha}, \quad t \in (0, 1], \ x, y \in \mathbb{R}. \tag{5.5}$$

Note that the right-hand side of (5.5) is of the form (2.17) with $d = 0$. Thus, the conditions of Proposition 4 are satisfied, and we can construct the Feynman–Kac semigroup for the related functional A_t and the transition density $p_t(x, y)$, and get the upper bound for the function $p_t^A(x, y)$ with $\rho_t \asymp t^{-1/\alpha}$, $t \in (0, 1]$.

To end this example, we remark that it is still possible to construct the upper bound for such $p_t(x, y)$ for $\alpha \in (0, 1)$ of the form $t^{-n/\alpha} f(xt^{-1/\alpha})$, but the function f in this upper bound might not be integrable; see [20] for details. Note that the upper bound (5.5) is non-integrable in $\mathbb{R}^n$ for $n \ge 2$.

Example 2. Consider the Lévy measure

$$\nu_0(A) = \int_{\mathbb{R}^n} \int_0^\infty \mathbb{1}_A(rv) r^{-1-\alpha}\, dr \mu_0(dv)\,, \quad \alpha \in (0, 2), \tag{5.6}$$

where $\alpha \in (0, 2)$, μ_0 is a finite symmetric non-degenerate (that is, not concentrated on a linear subspace of $\mathbb{R}^n$) measure on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^n$. Suppose that there exists $d > 0$ such that for small r we have

$$\nu_0\big(B(x, r)\big) \le Cr^d, \quad \|x\| = 1.$$

For $d + \alpha > n$, it is shown in [6] that the corresponding Lévy process X, $X_0 = x$, admits the transition probability density $p_t(x, y)$, which satisfies

$$p_t(x, y) \le ct^{-n/\alpha}\big(1 + \|y - x\| t^{-1/\alpha}\big)^{-d-\alpha}, \quad t > 0, \ x, y \in \mathbb{R}^n. \tag{5.7}$$

In the forthcoming paper [7], we construct a class of Lévy-type processes that admit the transition densities bounded from above by the left-hand side of (5.7). Thus, taking $\varpi \in \mathcal{K}_{n,\alpha}$, we may apply Proposition 4.

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A Appendix

Proof of Lemma 1. We follow the idea of the proof of [4, Lemma 11]. Without loss of generality, assume that ϖ is non-negative. Suppose first that $\varpi \in \mathcal{K}_{n,\alpha}$. Using integration by parts and the Fubini theorem, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \mathfrak{g}_s(x-y)\varpi(dy)ds \\
&\quad \asymp \int_0^t \int_{\mathbb{R}^n} s^{-n/\alpha}\big(1 \wedge \big(s^{1/\alpha}/\|x-y\|\big)\big)^{\alpha+d}\varpi(dy)ds \\
&\quad = \int_0^t s^{-n/\alpha}\varpi\big\{y:\ \|x-y\| \le s^{1/\alpha}\big\}ds \\
&\qquad + \int_0^t s^{-n/\alpha}\int_{\|x-y\|>s^{1/\alpha}}\left(\frac{s^{1/\alpha}}{\|x-y\|}\right)^{d+\alpha}\varpi(dy)ds \\
&\quad = \alpha\left(1+\frac{d+\alpha}{d+2\alpha-n}\right)\int_0^{t^{1/\alpha}}\frac{\varpi\{y:\ \|x-y\|\le v\}}{v^{n+1-\alpha}}dv \\
&\qquad + \frac{\alpha(d+\alpha)}{d+2\alpha-n}t^{\frac{d+2\alpha-n}{\alpha}}\int_{t^{1/\alpha}}^{\infty}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{d+1+\alpha}}dv. \qquad \text{(A.1)}
\end{aligned}$$

Since $\varpi \in \mathcal{K}_{n,\alpha}$, the first term tends to 0 as $t \to 0$. Further, since $d > n-\alpha$ and the measure ϖ is finite, we have

$$t^{\frac{d+2\alpha-n}{\alpha}}\sup_x \int_1^{\infty}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{d+1+\alpha}}dv \to 0, \quad t \to 0.$$

Let us show that

$$\sup_x t^{\frac{d+2\alpha-n}{\alpha}}\int_{t^{1/\alpha}}^{1}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{d+1+\alpha}}dv. \qquad \text{(A.2)}$$

Let $K_0 \equiv K_0(t) := [t^{-1/\alpha}]+1$; note that $K_0(t)t^{1/\alpha} \to 1$ as $t \to 0$. We have

$$\begin{aligned}
&t^{\frac{d-\epsilon+2\alpha-n}{\alpha}}\int_{t^{1/\alpha}}^{1}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{d+1+\alpha}}dv \\
&\quad \le \sum_{k=1}^{K_0}\left(\frac{1}{k}\right)^{(d-n+2\alpha)/\alpha}\int_{kt^{1/\alpha}}^{(k+1)t^{1/\alpha}}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{n+1-\alpha}}dv.
\end{aligned}$$

Since $d > n-\alpha$, we have $\sum_{k=1}^{\infty}k^{-(d-n+2\alpha)/\alpha} < \infty$. Since $\varpi \in \mathcal{K}_{n,\alpha}$, we have

$$\max_{1\le k\le K_0(t)}\sup_x \int_{kt^{1/\alpha}}^{(k+1)t^{1/\alpha}}\frac{\varpi\{y:\ \|x-y\|<v\}}{v^{n+1-\alpha}}dv \longrightarrow 0, \quad t \to 0.$$

Thus, we arrive at (A.2). This proves that (2.18) implies that $\varpi \in S_K$ with respect to $\mathfrak{g}_t(y-x)$.

The converse is straightforward. □

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