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Critical branching processes in a sparse random environment

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Abstract We introduce a branching process in a sparse random environment as an intermediate model between a Galton–Watson process and a branching process in a random environment. In the critical case we investigate the survival probability and prove Yaglom-type limit theorems, that is, limit theorems for the size of population conditioned on the survival event.

Keywords Branching process, functional limit theorem, random environment, survival probability
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1 Introduction and main results

The branching process is a random process starting with one individual, the initial ancestor, which produces offspring according to some random rule. The collection of offspring constitutes the first generation. Each individual of the first generation gives birth to a random number of children with the same offspring distribution as for the

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initial ancestor. The numbers of offspring of different individuals (including the initial ancestor) are independent. This process continues forever or until the population dies out. An interesting problem is the behavior of the long-time evolution of the process. Plainly, it depends on a particular rule that regulates giving birth to offspring. In the simplest case, when the offspring distribution is the same for all generations, the branching process is called the Galton–Watson process. We refer to [2] for numerous results concerning, for instance, long-term survival or extinction of such a process, the growth rate of the population, fluctuations of population sizes. Thanks to a simple tree structure, not only does the Galton–Watson process find numerous applications as a model of biological reproduction processes, but also in many other fields including computer science and physics.

The homogeneity of the Galton–Watson process reduces its applicability. In some cases it may happen that the population evolution conditions change randomly over time. This leads to the notion of branching process in random environment (BPRE) introduced by Smith and Wilkinson [13]. The BPRE is a population growth process, in which the individuals reproduce independently of each other with the offspring distribution picked randomly at each generation. More precisely, let ν be a random measure on the set of nonnegative integers $\mathbb{N}_0 := \{0, 1, 2, \dots, \}$. Then a sequence $(\nu_n)_{n\ge 1}$ of independent copies of ν can be interpreted as a random environment. The BPRE is then the sequence $Z' = (Z'_n)_{n\ge 0}$ defined by the recursive formula $Z'_{n+1} = \sum_{k=1}^{Z'_n} \xi_k^{(n)}$, where, given $(\nu_n)_{n\ge 1}$, $(\xi_k^{(n)})_{k\ge 1}$ are independent identically distributed (iid) and independent of Z'_n with the common distribution ν_{n+1}. We refer to the recent monograph by Kersting and Vatutin [11] for an overview of fundamental properties of this process.

We intend to study here *branching processes in sparse random environment* (BPSRE), in which homogeneity of the environment is modified on a sparse subset of $\mathbb{N}$. This is an intermediate model between Galton–Watson processes and the BPRE. To give a precise definition, let μ be a fixed probability measure on $\mathbb{N}_0$ and $\mathbb{Q} = ((d_k, \nu_k))_{k\ge 1}$ a sequence of independent copies of a random vector (d, ν), where d is a positive integer-valued random variable and ν is a random measure on $\mathbb{N}_0$ independent of d. First we choose a subset of integers marked by the positions of a standard random walk $(S_k)_{k\ge 0}$ defined by

$$S_0 = 0, \quad S_k = \sum_{j=1}^{k} d_j, \quad k \in \mathbb{N},$$

and then we impose random measures at the marked sites. The branching process in sparse random environment $\mathbb{Q}$ (BPSRE) is formally defined as follows:

$$Z_0 = 1, \quad Z_{n+1} = \sum_{j=1}^{Z_n} \xi_j^{(n)}, \quad n \in \mathbb{N}_0,$$

where, if $n = S_k$ for some $k \in \mathbb{N}$, then, given $\mathbb{Q}$, $\xi_j^{(n)}$ are independent random variables with the common distribution ν_{k+1}, which are also independent of Z_n. Otherwise, if $n \notin \{S_0, S_1, S_2, \dots\}$, then $\xi_j^{(n)}$ are independent random variables with the common distribution μ, which are also independent of Z_n. The process $(Z_n)_{n\geq 0}$ behaves like the Galton–Watson process, with the exception of some randomly chosen generations in which the offspring distribution is random.

We intend to describe how the additional randomness of the environment affects the behavior of the BPSRE. To this end, we focus on Yaglom-type results. For the Galton–Watson process in the critical case, that is, when the expected number of offspring is 1 (see (A2) below), it is known that the probability of survival up to the generation n is of the order $1/n$ and the population size conditioned to the survival set converges weakly to an exponential distribution (Section 9 in [2]). In contrast, in the critical case for the BPRE, that is, when the expectation of the logarithm of the number of offspring is 0 (see (A1)), the probability of survival up to the generation n is asymptotically $1/\sqrt{n}$, and the process conditioned to the survival event converges weakly to a Rayleigh distribution. We prove below in Theorems 1 and 2 that, although the environment is random on a sparse subset only, the behavior the BPSRE reminds that of a BPRE.

To close the introduction, we mention that closely related random walks in a sparse random environment, which is an intermediate model between the simple random walk and the random walk in a random environment, have been recently investigated in [4–6].

1.1 Notation and assumptions

Given a deterministic or random probability measure θ on $\mathbb{N}_0$, define the generating function

$$f_\theta(s) = \sum_{j=0}^{\infty} s^j \theta(\{j\}), \quad |s| \leq 1.$$

Denote by

$$A_\theta := f_\theta'(1) = \sum_{j=1}^{\infty} j\theta(\{j\})$$

its mean and by

$$\sigma_\theta := \frac{f_\theta''(1)}{(f_\theta'(1))^2} = \frac{1}{A_\theta^2} \sum_{j=2}^{\infty} j(j-1)\theta(\{j\})$$

its normalized second factorial moment. We shall also use a standardized truncated second moment defined by

$$\kappa(f_\theta; a) := \frac{1}{A_\theta^2} \sum_{j=a}^{\infty} j^2 \theta(\{j\}), \quad a \in \mathbb{N}_0.$$

To simplify our notation we shall write, for $k \geq 1$, A_k and σ_k instead of A_{ν_k} and σ_{ν_k}, respectively. Thus, in our setting $(A_k)_{k\geq 1}$ and $(\sigma_k)_{k\geq 1}$ are two (dependent) sequences of iid random variables. As usual, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ for $x \in \mathbb{R}$.

