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Perpetual cancellable American options with convertible features

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Abstract The major characteristic of the cancellable American options is the existing writer's right to cancel the contract prematurely paying some penalty amount. The main purpose of this paper is to introduce and examine a new subclass of such options for which the penalty which the writer owes for this right consists of three parts – a fixed amount, shares of the underlying asset, and a proportion of the usual option payment. We examine the asymptotic case in which the maturity is set to be infinity. We determine the optimal exercise regions for the option's holder and writer and derive the fair option price.

Keywords American options, game options, optimal strategies, convertible features, pricing
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1 Introduction

In the recent years and even more so after the financial crisis of 2008, there has been an increased interest in the international financial markets to derivatives which exhibit early exercise rights. In response to this interest many authors turn to the scientific side of these financial instruments. The American style derivatives are a very large

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and important class. They are examined in the presence of a credit risk in [14, 25]. An integral approach for the American put option evaluation is presented in [74]. A two-dimensional Lévy framework is used in [14]. A numerical method for solving the arising partial integro-differential equations in another jump model, namely Kou's jump-diffusion model, can be found in [28]. The pricing-hedging duality in a discrete market is considered in [3]. A HODIE (abbreviated from *high order via differential identity expansion*) finite difference approach is applied in [11]. The American option pricing is considered as an optimal stopping problem in [7, 17]. Options with a random start are considered in [6]. Perpetual lookback American options under a stochastic volatility are examined in [21]. Other stochastic volatility models based on the Heston framework are presented in [20, 49–51, 54, 59]. A Fourier–Padé method for pricing and hedging early exercise options – American, Bermuda, etc – is applied in [13]. Another Fourier approach is presented in [12]. American barrier options are considered in [19, 43]. A two-state regime-switching framework is considered in [46]. A wealth based model is developed in [4, 5]. American better-of options defined on two underlying assets are discussed in [33]. A deep neural network for deriving the American option prices as well as the corresponding deltas is applied in [15]. A pricing algorithm based on a maximization of the option's holder financial utility is presented in [68]. American strangle options are examined in [34, 35, 37, 64].

Another important class of early exercise derivatives are the so-called convertible bonds. Their importance is due to the mixed nature they exhibit – a debt which can be converted to stocks. A historical overview of the CoCo (abbreviated from *contingent convertible*) market trends is analyzed in [62]. A general evaluation method is presented in [8]. Some numerical methods – binomial trees and finite difference – for pricing CoCo bonds with and without credit risk are presented in [52, 53]. Convertible bonds under an uncertain market assumptions are explored in [72]. A modification of the CoCo bonds, named CoCoCat, is considered in [10]. They depend on some catastrophe stochastic event, not related to the financial market risks. Also, a financial decision technique is applied in [18] under an assumption of a possible default. An integral approach for pricing convertible bonds with puttable features is presented in [73]. A model in accordance with the Chinese convertible bonds' specifics is presented in [48]. Resettable convertible bonds are considered in [45]. The Heston stochastic volatility framework together with the Cox–Ingersoll–Ross interest rate term structure is used in [44]. Some specifics of the Western European markets are studied in [1, 2]. The relations of the CoCo bonds with different market characteristics as the underlying asset, credit default swap spreads, interest rates, implied volatilities, etc. are explored in [70]. The option featues of the convertible bonds are considered in [36]. The relation between the convertible bonds and stock returns is examined in [16].

The cancellable American options, first introduced in [38] as game or Israeli options, are a specific extension of the American style options. They exhibit a writer's right to cancel the contract earlier in addition to the existing such holder's right. The writer is obliged to pay some amount above the usual option payment when he decides to stop the contract. The general framework and a review on the topic are presented in [39]. A research of the call style options can be found in [27, 41, 63], whereas the put analogues are examined in [40, 42, 69]. Path dependent options, namely lookback and Asian, are considered in [30–32]. Cancellable options in the presence of a credit risk are studied in [24]. Transaction costs in a multi-asset model are admited in [57, 58]. The hedging problem is studied in [22, 23]. The cancellable options in a regime switching diffusion framework are examined in [47]. It turns out that the asymptotic perpetual case is very informative for the options' behavior when the maturity horizon is finite – see [29, 61, 66, 67].

A traditional assumption is that the penalty which the writer owes for his early canceling right is a constant during the option life. We abandon this assumption in the present paper considering a three-component penalty – a proportion of the usual option payment, some shares of the underlying asset, and a fixed amount. We introduce an additional discount factor in our model following the suggestion of [60]. First, it is the only deterministic factor which makes earlier exercising preferable when there is no maturity constrain. Second and more important, we can view this discount factor as a continuous dividend rate changing the parameters.

Our study begins exploring the so-called early exercise regions. They consist of these values of the underlying asset which make the immediate exercise optimal for one or the other option's participant. On the contrary, the continuation region consists of the points which give better opportunities for both of option's holder and writer. The boundaries between the optimal and continuation regions are known as early exercise or optimal boundaries. The facts that (A) the underlying asset is driven by a Markov process, (B) the lapse of maturity, and (C) the discount time dependence in the payment functions show that both exercise boundaries are flat. It turns out that the holder's optimal region for a put-style option has the form $(0, A)$ for some constant A less than the strike, $A < K$. The writer's exercise set is more variable. It can be the interval (B, K) for some constant $B \in (A, K)$, the singleton $\{K\}$ or even the empty set. The optimal regions for the call options are similar but in some sense inverse. The holder's one has the form (B, ∞), whereas the writer's set may again have three forms – the interval (K, A), the singleton $\{K\}$ or the empty set.

The approach we use to derive the exercise boundaries is based on maximizing the future utilities of both of the holder and writer. We assume first that one of the participants exercises when the underlying asset reaches some level and then obtain the optimal value for the other. We derive the equation which has to be satisfied by this optimal value. In such a way we look for the early exercise boundaries which suffice both of the option's holder and writer. Once we derive the exercise boundaries we use some Brownian motion's hitting properties to obtain the fair option prices. We investigate also the impact which the penalty coefficients have. As a rule, as higher they are, as the option is more similar to the corresponding noncancellable one. It turns out that the smooth fit principle always holds at the holder's boundary, but it is satisfied at the writer's one only when the writer's optimal set is an interval (not a singleton). We present also some numerical results for different values of the penalty components.

We have to mention that the call case in the absence of discounting is specific. As for the classical American options, early exercising is never optimal for the option's holder. This allows us to derive a closed form formula for the writer's optimal boundary as well as for the fair option price. It turns out that the writer's boundary is finite in the presence of the first or second penalty components (proportion of the usual option payment or shares of the underlying asset). Otherwise, if the penalty consists only of a fixed amount, then the writer's exercise boundary is infinite, too.

The plan of the paper is the following. In Section 2 we provide the base of our study. Section 3 presents the results for the call style options, whereas the put options are considered in Section 4. Some numerical results are presented in Section 5.

2 Preliminaries

Let the underlying asset be presented by a log-normal process S_t under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, where Q is the risk neutral measure:

$$S_t = xe^{(r-\frac{\sigma^2}{2})+\sigma B_t}. \tag{1}$$

We shall use a superscript if we need to mark the dependence on the initial value, namely S_t^x. The risk free and discount rates, r and λ, are assumed to be constants such that $\lambda \geq 0$ and $r + \lambda > 0$. Note that we do not impose positiveness of the risk free rate. The additional discount factor λ can be viewed also as a dividend rate – Proposition 2.3 from [66] says that if the asset pays a dividend with rate δ, then the (r, λ, δ)-model is equivalent to the $(r - \delta, \lambda + \delta, 0)$-model. We shall denote by K the strike price. Let the function $N_1(t, x)$ present the amount which the writer owes if the holder exercises the option in the moment t at the spot price $S_t = x$. Analogously, the function $N_2(t, x)$ defines the amount which the writer has to pay if he cancels the contract. Suppose that the penalty consists of three parts – the constant $\eta_1 \geq 1$ leads to a proportion of the usual option payment, $\eta_2 \geq 0$ is the number of shares, and $\eta_3 \geq 0$ is a fixed amount. Thus the functions $N_1(t, x)$ and $N_2(t, x)$ are

$$\begin{aligned} N_1(t, x) &= e^{-\lambda t}(x - K)^+, \\ N_2(t, x) &= e^{-\lambda t}\big(\eta_1(x - K)^+ + \eta_2 x + \eta_3\big) \end{aligned} \tag{2}$$

or

$$\begin{aligned} N_1(t, x) &= e^{-\lambda t}(K - x)^+, \\ N_2(t, x) &= e^{-\lambda t}\big(\eta_1(K - x)^+ + \eta_2 x + \eta_3\big), \end{aligned} \tag{3}$$

for the call or put style options, respectively. Thus the strategies $\tau^b \geq t$ and $\tau^s \geq t$ for the holder and writer, respectively, lead to the following value function at the point (t, x):

$$M\big(t, x; \tau^b, \tau^s\big) = E^{t,x}\big[e^{-r(\tau^b - t)} N_1\big(\tau^b, S_{\tau^b}\big) I_{\tau^b \leq \tau^s} + e^{-r(\tau^s - t)} N_2\big(\tau^s, S_{\tau^s}\big) I_{\tau^s < \tau^b}\big]. \tag{4}$$

Hence, the upper and lower value of this game are

$$V^*(t, x) = \inf_{\tau^b} \sup_{\tau^s} M\big(t, x; \tau^b, \tau^s\big), \quad V_*(t, x) = \sup_{\tau^s} \inf_{\tau^b} M\big(t, x; \tau^b, \tau^s\big). \tag{5}$$

We need now the following lemma.

Lemma 2.1. *The expectation of the sup-process, $H_t = \sup_{0\le u<t} S_u$, is*

$$\begin{cases} \lambda > 0: \; e^{-\lambda t}\big(1-\frac{\sigma^2}{2\lambda}\big)\overline{N}\big(\big(\frac{\lambda}{\sigma}-\frac{\sigma}{2}\big)\sqrt{t}\big) + \big(1+\frac{\sigma^2}{2\lambda}\big)N\big(\big(\frac{\lambda}{\sigma}+\frac{\sigma}{2}\big)\sqrt{t}\big), \\ \lambda = 0: \; 2N\big(\frac{\sigma\sqrt{t}}{2}\big) + \frac{\sigma^2 t}{2}N\big(\frac{\sigma\sqrt{t}}{2}\big) + \sigma\sqrt{t}n\big(\frac{\sigma\sqrt{t}}{2}\big), \end{cases} \tag{6}$$

where $n(\cdot)$, $N(\cdot)$, and $\overline{N}(\cdot)$ are the probability density, the cumulative distribution function and its complement of the standard normal distribution.

