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Multi-mixed fractional Brownian motions and Ornstein–Uhlenbeck processes

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Abstract The so-called multi-mixed fractional Brownian motions (mmfBm) and multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes are studied. These processes are constructed by mixing by superimposing or mixing (infinitely many) independent fractional Brownian motions (fBm) and fractional Ornstein–Uhlenbeck processes (fOU), respectively. Their existence as L^2 processes is proved, and their path properties, viz. long-range and short-range dependence, Hölder continuity, p-variation, and conditional full support, are studied.

Keywords Fractional Brownian motion, Gaussian processes, long-range dependence, multi-mixed fractional Brownian motion, multi-mixed fractional Ornstein–Uhlenbeck process, short-range dependence, stationary-increment processes, stationary processes

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1 Introduction and preliminaries

The first attempt to formulate the long-term memory of time series was in hydrology when Hurst (1951) and his colleagues were studying the fluctuation of the reservoir of the Nile river over a long period of time (see [16]). Later on, after the works of Mandelbrot (1968) in [24], it was clarified that this behavior of time series is because of including long-range depended noises called fractional Brownian motion

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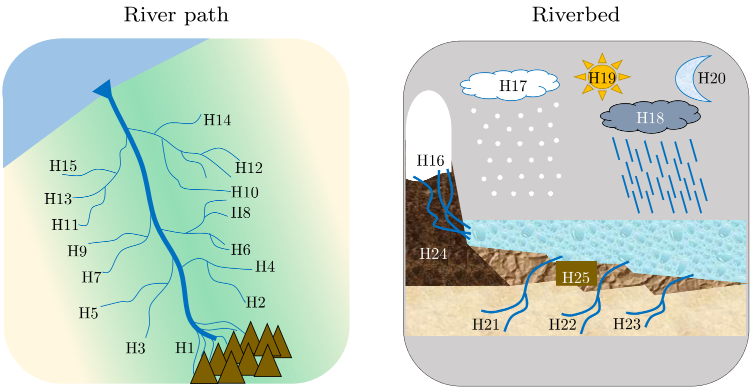
(fBm) B^H. Regarding sources of these noises, in hydrology, they are accumulated from factors such as waterfalls, glaciers melting, riverbed shape and material, slope and direction, width and depth, local temperature, etc. Moreover, we know a river (especially a large one like the Nile) is a combination of many sub-rivers, and each sub-river (or even a large river) is also a combination of many sources like streams, mountain glaciers, underground water reservoirs, and rainfall in general. Now, one may ask

> *How many sources of such noises are there for a river reservoir in reality?*

The answer of nature then, in practice, is infinity!

Table 1. Multi-mixed fBm arising from different Hurst exponents imposed to a river reservoir



So, let us consider the source i has the noise B^{H_i} with the weight of effect σ_i to the river reservoir, then the general noise of the river can be written as a linear combination

$$M_t = \sum_{i=1}^{\infty} \sigma_i B_t^{H_i}. \tag{1}$$

On the other hand, about the dynamic of a particle in a liquid, Langevin (1908) in [21] modeled the particle's velocity U with an equation which was wisely revised later on by Doob (1942) [9] as

$$\mathrm{d}U_t = -\lambda U_t \,\mathrm{d}t + \mathrm{d}M_t,$$

where $\lambda > 0$ is the mean reversion parameter and M is a noise, caused by a fluctuating force imposed by an impact of the molecules of the surrounding medium. If $U_0 = \xi$ then the unique solution of this equation is

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)} \,\mathrm{d}M_s.$$

First, this solution was given for all cases where M is semimartingale, then Cheridito et al. (2003) in [8] confirmed this solution for the case the noise process is an fBm

$M = B^H$. Now, let us think the liquid is not purely homogeneous, so the local surrounding molecules (particles) can have different imposing forces according to their different sizes, weights, density, or dynamic patterns. Hence, if molecule (particle) i

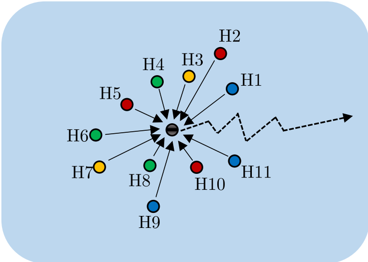


Fig. 1. Free particle movement in a liquid bombarded by multiple molecules (particles) imposing multi-mixed fBm noises

imposes the noise force B^{H_i} with the weight of effect σ_i to the free particle, then the Langevin equation takes the form

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}\left(\sum_{i=1}^{\infty}\sigma_i B_t^{H_i}\right). \tag{2}$$

In this article, our aim is to develop the analysis and some properties of the stochastic processes in equations (1) and (2). To do this, first, we review some mathematical concepts. The fractional Brownian motion (fBm) B^H, with parameter $H \in (0,1)$ called the Hurst index, is the unique (up to a multiplicative constant) centered H-self-similar stationary-increment Gaussian process. The fBm was first studied in [19]. The name fractional Brownian motion comes from the influential article [24]. For further information of the fBm, see the monographs [6, 25]. The covariance of the fBm with the Hurst index H is given by

$$r_H(t,s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

For $H = 1/2$ this process is well known as the Brownian motion (BM) or the Wiener process: $B^{\frac{1}{2}} = W$. As a stationary-increment process, the fBm has the spectral representation

$$r_H(t,s) = \int_{\mathbb{R}} \frac{(\mathrm{e}^{\mathrm{i}sx}-1)(\mathrm{e}^{\mathrm{i}tx}-1)}{x^2} f_H(x)\,\mathrm{d}x,$$

where

$$f_H(x) = \frac{\sin(\pi H)\Gamma(1+2H)}{2\pi}|x|^{1-2H}. \tag{3}$$

Here Γ is the complete gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$

see [28].

Let

$$\varrho_H(\delta; t) = \mathbb{E}\left[(B^H_\delta - B^H_0)(B^H_{t+\delta} - B^H_t)\right]$$

be the incremental autocovariance (with lag δ) of the fBm. For $t \to \infty$ we have the power decay

$$\varrho_H(\delta; t) \sim H(2H-1)\delta^2 t^{2H-2}.$$

This means that the increments of the fBm, called the fractional Gaussian noise (fGn), for $H > \frac{1}{2}$, are positively correlated and long-range dependent. However, for $H < \frac{1}{2}$ they are negatively correlated and short-range dependent.

In the Bm case $B^{\frac{1}{2}} = W$ we have independent increments, i.e. no dependence:

$$\varrho_{\frac{1}{2}}(\delta; t) = 0.$$

The fBm has almost surely Hölder continuous paths with any order $H - \varepsilon$ for any $\varepsilon > 0$. This follows, e.g., from Theorem 1 of [2].

In addition to Hölder continuity, we have the p-variation as a measure of the path regularity. For a process X and $p \in [1, \infty)$ for the partitions $\pi_n := \{t_k = \frac{k}{n}T : k = 0, 1, \dots, n\}$, if

$$V^p_T(X) := \lim_{n\to\infty} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p < \infty \quad \text{(limit in probability)},$$

then it is said X has equidistant p-variation on $[0, T]$, and its p-variation on $[0, T]$ is $V^p_T(X)$. For the fBm B^H then the p-variation is

$$V^p_T(B^H) = \begin{cases} \infty & ; \quad pH < 1 \\ T\mu_p & ; \quad pH = 1 \\ 0 & ; \quad pH > 1 \end{cases}$$

where μ_p is the pth moment of a standard Gaussian process, see [10, 11].