Throughout the paper we impose the following assumptions:

(A1) $\mathbb{E} \log A_1 = 0$, $\mathfrak{v}^2 := \mathrm{Var}\,(\log A_1) \in (0, \infty)$ and $\mathbb{E}(\log^- A_1)^4 < \infty$;

(A2) $A_\mu = 1$;

(A3) $\mathbb{E}d^{3/2} < \infty$ and we put $\mathtt{m} := \mathbb{E}d$;

(A4) $\mathbb{E}(\log^+ \kappa(f_\nu; a))^4 < \infty$ for some $a \in \mathbb{N}$.

1.2 Main results

Let $\tau_{\text{Sparse}} \in (0, \infty]$ be the extinction time of $(Z_n)_{n\geq 0}$, that is,

$$\tau_{\text{Sparse}} := \inf\{k \geq 0 : Z_k = 0\}.$$

The following observation is almost immediate.

Proposition 1. *Under the assumptions (A1)–(A2),* $\mathbb{P}\{\tau_{\text{Sparse}} < \infty\} = 1$.

In this paper we focus on the annealed analysis of BPSRE $(Z_n)_{n\geq 0}$, that is, on the behavior of $(Z_n)_{n\geq 0}$ averaged over all realizations of the environment. Our first main result is concerned with the (annealed) tail behavior of $\mathbb{P}\{\tau_{\text{Sparse}} > n\} = \mathbb{P}\{Z_n > 0\}$ as $n \to \infty$.

Theorem 1. *Assume (A1)–(A4). Then there exists* $C_{\text{Sparse}} \in (0, \infty)$ *such that*

$$\lim_{n\to\infty} \sqrt{n}\,\mathbb{P}\{\tau_{\text{Sparse}} > n\} = C_{\text{Sparse}}. \tag{1}$$

Our next result is an (annealed) Yaglom-type functional limit theorem for the process (Z_n). Recall that a Brownian meander, see [8], is a stochastic process $(B_+(t))_{t\in[0,1]}$ defined as follows. Let $(B(t))_{t\in[0,1]}$ be a standard Brownian motion and $\zeta := \sup\{t \in [0, 1] : B(t) = 0\}$ be its last visit to 0 on [0, 1]. Then

$$B_+(t) = \frac{1}{\sqrt{1-\zeta}}|B(\zeta + t(1-\zeta))|, \quad t \in [0, 1].$$

Theorem 2. *Assume (A1)–(A4). Then with* $(B_+(t))_{t\in[0,1]}$ *being a Brownian meander*

$$\text{Law}\left(\left(\frac{\log Z_{\lfloor nt\rfloor}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}}\right)_{t\in[0,1]} \,\middle|\, Z_n > 0\right) \Longrightarrow \text{Law}\left((B_+(t))_{t\in[0,1]}\right), \quad n \to \infty,$$

weakly on the space of probability measures on $D[0, 1]$ *endowed with the Skorokhod* J_1*-topology.*

Using formula (1.1) in [8] we obtain the following one-dimensional result.

Corollary 1. *Assume (A1)–(A4). Then, for every fixed* $t \in (0, 1]$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{\log Z_{\lfloor nt\rfloor}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}} \geq x \,\middle|\, Z_n > 0\right\} = \mathbb{P}\{B_+(t) \geq x\}, \quad x \geq 0. \tag{2}$$

The random variable $B_+(t)$ *has an absolutely continuous distribution with a bounded nonvanishing density on* $[0, \infty)$. *Furthermore,*

$$\mathbb{P}\{B_+(1) \leq x\} = 1 - e^{-x^2/2}, \quad x \geq 0,$$

so $B_+(1)$ *has the Rayleigh distribution.*

Remark 1. The assumption (A4) and the last part of the assumption (A1) can be weakened without changing the formulations of the main results. A version of (A4) appears as Assumption (C) in [11, Chapter 5]. It is a convenient general condition allowing for an (asymptotically) closed form of the survival probability and also validity of a functional limit theorem for a critical branching process in iid random environment. A more general version of Assumption (C) can be found in [1]. However, we prefer to sacrifice generality in favor of transparency and simplicity of the formulations.

2 Proofs

The proof of our main results consists of three steps. First, we analyze an embedded process $(Z_{S_n})_{n\geq 0}$ by finding its survival asymptotic and proving a counterpart of Theorem 2. Second, we deduce from the results obtained for $(Z_{S_n})_{n\geq 0}$ the corresponding statements for a randomly stopped process $(Z_{S_{\vartheta(n)}})_{n\geq 0}$, where $(\vartheta(n))_{n\geq 0}$ is the first passage time process for the random walk $(S_k)_{k\geq 0}$. At the last step, we show that $(Z_{S_{\vartheta(n)}})_{n\geq 0}$ is uniformly close to $(Z_n)_{n\geq 0}$.