Proof. The lemma can be obtained using the distribution of the sup-Brownian motion with drift μ and a variance coefficient σ^2:

$$Q(H_t < x) = N\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2x\mu}{\sigma^2}}\overline{N}\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right). \tag{7}$$

See Corollary 10.1 from [56] for the proof of equation (7). □

Having in mind Lemma 2.1, which leads to $E[\sup_{t\in[0,T]} e^{-rt}N_2(t,S_t)=0] < \infty$ and $Q(\lim_{t\to\infty} e^{-rt}N_2(t,S_t)=0) = 1$, we see that the conditions of Theorem 2.1 from [26] are satisfied except when $\lambda = 0$ together with $T = \infty$. This exception is considered separately in Section 3.4 for call options; see Proposition 4.5 for the puts (note that $r > 0$ when $\lambda = 0$). Therefore the above defined problem exhibits a Nash equilibrium, see also [55].

The value function can be defined as $V(t,x) = V_*(t,x) \equiv V^*(t,x)$. The optimal regions – Υ^b and Υ^s – and the optimal strategies – τ^b and τ^s – for the holder and writer, respectively, are

$$\begin{gathered} \Upsilon^b = \big\{(t,x) : V(t,x) = N_1(t,x)\big\} \text{ and } \Upsilon^s = \big\{(t,x) : V(t,x) = N_2(t,x)\big\}, \\ \tau^b = \inf\big\{t : S_t \in \Upsilon^b\big\} \text{ and } \tau^s = \inf\big\{t : S_t \in \Upsilon^s\big\}. \end{gathered} \tag{8}$$

We shall denote the continuation region by $\overline{\Upsilon}$. Suppose that ζ is a stopping time. We define the ζ-writer's/holder's optimal strategy – we denote them by $A(\zeta;x)$ and $B(\zeta;x)$ marking the dependence on the initial asset value – as the stopping time which minimizes/maximizes expected payoff (4) w.r.t. τ^s or τ^b, respectively. This way we deduce as a corollary the writer/holder optimal conditions, namely,

$$\begin{aligned} (t,x) \in \Upsilon^s \;&\to N_2(t,x) \le M\big(t,x;\zeta,B(\zeta;x)\big) \;\forall \text{ stopping times } \zeta, \\ (t,x) \in \Upsilon^b \;&\to N_1(t,x) \ge M\big(t,x;\zeta,A(\zeta;x)\big) \;\forall \text{ stopping times } \zeta. \end{aligned} \tag{9}$$

We need to restrict the writer's optimal set in some marginal cases to keep the generality of the presentation. In fact, we impose that the writer would not cancel the option immediately, even this is optimal for him, when some future strategy provides the same result. This assumption is not so restrictive from a financial point of view.

Condition 2.2. Let the option be out-of-the-money, $\lambda = 0$, and $\eta_3 = 0$. Suppose that $V(t,x) = N_2(t,x)$ and there exists a stopping time $\zeta > t$ a.s. such that $N_2(t,x) = M(t,x;\zeta,B(\zeta;x))$. Then $(t,x) \notin \Upsilon^s$.

3 Call options

We assume hereafter $t = 0$ – this is possible due to the Markovian property driving the asset price. We shall prove first a series of propositions for the optimal regions. Using them we shall obtain the optimal boundaries as well as the fair price.

3.1 Exercise regions

Proposition 3.1. *If $x < K$, then $x \in \overline{\Upsilon}$.*

Proof. Suppose that $x \notin \overline{\Upsilon}$. Obviously $x \notin \Upsilon^b$ and thus $x \in \Upsilon^s$. Let $\epsilon > 0$ be some small enough constant and τ be the smaller between the first hitting to the strike moment and ϵ. Since $e^{-rt}S_t$ is a Q-martingale, we derive

$$\begin{aligned} N_2(t,x) &\equiv \eta_2 x + \eta_3 = E^x\left[\eta_2 e^{-r\tau} S_\tau\right] + \eta_3 \\ &\geq E^x\left[\eta_2 e^{-(r+\lambda)\tau} S_\tau + \eta_3 e^{-(r+\lambda)\tau}\right] = E^x\left[e^{-r\tau} N_2(\tau, S_\tau)\right]. \end{aligned} \tag{10}$$

Note that $B(\tau; x) > \tau$, because $S_\zeta \leq K$ on every sample path at which $\zeta \leq \tau$ and therefore the exercise before τ is not optimal for the holder (he will receive nothing). Thus writer's optimal condition (9) is not true for the stopping time τ – see also Condition 2.2 for the marginal case $\lambda = 0$, $\eta_3 = 0$. Hence, $x \notin \Upsilon^s$. The contradiction finishes the proof. □

The following restriction on the penalty coefficients appears: if $\eta_3 \geq \eta_1 K$, then early canceling is never optimal for the writer. Hence, the option would be a pure American.

Proposition 3.2. *If $\eta_3 \geq \eta_1 K$, then $\Upsilon^s = \emptyset$.*

Proof. Suppose that $\eta_3 \geq \eta_1 K$ and $x \in \Upsilon^s$. Let us denote by $\overline{\eta}$ the price of the ordinary perpetual at-the-money American call option. Proposition 6.1 from [68] leads to $\overline{\eta} < K$. The strike cannot be writer optimal because the never canceling strategy, which makes the option pure American, leads to a better financial result for the writer than the immediate cancellation: $\eta_2 K + \eta_3 \geq \eta_2 K + \eta_1 K \geq K > \overline{\eta}$. On the other hand, Proposition 3.1 gives $(0, K) \notin \Upsilon^s$. Hence x is strictly above the strike $x > K$. For a positive ϵ and a constant K_1 such that $K < K_1 < x$, we define τ as the lower between asset's first hitting to the value K_1 and ϵ. Note that τ is a finite stopping time and $S_\tau > K$. Also, $S_{B(\tau;\cdot)} > K$, because in the opposite case the holder receives nothing. Let $\zeta = \tau \wedge B(\tau; \cdot)$ – note that it is finite. Hence,

$$\begin{aligned} &\eta_1(x-K) + \eta_2 x + \eta_3 \leq M\big(x; \tau, B(\tau;x)\big) \\ &= E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\tau;\cdot)}(S_{B(\tau;\cdot)} - K) I_{B(\tau;\cdot)\leq\tau} \\ + e^{-(r+\lambda)\tau}((\eta_1+\eta_2)S_\tau + (\eta_3 - \eta_1 K)) I_{\tau < B(\tau;\cdot)} \end{array}\right] \\ &\leq E^x\left[e^{-(r+\lambda)\zeta}\big((\eta_1+\eta_2)S_\zeta + (\eta_3 - \eta_1 K)\big)\right] \\ &= E^x\left[e^{-(r+\lambda)\zeta}(\eta_3 - \eta_1 K)\right] + E^x\left[e^{-(r+\lambda)\zeta}(\eta_1+\eta_2)S_\zeta\right] \\ &< (\eta_3 - \eta_1 K) + (\eta_1+\eta_2)E^x\left[e^{-r\zeta}S_\zeta\right] = \eta_1(x-K) + \eta_2 x + \eta_3. \end{aligned} \tag{11}$$

The contradiction finishes the proof. □

Hereafter we assume that $\eta_3 < \eta_1 K$. The following propositions hold.

Proposition 3.3. *Suppose that a larger than the strike constant x is writer optimal, $x \in \Upsilon^s$. Let y be another constant such that $K < y < x$. Then $y \in \Upsilon^s$.*

Proof. Let us fix some future moment T and a writer's strategy ζ with values between 0 and T. This leads to the ζ-holder's optimal strategy $B(\zeta; x)$. Let us define the function $f(\cdot)$ as $f(z) = M(0, z; \zeta, B(\zeta; z)) - N_2(0, z)$. Using the martingality of the discounted prices we derive

$$f(z) = \eta_1 K - \eta_3 \\ + E^z \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;z)} \max \begin{pmatrix} -(e^{\lambda B(\zeta;z)}(\eta_1+\eta_2) - 1)S_{B(\zeta;z)} - K, \\ -e^{\lambda B(\zeta;z)}(\eta_1+\eta_2)S_{B(\zeta;z)} \end{pmatrix} I_{B(\zeta;z)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \max \begin{pmatrix} -(\eta_1+\eta_2)(e^{\lambda\zeta} - 1)S_\zeta - \eta_1 K + \eta_3, \\ -(e^{\lambda\zeta}(\eta_1+\eta_2) - \eta_2)S_\zeta + \eta_3 \end{pmatrix} I_{\zeta<B(\zeta;z)} \end{array} \right].$$

We can see that this function is decreasing and therefore $f(y) > f(x) > 0$ since $x \in \Upsilon^s$. We finish the proof by taking $T \to \infty$. □

Proposition 3.4. *If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (0, K)$.*

Proof. See Propositions 3.7 from [66]. □

Proposition 3.5. *If $x \in \Upsilon^b$ and $y > x$, then $y \in \Upsilon^b$.*

Proof. The proof is very similar to the proof of Proposition 3.5 from [65] and we omit it. □

Proposition 3.6. *If $\lambda = 0$, then the holder's exercise region is empty.*

Proof. Note that Proposition 3.1 from [66] does not explore the form of the cancellation payment, but only the fact that $N_1(t, x) < N_2(t, x)$. □

Proposition 3.7. *If $r < 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.*

Proof. First, Proposition 3.1 states that all points below the strike are not writer optimal. Suppose that $K_1 \in \Upsilon^s$ for some $K_1 > K$. Proposition 3.3 gives us that the whole strip $(K, K_1) \in \Upsilon^s$. Let $x \in (K, K_1)$ and ζ be the first exit time from the strip (K, K_1). Therefore $B(\zeta; x) > \zeta$. Using the martingality of the discounted asset price and the inequality $\eta_3 < \eta_1 K$, we derive

$$\begin{aligned} & E^x\big[e^{-(r+\lambda)\zeta}\big(\eta_1(S_\zeta - K) + \eta_2 S_\zeta + \eta_3\big)I_{\zeta\le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}(S_\zeta - K)I_{B(\zeta;x)<\zeta}\big] \\ & = E^x\big[e^{-(r+\lambda)\zeta}\big(\eta_1(S_\zeta - K) + \eta_2 S_\zeta + \eta_3\big)\big] \\ & \le E^x\big[e^{-r\zeta}\big(\eta_1(S_\zeta - K) + \eta_2 S_\zeta + \eta_3\big)\big] < \eta_1(x - K) + \eta_2 x + \eta_3, \end{aligned} \tag{12}$$

which contradicts the writer's optimal condition (9). □

3.2 Pricing

Propositions 3.1, 3.3, and 3.5 indicate that the holder's exercise region has the form $\Upsilon^b = [B, \infty)$ for some constant $B > K$, whereas the writer's one is the interval $\Upsilon^s = [K, A]$ (A is a constant less than B, $K < A < B$), the singleton $\Upsilon^s = \{K\}$, or the empty set $\Upsilon^s \equiv \emptyset$.

