Parameter estimation for fractional mixed fractional Brownian motion based on discrete observations

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Abstract   The object of investigation is the mixed fractional Brownian motion of the form $X_t = \kappa B_t^{H_1} + \sigma B_t^{H_2}$, driven by two independent fractional Brownian motions B_1^H and B_2^H with Hurst parameters $H_1 < H_2$. Strongly consistent estimators of unknown model parameters $(H_1, H_2, \kappa^2, \sigma^2)^\top$ are constructed based on the equidistant observations of a trajectory. Joint asymptotic normality of these estimators is proved for $0 < H_1 < H_2 < \frac{3}{4}$.

1   Introduction

The paper is devoted to the statistical identification of the following stochastic process:

$$X_t = \kappa B_t^{H_1} + \sigma B_t^{H_2}, \quad t \geq 0, \tag{1}$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions with Hurst indices H_1 and H_2 respectively. The process (1) is known in the literature as *fractional mixed fractional Brownian motion* [16]. We aim to estimate all four unknown parameters $H_1, H_2, \kappa^2, \sigma^2$ based on equidistant observations $\{X_{kh}, k = 0, 1, 2, \dots\}$,

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$h > 0$. In order to get an identifiable model, we assume that $0 < H_1 < H_2 < 1$, $\sigma > 0$, and $\kappa > 0$ throughout the article.

The application of fractional Brownian motion is highly widespread for modelling real-world processes that evolve over time and have the properties of self-similarity and long- or short-range dependence, see e.g., [4, 5, 11, 24–26, 32]. Mixed models like (1) containing fractional Brownian motions with different Hurst indices are important from the perspective of practical applications, in particular, to economics and financial mathematics; the motion of a lower Hurst index usually corresponds to a component responsible for changes over a short period of time, and the motion of a larger one corresponds to a long-term component. The particular case of the process (1) with $H_1 = 1/2$ is known as mixed fractional Brownian motion. It was firstly proposed by Cheridito [3] for better financial modeling, its properties can be found in [33]. There are already existing practical applications of this model [9, 27, 30, 32]. The model (1) with arbitrary H_1 and H_2 was considered in [8, 16]. Its application to the pricing of credit default swaps was studied in [10]. A more general model, containing several fractional Brownian motions with different Hurst indices can be found in [28].

Various methods for parameter estimation in the mixed Brownian–fractional Brownian model (i.e., in the case $H_1 = 1/2$) were proposed in [29] (the maximum likelihood method), [31] (Powell's optimization algorithm), and [6] (approach based on power variations). The general case of $H_1, H_2 \in (0, 1)$ is less studied. We can mention only the papers [18, 19], where the following mixed model with a trend was considered:

$$X_t = \theta t + \kappa B_t^{H_1} + \sigma B_t^{H_2}, \quad t \geq 0. \tag{2}$$

The authors applied the maximum likelihood approach for the estimation of the unknown drift parameter θ in model (2) based on the continuous observations. They constructed an estimator, which involves a solution to the integral equation with weak polar kernel. An approach to parameter estimation based on the numerical solution to this equation was recently proposed in [21]. Estimation of the drift parameter by discrete observations was discussed in [20].

In Refs. [2, 7, 14, 17] the parameter estimation for model (2) with $H_1 = 1/2$ was studied. The simultaneous estimation of all four parameters of this model was studied in [7] by the maximum likelihood method, and in [14], where the approach of [6] was generalized and, moreover, ergodic-type estimators were proposed. Ref. [2] considered the estimation of the drift parameter θ assuming that H and σ are known and $\kappa = 1$. The drift parameter estimation for a more general model driven by a stationary Gaussian process was discussed in [17].

The goal of the present paper is to construct strongly consistent estimators and to study their joint asymptotic normality. We extend the approach of [14], based on the ergodic theorem and generalized method of moments. It is worth mentioning that asymptotic normality does not hold for $H_2 > \frac{3}{4}$, which is typical for models with fractional Brownian motion, see, e.g., [22]. The estimators remain strongly consistent but have non-Gaussian asymptotic distribution.

The paper is organized as follows. In Section 2 we construct strongly consistent estimators with the use of four statistics based on the various increments of the process X. Section 3 is devoted to asymptotic normality. First, we prove asymptotic

normality of constructed additional statistics and evaluate their asymptotic covariance matrix. Thereafter, we obtain the main result on the asymptotic normality of our estimators by applying the delta method. The theoretical results are illustrated by numerical study, which is presented in Section 4. All technical proofs together with some auxiliary results are collected in Section 5.