While the fBm has been proposed as a model for financial time series, modeling with it makes arbitrage possible, see [4]. To eliminate this problem, a generalization called mixed fractional Brownian motion (mfBm) was introduced in [7]. This is the mixture model

$$M^{a,b} = aB + bB^H,$$

where $a, b \in \mathbb{R}$ and B is a standard Brownian motion (Bm) independent of the fBm B^H. If $H > 1/2$, the mfBm has the path roughness governed by the Bm part and the long-range dependence governed by the fBm part. Hence, e.g., in pricing of financial derivatives the corresponding mixed Black–Scholes model yields the same option prices as the standard Brownian model, see [5].

A natural generalization of the mfBm is to consider two (or n) independent fBm mixtures, see [23]. In this paper, we study an independent infinite-mixture generalization that we call the multi-mixed fractional Brownian motion (mmfBm) with parameters σ_k, H_k, $k \in \mathbb{N}$:

$$M = \sum_{k=1}^{\infty} \sigma_k B^{H_k},$$

where B^{H_k}'s are independent fBm's with Hurst indices $H_k \in (0,1)$, and σ_k's are positive volatility constants satisfying $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. This study extends the work of [30].

For other kinds of generalizations of the fBm, see, e.g., [14, 22, 26, 27].

The fractional Ornstein–Uhlenbeck process (fOU) $U^{\lambda,H}$, with parameters $\lambda > 0$ and $H \in (0,1)$, is the stationary solution of the Langevin equation

$$\mathrm{d}U_t^{\lambda,H} = -\lambda U_t^{\lambda,H}\mathrm{d}t + \mathrm{d}B_t^H,$$

which is given by

$$U_t^{\lambda,H} = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B_s^H,$$

where $(B_s^H)_{s\le 0}$ is an independent copy of the fBm $(B_s^H)_{s\ge 0}$, see [8]. Note that the Langevin equation and its solution can be understood via integration by parts. As a stationary process, the fOU admits the spectral density

$$f_{\lambda,H}(x) = \frac{f_H(x)}{x^2+\lambda^2}, \tag{4}$$

where f_H is the spectral density of the driving fBm (3), see [3]. Denote, for $\alpha \in (-1,0)\cup(0,1)$,

$$\gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^x s^{\alpha-1}\mathrm{e}^{s}\,\mathrm{d}s, \tag{5}$$

$$\Gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty s^{\alpha-1}\mathrm{e}^{-s}\,\mathrm{d}s, \tag{6}$$

and $\gamma_0(x) = 1$, $\Gamma_0(x) = 0$. The functions γ_α and Γ_α are related to the incomplete Gamma functions and they can be calculated, e.g., by approximating the integrals with sums. The autocovariance function of the fOU process can be written as

$$\rho_{\lambda,H}(t) = \frac{\Gamma(1+2H)}{4}\frac{\mathrm{e}^{-\lambda t}}{\lambda^{2H}}\Big\{1+\gamma_{2H-1}(\lambda t)+\mathrm{e}^{2\lambda t}\Gamma_{2H-1}(\lambda t)\Big\}. \tag{7}$$

See Proposition 4.

A stationary process X with the autocovariance function satisfying

$$\rho(t) \sim c|t|^{-\alpha} \quad \text{as} \quad t\to\infty$$

where $0 \ne c \in \mathbb{R}$, and "$\sim$" means the ratio of left and right sides tends to 1, is called long-range dependent (having long memory) if $0 < \alpha \le 1$, and short-range dependent (having short memory) if $\alpha > 1$, see [18].

For $H = \frac{1}{2}$ we recover the well-known Bm case

$$\rho_{\lambda,\frac{1}{2}}(t) = \frac{\mathrm{e}^{-\lambda t}}{2\lambda}.$$

For $t \to \infty$ we have the power decay

$$\rho_{\lambda,H}(t) = \frac{1}{2}\sum_{n=1}^{N}\lambda^{-2n}\left(\prod_{j=0}^{2n-1}(2H-j)\right)t^{2H-2n} + O(t^{2H-2N-2}),$$

for $N = 1, 2, \dots$, i.e., the fOU process with $H > \frac{1}{2}$ is long-range dependent, and for $H \le \frac{1}{2}$ it is short-range dependent, see [8].

The Hölder continuity and p-variation of fOU is the same as for the mmfBm.

In this paper we study the multi-mixed fractional Ornstein–Uhlenbeck process (mmfOU) with parameters $\lambda > 0$ and σ_k, H_k, $k \in \mathbb{N}$, that is defined naturally as the stationary solution of Langevin equation with mmfBm as the driving noise:

$$\mathrm{d}U_t = -\lambda U_t \,\mathrm{d}t + \mathrm{d}M_t,$$

with

$$U_0 = \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \,\mathrm{d}M_s,$$

where $(M_s)_{s \le 0}$ is an independent copy of the mmfBm. This study develops the work of [17].

The rest of the paper is organized as follows. In Section 2 we define the multi-mixed fractional Brownian motions (mmfBm) and the associated multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes, prove their existence in $L^2(\Omega \times [0, T])$, and provide their basic properties. The long-range dependence of these processes are studied in Section 3. In Section 4 we analyze the Hölder continuity and p-variation of mmfBm's and mmfOU processes. The p-variations of these processes are calculated in Section 5. In Section 6 we show that the mmfBm's and mmfOU processes have the conditional full support property. Finally, In Section 7 some simulated paths of these processes are given.

2 Definitions and basic properties

Definition 1. Let σ_k, $k \in \mathbb{N}$, satisfy

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty, \tag{8}$$

and let H_k, $k \in \mathbb{N}$, satisfy

$$\begin{aligned} H_k &\ne H_l \quad \text{for } k \ne l, \\ H_{\inf} &= \inf_{k \in \mathbb{N}} H_k > 0 \\ H_{\sup} &= \sup_{k \in \mathbb{N}} H_k < 1. \end{aligned} \tag{9}$$

The multi-mixed fractional Brownian motion (mmfBm) is

$$M = \sum_{k=1}^{\infty} \sigma_k B^{H_k},$$

where B^{H_k}, $k \in \mathbb{N}$, are independent fBm's.

The following proposition shows the existence of the mmfBm.

Proposition 1. *The mmfBm M exists as a random function taking values in $L^2(\Omega \times [0, T])$ for all $T > 0$.*

Proof. Let $M^n = \sum_{k=1}^{n} \sigma_k B^{H_k}$. Clearly M^n takes values in $L^2(\Omega \times [0, T])$. Let $n, m \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned}
\|M^n - M^m\|^2_{L^2(\Omega\times[0,T])} &= \int_0^T \mathbb{E}\left[(M_t^n - M_t^m)^2\right] \mathrm{d}t \\
&= \int_0^T \mathbb{E}\left[\left(\sum_{k=m+1}^{n} \sigma_k B_t^{H_k}\right)^2\right] \mathrm{d}t \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 \mathbb{E}\left[(B_t^{H_k})^2\right] \mathrm{d}t \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 t^{2H_k} \,\mathrm{d}t \\
&= \sum_{k=m+1}^{n} \sigma_k^2 \frac{T^{1+2H_k}}{1+2H_k} \\
&\leq \sum_{k=m+1}^{n} \sigma_k^2 \max\left\{1, T^3\right\},
\end{aligned}$$

which shows that $(M^n)_{n\in\mathbb{N}}$ is the Cauchy sequence. Thus $M^n \to M$ in $L^2(\Omega\times[0, T])$ showing the existence. □

In the same way we see that the mmfBm $(M_t)_{t\geq 0}$ exists in the sense that $M_t^n \to M_t$ in $L^2(\Omega)$ for all $t \geq 0$.

The following is now obvious.