Suppose that the starting point x is between the boundaries A and B, $A < x < B$, and let us denote the option price as $f(A, B, x)$ under these assumptions. In such a way the pricing problem turns to a problem of first exit of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ from the strip (A_1, B_1) for

$$A_1 = \frac{\ln A - \ln x}{\sigma} < 0, \quad B_1 = \frac{\ln B - \ln x}{\sigma} > 0. \tag{13}$$

If we denote by τ^A and τ^B the first hitting moments to the values A_1 and B_1, then the option price can be derived as

$$\begin{aligned} f(A, B, x) &= M\big(x; \tau^A, \tau^B\big) \\ &= (B - K)E^x\big[e^{-(r+\lambda)\tau^B} I_{\tau^B \le \tau^A}\big] \\ &\quad + \big((\eta_1 + \eta_2)A - \eta_1 K + \eta_3\big)E^x\big[e^{-(r+\lambda)\tau^A} I_{\tau^A < \tau^B}\big]. \end{aligned} \tag{14}$$

Let the constants p and q be defined as

$$p = 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}, \quad q = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \tag{15}$$

We have that $p \ge q + 1$, equality holds only in the undiscounted case $\lambda = 0$. Assume now that discounting really exists, i.e. $\lambda > 0$ or equivalently $p > q + 1$. Using equations (45) and (46) from Appendix A we convert call price (14) to

$$\begin{aligned} f(A, B, x) &= \big((\eta_1 + \eta_2)A - \eta_1 K + \eta_3\big)e^{\psi A_1} \frac{\sinh(\sigma c B_1)}{\sinh(\sigma c(B_1 - A_1))} \\ &\quad + (B - K)e^{\psi B_1} \frac{\sinh(-\sigma c A_1)}{\sinh(\sigma c(B_1 - A_1))} \\ &= \big((\eta_1 + \eta_2)A - \eta_1 K + \eta_3\big)\left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + (B - K)\left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}. \end{aligned} \tag{16}$$

Let us fix the upper boundary B. After the substitution $a = \frac{A}{B}$, $k = \frac{K}{B}$, $\xi = \frac{\eta_3}{B}$, and $y = \frac{x}{B}$, formula (16) turns to

$$\begin{aligned} f(A, B, x) &= \frac{B}{y^q} \frac{((\eta_1 + \eta_2)a - \eta_1 k + \xi)a^q(1 - y^p) + (1 - k)(y^p - a^p)}{1 - a^p} \\ &= \frac{B}{y^q} \frac{-a^p(1 - k) + a^{q+1}(\eta_1 + \eta_2)(1 - y^p) - a^q(\eta_1 k - \xi)(1 - y^p) + y^p(1 - k)}{1 - a^p}. \end{aligned} \tag{17}$$

We have the order $0 \le \frac{\xi}{\eta_1} < k < a < 1$, because $\eta_3 < \eta_1 K$. Since the option's writer minimizes his financial result, we have to derive for which value of a the function

$$g(a;y) = \frac{-a^p(1-k)+a^{q+1}(\eta_1+\eta_2)(1-y^p)-a^q(\eta_1 k - \xi)(1-y^p)+y^p(1-k)}{1-a^p} \tag{18}$$

is smallest in the interval (0, 1). Its derivative is

$$g_a(a) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \begin{bmatrix} a^{p+1}(\eta_1+\eta_2)(p-q-1) - a^p(\eta_1 k - \xi)(p-q) \\ -a^{p-q}p(1-k) + a(q+1)(\eta_1+\eta_2) - q(\eta_1 k - \xi) \end{bmatrix}. \tag{19}$$

We prove in Proposition B.2 that function (19) has a unique root in the interval (0, 1) – we denote it by $a(B)$. It leads to the minimum of price function (17).

Let now fix the writer's boundary A. We have to find which value B maximizes function (16), since the option's holder maximizes his financial utility. Let us change the variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, $\xi = \frac{\eta_3}{A}$, and $y = \frac{x}{A}$. Now the order is $0 \le \frac{\xi}{\eta_1} < k \le 1 < b$. Thus option's price function (16) turns to

$$f(A,B,x) = \frac{A}{y^q} \frac{(\eta_1+\eta_2-\eta_1 k+\xi)(b^p-y^p)+(b-k)b^q(y^p-1)}{b^p-1}. \tag{20}$$

Hence, we have to derive the maximum of the function

$$\begin{aligned} g(b) &= \frac{(\eta_1+\eta_2-\eta_1 k+\xi)(b^p-y^p)+(b-k)b^q(y^p-1)}{b^p-1} \\ &= \frac{b^p(\eta_1+\eta_2-\eta_1 k+\xi)+b^{q+1}(y^p-1)-b^q k(y^p-1)-(\eta_1+\eta_2-\eta_1 k+\xi)y^p}{b^p-1}. \end{aligned}$$

Its derivative is

$$g_b(b) = \frac{y^p-1}{(b^p-1)^2} b^{q-1} \begin{bmatrix} -b^{p+1}(p-q-1)+b^p k(p-q) \\ +b^{p-q}p(\eta_1+\eta_2-\eta_1 k+\xi) - b(q+1) + qk \end{bmatrix}. \tag{21}$$

We show in Proposition B.3 that this function has just one root in the interval $(1,\infty)$ except in the marginal case $\eta_2 = \eta_3 = 0$, which is examined in [65]. We shall denote the root by $b(A)$. Thus pricing function (20) has a maximum for $B = b(A)A$.

Let us denote the true boundaries (if they exist) by A^* and B^*. We search the potential writer's optimal boundary as the unique solution $\overline{A}$ of the equation $yb(y)a(yb(y)) = y$ or equivalently

$$b(y)a\big(yb(y)\big) = 1. \tag{22}$$

If $\overline{A} \ge K$, then this is the true boundary, i.e. $A^* = \overline{A}$ and $B^* = b(\overline{A})\overline{A}$. It may happen that $\overline{A} < K$, because we have changed the original payment functions in formula (14) from $N_1(t,x) = e^{-\lambda t}(x-K)^+$ and $N_2(t,x) = e^{-\lambda t}(\eta_1(x-K)^+ + \eta_2 x + \eta_3)$ to $N_1(t,x) = e^{-\lambda t}(x-K)$ and $N_2(t,x) = e^{-\lambda t}(\eta_1(x-K) + \eta_2 x + \eta_3)$, respectively. Note that we are just in this case when $r < 0$ due to Proposition 3.7. Hence, if $\overline{A} < K$ we have to recognize whether the writer's exercise region is the singleton $\{K\}$ or the empty set. Suppose that the initial asset price is the strike, $x = K$. The writer has the alternatives to cancel immediately or to do nothing. In the first case he has to pay amount of $\eta_2 K + \eta_3$, whereas in the second one the option turns to ordinary American. It is shown in [68], Proposition 6.1, that its price is

$$\overline{\eta} = \frac{K}{\gamma}\left(\frac{\gamma - 1}{\gamma}\right)^{\gamma-1} \tag{23}$$

for $\gamma = p - q$. We conclude that if $\eta_2 K + \eta_3 \leq \overline{\eta}$, then the writer would prefer to cancel the option immediately, i.e. $\Upsilon^s = \{K\}$ and thus $A^* = K$ and $B^* = b(K)K$. Otherwise, if $\eta_2 K + \eta_3 > \overline{\eta}$, then $\Upsilon^s = \emptyset$, which means that the option is ordinary American, A^* does not exist, and $B^* = \frac{\gamma}{\gamma-1}K$, see Proposition 6.1 from [68].

Note at last that if the writer's exercise region is not empty and the asset starts below the strike, $x < K$, then the writer cancels when the asset hits the strike and this strategy leads to the option price

$$\begin{aligned} &E^x\left[e^{-(r+\lambda)\tau}\left(\eta_1(S_\tau - K)^+ + \eta_2 K + \eta_3\right)I_{\tau<\infty}\right] \\ &= (\eta_2 K + \eta_3)E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] = (\eta_2 K + \eta_3)\left(\frac{x}{K}\right)^{\gamma} \end{aligned} \tag{24}$$

due to Proposition A.1.

Remark 1. If $\eta_2 = \eta_3 = 0$, then we have an option with a proportional penalty – we refer to Theorem 4.1 from [65]. There all points below the strike are considered as writer optimal, since the writer owes nothing. On the other hand, these points can be viewed also as belonging to the continuation region, since the first hitting to the strike strategy gives the same results – we can apply Condition 2.2. Let us mention, that if $r \leq 0$, then $\Upsilon^s = \{K\}$ and $\Upsilon^b = (K, \infty)$. If $r > 0$, then the results are similar to the general case presented below.

We summarize the derived results in the following theorem.

Theorem 3.8. *Let $\lambda > 0$ and $\eta_2 + \eta_3 > 0$. If $\eta_3 \geq \eta_1 K$, then $\Upsilon^s = \emptyset$ and the option is ordinary American – for more details about these options see Theorem 6.1 from [68]. If $\eta_3 < \eta_1 K$ in addition to $\eta_2 + \eta_3 > 0$, then $\overline{A}$ is defined as the solution of equation (22) and the following statements hold.*

If $\overline{A} \geq K$, then $A^* = \overline{A}$ and $B^* = \overline{A}b(\overline{A})$. The exercise regions for the writer and holder are $\Upsilon^s = [K, A^*]$ and $\Upsilon^b = [B^*, \infty)$, respectively. The option price $V(x)$ is given by equation (24) when $x \leq K$; $V(x) = (\eta_1 + \eta_2)x - \eta_1 K + \eta_3$ when $K < x < A^*$; $V(x) = x - K$ when $x > B^*$; and $V(x)$ is given by formula (16) when $A^* \leq x \leq B^*$.

If $\overline{A} < K$ and $\eta_2 K + \eta_3 \leq \overline{\eta}$, $\overline{\eta}$ is given by equation (23), then $A^* = K$ and $B^* = Kb(K)$. The exercise regions are $\Upsilon^s = \{K\}$ and $\Upsilon^b = [\overline{B}, \infty)$. The option price $V(x)$ is determined as in the previous case.

If $\overline{A} < K$ and $\eta_2 K + \eta_3 > \overline{\eta}$, then the option is again ordinary American.