2.1 Analysis of the embedded process

Observe that $(Z_{S_k})_{k\geq 0}$ is a branching process in iid random environment $\widetilde{\mathbb{Q}} = (\widetilde{\nu}_k)_{k\geq 1}$ which can be explicitly described as follows. Let $((\widetilde{Z}_j^{(i)})_{j\geq 0})_{i\geq 0}$ be a sequence of independent copies of a critical Galton–Watson process $(\widetilde{Z}_j)_{j\geq 0}$ in deterministic environment with the offspring distribution μ and $\widetilde{Z}_0 = 1$. Suppose that $((\widetilde{Z}_j^{(i)})_{j\geq 0})_{i\geq 0}$ is independent of the environment $\mathbb{Q}$. Then

$$\widetilde{\nu}_k(\{j\}) = \sum_{l=0}^{\infty} \nu_k(\{l\})\mathbb{P}\left\{\sum_{i=1}^{l} \widetilde{Z}_{d_k-1}^{(i)} = j\right\}, \quad k, j \in \mathbb{N}_0. \tag{3}$$

Let $\widetilde{\nu}$ be a generic copy of iid random measures $(\widetilde{\nu}_k)_{k\geq 1}$. Put

$$\widetilde{g}(s) := \mathbb{E}s^{\widetilde{Z}_{d-1}}, \quad |s| \leq 1,$$

where d is assumed independent of $(\widetilde{Z}_k)_{k\geq 0}$. Equality (3) entails that the generating function of the random measure $\widetilde{\nu}$ is given by

$$f_{\widetilde{\nu}}(s) = f_\nu(\widetilde{g}(s)), \quad |s| \leq 1.$$

Since $\widetilde{g}'(1) = \mathbb{E}\widetilde{Z}_{d-1} = 1$, the latter formula immediately implies that

$$A_{\widetilde{\nu}_k} = f'_{\widetilde{\nu}_k}(1) = f'_{\nu_k}(1) = A_{\nu_k} = A_k, \quad k \in \mathbb{N}_0. \tag{4}$$

Further,

$$\sigma_{\widetilde{\nu}_k} = \frac{f''_{\widetilde{\nu}_k}(1)}{(f'_{\widetilde{\nu}_k}(1))^2} = \frac{f''_{\nu_k}(1) + f'_{\nu_k}(1)\widetilde{g}''(1)}{(f'_{\nu_k}(1))^2}$$
$$= \sigma_{\nu_k} + \frac{\sigma_\mu(\mathbb{E}d-1)}{A_{\nu_k}} = \sigma_k + \frac{\sigma_\mu(\mathbb{E}d-1)}{A_k}, \quad k \in \mathbb{N}_0,$$

where we have used that

$$\widetilde{g}''(1) = \mathbb{E}\widetilde{Z}_{d-1}(\widetilde{Z}_{d-1} - 1) = \sigma_\mu(\mathbb{E}d - 1), \tag{5}$$

see, for instance, Chapter I.2 in [2] for the last equality. Note that (4) guarantees that

$$\mathbb{E}\log A_{\widetilde{\nu}_1} = \mathbb{E}\log A_{\nu_1} = \mathbb{E}\log A_1 = 0,$$

which means that the embedded process $(Z_{S_n})_{n\geq 0}$ is **critical**. In particular,

$$\tau_{\text{Embed}} := \inf\{k \geq 0 : Z_{S_k} = 0\} < \infty \quad \text{a.s.} \tag{6}$$

Recall that we denote by $\kappa(f_\theta; a)$ the truncated second moment of a measure θ.

Lemma 1. *Let* $a_* \in \mathbb{N}_0$ *and assume that*

$$\mathbb{E}(\log^+ \kappa(f_\nu; a_*))^4 < \infty \quad \textit{and} \quad \mathbb{E}(\log^- A_\nu)^4 < \infty.$$

Then $\mathbb{E}(\log^+ \kappa(f_{\widetilde{\nu}}; a_*))^4 < \infty$.

Proof. We start by writing

$$\kappa(f_{\widetilde{\nu}}; a) = \frac{1}{A_{\widetilde{\nu}}^2}\sum_{j=a}^{\infty} j^2 \sum_{l=0}^{\infty} \nu(\{l\})\mathbb{P}\left\{\sum_{i=1}^{l} \widetilde{Z}_{d-1}^{(i)} = j\right\}$$
$$= \frac{1}{A_\nu^2}\sum_{l=0}^{\infty}\nu(\{l\})\mathbb{E}\left(\left(\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\right)^2 \mathbb{1}_{\left\{\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\geq a\right\}}\right).$$

In view of (5), for all $a \in \mathbb{N}$,

$$\mathbb{E}\left(\left(\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\right)^2 \mathbb{1}_{\left\{\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\geq a\right\}}\right) \leq \mathbb{E}\left(\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\right)^2 \leq C_1 \mathtt{m} l^2,$$

where $\mathtt{m} = \mathbb{E}d$ and $C_1 > 0$ is a constant. Thus,

$$\frac{1}{A_\nu^2}\sum_{l=a_*}^{\infty}\nu(\{l\})\mathbb{E}\left(\left(\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\right)^2 \mathbb{1}_{\left\{\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\geq a_*\right\}}\right)$$
$$\leq \frac{C_1\mathtt{m}}{A_\nu^2}\sum_{l=a_*}^{\infty} l^2\nu(\{l\}) = C_1\mathtt{m}\kappa(f_\nu; a_*).$$

Since $\mathbb{E}(\log^+ \kappa(f_\nu; a_*))^4 < \infty$, it suffices to check that

$$\mathbb{E}\left(\log^+ \frac{1}{A_\nu^2}\sum_{l=0}^{a_*}\nu(\{l\})\mathbb{E}\left(\left(\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\right)^2 \mathbb{1}_{\left\{\sum_{i=1}^{l}\widetilde{Z}_{d-1}^{(i)}\geq a_*\right\}}\right)\right)^4 < \infty.$$

The inner expectation is equal to 0 if $l = 0$ and uniformly bounded by a constant $C_2 > 0$ for all $l = 1, \dots, a_*$. It remains to note that

$$\mathbb{E}\left(\log^+ \frac{C_2}{A_\nu^2}\sum_{l=1}^{a_*}\nu(\{l\})\right)^4 \le \mathbb{E}\left(\log^+ \frac{C_2}{A_\nu^2}\sum_{l=1}^{\infty} l\nu(\{l\})\right)^4 \le \mathbb{E}\left(\log^+ \frac{C_2}{A_\nu}\right)^4$$

$$\le C_3\mathbb{E}\left(\log^+ \frac{1}{A_\nu}\right)^4 + C_4 = C_3\mathbb{E}\left(\log^- A_\nu\right)^4 + C_4 < \infty$$

for some $C_3 > 0$ and $C_4 \ge 0$. □

Using Theorem 5.1 on p. 107 in [11] we obtain the following result.