Proposition 2. *The mmfBm has stationary increments, its covariance function is*

$$r(t, s) = \sum_{k=1}^{\infty} \sigma_k^2 r_{H_k}(s, t) = \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 \left[|t|^{2H_k} + |s|^{2H_k} - |t-s|^{2H_k}\right], \tag{10}$$

and it admits the spectral density

$$f(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{H_k}(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi H_k)\Gamma(1+2H_k)}{2\pi} \sigma_k^2 |x|^{1-2H_k}. \tag{11}$$

Definition 2. The multi-mixed fractional Ornstein–Uhlenbeck process (mmfOU) U with parameter $\lambda > 0$ is the stationary solution of the Langevin equation

$$\mathrm{d}U_t = -\lambda U_t \mathrm{d}t + \mathrm{d}M_t, \tag{12}$$

where the equation is understood in the integration by parts sense.

Proposition 3. *On $L^2(\Omega \times [0,T])$, the mmfOU can be represented as the integral*

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s,$$

where the integral is understood in the integration by parts sense, and

$$\xi = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\,\mathrm{d}M_s,$$

where $(M_s)_{s\le 0}$ is an independent copy of the mmfBm $(M_s)_{s\ge 0}$.

Proof. Let $M^n = \sum_{k=1}^n \sigma_k B^{H_k}$. Then, the stationary solution of the Langevin equation

$$\mathrm{d}U_t^n = -\lambda U_t^n \mathrm{d}t + \mathrm{d}M_t^n$$

is given by

$$U_t^n = \mathrm{e}^{-\lambda t}\xi_n + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s^n,$$

where

$$\xi_n = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\,\mathrm{d}M_s^n.$$

Then, with integration by parts

$$\begin{aligned}\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}M_s^n &= \mathrm{e}^{\lambda t}M_t^n - \lambda\int_0^t \mathrm{e}^{\lambda s}M_s^n\,\mathrm{d}s\\ &\to \mathrm{e}^{\lambda t}M_t - \lambda\int_0^t \mathrm{e}^{\lambda s}M_s\,\mathrm{d}s = \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}M_s,\end{aligned}$$

because $M^n \to M$ in $L^2(\Omega\times[0,T])$. With the same arguments $\xi_n \to \xi$ in $L^2(\Omega)$. This yields $U^n \to U$ in $L^2(\Omega\times[0,T])$. □

Lemma 1. *For $0 \ne p \in (-1,1)$, $\lambda > 0$, $t > 0$,*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\frac{|x|^p}{\lambda^2+x^2}\,\mathrm{d}x = \frac{\pi\mathrm{e}^{-\lambda t}}{2\cos(\frac{p\pi}{2})\lambda^{1-p}}\left\{1+\gamma_{-p}(\lambda t)+\mathrm{e}^{2\lambda t}\Gamma_{-p}(\lambda t)\right\}, \tag{13}$$

where γ_{-p} and Γ_{-p} are given by (5) and (6).

Proof. Recall that for the Fourier transform

$$\mathscr{F}(f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}f(t)\,\mathrm{d}t$$

we have the convolution theorem

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}tx}\mathscr{F}(f)(x)\mathscr{F}(g)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(t-\xi)g(\xi)\,\mathrm{d}\xi. \tag{14}$$

Moreover, we have

$$\mathscr{F}\left(e^{-\lambda|t|}\right)=\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\lambda^2+x^2},\tag{15}$$

$$\mathscr{F}\left(|t|^\alpha\right)=\sqrt{\frac{2}{\pi}}\cdot\Gamma(\alpha+1)\cos\left(\frac{(\alpha+1)\pi}{2}\right)|x|^{-(\alpha+1)}.\tag{16}$$

The first formula (15) is valid for $\lambda > 0$. The second formula (16) is valid for $-1 < \alpha < 0$. For $-2 < \alpha < -1$, because of the function $|t|^\alpha$, some singular terms arise at the origin. Nevertheless, it admits a unique meromorphic extension as a tempered distribution, also denoted $|t|^\alpha$ as a homogeneous distribution on all real line $\mathbb{R}$ including the origin (see [13]). So, we use that extension and formula (16) will be valid for all $-1 \neq \alpha \in (-2,0)$. So, using $f(t)=e^{-\lambda|t|}$ and $g(t)=|t|^\alpha$ in (14) we obtain

$$\begin{aligned}
&\frac{2}{\pi}\cdot\Gamma(\alpha+1)\cos\left(\frac{(\alpha+1)\pi}{2}\right)\lambda\int_{-\infty}^{\infty}\mathrm{e}^{itx}\frac{|x|^{-(\alpha+1)}}{\lambda^2+x^2}\,\mathrm{d}x\\
&\quad=\int_{-\infty}^{\infty}|\xi|^\alpha\mathrm{e}^{-\lambda|t-\xi|}\,\mathrm{d}\xi\\
&\quad=\int_{-\infty}^{0}(-\xi)^\alpha\mathrm{e}^{-\lambda(t-\xi)}\,\mathrm{d}\xi\\
&\qquad+\int_0^t\xi^\alpha\mathrm{e}^{-\lambda(t-\xi)}\,\mathrm{d}\xi\\
&\qquad+\int_t^\infty\xi^\alpha\mathrm{e}^{-\lambda(\xi-t)}\,\mathrm{d}\xi\\
&\quad=\frac{\mathrm{e}^{-\lambda t}}{\lambda^{(\alpha+1)}}\int_0^\infty u^\alpha\mathrm{e}^{-u}\,\mathrm{d}u\\
&\qquad+\frac{\mathrm{e}^{-\lambda t}}{\lambda^{(\alpha+1)}}\int_0^{\lambda t}u^\alpha\mathrm{e}^{u}\,\mathrm{d}u\\
&\qquad+\frac{\mathrm{e}^{\lambda t}}{\lambda^{(\alpha+1)}}\int_{\lambda t}^\infty u^\alpha\mathrm{e}^{-u}\,\mathrm{d}u\\
&\quad=\frac{\mathrm{e}^{-\lambda t}\Gamma(\alpha+1)}{\lambda^{(\alpha+1)}}\left\{1+\gamma_{(\alpha+1)}(\lambda t)+\mathrm{e}^{2\lambda t}\Gamma_{(\alpha+1)}(\lambda t)\right\}.
\end{aligned}$$

Now, choosing $p=-(\alpha+1)$ proves (13). □

Proposition 4 follows from Lemma 1 (see also [20]).

Proposition 4. *The covariance function of the fOU is*

$$\begin{aligned}
\rho_{\lambda,H}(t)&=\mathbb{E}[U_s^{\lambda,H}U_{s+t}^{\lambda,H}]\\
&=\frac{\Gamma(1+2H)}{4}\frac{\mathrm{e}^{-\lambda t}}{\lambda^{2H}}\left\{1+\gamma_{2H-1}(\lambda t)+\mathrm{e}^{2\lambda t}\Gamma_{2H-1}(\lambda t)\right\}.
\end{aligned}\tag{17}$$

Proposition 5. *The covariance function of the mmfOU is*

$$\begin{aligned}\rho_\lambda(t) &= \mathbb{E}[U_s U_{s+t}] \\ &= \sum_{k=1}^{\infty} \sigma_k^2 \frac{\Gamma(1+2H_k)\mathrm{e}^{-\lambda t}}{4\lambda^{2H_k}} \left\{ 1 + \gamma_{2H_k-1}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{2H_k-1}(\lambda t) \right\}, \end{aligned} \tag{18}$$

and it admits the spectral density

$$f_\lambda(x) = \sum_{k=1}^{\infty} \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1+2H_k)}{2\pi} \frac{|x|^{1-2H_k}}{x^2+\lambda^2}. \tag{19}$$

Proof. Let U^n be like in the proof of Proposition 3, then

$$f_{\lambda,n}(x) = \sum_{k=1}^{n} \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1+2H_k)}{2\pi} \frac{|x|^{1-2H_k}}{x^2+\lambda^2},$$

and $f_{\lambda,n}(x) \to f_\lambda(x)$ because $U^n \to U$ in $L^2(\Omega \times [0,T])$. This proves (19). Similarly, (18) follows by Proposition 4. □