3.3 Smooth fit principle

Let us discuss now the smooth fit principle, i.e. when the derivative of the value function $V(x)$ is continuous at the optimal boundaries. We have that $V'(x) = 1$ for

$x \in \Upsilon^b$ and $V'(x) = \eta_1 + \eta_2$ for $x \in \Upsilon^s$. If $x > K$ and $x \in \overline{\Upsilon}$, then we derive $V'(x)$ differentiating formula (16):

$$V'(x) = \frac{(B^* - K)B^{*q}(x^p(p-q) + qA^{*p})}{x^{q+1}(B^{*p} - A^{*p})} - \frac{((\eta_1+\eta_2)A^* - \eta_1 K + \eta_3)A^{*q}(x^p(p-q) + qB^{*p})}{x^{q+1}(B^{*p} - A^{*p})}. \tag{25}$$

Let us check the smooth fit at the holder's boundary. Using again the change of variables $b^* = \frac{B^*}{A^*}$, $k = \frac{K}{A^*}$, $\xi = \frac{\eta_3}{A^*}$, and $y = \frac{x}{A^*}$, we derive for derivative (25) at the point b^*

$$V'(b^*) = \frac{(b^*-k)(b^{*p}(p-q)+q) - ((\eta_1+\eta_2) - \eta_1 k + \xi)(b^{*p}(p-q) + qb^{*p})}{b^*(b^{*p}-1)}. \tag{26}$$

We can easily check that $V'(b^*) = 1$, because b^* is the root of function (49). Hence, there is a smooth fit at the holder's boundary. Analogously, we can establish the smooth fit at the writer's boundary A^* when $A^* = \overline{A} \geq K$ having in mind that we use the root of function (48) to derive the value of $\overline{A}$. Otherwise, if $\overline{A} < K$, then we have not smooth fitting at the writer's boundary, namely the strike, because it is not B^*-writer optimal. Note that B^* is K-holder optimal which confirms the smooth fit at B^*.

On the other hand, all points below the strike belong to the continuation region. Suppose that the writer's optimal region is not empty. We cannot expect lower smooth fit at the strike because the writer's payoff function $N_2(t, x)$ is not smooth namely at the strike.

We can summarize: we have always smooth fit at the holder's boundary, but only when $\overline{A} \geq K$ at the writer's one.

3.4 *Absence of discounting*

Suppose now, that $\lambda = 0$ or equivalently $p = q + 1$. We have $r > 0$, because the total discount factor is positive. Proposition 3.6 shows that it is never optimal for the holder to exercise the option. Hence, his boundary is infinitely large. This conclusion is supported by the fact that derivative (21) is always positive, which is proven in Proposition B.3. Thus the holder maximizes his utility for $B = \infty$. Suppose that the writer's optimal boundary is a constant $A \geq K$ and the underling asset starts above it, $x > A$. Let us denote the first hitting moment of the asset to the level A by ζ and the price of a down-and-out barrier option with strike K and barrier A by $C_{DO}(x, A, K)$. Therefore, the option price can be presented as the following dependent on A function

$$\begin{aligned} F(A) &= E^x\left[e^{-r\zeta}\left(\eta_1(S_\zeta - K)^+ + \eta_2 S_\zeta + \eta_3\right)I_{\zeta<\infty}\right] \\ &\quad + \lim_{T\to\infty} E^x\left[e^{-rT}(S_T - K)^+ I_{T<\zeta}\right] \\ &= \left((\eta_1+\eta_2)A - \eta_1 K + \eta_3\right)E^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] + \lim_{T\to\infty} C_{DO}(x, A, K). \end{aligned} \tag{27}$$

Using Proposition A.1 and equation (10.45) from [71] we obtain

$$E^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}, \quad \lim_{T\to\infty} C_{DO}(x, A, K) = x\left(1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}\right). \tag{28}$$

Substituting equations (28) into (27) we derive for the option price

$$F(A) = x + \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}} \big(A(\eta_1 + \eta_2 - 1) + \eta_3 - \eta_1 K\big). \tag{29}$$

Its derivative is $F'(A) = \frac{1}{A\sigma^2}(\frac{A}{x})^{\frac{2r}{\sigma^2}} h(A)$, where the function $h(A)$ is

$$h(A) = A(\eta_1 + \eta_2 - 1)\big(2r + \sigma^2\big) + 2r(\eta_3 - \eta_1 K). \tag{30}$$

It is linear and increasing with root

$$M = \frac{2r(\eta_1 K - \eta_3)}{(\eta_1 + \eta_2 - 1)(2r + \sigma^2)}. \tag{31}$$

Note that it is positive. We have two cases to examine – suppose first that $M \leq K$. Thus derivative $F'(A)$ is always positive for $A > K$. Hence, the price function (29) is increasing and therefore its minimum is for $A = K$. We have to recognize whether the writer's exercise region is the singleton $\{K\}$ or the empty set. Suppose that the option is at-the-money, i.e. the initial asset price is the strike. The writer has the alternatives to cancel the option immediately or to do nothing. If he chooses the first one, then he has to pay $\eta_2 K + \eta_3$. The second alternative turns the option to European, since the holder will never exercise earlier, too. Its price is just the initial asset value – we can see this if we take the limit $T \to \infty$ in the Black–Scholes formula. Hence, $\Upsilon \equiv \{K\}$ when $\eta_2 K + \eta_3 \leq K$, and it is the empty set otherwise. Thus, if $\eta_2 K + \eta_3 \leq K$, the option price (29) takes the form

$$x + \left(\frac{K}{x}\right)^{\frac{2r}{\sigma^2}} \big(K(\eta_2 - 1) + \eta_3\big) \tag{32}$$

when $x \geq K$. Note that function (32) is increasing, since $K(\eta_2 - 1) + \eta_3 < 0$ and therefore its minimum is for $x = K$ and it is $\eta_2 K + \eta_3$. When $x < K$ we use Proposition A.1 to derive the option price as

$$(\eta_2 K + \eta_3) E^x\big[e^{-r\zeta} I_{\zeta<\infty}\big] = \frac{(\eta_2 K + \eta_3)x}{K}. \tag{33}$$

If $\eta_2 K + \eta_3 > K$, early exercising is never optimal neither for the writer nor for the holder. Hence, the option turns to a European one and its price is the initial asset price x.

Suppose now that $M > K$ and therefore function (30) is negative for $A \in (K, M)$ and positive for $A > M$. Therefore price function (29) has a minimum for $A = M$. This means that the exercise boundary is given by formula (31) and the writer's exercise region is the interval (K, M). Hence, the option price is given by formula (33) when $x < K$. Also, if $x \geq M$, then option price formula (29) turns to

$$x\left(1 - \frac{\sigma^2(\eta_1 + \eta_2 - 1)}{2r}\left(\frac{2r(\eta_1 K - \eta_3)}{x(\eta_1 + \eta_2 - 1)(2r + \sigma^2)}\right)^{\frac{2r}{\sigma^2}+1}\right). \tag{34}$$

We can summarize the results above in the following theorem.

Theorem 3.9. *Let $\lambda = 0$ and suppose first that $M < K$, where the constant M is defined by formula (31). If $\eta_2 K + \eta_3 > K$, then early exercising is never optimal for both participants and the option price is $V(x) = x$. Otherwise, if $\eta_2 K + \eta_3 \leq K$, then the writer's exercise region is the strike. The option price $V(x)$ is given by equation (32) when $x \geq K$ and by equation (33) when $x < K$.*

If $M \geq K$, then the writer's exercise region is $\Upsilon^s = [K, M]$. The price $V(x)$ is given by statement (33) when $x < K$; by (34) when $M < x$; and $V(x) = (\eta_1 + \eta_2)x - \eta_1 K + \eta_3$ when $K \leq x \leq M$.

4 Put style options

We turn to the cancellable put options considering payments structures (3). We work in a similar manner giving only the differences with the call case. Analogously to Proposition 3.1 we can prove that all points above the strike are not optimal for both participants:

Proposition 4.1. *If $x > K$, then $x \in \overline{\Upsilon}$.*

The following restriction for the penalty coefficients stands.

Proposition 4.2. *If $\eta_2 \geq \eta_1$, then $\Upsilon^s \equiv \emptyset$.*

Proof. Suppose that $\eta_2 \geq \eta_1$ and $x \in \Upsilon^s$. Proposition 4.1 gives us that $x \leq K$. Note that the price of the ordinary perpetual at-the-money American put option, denoted by $\overline{\eta}$, is less the strike, $\overline{\eta} < K$. The point $x = K$ cannot be writer's optimal because the writer has to pay $\eta_2 K + \eta_3 \geq K > \overline{\eta}$ (note that $\eta_2 \geq \eta_1 \geq 1$), i.e. the strategy of never canceling, which leads to a pure American option, is better for him. Hence $x < K$. We continue in the same way as in Proposition 3.2 turning contradictory inequality (11) to

$$\begin{aligned}
\eta_1(K - x) + \eta_2 x + \eta_3 &\leq M\big(x; \tau, B(\tau; x)\big) \\
&= E^x \left[\begin{aligned} &e^{-(r+\lambda)B(\tau;\cdot)}(K - S_{B(\tau;\cdot)}) I_{B(\tau;\cdot)\leq\tau} \\ &+ e^{-(r+\lambda)\tau}(\eta_1 K + \eta_3 + (\eta_2 - \eta_1)S_\tau) I_{\tau < B(\tau;\cdot)} \end{aligned} \right] \\
&\leq E^x \big[e^{-(r+\lambda)\zeta} \big(\eta_1 K + \eta_3 + (\eta_2 - \eta_1) S_\zeta\big)\big] \\
&< \eta_1(K - x) + \eta_2 x + \eta_3. \qquad \square
\end{aligned}$$

We assume hereafter that $\eta_2 < \eta_1$. The following statements describe the shape of the exercise boundaries.