Proposition 2. *Assume (A1), (A2), (A4) and* $\mathbb{E}d < \infty$. *Then*

$$\mathbb{P}\{Z_{S_n} > 0\} \sim \frac{C_{\text{Embed}}}{\sqrt{n}}, \quad n \to \infty$$

for some constant $C_{\text{Embed}} > 0$.

Furthermore, Theorem 5.6 on p. 126 in [11] entails the proposition.

Proposition 3. *Assume (A1), (A2), (A4) and* $\mathbb{E}d < \infty$ *and* $\mathbb{E}(\kappa(f_\nu; a))^4 < \infty$ *for some* $a \in \mathbb{N}_0$. *Then, with* $(B_+(t))_{t\in[0,1]}$ *being the Brownian meander,*

$$\text{Law}\left(\left(\frac{\log Z_{S_{\lfloor nt\rfloor}}}{\mathfrak{v}\sqrt{n}}\right)_{t\in[0,1]} \,\middle|\, Z_{S_n} > 0\right) \implies \text{Law}\left((B_+(t))_{t\in[0,1]}\right), \quad n \to \infty,$$

weakly on the space of probability measures on $D[0, 1]$ *endowed with the Skorokhod* J_1*-topology.*

The corollary given next follows from formula (1.1) in [8].

Corollary 2. *Under the assumptions of Proposition 3, for every fixed* $t \in (0, 1]$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{\log Z_{S_{\lfloor nt\rfloor}}}{\mathfrak{v}\sqrt{n}} \ge x \,\middle|\, Z_{S_n} > 0\right\} = \mathbb{P}\{B_+(t) \ge x\}, \quad x \ge 0. \tag{7}$$

The random variable $B_+(t)$ *has an absolutely continuous distribution with a bounded nonvanishing density on* $[0, \infty)$.

Propositions 2 and 3 are the key ingredients for the proof of our main results.

2.2 *Proof of Proposition 1 and Theorem 1*

Recall that $\tau_{\text{Embed}} = \inf\{k \ge 0 : Z_{S_k} = 0\}$ is the extinction time of the embedded process $(Z_{S_k})_{k\ge 0}$ and note that

$$\mathbb{P}\{\tau_{\text{Sparse}} < \infty\} \ge \mathbb{P}\{\tau_{\text{Embed}} < \infty\} = 1,$$

where the equality is justified by (6). This proves Proposition 1.

For $n \in \mathbb{N}_0$, define the first passage time $\vartheta(n)$ by

$$\vartheta(n) := \inf\{k \ge 0 : S_k > n\}. \tag{8}$$

Note that

$$\mathbb{P}\{Z_{S_{\vartheta(n)}} > 0\} \le \mathbb{P}\{Z_n > 0\} \le \mathbb{P}\{Z_{S_{\vartheta(n)-1}} > 0\}, \quad n \in \mathbb{N}_0.$$

In view of the strong law of large numbers for $\vartheta(n)$, which reads

$$\frac{\vartheta(n)}{n} \to \frac{1}{\mathtt{m}}, \quad n \to \infty, \quad \text{a.s.},$$

and Proposition 2, it is natural to expect that

$$\mathbb{P}\{Z_{S_{\vartheta(n)}} > 0\} \sim \frac{\mathtt{m}^{1/2} C_{\text{Embed}}}{\sqrt{n}} \sim \mathbb{P}\{Z_{S_{\vartheta(n)-1}} > 0\}, \quad n \to \infty. \tag{9}$$

Checking of relation (9) is clearly sufficient for a proof of Theorem 1. Furthermore, (9) would demonstrate that

$$C_{\text{Sparse}} = \mathtt{m}^{1/2} C_{\text{Embed}}. \tag{10}$$

Observe that

$$\mathbb{P}\{Z_{S_{\vartheta(n)}} > 0\} = \mathbb{P}\{\tau_{\text{Embed}} > \vartheta(n)\}$$
$$= \mathbb{P}\{\tau_{\text{Embed}} - 1 \ge \vartheta(n)\} = \mathbb{P}\{S_{\tau_{\text{Embed}}-1} > n\},$$

and, similarly,

$$\mathbb{P}\{Z_{S_{\vartheta(n)-1}} > 0\} = \mathbb{P}\{\tau_{\text{Embed}} > \vartheta(n) - 1\}$$
$$= \mathbb{P}\{\tau_{\text{Embed}} \ge \vartheta(n)\} = \mathbb{P}\{S_{\tau_{\text{Embed}}} > n\}.$$

The desired relation (9) follows from Theorem 3.1 in [12] applied with $r = 3/2$ provided we can check that

$$n\mathbb{P}\{d > n\} = o(\mathbb{P}\{\tau_{\text{Embed}} > n\}) = o(\mathbb{P}\{Z_{S_n} > 0\}), \quad n \to \infty.$$

By Proposition 2, this is equivalent to

$$\mathbb{P}\{d > n\} = o(n^{-3/2}), \quad n \to \infty,$$

which is secured by assumption (A3). This completes the proof of Theorem 1.

Remark 2. It is plausible that the assumption $\mathbb{E}d^{3/2} < \infty$ in Theorem 1 can be replaced by $\mathbb{E}d < \infty$. According to Proposition 2 we still have in this case

$$\mathbb{P}\{\tau_{\text{Embed}} > n\} \sim \frac{C_{\text{Embed}}}{\sqrt{n}}, \quad n \to \infty. \tag{11}$$

For the claim of Theorem 1 to be true it is sufficient that

$$\mathbb{P}\{S_{\tau_{\text{Embed}}} > n\} \sim \frac{C_{\text{Sparse}}}{\sqrt{n}}, \quad n \to \infty. \tag{12}$$

The major difficulty in proving that (11) together with $\mathbb{E}d < \infty$ implies (12) is dependence of τ_{Embed} and $(S_k)_{k\in\mathbb{N}}$. If these quantities were independent, then (12) would hold, see Proposition 4.3 in [9]. In our setting, τ_{Embed} and $(S_k)_{k\in\mathbb{N}}$ are not independent but τ_{Embed} *does not depend on the future* of $(S_k)_{k\in\mathbb{N}}$ in the sense of [7], see Section 7 therein. However, we have not been able to prove any version of the aforementioned Proposition 4.3 in [9] suitable for our purposes.