Remark 1. Proposition 4 represents the covariance function $\rho_{\lambda,H}(t)$ in a form involving special functions. However, these special complex functions are usually not suitable for numerical computations. For example, in [3], Lemma B.1, the following representation was used for $H > \frac{1}{2}$:

$$\begin{aligned}\rho_{\lambda,H}(t) &= H\Gamma(2H)\frac{e^{-\lambda t}}{\lambda^{2H}} \left\{ \frac{1+e^{2\lambda t}}{2} - \frac{\lambda}{\Gamma(2H-1)} I_{\lambda,H}(t) \right\}, \\ I_{\lambda,H}(t) &= \int_0^t \int_0^{\lambda v} e^{2\lambda v} e^{-s} s^{2H-2}\,\mathrm{d}s\mathrm{d}v. \end{aligned}$$

The double integral above seems reasonable enough, but yields slow numerical calculation in practice. This can be improved by calculating the inner integral as follows:

$$\begin{aligned} I_{\lambda,H}(t) &= \int_0^{\lambda t} \int_{s/\lambda}^{t} e^{2\lambda v} e^{-s} s^{2H-2}\,\mathrm{d}s\mathrm{d}v \\ &= \frac{1}{2\lambda} \int_0^{\lambda t} s^{2H-2}(e^{2\lambda t - s} - e^s)\,\mathrm{d}s \\ &= \frac{e^{\lambda t}}{\lambda} \int_0^{\lambda t} s^{2H-2} \sinh(\lambda t - s)\,\mathrm{d}s. \end{aligned}$$

Consequently,

$$\rho_{\lambda,H}(t) = \frac{\Gamma(2H+1)}{2\lambda^{2H}} \left\{ \cosh(\lambda t) - \frac{1}{\Gamma(2H-1)} \int_0^{\lambda t} s^{2H-2} \sinh(\lambda t - s)\, ds \right\}. \tag{20}$$

For the case $H < 1/2$ we use the following developed version of Lemma 5.1 in [15] for $\alpha > -1$. The proof is similar.

Lemma 2. *For* $\alpha > -1$

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-(x+y)} |x-y|^\alpha \, \mathrm{d}x \mathrm{d}y = \Gamma(\alpha+1).$$

Theorem 1. *For the fOU process* $U^{\lambda,H}$*, we have*

$$\rho_{\lambda,H}(t) = \frac{\Gamma(2H+1)}{2\lambda^{2H}} \left\{ \cosh(\lambda t) - \frac{1}{\Gamma(2H)} \int_0^{\lambda t} s^{2H-1} \cosh(\lambda t - s)\, ds \right\}, \quad (21)$$

and so for mmfOU process we have

$$\rho_\lambda(t) = \sum_{k=0}^\infty \sigma_k^2 \frac{\Gamma(2H_k+1)}{2\lambda^{2H_k}} \left\{ \cosh(\lambda t) - \frac{1}{\Gamma(2H_k)} \int_0^{\lambda t} s^{2H_k-1} \cosh(\lambda t - s)\, ds \right\}.$$

Proof. For $H = 1/2$, the right-hand side of (21) is $\mathrm{e}^{-\lambda t}/2\lambda$, equal to the autocovariance of the classical Ornstein–Uhlenbeck process with respect to the standard Brownian motion. For $H > 1/2$, we obtain (21) from (20) via integration by parts. To prove it for $H < 1/2$, we will apply the same approach as in the proof of *Lemma B.1* in [3]

$$\begin{aligned} \rho_{\lambda,H}(t) &= \mathbb{E}[U_t^{\lambda,H} U_0^{\lambda,H}] \\ &= \mathbb{E}\left[\int_{-\infty}^0 \mathrm{e}^{\lambda u} \, \mathrm{d}B_u^H \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} \, \mathrm{d}B_v^H \right] \\ &= \mathrm{e}^{-\lambda t} \left\{ \mathbb{V}ar(U_0^{\lambda,H}) + \mathbb{E}\left[\int_{-\infty}^0 \mathrm{e}^{\lambda u} \, \mathrm{d}B_u^H \int_0^t \mathrm{e}^{\lambda v} \, \mathrm{d}B_v^H \right] \right\}. \end{aligned}$$

To obtain the term $\mathbb{V}ar(U_0^{\lambda,H})$ in a closed form, [3] referred to *Lemma 5.2* in [15]; however, such form was only obtained for $H \geq 1/2$, and so we need to extend their result for $H < 1/2$.

Since

$$U_0^{\lambda,H} = \int_{-\infty}^0 \mathrm{e}^{\lambda u} \, \mathrm{d}B_u^H = -\lambda \int_{-\infty}^0 \mathrm{e}^{\lambda u} B_u^H \, \mathrm{d}u,$$

we have

$$\begin{aligned} \mathbb{V}ar(U_0^{\lambda,H}) &= \mathbb{V}ar\left[-\lambda \int_{-\infty}^0 \mathrm{e}^{\lambda u} B_u^H \, \mathrm{d}u \right] \\ &= \lambda^2 \, \mathbb{V}ar\left[\int_0^\infty \mathrm{e}^{-\lambda u} B_u^H \, \mathrm{d}u \right] \\ &= \lambda^2 \mathbb{E}\left[\left(\int_0^\infty \mathrm{e}^{-\lambda u} B_u^H \, \mathrm{d}u \right)^2 \right] \\ &= \lambda^2 \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda(u+v)} B_u^H B_v^H \, \mathrm{d}u \mathrm{d}v \right] \end{aligned}$$

$$= \frac{\lambda^2}{2} \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda(u+v)} \cdot \left\{u^{2H} + v^{2H} - |u-v|^{2H}\right\} \mathrm{d}u\mathrm{d}v$$
$$= \frac{\lambda^2}{2} \left\{2 \left(\int_0^\infty \mathrm{e}^{-\lambda u}\,\mathrm{d}u\right)\left(\int_0^\infty \mathrm{e}^{-\lambda v} v^{2H}\,\mathrm{d}v\right) - \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda(u+v)} |u-v|^{2H}\,\mathrm{d}u\mathrm{d}v\right\}.$$

Now choosing $x = \lambda u$, $y = \lambda v$ by Lemma 2 we have

$$\begin{aligned}\mathbb{V}ar(U_0^{\lambda,H}) &= \frac{\lambda^{-2H}}{2}\left\{2\int_0^\infty \mathrm{e}^{-y} y^{2H}\,\mathrm{d}y - \int_0^\infty \int_0^\infty \mathrm{e}^{-(x+y)}|x-y|^{2H}\,\mathrm{d}x\mathrm{d}y\right\} \\ &= \frac{\lambda^{-2H}}{2}\Big[2\Gamma(2H+1) - \Gamma(2H+1)\Big] \\ &= \lambda^{-2H} H\Gamma(2H). \end{aligned} \tag{22}$$