We use the following notation. The symbol **cov** stands for the covariance of two random variables and for the covariance matrix of a random vector. In the paper, all the vectors are column ones. The superscript $\top$ denotes transposition. Weak convergence in distribution is denoted by $\overset{d}{\to}$.

2 Construction of strongly consistent estimators

Let $h > 0$ be fixed. Assume that the process X_t defined in (1) is observed at points $t_k = kh, k = 0, 1, 2, \ldots$. Let us denote the increment

$$\Delta X_k := X_{(k+1)h} - X_{kh} = \kappa \Delta B_k^{H_1} + \sigma \Delta B_k^{H_2}.$$

The estimation procedure will be based on the following result.

Lemma 1. *The Gaussian sequence* $\Delta X_k, k = 0, 1, \ldots$, *is stationary and ergodic.*

This lemma allows us to apply the ergodic theorem for the construction of estimators. Namely, if $g \colon \mathbb{R}^{l+1} \to \mathbb{R}$ is a Borel function such that $\mathsf{E} \, |g(\Delta X_0, \Delta X_1, \ldots, \Delta X_l)| < \infty$, then

$$\frac{1}{N} \sum_{k=0}^{N-1} g(\Delta X_k, \Delta X_{k+1}, \ldots, \Delta X_{k+l}) \to \mathsf{E} \, g(\Delta X_0, \Delta X_1, \ldots, \Delta X_l) \qquad (3)$$

a. s., as $N \to \infty$. The main idea is to obtain four different convergences by choosing different functions g, and then to construct the estimators by solving the corresponding system of four equations. To this end, we introduce the following four statistics

$$\xi_N := \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh} \right)^2, \quad \eta_N := \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+2)h} - X_{kh} \right)^2,$$

$$\zeta_N := \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+4)h} - X_{kh} \right)^2, \quad \phi_N := \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+8)h} - X_{kh} \right)^2. \qquad (4)$$

Denote also $\vartheta = (H_1, H_2, \kappa^2, \sigma^2)^\top$.

Lemma 2. *The following a.s. convergences hold as* $N \to \infty$:

(i) $\xi_N \to \mathsf{E}\,\xi_0 = \kappa^2 h^{2H_1} + \sigma^2 h^{2H_2} =: f_1(\vartheta)$,

(ii) $\eta_N \to \mathsf{E}\,\eta_0 = \kappa^2 h^{2H_1} 2^{2H_1} + \sigma^2 h^{2H_2} 2^{2H_2} =: f_2(\vartheta)$,

(iii) $\zeta_N \to \mathsf{E}\,\zeta_0 = \kappa^2 h^{2H_1} 2^{4H_1} + \sigma^2 h^{2H_2} 2^{4H_2} =: f_3(\vartheta)$,

(iv) $\phi_N \to \mathsf{E}\,\phi_0 = \kappa^2 h^{2H_1} 2^{6H_1} + \sigma^2 h^{2H_2} 2^{6H_2} =: f_4(\vartheta)$.

Let us introduce the notation

$$f(\vartheta) := (f_1(\vartheta), f_2(\vartheta), f_3(\vartheta), f_4(\vartheta))^\top, \tag{5}$$

$$\tau_N = (\xi_N, \eta_N, \zeta_N, \phi_N)^\top, N \in \mathbb{N}, \quad \tau_0 = (\mathsf{E}\,\xi_0, \mathsf{E}\,\eta_0, \mathsf{E}\,\zeta_0, \mathsf{E}\,\phi_0)^\top. \tag{6}$$

Then $f(\vartheta) = \tau_0$. Therefore, in order to obtain estimators for parameters $(H_1, H_2, \kappa^2, \sigma^2)^\top$ we need to construct function q such that $q(\tau_0) = f^{-1}(\tau_0)$, and then required estimators would be defined as $(\widehat{H}_N^{(1)}, \widehat{H}_N^{(2)}, \hat{\kappa}_N^2, \hat{\sigma}_N^2)^\top := q(\tau_N)$. In other words, in order to construct strongly consistent estimators, we need to solve the estimation equation

$$f(\vartheta) = \tau_N \tag{7}$$

with respect to unknown parameters. The solution to (7) can be found in explicit form, see Subsection 5.3 for details. This leads to the following result.