Proposition 4.3. *If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (K, \infty)$.*

Proof. See Proposition 2.6 from [67]. □

Proposition 4.4. *The following two statements hold: (A) if $x \in \Upsilon^b$ and $y < x$, then $y \in \Upsilon^b$ and (B) if $x < K$, $x \in \Upsilon^s$, and $x < y < K$, then $y \in \Upsilon^s$.*

Proof. We refer again to Proposition 3.5 from [65] for the proof of the first part. Let us turn to the second statement of the proposition. Suppose that a point y from the interval (x, K) is not writer optimal, $y \notin \Upsilon^s$. If $y \in \Upsilon^b$, the first part of the proposition leads to $x \in \Upsilon^b$, which is impossible. Hence $y \in \overline{\Upsilon}$. Something more, all points between x and y are not holder optimal. Suppose also, that all points above y are not writer optimal. Therefore they belong to the continuation region. Let us examine the writer's minimization problem if the initial asset price is $S_0 = y$. Let τ_z be first hitting of the asset to the value z. The writer has to minimize the following term in the interval $z \in (0, y)$

$$\begin{aligned} h(z) &= E^y\left[e^{-(r+\lambda)\tau_z}\left(\eta_1(K - S_{\tau^z}) + \eta_2 S_{\tau^z} + \eta_3\right)\right] \\ &= (\eta_1 - \eta_2)E^y\left[e^{-(r+\lambda)\tau_z}\left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - S_{\tau^z}\right)\right]. \end{aligned} \tag{35}$$

Function (35) is the payment of $(\eta_1 - \eta_2)$ ordinary American put options with strike $\frac{\eta_1 K+\eta_3}{\eta_1-\eta_2}$; note that $\eta_1 - \eta_2 > 0$. This function first increases to a maximum and then decreases; for the proof see Theorem 6.2 from [68]. Hence, its minimum is either for $z = 0$ or for $z = y$. The second one contradicts to $y \in \overline{\Upsilon}$, whereas the first one contradicts to $x \in \Upsilon^s$.

Suppose now, that some point between y and K is writer optimal. Therefore there exist points $B < C$ such that $\{B, C\} \in \Upsilon^s$ and the interval between them is a part of the continuation region, $(B, C) \in \overline{\Upsilon}$. We can think that $B < y < C$. Let us denote by ζ_B and ζ_C the first hitting moments of the underlying asset to the levels B and C, respectively, and by ζ the lesser of them, $\zeta = \zeta_B \wedge \zeta_C$. Note that $\zeta < B(\zeta, y)$. Therefore

$$\eta_1 K + \eta_3 - (\eta_1 - \eta_2)y > E^y\left[e^{-(r+\lambda)\zeta}\left(\eta_1 K + \eta_3 - (\eta_1 - \eta_2)S_\zeta\right)\right]. \tag{36}$$

Let us define a new cancellable option with strike $\frac{\eta_1 K+\eta_3}{\eta_1-\eta_2}$ and without penalty. We shall denote by Υ_1^s, Υ_1^b, and $\overline{\Upsilon}_1$ the corresponding regions and by $A_1(\cdot)$ and $B_1(\cdot)$ the writer's and holder's optimal strategies, respectively. The fact that $x \in \Upsilon^s$ means that the writer prefers to cancel the option immediately provided that he may pay less if the holder exercises. This means that the writer will prefer to stop the contract immediately again if there is no possibility for a lower payment when the holder exercises the option. Hence $x \in \Upsilon_1^s$ and thus $y \notin \Upsilon_1^b$, because the opposite would contradict to the first part of the proposition. Note that the conclusion above is true for all points between B and C. Suppose that $y \in \Upsilon_1^s$ and let us examine the strategy ζ. We have that $\zeta < B_1(\zeta, y)$, since $(B, C) \notin \Upsilon_1^b$. Therefore

$$\begin{aligned} &(\eta_1 - \eta_2)\left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - y\right) \\ &\leq (\eta_1 - \eta_2)E^y\left[e^{-(r+\lambda)(\zeta\wedge B_1(\zeta_C;y))}\left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - S_{\zeta\wedge B_1(\zeta;y)}\right)\right] \\ &= E^y\left[e^{-(r+\lambda)\zeta}\left(\eta_1 K + \eta_3 - (\eta_1 - \eta_2)S_\zeta\right)\right], \end{aligned} \tag{37}$$

which contradicts to inequality (36). Thus $y \in \overline{\Upsilon}_1$, which is impossible due to Proposition 4.3. The last contradiction finishes the proof. □

Proposition 4.5. *If $r > 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.*

Proof. The proof is similar to the proof of Proposition 3.7. Supposing the opposite, we construct ζ as the first exit from the strip (K_1, K). Assuming $x \in (K_1, K)$ we modify inequality (12) to

$$\begin{aligned}
&E^x\big[e^{-(r+\lambda)\zeta}\big(\eta_1(K-S_\zeta)+\eta_2 S_\zeta+\eta_3\big)I_{\zeta\le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}(K-S_\zeta)I_{B(\zeta;x)<\zeta}\big] \\
&= E^x\big[e^{-(r+\lambda)\zeta}\big(\eta_1(K-S_\zeta)+\eta_2 S_\zeta+\eta_3\big)\big] \\
&\le E^x\big[e^{-r\zeta}\big(\eta_1(K-S_\zeta)+\eta_2 S_\zeta+\eta_3\big)\big] < \eta_1(K-x)+\eta_2 x+\eta_3.
\end{aligned}$$

Hence, the point x cannot be writer's optimal. □

Let us turn to the pricing problem. We shall use an approach similar to those presented in Section 3.2 to obtain the equations which are solved by the optimal boundaries. Propositions 4.1 and 4.4 indicate that the holder's exercise region has the form $\Upsilon^b = (0, A]$ for some constant A, whereas the writer's set has one of the following three forms: $\Upsilon^s = [B, K]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$.

If $\eta_2 = \eta_3 = 0$ we have an option with multiplied penalty. These options are examined in [65]. Suppose now that $\eta_2+\eta_3 > 0$ and $A < x < B < K$. Let us denote again by τ^A and τ^B the first hitting moments of the underlying asset to the values A and B, respectively. The pricing function of the option can be written as

$$\begin{aligned}
f(A,B,x) &= E\left[\begin{matrix} e^{-(r+\lambda)\tau^B}(\eta_1 K-(\eta_1-\eta_2)S_{\tau^B}+\eta_3)I_{\tau^B\le\tau^A} \\ +\,e^{-(r+\lambda)\tau^A}(K-S_{\tau^A})I_{\tau^A<\tau^B}\end{matrix}\right] \\
&= (K-A)\left(\frac{A}{x}\right)^q\frac{B^p-x^p}{B^p-A^p} + \big(\eta_1 K-(\eta_1-\eta_2)B+\eta_3\big)\left(\frac{B}{x}\right)^q\frac{x^p-A^p}{B^p-A^p}. \qquad (38)
\end{aligned}$$

For the meaning of p and q, see equations (15). Note that $p \ge q+1$ and the equality is reached when $\lambda = 0$. First, let us fix the boundary B. The change of variables we use, $a = \frac{A}{B}$, $k = \frac{K}{B}$, $y = \frac{x}{B}$, and $\xi = \frac{\eta_3}{B}$, leads to an order $0 < a < 1 \le k$. Thus pricing function (38) can be transformed to $f(A,B,x) = \frac{B}{y^q}g(a)$ where the function $g(a)$ is

$$\begin{aligned}
g(a) &= \frac{(k-a)a^q(1-y^p)+(\eta_1 k-\eta_1+\eta_2+\xi)(y^p-a^p)}{1-a^p} \\
&= \frac{-a^p(\eta_1 k-\eta_1+\eta_2+\xi)-a^{q+1}(1-y^p)+a^q k(1-y^p)+y^p(\eta_1 k-\eta_1+\eta_2+\xi)}{1-a^p}.
\end{aligned} \qquad (39)$$

It is proven in Appendix B, Proposition B.6, that its derivative

$$g_a(a;y) = \frac{1-y^p}{(1-a^p)^2}a^{q-1}\left[\begin{matrix}-a^{p+1}(p-q-1)+a^p k(p-q) \\ -\,a^{p-q}p(\eta_1 k-\eta_1+\eta_2+\xi)-a(q+1)+qk\end{matrix}\right]$$

has a unique root in the interval $(0, 1)$ which leads to the maximum of the price function. We shall denote the root by $a(B)$. Hence, if the writer's strategy is to cancel when the asset reaches the level B, then the holder's strategy is to exercise at level $Ba(B)$.

Let us fix now the value A in formula (38). We change the variables to $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta_3}{A}$. Therefore we have to examine $b > 1$. Note that $k > 1$. Price function (38) turns to $f(A, B, x) = \frac{A}{y^q} g(b)$ where

$$g(b) = \frac{b^p(k-1) - b^{q+1}(\eta_1 - \eta_2)(y^p - 1) + b^q(\eta_1 k + \xi)(y^p - 1) - (k-1)y^p}{b^p - 1}.$$

Its derivative is

$$g_b(b; y) = \frac{y^p - 1}{(b^p - 1)^2} b^{q-1} \begin{bmatrix} b^{p+1}(p-q-1)(\eta_1 - \eta_2) - b^p(\eta_1 k + \xi)(p-q) \\ + b^{p-q} p(k-1) + b(q+1)(\eta_1 - \eta_2) - q(\eta_1 k + \xi) \end{bmatrix}. \tag{40}$$

Suppose first $\lambda > 0$ or equivalently $p > q + 1$. It is proven in Proposition B.4 that derivative (40) has a unique root larger than one. It leads to the minimum of the price function and we shall denote it by $b(A)$. Hence, our candidate for the writer's boundary is the solution $\overline{B}$ of the equation $Ba(y)b(ya(y)) = B$ or equivalently

$$a(y)b\big(ya(y)\big) = 1. \tag{41}$$

We shall denote the true holder's and writer's boundaries by A^* and B^*. If $\overline{B} \le K$, then $B^* = \overline{B}$ and $A^* = B^* a(B^*)$. If $\overline{B} > K$, then we need to recognize when $\Upsilon^s = \{K\}$ and when $\Upsilon^s = \emptyset$. Similarly to the call case, we conclude that $\Upsilon^s = \{K\}$ when $\eta_2 K + \eta_3 \le \overline{\eta}$ and $\Upsilon^s = \emptyset$ when $\eta_2 K + \eta_3 > \overline{\eta}$, where $\overline{\eta}$ is the price of the corresponding noncancellable at-the-money American option. We derive its value via Theorem 6.2 from [68]

$$\overline{\eta} = \frac{K}{q+1}\left(\frac{q}{q+1}\right)^q. \tag{42}$$

If $\lambda = 0$, then derivative (40) is negative for $b > 1$ due to Proposition B.4 from Appendix B. Therefore $\overline{B} = \infty$, particularly $\overline{B} > K$, and hence the writer's exercise region is either empty or the singleton $\{K\}$. This case is examined above. Note that the same result can be establish via Proposition 4.5: $r > 0$, since $r + \lambda > 0$ and $\lambda = 0$.

Finally, if we suppose that the writer's optimal region is not empty and the asset starts above the strike $x > K$, then the optimal writer's strategy is first hitting to the strike. We use Proposition A.1 to obtain the option price as

$$E^x\big[e^{-(r+\lambda)\tau}\big(\eta_1(S_\tau - K)^+ + \eta_2 S_\tau + \eta_3\big)I_{\tau<\infty}\big] = (\eta_2 K + \eta_3)\left(\frac{K}{x}\right)^q. \tag{43}$$

We summarize the derived results in the following theorem.