On the other hand, as in Lemma 2.1 in [8] and the proof of Lemma B.1 in [3], using formula

$$\gamma_\ell(z,x) = \frac{\gamma_\ell(z+1,x) + x^z \mathrm{e}^{-x}}{z},$$

where γ_ℓ is the well-known lower Gamma function, for $H < 1/2$ we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{-\infty}^0 \mathrm{e}^{\lambda u}\,\mathrm{d}B_u^H \int_0^t \mathrm{e}^{\lambda v}\,\mathrm{d}B_v^H\right] \\
&= H(2H-1)\int_{-\infty}^0 \int_0^t \mathrm{e}^{-\lambda(u+v)}|u-v|^{2H-2}\,\mathrm{d}u\mathrm{d}v \\
&= \mathbb{V}ar(U_0^{\lambda,H})\left\{\frac{\mathrm{e}^{2\lambda t}-1}{2} - \frac{\lambda}{\Gamma(2H-1)}\int_0^t \mathrm{e}^{2\lambda v}\int_0^{\lambda v} \mathrm{e}^{-s} s^{2H-2}\,\mathrm{d}s\mathrm{d}v\right\} \\
&= \mathbb{V}ar(U_0^{\lambda,H})\left\{\frac{\mathrm{e}^{2\lambda t}-1}{2} - \frac{\lambda}{\Gamma(2H-1)}\int_0^t \mathrm{e}^{2\lambda v}\gamma_\ell(2H-1,\lambda v)\,\mathrm{d}v\right\} \\
&= \mathbb{V}ar(U_0^{\lambda,H})\left\{\frac{\mathrm{e}^{2\lambda t}-1}{2} - \frac{\lambda}{\Gamma(2H)}\int_0^t \mathrm{e}^{2\lambda v}\gamma_\ell(2H,\lambda v)\,\mathrm{d}v - \frac{\lambda^{2H}}{\Gamma(2H)}\int_0^t \mathrm{e}^{\lambda v} v^{2H-1}\,\mathrm{d}v\right\} \\
&= \mathbb{V}ar(U_0^{\lambda,H})\left\{\frac{\mathrm{e}^{2\lambda t}-1}{2}\right.
\end{aligned}$$

$$- \frac{\lambda}{\Gamma(2H)} \int_0^t \mathrm{e}^{2\lambda v} \int_0^{\lambda v} \mathrm{e}^{-s} s^{2H-1} \,\mathrm{d}s \mathrm{d}v$$
$$- \frac{\lambda^{2H}}{\Gamma(2H)} \int_0^t \mathrm{e}^{\lambda v} v^{2H-1} \,\mathrm{d}v \bigg\}. \tag{23}$$

Using (22) and (23), with similar arguments as we did for (20), we obtain (21). □

3 Long-range dependence

The increments of fBm are a well-known stationary process, that is long-range dependent (LRD) if $H > 1/2$, see [18]. Motivated by this, we consider the LRD for the increments of the mmfBm

$$\Delta_\delta M_t = \sum_{k=1}^{\infty} \sigma_k \Delta_\delta B_t^{H_k},$$

with covariance function

$$\varrho(\delta; t) = \mathbb{E}\big[\Delta_\delta M_{s+t} \Delta_\delta M_s\big],$$

where $\delta > 0$ is the lag and $\Delta_\delta x_t = x_{t+\delta} - x_t$ for a process x.

Theorem 2. *For $t \to \infty$,*

$$\varrho(\delta; t) \sim \delta^2 \sum_{k=1}^{\infty} \sigma_k^2 H_k (2H_k - 1) t^{2H_k-2} = O\big(t^{2H_{\sup}-2}\big). \tag{24}$$

So the mmfBm increment process $\Delta_\delta M_t$ is LRD if and only if $H_k > 1/2$ for some $k \geq 0$.

Proof. By using the generalized binomial theorem,

$$\begin{aligned}
\varrho(\delta; t) &= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 \Big\{ (t+\delta)^{2H_k} + (t-\delta)^{2H_k} - 2t^{2H_k} \Big\} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 t^{2H_k} \bigg\{ \Big(1 + \frac{\delta}{t}\Big)^{2H_k} + \Big(1 - \frac{\delta}{t}\Big)^{2H_k} - 2 \bigg\} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 t^{2H_k} \bigg\{ \sum_{r=0}^{\infty} \binom{2H_k}{r} \Big(\frac{\delta}{t}\Big)^r + \sum_{r=0}^{\infty} \binom{2H_k}{r} (-1)^r \Big(\frac{\delta}{t}\Big)^r - 2 \bigg\} \\
&\sim \delta^2 \sum_{k=1}^{\infty} \sigma_k^2 H_k (2H_k - 1) t^{2H_k-2}.
\end{aligned} \tag{25}$$

Since

$$\sigma_k^2 H_k (2H_k - 1) t^{2H_k-2} \leq \sigma_k^2,$$

the series (25) is uniformly convergent. So we have

$$\lim_{t\to\infty}\sum_{k=1}^{\infty}\sigma_k^2 H_k(2H_k-1)t^{2H_k-2}=\sum_{k=1}^{\infty}\lim_{t\to\infty}\sigma_k^2 H_k(2H_k-1)t^{2H_k-2}.$$

This yields (24). □

To investigate LRD for the mmfOU process, we first need some lemmas.

The following theorem shows that similar to the mmfBm increment process, the long-range dependence of the mmfOU is governed by the long-range dependence of the largest Hurst index in the driving mmfBm.

Theorem 3. *For $t\to\infty$ and each $N=1,2,\dots,$*

$$\rho_\lambda(t)=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{N}\sigma_k^2\lambda^{-2n}\left(\prod_{j=0}^{2n-1}(2H_k-j)\right)t^{2H_k-2n}+O(t^{2H_{\sup}-2N-2}). \quad (26)$$

So the mmfOU process U is LRD if and only if $H_k>1/2$ for some $k\geq 0$.

Proof. By the proof of Lemma 2.2 and Theorem 2.3 in [8]

$$\begin{aligned}
\rho_\lambda(t)&=\mathbb{E}\left[\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\,\mathrm{d}M_u\int_{-\infty}^{t}\mathrm{e}^{-\lambda(t-v)}\,\mathrm{d}M_v\right]\\
&=\mathrm{e}^{-\lambda t}\mathbb{E}\left[\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\,\mathrm{d}M_u\int_{-\infty}^{1/\lambda}\mathrm{e}^{\lambda v}\,\mathrm{d}M_v\right]\\
&\quad+\mathrm{e}^{-\lambda t}\sum_{i=1}^{\infty}\sigma_i^2H_i(2H_i-1)\\
&\quad\times\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\left(\int_{1/\lambda}^{t}\mathrm{e}^{\lambda v}(v-u)^{2H_i-2}\,\mathrm{d}v\right)\mathrm{d}u\\
&=O(\mathrm{e}^{-\lambda t})\\
&\quad+\frac{1}{2}\sum_{i=1}^{\infty}\sigma_i^2\frac{H_i(2H_i-1)}{\lambda^{2H_i}}\left\{\mathrm{e}^{-\lambda t}\int_{1}^{\lambda t}\mathrm{e}^{y}y^{2H_i-2}\,\mathrm{d}y\right.\\
&\quad\left.+\mathrm{e}^{\lambda t}\int_{\lambda t}^{\infty}\mathrm{e}^{-y}y^{2H_i-2}\,\mathrm{d}y\right\}\\
&\leq O(\mathrm{e}^{-\lambda t})\\
&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{N}\sigma_k^2\lambda^{-2n}\left(\prod_{j=0}^{2n-1}(2H_k-j)\right)t^{2H_k-2n}\\
&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}\\
&\quad\times\left[\mathrm{e}^{-\frac{\lambda t}{2}}+(1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right].
\end{aligned} \quad (27)$$

Now, for $t \in [1, \infty)$,

$$\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}\mathrm{e}^{-\frac{\lambda t}{2}} < \Lambda_N$$

$$\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}(1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3} < \Pi_N,$$

where

$$\Lambda_N = H_{\sup}\frac{\max\left(|2H_{\inf}-1|, |2H_{\sup}-1|\right)}{\max\left(\lambda^{2H_{\inf}}, \lambda^{2H_{\sup}}\right)}\left|(2H_{\inf}-2)\cdots(2H_{\inf}-2-2N)\right|,$$

$$\Pi_N = H_{\sup}\frac{\max\left(|2H_{\inf}-1|, |2H_{\sup}-1|\right)}{\lambda^{2N+3}}\left|(2H_{\inf}-2)\cdots(2H_{\inf}-2-2N)\right| \times (1+2^{2H_{\sup}-2N-3}).$$

So, as $\sum_{k=1}^{\infty}\sigma_k^2 < \infty$, the series in the right-hand side of the inequality (27) is uniformly convergent on $t \in [1, \infty)$. Hence

$$\begin{aligned}\lim_{t\to\infty}\sum_{k=1}^{\infty}\sigma_k^2&\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}\\&\times\left[\mathrm{e}^{-\frac{\lambda t}{2}}+(1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right]\\=\sum_{k=1}^{\infty}\sigma_k^2&\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}\\&\times\lim_{t\to\infty}\left[\mathrm{e}^{-\frac{\lambda t}{2}}+(1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right].\end{aligned}$$

This proves (26). □

4 Continuity

Definition 3. Let $X = (X_t)$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If

$$H := \sup\left\{\nu > 0 : \sup_{s,t}\frac{X_t - X_s}{|t-s|^{\nu}} < \infty\right\} < \infty,$$

the process X is called Hölder continuous with index H, and H is its Hölder index.