Theorem 1. *Let* $0 < H_1 < H_2 < 1$. *The statistic*

$$\widehat{\vartheta} := (\widehat{H}_N^{(1)}, \widehat{H}_N^{(2)}, \hat{\kappa}_N^2, \hat{\sigma}_N^2)^\top, \tag{8}$$

where

$$\widehat{H}_N^{(1)} = \frac{1}{2}\log_{2+}\left(\frac{\eta_N\zeta_N - \xi_N\phi_N + d_N}{2(\eta_N^2 - \xi_N\zeta_N)}\right), \tag{9}$$

$$\widehat{H}_N^{(2)} = \frac{1}{2}\log_{2+}\left(\frac{\eta_N\zeta_N - \xi_N\phi_N - d_N}{2(\eta_N^2 - \xi_N\zeta_N)}\right), \tag{10}$$

$$\hat{\kappa}_N^2 = \left(\frac{2(\eta_N^2 - \xi_N\zeta_N)}{\eta_N\zeta_N - \xi_N\phi_N + d_N}\right)^{\log_2 h}\frac{\xi_N d_N + 2\eta_N^3 - 3\xi_N\eta_N\zeta_N + \xi_N^2\phi_N}{2d_N}, \tag{11}$$

$$\hat{\sigma}_N^2 = \left(\frac{2(\eta_N^2 - \xi_N\zeta_N)}{\eta_N\zeta_N - \xi_N\phi_N - d_N}\right)^{\log_2 h}\frac{\xi_N d_N - 2\eta_N^3 + 3\xi_N\eta_N\zeta_N - \xi_N^2\phi_N}{2d_N} \tag{12}$$

with

$$d_N := \left(\xi_N^2\phi_N^2 - 6\xi_N\eta_N\zeta_N\phi_N - 3\eta_N^2\zeta_N^2 + 4\eta_N^3\phi_N + 4\xi_N\zeta_N^3\right)_+^{1/2} \tag{13}$$

is a strongly consistent estimator of $\vartheta = (H_1, H_2, \kappa^2, \sigma^2)^\top$.
    Here

$$\log_{2+} x = \begin{cases} \log_2 x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \qquad (x)_+^{1/2} = \begin{cases} \sqrt{x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Remark 1. The constructed estimators can be generalized for mixed model with trend (2) by introducing the estimator $\frac{X_{Nh}}{N} = \frac{1}{N}\sum_{k=0}^{N-1}(X_{(k+1)h} - X_{kh})$ to define the estimator of θ. By subtracting $\frac{X_{Nh}}{N}$ from previously constructed statistics (4) we can obtain a system of equations similar to (7).

3 Asymptotic normality

We will start with a study of the asymptotic properties of the vector $\tau_N = (\phi_N, \xi_N, \eta_N, \zeta_N)$. The previously defined stationary Gaussian sequence ΔX_k has the autocovariance function

$$\tilde{\rho}(i) = \mathbf{cov}(\Delta X_0, \Delta X_i) = \kappa^2 h^{2H_1}\rho(i, H_1) + \sigma^2 h^{2H_2}\rho(i, H_2), \quad i \in \mathbb{Z}, \quad (14)$$

where

$$\rho(i, H) := \frac{1}{2}\left(|i+1|^{2H} - 2|i|^{2H} + |i-1|^{2H}\right) \quad (15)$$

denotes the autocovariance function of the stationary sequence $\{B_{k+1}^H - B_k^H, k \geq 0\}$, which is known as fractional Gaussian noise.

Theorem 2. *Let* $0 < H_1 < H_2 < \frac{3}{4}$. *The vector* τ_N *defined by* (6) *is asymptotically normal, namely,*

$$\sqrt{N}(\tau_N - \tau_0) = \sqrt{N}\begin{pmatrix} \xi_N - \mathsf{E}\,\xi_0 \\ \eta_N - \mathsf{E}\,\eta_0 \\ \zeta_N - \mathsf{E}\,\zeta_0 \\ \phi_N - \mathsf{E}\,\phi_0 \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}(\vec{0}, \widetilde{\Sigma})$$

with the asymptotic covariance matrix $\widetilde{\Sigma}$, *which can be presented explicitly as*

$$\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{13} & \widetilde{\Sigma}_{14} \\ \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{22} & \widetilde{\Sigma}_{23} & \widetilde{\Sigma}_{24} \\ \widetilde{\Sigma}_{13} & \widetilde{\Sigma}_{23} & \widetilde{\Sigma}_{33} & \widetilde{\Sigma}_{34} \\ \widetilde{\Sigma}_{14} & \widetilde{\Sigma}_{24} & \widetilde{\Sigma}_{34} & \widetilde{\Sigma}_{44} \end{pmatrix},$$

where

$$\widetilde{\Sigma}_{11} = 2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)^2, \quad \widetilde{\Sigma}_{12} = \sum_{i=-\infty}^{\infty}\tilde{\rho}(i)\Big(4\tilde{\rho}(i) + 4\tilde{\rho}(i+1)\Big),$$