Theorem 4.6. *If $\eta_2 \ge \eta_1$, then the option is ordinary American. Suppose now $\eta_2 < \eta_1$ and $\eta_2 + \eta_3 > 0$.*[1] *If $\overline{B} \le K$, then $B^* = \overline{B}$ and $A^* = B^* a(B^*)$; thus the exercise*

[1] If $\eta_2 = \eta_3 = 0$ we refer to Theorem 6.1 from [65]. See also Remark 1. Let us mention that if $r \ge 0$, then $\Upsilon^s = \{K\}$ and $\Upsilon^b = (0, K)$. The rest of the results are similar to the general case.

regions for the writer and holder are $\Upsilon^s = [B^*, K]$ *and* $\Upsilon^b = [0, A^*)$*, respectively. The option price* $V(x)$ *is given by equation* (43) *when* $x \geq K$*; by formula* (38) *when* $\overline{A} \leq x \leq \overline{B}$*; it is* $V(x) = -(\eta_1 - \eta_2)x + \eta_1 K + \eta_3$ *when* $\overline{B} < x < K$*; and* $V(x) = K - x$ *when* $x < \overline{A}$*.*

If $K < \overline{B}$ *and* $\eta_2 K + \eta_3 \leq \overline{\eta}$*,* $\overline{\eta}$ *is given by equation* (42)*, then* $B^* = K$ *and* $A^* = Ka(K)$*. The exercise regions are* $\Upsilon^s = \{K\}$ *and* $\Upsilon^b = (0, \overline{A}]$*. The option price is determined as in the previous case.*

If $K < \overline{B}$ *and* $\eta_2 K + \eta_3 > \overline{\eta}$*, then the option is again ordinary American.*

Remark 2. Analogously to the results from Section 3.3 we always establish a smooth fit at the holder's boundary, but only when $\overline{B} < K$ at the writer's one.

5 Numerical results

Now we discuss some numerical examples based on the theoretical results presented above.

5.1 Call options

As we have seen above, the penalty coefficients η_1, η_2, and η_3 influence significantly the option behavior. Roughly said, the option looks more like the corresponding ordinary American call when they are larger. We shall see for which values the cancellable feature has its impact. First, Proposition 3.2 says that η_3 is limited by the inequality $\eta_3 < \eta_1 K$. It turns out that this evaluation is too weak. We know that if the writer's optimal set is not empty, then the strike belongs to it. Hence, $\eta_2 K + \eta_3 < \overline{\eta}$, where $\overline{\eta}$ is given by equation (23). Obviously, this equation is stronger due to $\overline{\eta} < K$. Otherwise, the assumption $\Upsilon^s = \emptyset$ means that the nonuse of the canceling right is the best writer's strategy, particularly better than the immediate exercise. So, the inequality $\eta_2 K + \eta_3 \geq \overline{\eta}$ holds and hence it determines whether the option is ordinary American or cancellable. Also, we can see that if the number of shares, η_2, is larger than $\frac{1}{\gamma}(\frac{\gamma-1}{\gamma})^{\gamma-1}$, then the option is ordinary American ($\gamma = p - q$). Otherwise, the value $K[\frac{1}{\gamma}(\frac{\gamma-1}{\gamma})^{\gamma-1} - \eta_2]$ is critical for the fixed amount η_3: if it is larger, then the option is ordinary American; otherwise it is a real cancellable option. We have to mention an important fact that the option's essence does not depend on the coefficient η_1 – it influences whether the writer's optimal set is only the strike once we know that the option is real cancellable.

Having in mind the previous restrictions, we examine call options with the following parameters: the risk free rate $r = 0.05$, the discount factor $\lambda = 0.01$, the volatility $\sigma = 0.3$, the strike $K = \$5$, and the initial asset value $x = \$20$. We vary the penalty parts as $\eta_1 \in (1, 1.1)$, $\eta_2 \in (0, 0.3)$, and $\eta_3 \in (0, 1)$. When we fix some of these penalties, we use the values $\eta_1 = 1.05$, $\eta_2 = 0.2$, and $\eta_3 = 0.5$.

The behavior of the optimal boundaries w.r.t. the three components of the penalty is presented in Figure 1. We can see that the writer's boundary decreases to the strike when the penalty coefficients increase – we mark by red color the critical values. Note that this boundary vanishes when the penalties are large enough. Also, the holder's boundary is an increasing function and it tends to the American optimal boundary.

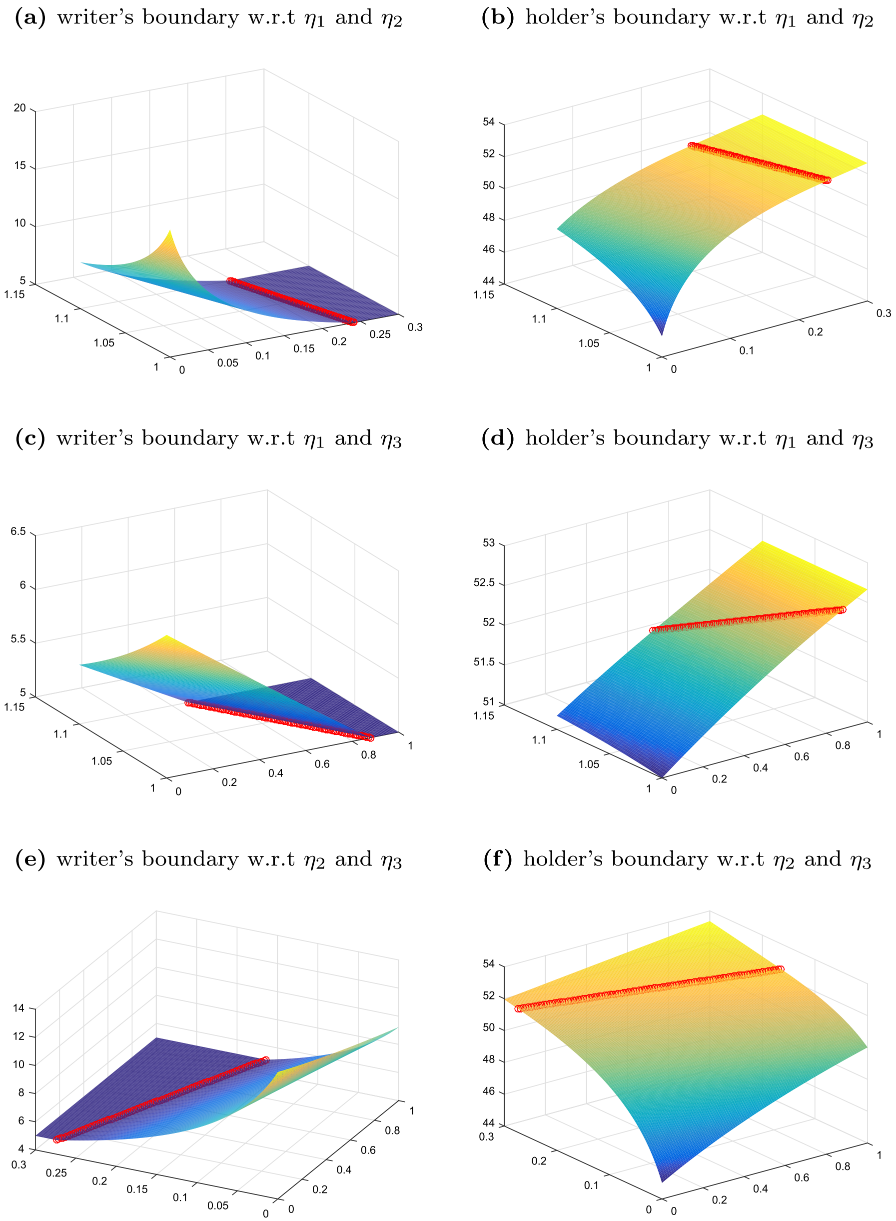


Fig. 1. Call options boundaries

We present the call prices at Figures 3a, 3b, and 3c varying the three different penalty coefficients.

The results for some particular options are reported in Table 1. There can be seen the option prices – the second line – as well as the optimal boundaries. The writer's

Table 1. Call option prices and optimal boundaries

η_2	0.05	0.1	0.15	0.2
penalty cefficient $\eta_1 = 1$				
$\eta_3 = 0.25$	{10.9835; 47.9751} \$15.6691	{8.4011; 49.8094} \$15.8449	{6.9288; 50.8208} \$15.9507	{5.9432; 51.4786} \$16.0229
$\eta_3 = 0.5$	{10.0752; 49.0768} \$15.7722	{7.7986; 50.5301} \$15.9196	{6.4656; 51.3630} \$16.0100	{5.5627; 51.9147} \$16.0723
$\eta_3 = 0.75$	{9.2236; 50.0061} \$15.8649	{7.2169; 51.1625} \$15.9879	{6.0134; 51.8470} \$16.0646	{5.1889; 52.3078} \$16.1179
$\eta_3 = 1$	{8.4209; 50.7978} \$15.9482	{6.6546; 51.7185} \$16.0500	{5.5718; 52.2787} \$16.1145	{5; 52.6652} \$16.1602
penalty cefficient $\eta_1 = 1.05$				
$\eta_3 = 0.25$	{9.0261; 48.9852} \$15.7633	{7.4034; 50.2132} \$15.8863	{6.3308; 50.9953} \$15.9696	{5.5558; 51.5439} \$16.0302
$\eta_3 = 0.5$	{8.4011; 49.8094} \$15.8449	{6.9288; 50.8208} \$15.9507	{5.9432; 51.4786} \$16.0229	{5.2260; 51.9456} \$16.0759
$\eta_3 = 0.75$	{7.7986; 50.5301} \$15.9196	{6.4656; 51.3630} \$16.0100	{5.5627; 51.9147} \$16.0723	{5.1889; 52.3078} \$16.1179
$\eta_3 = 1$	{7.2169; 51.1625} \$15.9879	{6.0134; 51.8470} \$16.0646	{5.1889; 52.3078} \$16.1179	{5; 52.6652} \$16.1602
penalty cefficient $\eta_1 = 1.1$				
$\eta_3 = 0.25$	{7.8900; 49.5313} \$15.8169	{6.7256; 50.4598} \$15.9122	{5.8903; 51.1023} \$15.9813	{5.2554; 51.5765} \$16.0339
$\eta_3 = 0.5$	{7.4034; 50.2132} \$15.8863	{6.3308; 50.9953} \$15.9696	{5.5558; 51.5439} \$16.0302	{5; 51.9526} \$16.0767
$\eta_3 = 0.75$	{6.9288; 50.8208} \$15.9507	{5.9432; 51.4786} \$16.0229	{5.2260; 51.9456} \$16.0759	{5; 52.3120} \$16.1184
$\eta_3 = 1$	{6.4656; 51.3630} \$16.0100	{5.5627; 51.9147} \$16.0723	{5; 52.3120} \$16.1184	{5; 52.6652} \$16.1602
penalty cefficient $\eta_1 = 1.2$				
$\eta_3 = 0.25$	{6.5741; 50.0836} \$15.8729	{5.8489; 50.7178} \$15.9396	{5.2793; 51.2001} \$15.9920	{5; 51.5866} \$16.0351
$\eta_3 = 0.5$	{6.2297; 50.6170} &15.9288	{5.5504; 51.1660} \$15.9882	{5.0151; 51.5866} \$16.0350	{5; 51.9526} \$16.0767
$\eta_3 = 0.75$	{5.8903; 51.1023} &15.9813	{5.2554; 51.5765} \$16.0339	{5; 51.9526} \$16.0767	{5; 52.3120} \$16.1184
$\eta_3 = 1$	{5.5558; 51.5439} \$16.0302	{5; 51.9526} \$16.0767	{5; 52.3120} \$16.1184	{5; 52.6652} \$16.1602

boundary is the first value at the first line, whereas the holder's one is given at the second place. We vary the three parts of the penalty among $\eta_1 \in \{1; 1.05; 1.1; 1.2\}$, $\eta_2 \in \{0.05; 0.1; 0.15, 0.2\}$, and $\eta_3 \in \{0.25; 0.5; 0.75; 1\}$.