Theorem 4. *Both mmfBm and mmfOU have Hölder index* $H_{\inf}$.

Proof. For $\epsilon > 0$ and $|t-s| < 1$, the mmfBm satisfies

$$\mathbb{E}\left[(M_t - M_s)^2\right] = \sum_{k=1}^{\infty}\sigma_k^2|t-s|^{2H_k} \le \left(\sum_{k=1}^{\infty}\sigma_k^2\right)|t-s|^{2H_{\inf}-\epsilon} = C_0|t-s|^{2H_{\inf}-\epsilon},$$

where $C_0 := \sum_{k=1}^{\infty} \sigma_k^2 > 0$. Thus, the claim follows from Theorem 1 of [2]. On the other hand, for some $j \geq 1$ we have $H_{\inf} \leq H_j < H_{\inf} + \epsilon$ and so the fBm B^{H_j} is not $(H_{\inf} + \epsilon)$-Hölder continuous. Hence the process $M = \sigma_j B^{H_j} + \sum_{k \neq j} \sigma_k B^{H_k}$ is not $(H_{\inf} + \epsilon)$-Hölder continuous. This proves the claim for mmfBm.

For the mmfOU, we apply Corollary 2 of [2]. That states the process U_t is Hölder-continuous of any order $0 < a < H_{\inf}$ if and only if for each $0 < \epsilon < 2H_{\inf}$, there is some $0 < \delta < 1$ such that

$$\int_0^{\infty} (1 - \cos(sx)) f_\lambda(x) dx < C_\epsilon s^{2H_{\inf} - \epsilon}, \quad s \in (0, \delta). \tag{28}$$

This is equivalent to

$$\int_0^{\infty} \frac{(1 - \cos(sx))}{s^{2H_{\inf} - \epsilon}} f_\lambda(x) dx < C_\epsilon, \quad s \in (0, \delta).$$

To show this, here for $s < 1$ we have

$$\begin{aligned}
&\int_0^{\infty} \frac{(1 - \cos(sx))}{s^{2H_k - \epsilon}} f_{\lambda, H_k}(x) dx \\
&= s^\epsilon c_{H_k} \int_0^{\infty} (1 - \cos(sx)) \frac{x \cdot (sx)^{-2H_k}}{\lambda^2 + x^2} dx \\
&= s^\epsilon c_{H_k} \int_0^{\infty} (1 - \cos u) \frac{u^{1-2H_k}}{s^2\lambda^2 + u^2} du \quad (u = sx) \\
&\leq c_{H_k} \int_0^{\infty} (1 - \cos u) \frac{u^{1-2H_k}}{s^2\lambda^2 + u^2} du \quad (0 < s < 1) \\
&\leq c_{H_k} \left\{ \int_0^{\epsilon} (1 - \cos u) \frac{u^{1-2H_k}}{u^2} du + \int_\epsilon^{\infty} \frac{u^{1-2H_k}}{u^2} du \right\} \\
&= c_{H_k} \left\{ \int_0^{\epsilon} \frac{2\sin^2(\frac{u}{2})}{u^2} u^{1-2H_k} du + \int_\epsilon^{\infty} u^{-1-2H_k} du \right\} \\
&\leq c_{H_k} \left\{ \int_0^{\epsilon} \frac{1}{2} u^{1-2H_k} du + \int_\epsilon^{\infty} u^{-1-2H_k} du \right\} \\
&= c_{H_k} \left\{ \frac{\epsilon^{2-2H_k}}{4(1 - H_k)} + \frac{\epsilon^{-2H_k}}{2H_k} \right\} =: C_{\epsilon, H_k} < \infty.
\end{aligned}$$

Therefore,

$$\int_0^{\infty} (1 - \cos(sx)) f_{\lambda, H_k}(x) dx \leq C_{\epsilon, H_k} s^{2H_k - \epsilon} \leq C_{\epsilon, H_k} s^{2H_{\inf} - \epsilon}. \tag{29}$$

Also, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \sigma_k^2 C_{\epsilon, H_k} &= \sum_{k=1}^{\infty} \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1 + 2H_k)}{2\pi} \left\{ \frac{\epsilon^{2-2H_k}}{4(1 - H_k)} + \frac{\epsilon^{-2H_k}}{2H_k} \right\} \\
&\leq \frac{\Gamma(3)}{2\pi} \left\{ \frac{\epsilon^{2-2H_{\sup}}}{4(1 - H_{\sup})} + \frac{\epsilon^{-2H_{\inf}}}{2H_{\inf}} \right\} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) =: C_\epsilon < \infty,
\end{aligned} \tag{30}$$

if and only if $0 < H_{\inf} \leq H_{\sup} < 1$ and $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. Now, (29) and (30) yield (28). Moreover, for some $j \geq 1$ we have $H_{\inf} \leq H_j < H_{\inf} + \epsilon$ and so the fOU U^{H_j} is not $(H_{\inf} + \epsilon)$-Hölder continuous. Hence the process $U = \sigma_j U^{H_j} + \sum_{k \neq j} \sigma_k U^{H_k}$ is not $(H_{\inf} + \epsilon)$-Hölder continuous. This proves the claim for mmfOU. □

5 Variation

Recall that the p-variation of fBm with $H \in (0, 1)$ on the time-interval $[0, T]$ is given in Definition 3.4 of [29] as

$$V_T^p(B^H) = \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} |\Delta B_{t_k}^H|^p = \begin{cases} \infty & ; \quad pH < 1 \\ T\mu_p & ; \quad pH = 1 \\ 0 & ; \quad pH > 1 \end{cases}$$

where $\pi_n = \{t_k = \frac{k}{n}\}_{k=0}^n$ is a partition of $[0, T]$, and μ_p is the pth absolute moment of a standard Gaussian process, and the limit is taken in probability. With the same argument, it is easy to check that for the mixed fractional Brownian motion (mfBm) $Y = aB + bB^H$ the p-variation is

$$V_T^p(Y) = \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} |\Delta Y_{t_k}|^p = \begin{cases} \infty & ; \quad p \min(1/2, H) < 1 \\ Ta^p \mu_p & ; \quad H > 1/2, \ p/2 = 1 \\ T(a^2 + b^2)^{p/2} \mu_p & ; \quad H = 1/2, \ p/2 = 1 \\ Tb^p \mu_p & ; \quad H < 1/2, \ pH = 1 \\ 0 & ; \quad p \min(1/2, H) > 1 \end{cases}$$

where $a, b > 0$, and B is the standard Brownian motion, and B^H is a standard fBm independent from B. Now, for the p-variation of the mmfBm we have the next theorem.