Remark 3. The tail behavior of τ_{Sparse} is rather elusive in the case where $\mathbb{E}d = \infty$ and/or $\mathbb{E}(\log A_1)^2 = \infty$. We shall only sketch difficulties that arise in this scenario. Let $(\widehat{T}_n)_{n\in\mathbb{N}}$ be a standard zero-delayed random walk with steps $\log A_k$, $k \in \mathbb{N}$, see (4). It is known, see Section 5.8 in [11], that the existence of the limit

$$\lim_{n\to\infty} \mathbb{P}\{\widehat{T}_n > 0\} = \rho \in (0,1) \tag{13}$$

and some further technical assumptions akin to (A4) imply that

$$\mathbb{P}\{\tau_{\text{Embed}} > n\} \sim \frac{\ell(n)}{n^{1-\rho}}, \quad n \to \infty,$$

for some ℓ slowly varying at infinity. Assume further that the distribution of d belongs to the domain of attraction of an α-stable law, $\alpha \in (0,1)$. Recall that $\mathbb{P}\{d > n\}\vartheta(n)$ converges in distribution as $n \to \infty$ to a random variable W, say, with the Mittag-Leffler distribution with parameter α, see, for instance, Theorem 7 in [10]. Therefore, one is tempted to write

$$\begin{aligned}\mathbb{P}\{Z_n > 0\} \sim \mathbb{P}\{\tau_{\text{Embed}} \geq \vartheta(n)\} \sim \mathbb{P}\{\tau_{\text{Embed}} > \vartheta(n)\}\\ \approx \frac{\ell(1/\mathbb{P}\{d \geq n\})}{(\mathbb{P}\{d \geq n\})^{\rho-1}}\mathbb{E}W^{\rho-1}, \quad n \to \infty. \end{aligned}\tag{14}$$

The random variable W has the same distribution as $\mathcal{S}_\alpha^{-\alpha}$, where $\mathcal{S}_\alpha$ is a random variable with an α-stable distribution concentrated on the positive halfline. Since $\mathbb{E}\mathcal{S}_\alpha^r < \infty$ for all $r \in (0,\alpha)$, we infer $\mathbb{E}W^{\rho-1} < \infty$. Hence, the right-hand side of (14) is regularly varying at ∞ of index $(\rho-1)\alpha$. However, this result looks quite dubious, since for an extremely sparse environment we expect the survivial probability to be of order close to $1/n$, as for the critical Galton–Watson process. The problem here is again the dependence of τ_{Embed} and $\vartheta(n)$ which makes the 'natural' asymptotics $\approx$ in (14) doubtful. We do not have any reasonable conjecture for the asymptotic behavior of $\mathbb{P}\{\tau_{\text{Embed}} \geq \vartheta(n)\}$.

2.3 *Proof of Theorem 2*

We start by noting that $\mathtt{m} < \infty$ together with the strong law of large numbers for $(\vartheta(n))$ imply

$$\sup_{t\in[0,1]}\left|\frac{\vartheta(\lfloor nt\rfloor)-1}{n} - \frac{t}{\mathtt{m}}\right| \to 0, \quad n\to\infty, \quad \text{a.s.}$$

Thus, the weak convergence claimed in Proposition 3 can be strengthened to the joint convergence

$$\begin{aligned}\text{Law}\left(\left(\left(\frac{\log Z_{S_{\lfloor \mathtt{m}^{-1}nt\rfloor}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}}, \frac{\vartheta(\lfloor nt\rfloor)-1}{\mathtt{m}^{-1}n}\right)_{t\in[0,1]} \middle| Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}} > 0\right)\right)\\ \Longrightarrow \text{Law}\left((B_+(t),t)_{t\in[0,1]}\right), \quad n\to\infty,\end{aligned}$$

which holds weakly on the space of probability measures on $D[0,1]\times D[0,1]$ endowed with the product J_1-topology. Using the continuous mapping theorem in combination with continuity of the composition (see, for instance, Theorem 13.2.2 in [14]) we infer

$$\mathrm{Law}\left(\left(\left(\frac{\log^+ Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}}\right)_{t\in[0,1]}\,\middle|\, Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\right)\right) \Longrightarrow \mathrm{Law}\left((B_+(t))_{t\in[0,1]}\right),\quad n\to\infty,\tag{15}$$

weakly on the space of probability measures on $D[0,1]$. We have replaced log by $\log^+$ in (15) because the event $\{Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}$ does not entail the event

$$\left\{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}>0 \text{ for all } t\in[0,1]\right\}.$$

Now we check that (15) secures

$$\mathrm{Law}\left(\left(\left(\frac{\log Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}}\right)_{t\in[0,1]}\,\middle|\, Z_n>0\right)\right) \Longrightarrow \mathrm{Law}\left((B_+(t))_{t\in[0,1]}\right),\quad n\to\infty.\tag{16}$$

By Proposition 2, Theorem 1 and (10),

$$\mathbb{P}\{Z_n>0\}\ \sim\ \mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}\ \sim\ \frac{C_{\mathrm{Sparse}}}{\sqrt{n}},\quad n\to\infty.\tag{17}$$

Thus, the limit relation (16) follows once we can prove that

$$\lim_{n\to\infty}\sqrt{n}\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0,\, Z_n=0\}=\lim_{n\to\infty}\sqrt{n}\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}=0,\, Z_n>0\}=0.\tag{18}$$