5.2 *Put options*

Analogously to the call case, we can see that the inequality $\eta_2 K + \eta_3 < \overline{\eta}$ determines when the option is ordinary American or cancellable – we have a cancellable option when it holds and pure American otherwise. Note that $\overline{\eta}$ is given by equation (42). We conclude that we have a real cancellable option if (A) the number of shares is less than $\frac{1}{q+1}(\frac{q}{q+1})^q$ and (B) the fixed amount η_3 is less than $K[\frac{1}{q+1}(\frac{q}{q+1})^q - \eta_2]$. If one of these conditions does not hold, then we have a noncancellable American option. Let us discuss the role of the penalty coefficient η_1. Proposition 4.2 says that a necessary condition the option to be real cancellable is $\eta_2 < \eta_1$. On the other hand, inequality $\eta_2 K + \eta_3 < \overline{\eta}$ is stronger, since $\overline{\eta} < K$ and therefore $\eta_2 < 1 \leq \eta_1$. Hence, as in the call case, the coefficient η_1 influences whether the writer's optimal set is the strike, but not whether we have a real cancellable option or pure American.

Taking into account the previous limitations, we consider put options with the following parameters: the risk free rate $r = -0.03$, the discount factor $\lambda = 0.05$, the volatility $\sigma = 0.3$, the strike $K = \$10$, and the initial asset value $x = \$5$. The penalties are taken as before: $\eta_1 \in (1, 1.1)$, $\eta_2 \in (0, 0.3)$, and $\eta_3 \in (0, 1)$. When we fix some of them, we use the values $\eta_1 = 1.05$, $\eta_2 = 0.2$, and $\eta_3 = 0.5$.

We present the optimal boundaries in Figure 2 fixing one of the penalty parts and varying the others. As we expected, the writer's boundary increases w.r.t. the penalties and goes to the strike. The meaning of the red points is preserved – they mark namely the values for which the writer's boundary turns to the strike. Also, we can see that the holder's boundaries are decreasing functions. The resulting price behavior is presented in Figures 3d, 3e, and 3f.

We report some results for option prices and the related exercise boundaries in Table 1. The optimal boundaries are placed at the first line – the holder's boundary is first; the writer's one is second. The obtained prices are presented at the second line. The three parts of the penalties are again among $\eta_1 \in \{1; 1.05; 1.1; 1.2\}$, $\eta_2 \in \{0.05; 0.1; 0.15, 0.2\}$, and $\eta_3 \in \{0.25; 0.5; 0.75; 1\}$.

A Some Laplace transforms

The following results are reported in [9], pages 223 and 233.

Proposition A.1. *Let τ be the first hitting time of a Brownian motion with drift μ to the level a. Then*

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = \begin{cases} e^{-(\sqrt{\mu^2+2y}-\mu)a} & \text{if } a > 0, \\ e^{(\sqrt{\mu^2+2y}+\mu)a} & \text{if } a < 0. \end{cases} \tag{44}$$

Also, the Laplace transforms of the first exit time from a strip (a, b) are

$$E\left[e^{-y\tau} I_{\tau=a}\right] = e^{\mu a} \frac{\sinh(b\sqrt{2y+\mu^2})}{\sinh((b-a)\sqrt{2y+\mu^2})}, \tag{45}$$

$$E\left[e^{-y\tau} I_{\tau=b}\right] = e^{\mu b} \frac{\sinh(-a\sqrt{2y+\mu^2})}{\sinh((b-a)\sqrt{2y+\mu^2})}. \tag{46}$$

B Uniqueness of the solutions

Let p and q be defined as in equations (15).

Lemma B.1. *Let $\eta > 1$ and $k < 1$. The function $\overline{h}(a)$, defined on the interval $(0, 1)$ as*

$$\overline{h}(a) = a^{p+1}\eta(p-q-1) - a^p \eta k(p-q) - a^{p-q} p(1-k) + a(q+1)\eta - q\eta k, \tag{47}$$

starts from a negative value, increases having a root, and then stays positive.

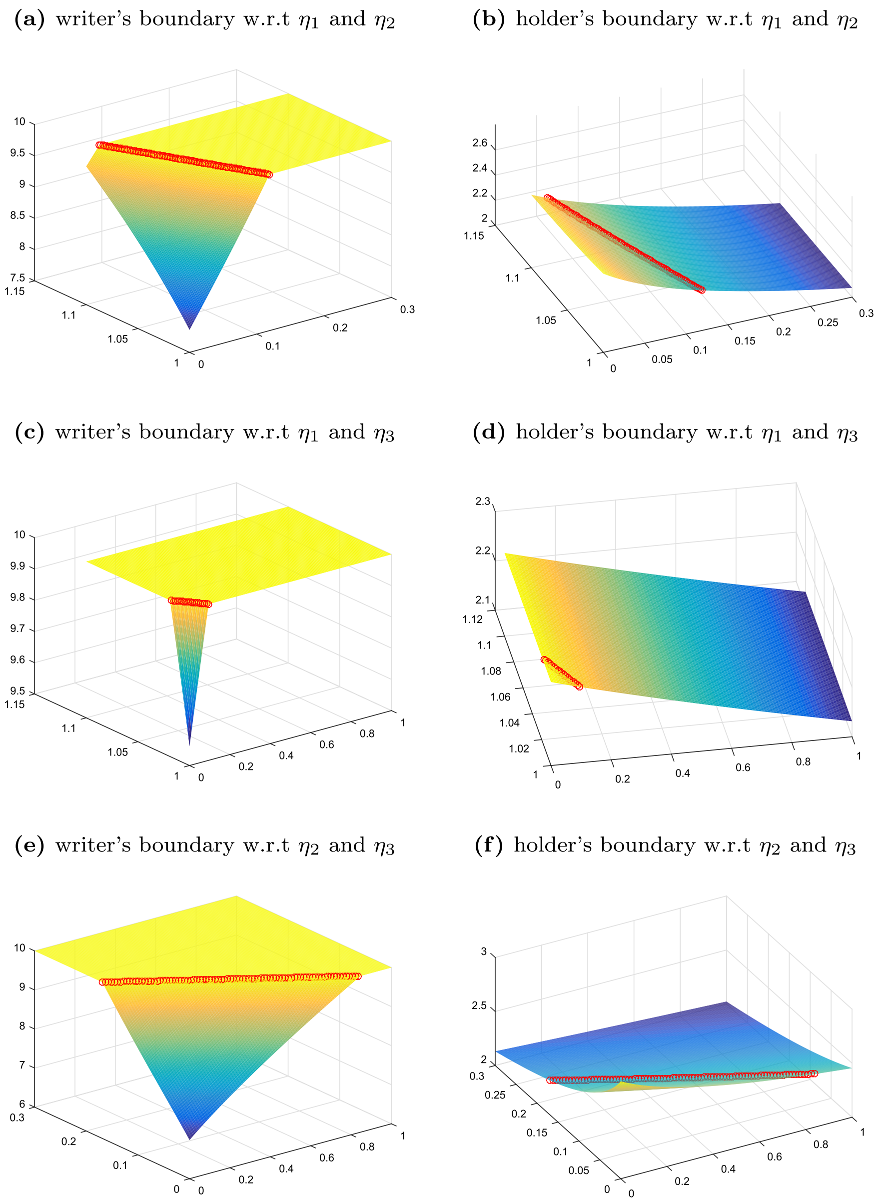


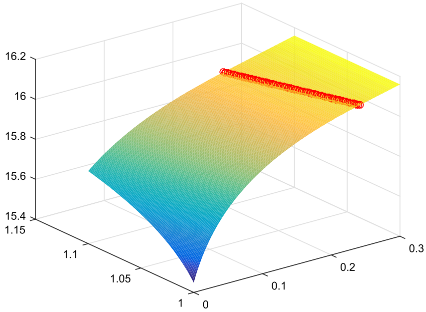
Fig. 2. Put options boundaries

Proof. See Appendix B.1 in [65]. □

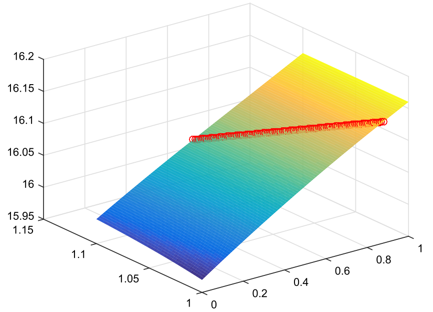
Proposition B.2. *Let* $\eta_1 \geq 1$, $\eta_2 \geq 0$, *and* $0 \leq \frac{\xi}{\eta_1} < k < 1$. *The function*

$$\begin{aligned} h(a) = {} & a^{p+1}(\eta_1+\eta_2)(p-q-1) - a^p(\eta_1 k - \xi)(p-q) \\ & - a^{p-q} p(1-k) + a(q+1)(\eta_1+\eta_2) - q(\eta_1 k - \xi) \end{aligned} \tag{48}$$