Theorem 5. *For $p > 0$, the p-variations of the mmfBm M and the mmfOU U on the time-interval $[0, T]$ are equal and*

$$V_T^p(M) = V_T^p(U) = \begin{cases} \infty & ; \quad pH_{\inf} < 1 \\ T\left(\sum_{H_i = H_{\inf}} \sigma_i^2 \right)^{p/2} \mu_p & ; \quad pH_{\inf} = 1 \\ 0 & ; \quad pH_{\inf} > 1 \end{cases} \tag{31}$$

Proof. For the mmfBm M, we have

$$\begin{aligned} v_{\pi_n}^p(M) &:= \sum_{t_k \in \pi_n} |\Delta M_{t_k}|^p \\ &= \sum_{t_k \in \pi_n} \left| \sum_{i=1}^{\infty} \sigma_i^2 (\Delta t_k)^{2H_i} \right|^{p/2} \left| \frac{\Delta M_{t_k}}{[\sum_{i=1}^{\infty} \sigma_i^2 (\Delta t_k)^{2H_i}]^{1/2}} \right|^p \\ &\overset{d}{=} \left(\sum_{i=1}^{\infty} \sigma_i^2 T^{2H_i} n^{2/p - 2H_i} \right)^{p/2} \cdot \frac{1}{n} \sum_{k=1}^{n} |Z_k|^p \end{aligned}$$

as $|\pi_n| \to 0$, or equivalently $n \to \infty$. Here Z_k is a standard Gaussian process and so by the proof of Lemma 3.7 in [29]

$$\frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \mu_p,$$

as $n \to \infty$, where μ_p is the pth absolute moment of the standard Gaussian process. Now, if $pH_{\inf} < 1$ then $H_{\inf} < 1/p$, and so there exists some $j \geq 1$ that $H_j < 1/p$, and so $2/p - 2H_j > 0$. Therefore

$$v^p_{\pi_n}(M) \geq \left(\sigma_j^2 T^{2H_j} n^{2/p-2H_j}\right)^{p/2} \cdot \frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \infty.$$

On the other hand, if $pH_{\inf} \geq 1$, for $x \in (1,\infty)$

$$\sigma_i^2 T^{2H_i} x^{2/p-2H_i} \leq \sigma_i^2 T^2,$$

and because $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$, the $\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i} x^{2/p-2H_i}$ is uniformly convergent on $x \in [1,\infty)$. So for $pH_{\inf} \geq 1$,

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i} n^{2/p-2H_i} = \sum_{i=1}^{\infty}\lim_{n\to\infty}\sigma_i^2 T^{2H_i} n^{2/p-2H_i}.$$

This yields the values mentioned in (31) are correct for the p-variation of M. For the mmfOU U, as it is stationary, we have

$$\begin{aligned}
v^p_{\pi_n}(U) &:= \sum_{t_k\in\pi_n}|\Delta U_{t_k}|^p \\
&\overset{d}{=} \sum_{k=1}^{n}\left(\mathbb{V}ar[\Delta U_{t_1}]\right)^{p/2}|Z_k|^p \\
&= n\left(\mathbb{V}ar[U_{\frac{T}{n}} - U_0]\right)^{p/2} \cdot \frac{1}{n}\sum_{k=1}^{n}|Z_k|^p.
\end{aligned}$$

As $\frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \mu_p$ for $n \to \infty$, the problem reduces to the limit

$$\lim_{n\to\infty} n\left(\mathbb{V}ar[U_{\frac{T}{n}} - U_0]\right)^{p/2}.$$

To find it, again because U is stationary, and using the proof of Theorem 1 we have

$$\begin{aligned}
\mathbb{V}ar[U_{\frac{T}{n}} - U_0] &= \mathbb{V}ar\, U_{\frac{T}{n}} + \mathbb{V}ar\, U_0 - 2\,\mathrm{C}ov\left(U_{\frac{T}{n}}, U_0\right) \\
&= 2\,\mathbb{V}ar\, U_0 - 2\,\mathrm{C}ov\left(U_{\frac{T}{n}}, U_0\right) \\
&= 2\sum_{i=1}^{\infty}\sigma_i^2\lambda^{-2H_i} H_i\Gamma(2H_i)
\end{aligned}$$

$$
\begin{aligned}
&- 2\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{2\lambda^{2H_i}}\bigg\{\cosh\Big(\frac{\lambda T}{n}\Big)\\
&- \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\Big(\frac{\lambda T}{n}-s\Big)\,\mathrm{d}s\bigg\}\\
= &\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{\lambda^{2H_i}}\bigg\{1-\cosh\Big(\frac{\lambda T}{n}\Big)\\
&+ \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\Big(\frac{\lambda T}{n}-s\Big)\,\mathrm{d}s\bigg\}.
\end{aligned}
$$

For the large values of n, the final series in the right-hand side above is uniformly convergent. So, the $\lim_{n\to\infty}$ and $\sum_{i=1}^{\infty}$ could change places. This yields

$$
\begin{aligned}
&\lim_{n\to\infty} n\Big(\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2}\\
&= \lim_{n\to\infty}\Big(n^{2/p}\,\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2}\\
&= \Bigg(\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{\lambda^{2H_i}}\cdot\lim_{n\to\infty} n^{2/p}\bigg\{1-\cosh\Big(\frac{\lambda T}{n}\Big)\\
&\quad+ \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\Big(\frac{\lambda T}{n}-s\Big)\,\mathrm{d}s\bigg\}\Bigg)^{p/2}.
\end{aligned}
$$

Now for $t\to 0$, by the Taylor expansion

$$1-\cosh t = -\sum_{r=1}^{\infty}\frac{t^{2r}}{(2r)!},$$

and via integration by parts

$$\int_0^t s^{2H_i-1}\cosh(t-s)\,\mathrm{d}s = \frac{t^{2H_i}}{2H_i}+\frac{1}{2H_i}\int_0^t s^{2H_i}\sinh(t-s)\,\mathrm{d}s.$$

Again for $t\to 0$, by the Taylor expansion,

$$\int_0^t s^{2H_i}\sinh(t-s)\,\mathrm{d}s \le \int_0^t t^{2H_i}\sinh t\,\mathrm{d}s = t^{2H_i+1}\sinh t = \sum_{r=1}^{\infty}\frac{t^{2r+2H_i}}{(2r-1)!}.$$

These yield for $t\to 0$

$$1-\cosh t+\frac{1}{\Gamma(2H_i)}\int_0^t s^{2H_i-1}\cosh(t-s)\,\mathrm{d}s \sim \frac{t^{2H_i}}{2H_i+1}.$$

Therefore

$$\lim_{n\to\infty} n\Big(\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2} = \Bigg(\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i}\lim_{n\to\infty} n^{2/p-2H_i}\Bigg)^{p/2},$$

this proves (31). □

6 Conditional full support

As explained in [5], in mathematical finance models one of the must require features is the so-called Conditional Full Support (CFS) to avoid simple kind of arbitrage. This means that, given the information up to any time $\tau \in [0, T]$, the process is inherently free enough to go anywhere after time τ with positive probability. This motivates us to study the CFS property of the mmfBm and mmfOU processes but first we restate the precise definition of CFS from [12].

Definition 4. Let $X = (X_t)_{0\le t\le T}$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)$ be its natural filtration. The process X is said to have CFS if, for all $t \in [0, T]$, the conditional law of $(X_u)_{t\le u\le T}$ given $(\mathcal{F}_t)$, almost surely has support $C_{X_t}[t, T]$, where $C_x[t, T]$ is the space of continuous functions f on $[t, T]$ satisfying $f(t) = x$. Equivalently, this means that, for all $t \in [0, T]$, $f \in C_0[t, T]$, and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t\le u\le T} |X_u - X_t - f(u)| < \varepsilon \,\middle|\, \mathcal{F}_t\right) > 0,$$

almost surely.