In view of (17), it suffices to show that

$$\lim_{n\to\infty}\mathbb{P}\{Z_n>0\,|\,Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}=1.$$

Fix any $\varepsilon>0$. The assumption (A3) implies that

$$\mathbb{P}\{|S_n-\mathtt{m}n|\geq\varepsilon n\}=o(n^{-1/2}),\quad n\to\infty,$$

by Theorem 4 in [3]. Thus,

$$\begin{aligned}
&\mathbb{P}\{Z_n>0\,|\,Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}\\
&\quad=\mathbb{P}\{Z_n>0,\, S_{\lfloor \mathtt{m}^{-1}(1+\varepsilon)n\rfloor}>n\,|\,Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}+o(1)\\
&\quad\geq\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}(1+\varepsilon)n\rfloor}}>0,\, S_{\lfloor \mathtt{m}^{-1}(1+\varepsilon)n\rfloor}>n\,|\,Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}+o(1)\\
&\quad=\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}(1+\varepsilon)n\rfloor}}>0\,|\,Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}+o(1)\\
&\quad=\frac{\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}(1+\varepsilon)n\rfloor}}>0\}}{\mathbb{P}\{Z_{S_{\lfloor \mathtt{m}^{-1}n\rfloor}}>0\}}+o(1)
\end{aligned}$$

$$\to (1+\varepsilon)^{-1/2}, \quad n \to \infty,$$

where we have used Proposition 2 for the last passage. Sending $\varepsilon \to 0$ gives (18).

To finish the proof of Theorem 2 it remains to check that, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,1]} \left| \frac{\log Z_{\lfloor nt\rfloor} - \log Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}} \right| > \varepsilon \,\middle|\, Z_n > 0 \right\} = 0. \tag{19}$$

To this end, we need an auxiliary lemma.

Lemma 2. *Assume (A2),* $\mathbb{E}d < \infty$ *and that* d *is independent of* $(\widetilde{Z}_j)_{j\geq 0}$. *Then*

$$\mathbb{E}\left(\max_{0\leq k\leq d} \widetilde{Z}_k \right) \leq 1 + \mathbb{E}d < \infty.$$

Proof. The proof follows from the chain of relations

$$\mathbb{E}\left(\max_{0\leq k\leq d} \widetilde{Z}_k \right) \leq \mathbb{E}\left(\sum_{k\geq 0} \widetilde{Z}_k \, \mathbb{1}_{\{d\geq k\}} \right) = \sum_{k\geq 0} \mathbb{E}\widetilde{Z}_k \cdot \mathbb{P}\{d \geq k\} = 1 + \mathbb{E}d.$$

□

In order to prove (19) we first show that

$$\lim_{n\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,1]} \frac{\log Z_{\lfloor nt\rfloor} - \log Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1}n}} > \varepsilon \,\middle|\, Z_n > 0 \right\} = 0. \tag{20}$$

Note that

$$Z_{\lfloor nt\rfloor} = \sum_{j=1}^{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} \widetilde{Z}^{(j)}_{\lfloor nt\rfloor - S_{\vartheta(\lfloor nt\rfloor)-1}}(S_{\vartheta(\lfloor nt\rfloor)-1}), \quad t \in [0,1], \quad n \in \mathbb{N}, \tag{21}$$

where $(\widetilde{Z}^{(j)}_k(m))_{k\geq 0}$ is the Galton–Watson process initiated by the j-th individual in the generation m. On the event $\{Z_n > 0\}$,

$$\frac{Z_{\lfloor nt\rfloor}}{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} = \frac{\sum_{j=1}^{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} \widetilde{Z}^{(j)}_{\lfloor nt\rfloor - S_{\vartheta(\lfloor nt\rfloor)-1}}(S_{\vartheta(\lfloor nt\rfloor)-1})}{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}, \quad t \in [0,1], \quad n \in \mathbb{N},$$

and thereupon

$$\sup_{t\in[0,1]} \frac{Z_{\lfloor nt\rfloor}}{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} \leq \sup_{1\leq k\leq \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}^{(j)}_i(S_{k-1})}{Z_{S_{k-1}}}, \quad n \in \mathbb{N}.$$

Instead of (20), we shall prove a stronger relation

$$\lim_{n\to\infty} \mathbb{P}\left\{ \sum_{1\leq k\leq \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}^{(j)}_i(S_{k-1})}{Z_{S_{k-1}}} > \varepsilon n^3 \,\middle|\, Z_n > 0 \right\} = 0. \tag{22}$$

By Markov's inequality in combination with $\mathbb{P}\{Z_n > 0\} \geq (1/C_5)n^{-1/2}$ for some $C_5 > 0$ and large n,

$$\mathbb{P}\left\{\sum_{1\leq k\leq \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}_i^{(j)}(S_{k-1})}{Z_{S_{k-1}}} > \varepsilon n^3 \,\Big|\, Z_n > 0\right\}$$

$$\leq \varepsilon^{-1} n^{-3} \sum_{k=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{S_{k-1}\leq n\}} \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}_i^{(j)}(S_{k-1}) \,\Big|\, Z_n > 0\right)$$

$$\leq C_5 \varepsilon^{-1} n^{-5/2} \sum_{k=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{S_{k-1}\leq n,\, Z_n>0\}} \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}_i^{(j)}(S_{k-1})\right)$$

$$\leq C_5 \varepsilon^{-1} n^{-5/2} \sum_{k=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{S_{k-1}\leq n,\, Z_{S_{k-1}}>0\}} \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \max_{0\leq i\leq d_k} \widetilde{Z}_i^{(j)}(S_{k-1})\right)$$

$$= C_5 \varepsilon^{-1} n^{-5/2} \left(\mathbb{E} \max_{0\leq i\leq d} \widetilde{Z}_i\right) \sum_{k=1}^{\infty} \mathbb{P}\{S_{k-1} \leq n\} = O(n^{-3/2}), \quad n \to \infty.$$