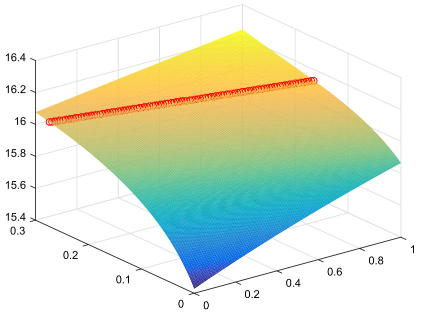
(a) Call prices w.r.t η_1 and η_2



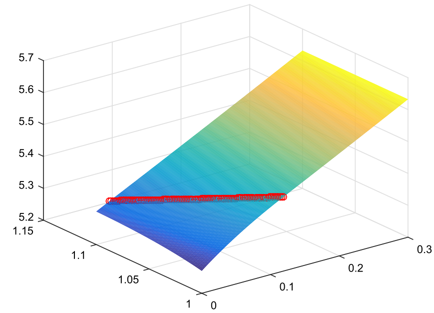
(b) Call prices w.r.t η_1 and η_3



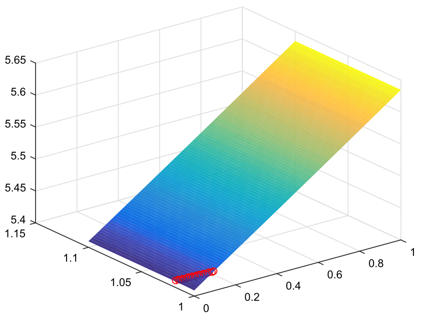
(c) Call prices w.r.t η_2 and η_3



(d) Put prices w.r.t η_1 and η_2



(e) Put prices w.r.t η_1 and η_3



(f) Put prices w.r.t η_2 and η_3

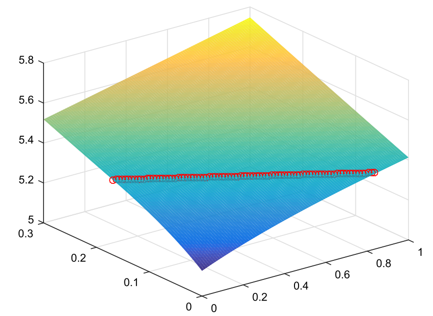


Fig. 3. Options prices

has a unique root in the interval $(0, 1)$.

Proof. First, note that if $\eta_1 = 1$ and $\eta_2 = 0$, we have an option with a constant penalty. Hence, we can use Appendix B.1 of [66]. Suppose now that $\eta_1 + \eta_2 > 1$. We can decompose function (48) as $h(a) = \overline{h}(a) + \widetilde{h}(a)$ for $\widetilde{h}(a) = (\eta_2 k + \xi)(a^p(p-q) +$

Table 2. Put option prices and optimal boundaries

η_2	0.05	0.1	0.15	0.2
penalty cefficient $\eta_1 = 1$				
$\eta_3 = 0.25$	{2.5588; 7.6681} \$5.2703	{2.4126; 8.6126} \$5.3475	{2.3100; 9.6050} \$5.4102	{2.2308; 10} \$5.4650
$\eta_3 = 0.5$	{2.4656; 8.7271} \$5.3480	{2.3541; 9.6389} \$15.9196	{2.2686; 10} \$5.4663	{2.1955; 10} \$5.5212
$\eta_3 = 0.75$	{2.4022; 9.6654} \$5.4120	{2.3093; 10} \$5.4673	{2.2308; 10} \$5.5224	{2.1625; 10} \$5.5778
$\eta_3 = 1$	{2.3534; 10} \$5.4680	{2.2686; 10} \$5.5233	{2.1955; 10} \$5.5789	{2.1315; 10} \$5.6348
penalty cefficient $\eta_1 = 1.05$				
$\eta_3 = 0.25$	{2.5247; 8.8198} \$5.2946	{2.4022; 9.6654} \$5.3564	{2.3093; 10} \$5.4108	{2.2308; 10} \$5.4650
$\eta_3 = 0.5$	{2.4551; 9.6858} \$5.3572	{2.3534; 10} \$5.4119	{2.2686; 10} \$5.4663	{2.1955; 10} \$5.5212
$\eta_3 = 0.75$	{2.4015; 10} \$5.4126	{2.3093; 10} \$5.4673	{2.2308; 10} \$5.5224	{2.1625; 10} \$5.5778
$\eta_3 = 1$	{2.3534; 10} \$5.4680	{2.2686; 10} \$5.5233	{2.1955; 10} \$5.5789	{2.1315; 10} \$5.6348
penalty cefficient $\eta_1 = 1.1$				
$\eta_3 = 0.25$	{2.5139; 9.7012} \$5.3032	{2.4015; 10} \$5.3570	{2.3093; 10} \$5.4108	{2.2308; 10} \$5.4650
$\eta_3 = 0.5$	{2.4544; 10} \$5.3579	{2.3534; 10} \$5.4119	{2.2686; 10} \$5.4663	{2.1955; 10} \$5.5212
$\eta_3 = 0.75$	{2.4015; 10} \$5.4126	{2.3093; 10} \$5.4673	{2.2308; 10} \$5.5224	{2.1625; 10} \$5.5778
$\eta_3 = 1$	{2.3534; 10} \$5.4680	{2.2686; 10} \$5.5233	{2.1955; 10} \$5.5789	{2.1315; 10} \$5.6348
penalty cefficient $\eta_1 = 1.2$				
$\eta_3 = 0.25$	{2.5132; 10} \$5.3038	{2.4015; 10} \$5.3570	{2.3093; 10} \$5.4108	{2.2308; 10} \$5.4650
$\eta_3 = 0.5$	{2.4544; 10} &\$5.3579	{2.3534; 10} \$5.4119	{2.2686; 10} \$5.4663	{2.1955; 10} \$5.5212
$\eta_3 = 0.75$	{2.4015; 10} &\$5.4126	{2.3093; 10} \$5.4673	{2.2308; 10} \$5.5224	{2.1625; 10} \$5.5778
$\eta_3 = 1$	{2.3534; 10} \$5.4680	{2.2686; 10} \$5.5233	{2.1955; 10} \$5.5789	{2.1315; 10} \$5.6348

q) and $\overline{h}(a)$ defined as (47) for $\eta = \eta_1 + \eta_2$. We finish the proof using the inequality $h(0) = -q(\eta_1 k - \xi) < 0$, Lemma B.1, and the fact that $\widetilde{h}(a)$ is an increasing positive function. □

Proposition B.3. *Let $\eta_1 \geq 1$, $\eta_2 \geq 0$, $0 \leq \frac{\xi}{\eta_1} < k \leq 1$, and the function $h(\cdot)$ be defined as*

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q} p(\eta_1 + \eta_2 - \eta_1 k + \xi) \\ - b(q+1) + qk \tag{49}$$

in the interval $b \in [1, \infty)$. If $p = q + 1$, then function (49) is positive. If $p > q + 1$, $k = 1$, $\eta_2 = \eta_3 = 0$, and $r < 0$, then function (49) is negative. In all other cases function (49) has just one root larger than one.

Proof. When $\eta_2 = \eta_3 = 0$ we refer to Appendix B.2 of [65]. The proof when $\eta_2 + \eta_3 > 0$ is very similar to the the case $\{\eta_1 = 1, \eta_2 = 0, \ \eta_3 > 0\}$, which is examined in Appendix B.2 of [66] and thus we omit it. $\square$

Proposition B.4. *Let $k > 1$, $\eta_1 > \eta_2 \geq 0$, $\eta_1 \geq 1$, and $\xi \geq 0$. If $p > q + 1$, then the function*

$$\begin{aligned} h(b) &= b^{p+1}(p-q-1)(\eta_1 - \eta_2) - b^p(\eta_1 k + \xi)(p-q) \\ &\quad + b^{p-q} p(k-1) + b(q+1)(\eta_1 - \eta_2) - q(\eta_1 k + \xi) \end{aligned} \tag{50}$$

has a unique root larger than one. Otherwise, if $p = q + 1$, then function (50) is negative for $b > 1$.

Proof. We rewrite function (50) as $h(b) = (\eta_1 - \eta_2)\overline{h}(b)$ for

$$\begin{aligned} \overline{h}(b) &= b^{p+1}(p-q-1) - b^p(\overline{k} + \overline{\xi})(p-q) + b^{p-q} p(\overline{k} - 1) + b(q+1), \\ &\quad - q(\overline{k} + \overline{\xi}), \\ \overline{k} &= 1 + \frac{k-1}{\eta_1 - \eta_2}, \\ \overline{\xi} &= \frac{(k-1)(\eta_1 - 1) + \eta_2 + \xi}{\eta_1 - \eta_2}. \end{aligned} \tag{51}$$

Note that $\eta_1 > \eta_2$, $\overline{k} > 1$, and $\overline{\xi} > 0$. The desired result is proven for the function $\overline{h}(b)$ in Appendix B.1 of [67]. $\square$

Lemma B.5. *Let $k \geq 1$ and $\xi \geq 0$. The function $\overline{h}(a)$, defined on the interval $(0, 1)$ as*

$$\overline{h}(a) = -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q} p(k-1+\xi) - a(q+1) + qk, \tag{52}$$

starts from a positive value, decreases having a root, and then stays negative.

Proof. See Appendix B.2 in [67]. $\square$

Proposition B.6. *Let $k \geq 1$, $\eta_1 > \eta_2 \geq 0$, $\eta_1 \geq 1$, $\xi \geq 0$, and the function*

$$\begin{aligned} h(a) &= -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q} p(\eta_1 k - \eta_1 + \eta_2 + \xi) \\ &\quad - a(q+1) + qk \end{aligned} \tag{53}$$

be defined on the interval $a \in (0, 1)$. If $\eta_2 + \eta_3 = 0$, then function (53) is positive when $\{p = q + 1, k = 1\}$ or $\{p > q + 1, k = 1, r \geq 0\}$. In the rest of the cases, namely, when $\{p = q+1, k > 1\}$, $\{p > q+1, k = 1, r < 0\}$, and $\{p > q+1, k > 1\}$, function (53) has unique root.

Otherwise, if $\eta_2 + \eta_3 > 0$, then function (53) has just one root. Also $h(a)$ is positive before the root and negative after it.

Proof. The proposition is proven in Appendix B.3 of [65] when $\eta_2+\eta_3 = 0$. Suppose now that $\eta_2+\eta_3 > 0$. We can decompose $h(a) = \overline{h}(a)+\widetilde{h}(a)$ where function $\overline{h}(a)$ is defined by formula (52) and $\widetilde{h}(a) = -a^{p-q} p((k-1)(\eta_1-1)+\eta_2)$. We complete the proof using Lemma B.5 and observing that $h(0) = qk > 0$ and $\widetilde{h}(a)$ is a decreasing negative function. $\square$

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