Theorem 6. *Both the mmfBm and the mmfOU have conditional full support.*

Proof. It is easy to check that

$$f(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{H_k}(x) \ge \begin{cases} \frac{\varepsilon_H \Gamma(1)}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)|x|^{1-2H_{\inf}} & : \quad |x| \le 1 \\ \frac{\varepsilon_H \Gamma(1)}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)|x|^{1-2H_{\sup}} & : \quad |x| \ge 1 \end{cases} =: h(x),$$

where

$$\varepsilon_H := \inf\Big\{\sin(\pi H_k)\Big\}_{k\ge 1} = \inf\Big\{\sin(\pi H_{\inf}), \sin(\pi H_{\sup})\Big\}.$$

Since $0 < H_{\inf} \le H_{\sup} < 1$, $\varepsilon_H > 0$. Thus $h(x) > 0$ for $x \ne 0$. Therefore, for any $x_0 > 1$ we have

$$\begin{aligned} \int_{x_0}^{\infty} \frac{\log f(x)}{x^2}\,\mathrm{d}x &\ge \int_{x_0}^{\infty} \frac{\log h(x)}{x^2}\,\mathrm{d}x \\ &= \log\left\{\frac{\varepsilon_H}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)\right\} \int_{x_0}^{\infty} \frac{\mathrm{d}x}{x^2} \\ &\quad + (1 - 2H_{\sup}) \int_{x_0}^{\infty} \frac{\log x}{x^2}\,\mathrm{d}x > -\infty, \end{aligned}$$

and by Theorem 2.1 of [12] this proves that M has conditional full support.

For mmfOU it is easy to check that

$$f_\lambda(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{\lambda,H_k}(x) \geq \begin{cases} \frac{\varepsilon_H \Gamma(1)}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) \frac{|x|^{1-2H_{\inf}}}{\lambda^2 + x^2} & : \quad |x| \leq 1 \\ \frac{\varepsilon_H \Gamma(1)}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) \frac{|x|^{1-2H_{\sup}}}{\lambda^2 + x^2} & : \quad |x| \geq 1 \end{cases} =: h(x),$$

where

$$\varepsilon_H := \inf \Big\{ \sin(\pi H_k) \Big\}_{k \geq 1} = \inf \Big\{ \sin(\pi H_{\inf}), \sin(\pi H_{\sup}) \Big\}.$$

Since $0 < H_{\inf} \leq H_{\sup} < 1$, we have $\varepsilon_H > 0$. Consequently, $h(x) > 0$ for $x \neq 0$. Therefore, for any $x_0 > 1$ we have that

$$\begin{aligned} \int_{x_0}^{\infty} \frac{\log f_\lambda(x)}{x^2} \, \mathrm{d}x &\geq \int_{x_0}^{\infty} \frac{\log h(x)}{x^2} \, \mathrm{d}x \\ &= \log \left\{ \frac{\varepsilon_H}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) \right\} \int_{x_0}^{\infty} \frac{\mathrm{d}x}{x^2} \\ &\quad + (1 - 2H_{\sup}) \int_{x_0}^{\infty} \frac{\log x}{x^2} \, \mathrm{d}x \\ &\quad - \int_{x_0}^{\infty} \frac{\log(\lambda^2 + x^2)}{x^2} \, \mathrm{d}x > -\infty. \end{aligned}$$

The claim follows now from Theorem 2.1 of [12]. □

7 Sample paths

Here we aim to present some replications of the mmfOU and its related mmfBm with different limitations for its Hurst exponents. Obviously the limitations of the Hurst exponents characterize the roughness of the sample paths. In each of these replications, the mmfOU is given on $N = 1000$ equidistant points $t_k = k/(N-1)$ of the time interval [0, 1], with $n = 10$ equidistant Hurst exponents $H_i = H_{\inf} + (i-1)(H_{\sup} - H_{\inf})/(n-1)$ on the Hurst interval $[H_{\inf}, H_{\sup}]$. Also, the coefficients $\sigma_i = i^{-1}, i!^{-1}, \mathrm{e}^{-i}$ are used and indicated in each figure. In all paths here $\lambda = 1$.

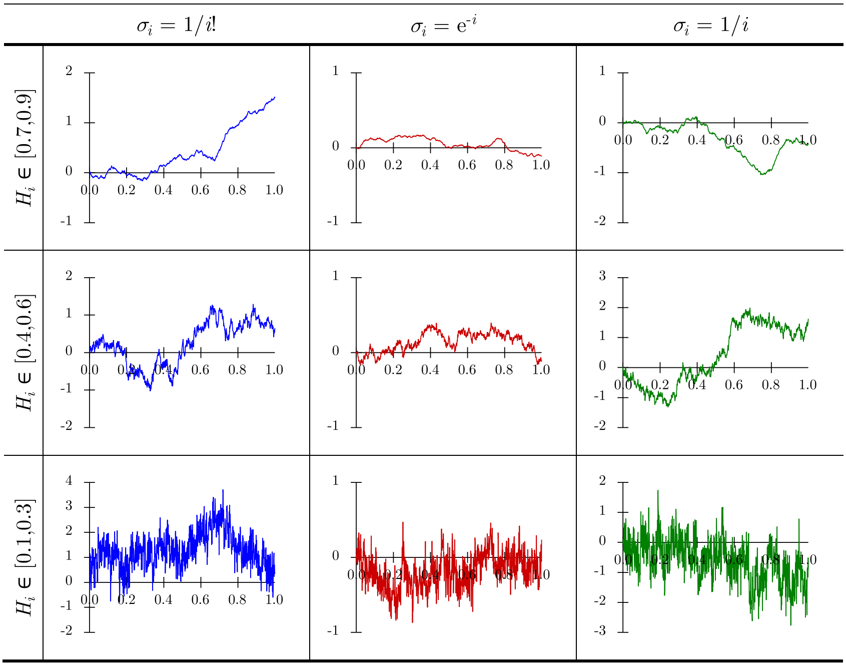


Fig. 2. Sample paths of mmfBm with equidistant time points and equidistant Hurst parameters

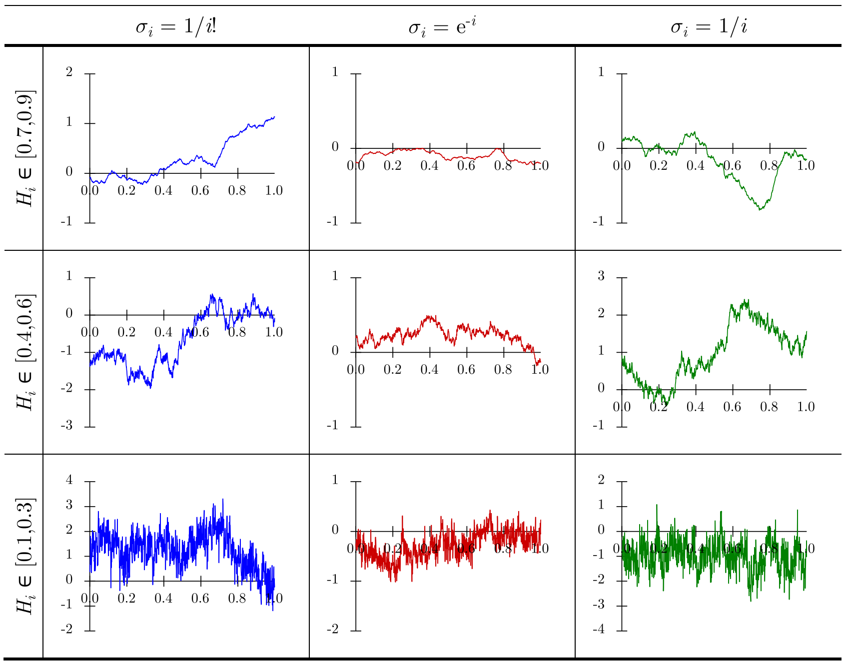


Fig. 3. Sample paths of mmfOU with equidistant time points and equidistant Hurst parameters

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