To justify the penultimate equality, observe that, given $(Z_{S_{k-1}}, S_{k-1})$, the sequences

$$(\widetilde{Z}_i^{(1)}(S_{k-1}))_{i\geq 0}, \ldots, (\widetilde{Z}_i^{(Z_{S_{k-1}})}(S_{k-1}))_{i\geq 0}$$

are independent copies of the critical Galton–Watson process $(\widetilde{Z}_i)_{i\geq 0}$. The last equality is a consequence of Lemma 2 and the elementary renewal theorem which states that

$$\sum_{k=1}^{\infty} \mathbb{P}\{S_{k-1} \leq n\} = \mathbb{E}\vartheta(n) \sim \frac{n}{\mathtt{m}}, \quad n \to \infty.$$

We shall now check that

$$\lim_{n\to\infty} \mathbb{P}\left\{\inf_{t\in[0,1]} \frac{\log Z_{\lfloor nt\rfloor} - \log Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}{\mathfrak{v}\sqrt{\mathtt{m}^{-1} n}} < -\varepsilon \,\Big|\, Z_n > 0\right\} = 0. \tag{23}$$

Using again decomposition (21), we write on the event $\{Z_n > 0\}$

$$\inf_{t\in[0,1]} \frac{Z_{\lfloor nt\rfloor}}{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} = \inf_{t\in[0,1]} \frac{\sum_{j=1}^{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}} \widetilde{Z}^{(j)}_{\lfloor nt\rfloor - S_{\vartheta(\lfloor nt\rfloor)-1}}(S_{\vartheta(\lfloor nt\rfloor)-1})}{Z_{S_{\vartheta(\lfloor nt\rfloor)-1}}}$$

$$\geq \inf_{1\leq k\leq \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \min_{0\leq i\leq d_k} \widetilde{Z}_i^{(j)}(S_{k-1})}{Z_{S_{k-1}}}$$

$$\geq \inf_{1\leq k\leq \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}}}{Z_{S_{k-1}}}, \quad n \in \mathbb{N}.$$

As in the proof of (20), we shall prove a relation which is stronger than (23), namely,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \inf_{1\le k\le \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}}}{Z_{S_{k-1}}} < \varepsilon n^{-3} \,\middle|\, Z_n > 0 \right\} = 0. \tag{24}$$

Since $\mathbb{P}\{Z_{S_{\vartheta(n)}} > 0\} \sim \mathbb{P}\{Z_n > 0\}$ as $n \to \infty$, by (9), and $\{Z_{S_{\vartheta(n)}} > 0\}$ entails $\{Z_n > 0\}$, relation (24) is equivalent to

$$\lim_{n\to\infty} \mathbb{P}\left\{ \inf_{1\le k\le \vartheta(n)} \frac{\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}}}{Z_{S_{k-1}}} < \varepsilon n^{-3} \,\middle|\, Z_{S_{\vartheta(n)}} > 0 \right\} = 0. \tag{25}$$

Observe that on the event $\{Z_{S_{\vartheta(n)}} > 0\}$

$$\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} > 0, \quad k \le \vartheta(n),$$

since otherwise the population does not survive up to time $S_{\vartheta(n)}$. Using this and the union bound yields

$$\begin{aligned}
&\mathbb{P}\left\{ \inf_{1\le k\le \vartheta(n)} \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \varepsilon n^{-3} \,\middle|\, Z_{S_{\vartheta(n)}} > 0 \right\} \\
&\quad \le \sum_{k\ge 1} \mathbb{P}\left\{ 0 < \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \varepsilon n^{-3}, k \le \vartheta(n) \,\middle|\, Z_{S_{\vartheta(n)}} > 0 \right\}.
\end{aligned}$$

Invoking $\mathbb{P}\{Z_{S_{\vartheta(n)}} > 0\} \ge (1/C_6)n^{-1/2}$ for some $C_6 > 0$ and large n, we obtain, for such n,

$$\begin{aligned}
&\sum_{k\ge 1} \mathbb{P}\left\{ 0 < \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \frac{\varepsilon}{n^3}, k \le \vartheta(n) \,\middle|\, Z_{S_{\vartheta(n)}} > 0 \right\} \\
&\le C_6\sqrt{n} \sum_{k\ge 1} \mathbb{P}\left\{ 0 < \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \frac{\varepsilon}{n^3}, k \le \vartheta(n), Z_{S_{\vartheta(n)}} > 0 \right\} \\
&= C_6\sqrt{n} \sum_{k\ge 1} \mathbb{P}\left\{ 0 < \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \frac{\varepsilon}{n^3}, S_{k-1} \le n, Z_{S_{\vartheta(n)}} > 0 \right\} \\
&\le C_6\sqrt{n} \sum_{k\ge 1} \mathbb{P}\left\{ 0 < \frac{1}{Z_{S_{k-1}}} \sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \frac{\varepsilon}{n^3}, S_{k-1} \le n, Z_{S_{k-1}} > 0 \right\}.
\end{aligned} \tag{26}$$

Let $\widetilde{p} := \mathbb{P}\{\widetilde{Z}_d > 0\}$ be the probability of the event that the critical Galton–Watson process $(\widetilde{Z}_k)_{k\ge 0}$ survives up to random time d independent of $(\widetilde{Z}_k)_{k\ge 0}$. Obviously, $\widetilde{p} \in (0, 1)$. Given $(S_{k-1}, Z_{S_{k-1}})$, the sum

$$\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}}$$

has a binomial distribution with parameters $(Z_{S_{k-1}}, \widetilde{p})$.

In what follows we denote by $\text{Bin}(N, p)$ a random variable with a binomial distribution with N and p interpreted as the number of independent trials and a success probability, respectively. The next lemma provides a uniform in N estimate for $\mathbb{P}\{0 < N^{-1}\text{Bin}(N, p) \le x\}$, which is useful when x is close to zero.

Lemma 3. *For all $N \in \mathbb{N}$ and $x \in (0, p)$,*

$$\mathbb{P}\{0 < N^{-1}\text{Bin}(N, p) \le x\} \le \frac{p(1-p)x}{(p-x)^2}.$$

Proof. Plainly, $\mathbb{P}\{0 < N^{-1}\text{Bin}(N, p) \le x\} = 0$ if $x < 1/N$. If $x \ge 1/N$, then by Chebyshev's inequality

$$\begin{aligned}
&\mathbb{P}\{0 < N^{-1}\text{Bin}(N, p) \le x\} \le \mathbb{P}\{\text{Bin}(N, p) \le Nx\} \\
&= \mathbb{P}\{\text{Bin}(N, 1-p) - N(1-p) \ge N(p-x)\} \le \frac{p(1-p)}{(p-x)^2}\frac{1}{N} \le \frac{p(1-p)}{(p-x)^2}x.
\end{aligned}$$

□

Using Lemma 3 we estimate the summands in (26) as follows. For $k \ge 1$ and n large enough,

$$\begin{aligned}
&\mathbb{P}\left\{0 < \frac{1}{Z_{S_{k-1}}}\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \varepsilon n^{-3}, S_{k-1} \le n, Z_{S_{k-1}} > 0\right\} \\
&= \mathbb{P}\left\{0 < \frac{1}{Z_{S_{k-1}}}\sum_{j=1}^{Z_{S_{k-1}}} \mathbb{1}_{\{\widetilde{Z}_{d_k}^{(j)}(S_{k-1})>0\}} < \varepsilon n^{-3} \,\middle|\, S_{k-1} \le n, Z_{S_{k-1}} > 0\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{P}\{S_{k-1} \le n, Z_{S_{k-1}} > 0\} \\
&\le \frac{\widetilde{p}(1-\widetilde{p})\varepsilon}{(\widetilde{p} - \varepsilon n^{-3})} n^{-3}\mathbb{P}\{S_{k-1} \le n, Z_{S_{k-1}} > 0\} \\
&\le \frac{\widetilde{p}(1-\widetilde{p})\varepsilon}{(\widetilde{p} - \varepsilon n^{-3})} n^{-3}\mathbb{P}\{S_{k-1} \le n\}.
\end{aligned}$$

Summarizing, the probability on the left-hand side of (25) is bounded from above by

$$C_6 n^{1/2}\frac{\widetilde{p}(1-\widetilde{p})\varepsilon}{(\widetilde{p} - \varepsilon n^{-3})} n^{-3}\sum_{k\ge1}\mathbb{P}\{S_{k-1} \le n\} = O(n^{-3/2}), \quad n \to \infty,$$

thereby finishing the proof of (25) and Theorem 2.

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References

[1] Afanasyev, V., Geiger, J., Kersting, G., Vatutin, V.: Criticality for branching processes in random environment. Ann. Probab. **33**(2), 645–673 (2005). doi:https://doi.org/10.1214/009117904000000928. MR2123206

[2] Athreya, K., Ney, P.: Branching Processes. Die Grundlehren der mathematischen Wissenschaften, Band 196, p. 287. Springer (1972). MR0373040

[3] Baum, L., Katz, M.: Convergence rates in the law of large numbers. Trans. Am. Math. Soc. **120**, 108–123 (1965). doi:https://doi.org/10.2307/1994170. MR198524

[4] Buraczewski, D., Dyszewski, P., Kołodziejska, A.: Weak quenched limit theorems for a random walk in a sparse random environment. arXiv preprint arXiv:2301.00478 (2023).

[5] Buraczewski, D., Dyszewski, P., Iksanov, A., Marynych, A., Roitershtein, A.: Random walks in a moderately sparse random environment. Electron. J. Probab. **24**, 69, 44 (2019). doi:https://doi.org/10.1214/19-EJP330. MR3978219

[6] Buraczewski, D., Dyszewski, P., Iksanov, A., Marynych, A.: Random walks in a strongly sparse random environment. Stoch. Process. Appl. **130**(7), 3990–4027 (2020). doi:https://doi.org/10.1016/j.spa.2019.11.007. MR4102257

[7] Denisov, D., Foss, S., Korshunov, D.: Asymptotics of randomly stopped sums in the presence of heavy tails. Bernoulli **16**(4), 971–994 (2010). doi:https://doi.org/10.3150/10-BEJ251. MR2759165

[8] Durrett, R., Iglehart, D., Miller, D.: Weak convergence to Brownian meander and Brownian excursion. Ann. Probab. **5**(1), 117–129 (1977). doi:https://doi.org/10.1214/aop/1176995895. MR436353

[9] Faÿ, G., González-Arévalo, B., Mikosch, T., Samorodnitsky, G.: Modeling teletraffic arrivals by a Poisson cluster process. Queueing Syst. **54**(2), 121–140 (2006). doi:https://doi.org/10.1007/s11134-006-9348-z. MR2268057

[10] Feller, W.: Fluctuation theory of recurrent events. Trans. Am. Math. Soc. **67**, 98–119 (1949). doi:https://doi.org/10.2307/1990420. MR32114

[11] Kersting, G., Vatutin, V.: Discrete Time Branching Processes in Random Environment. John Wiley & Sons (2017). MR3587982

[12] Robert, C., Segers, J.: Tails of random sums of a heavy-tailed number of light-tailed terms. Insur. Math. Econ. **43**(1), 85–92 (2008). doi:https://doi.org/10.1016/j.insmatheco.2007.10.001. MR2442033

[13] Smith, W., Wilkinson, W.: On branching processes in random environments. Ann. Math. Stat. **40**, 814–827 (1969). doi:https://doi.org/10.1214/aoms/1177697589. MR246380

[14] Whitt, W.: Stochastic-process Limits: An Introduction to Stochastic-process Limits and Their Application to Queues. Springer Series in Operations Research, p. 602. Springer (2002). MR1876437