$$\widetilde{\Sigma}_{22} = \sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(12\tilde{\rho}(i) + 16\tilde{\rho}(i+1) + 4\tilde{\rho}(i+2)\Big),$$

$$\widetilde{\Sigma}_{13} = \sum_{i=-\infty}^{\infty}\tilde{\rho}(i)\Big(8\tilde{\rho}(i) + 12\tilde{\rho}(i+1) + 8\tilde{\rho}(i+2) + 4\tilde{\rho}(i+3)\Big),$$

$$\widetilde{\Sigma}_{23} = \sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(28\tilde{\rho}(i) + 48\tilde{\rho}(i+1) + 32\tilde{\rho}(i+2) + 16\tilde{\rho}(i+3) + 4\tilde{\rho}(i+4)\Big),$$

$$\widetilde{\Sigma}_{33} = \sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(88\tilde{\rho}(i) + 160\tilde{\rho}(i+1) + 124\tilde{\rho}(i+2) + 80\tilde{\rho}(i+3)$$

$$+ 40\tilde{\rho}(i+4) + 16\tilde{\rho}(i+5) + 4\tilde{\rho}(i+6)\Big),$$

$$\widetilde{\Sigma}_{14} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(16\tilde{\rho}(i) + 28\tilde{\rho}(i+1) + 24\tilde{\rho}(i+2) + 20\tilde{\rho}(i+3)$$
$$+ 16\tilde{\rho}(i+4) + 12\tilde{\rho}(i+5) + 8\tilde{\rho}(i+6) + 4\tilde{\rho}(i+7)\Big),$$

$$\widetilde{\Sigma}_{24} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(60\tilde{\rho}(i) + 112\tilde{\rho}(i+1) + 96\tilde{\rho}(i+2) + 80\tilde{\rho}(i+3)$$
$$+ 64\tilde{\rho}(i+4) + 48\tilde{\rho}(i+5) + 32\tilde{\rho}(i+6) + 16\tilde{\rho}(i+7) + 4\tilde{\rho}(i+8)\Big),$$

$$\widetilde{\Sigma}_{34} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(216\tilde{\rho}(i) + 416\tilde{\rho}(i+1) + 376\tilde{\rho}(i+2) + 320\tilde{\rho}(i+3)$$
$$+ 256\tilde{\rho}(i+4) + 192\tilde{\rho}(i+5) + 132\tilde{\rho}(i+6) + 80\tilde{\rho}(i+7)$$
$$+ 40\tilde{\rho}(i+8) + 16\tilde{\rho}(i+9) + 4\tilde{\rho}(i+10)\Big),$$

$$\widetilde{\Sigma}_{44} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(688\tilde{\rho}(i) + 1344\tilde{\rho}(i+1) + 1260\tilde{\rho}(i+2) + 1136\tilde{\rho}(i+3)$$
$$+ 984\tilde{\rho}(i+4) + 816\tilde{\rho}(i+5) + 644\tilde{\rho}(i+6) + 480\tilde{\rho}(i+7)$$
$$+ 336\tilde{\rho}(i+8) + 224\tilde{\rho}(i+9) + 140\tilde{\rho}(i+10) + 80\tilde{\rho}(i+11)$$
$$+ 40\tilde{\rho}(i+12) + 16\tilde{\rho}(i+13) + 4\tilde{\rho}(i+14)\Big).$$

Remark 2. The condition $H < \frac{3}{4}$ is essential since otherwise the series in Theorem 2 do not converge. Moreover, for $H > \frac{3}{4}$, the asymptotic normality of τ_N does not hold; using the results of [15], it is possible to establish the convergence of the suitably normalized vector $\tau_N - \tau_0$ to a special non-Gaussian distribution.

Now we are ready to provide the main result on asymptotic normality of our estimator (8).

Theorem 3. *Let $0 < H_1 < H_2 < \frac{3}{4}$. The estimator $\hat{\vartheta}_N$ is asymptotically normal, namely,*

$$\sqrt{N}\left(\hat{\vartheta}_N - \vartheta\right) = \sqrt{N}\begin{pmatrix}\widehat{H}_N^{(1)} - H_1 \\ \widehat{H}_N^{(2)} - H_2 \\ \hat{\kappa}_N^2 - \kappa^2 \\ \hat{\sigma}_N^2 - \sigma^2\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\vec{0}, \Sigma^0\right)$$

with the asymptotic covariance matrix Σ^0 that can be found by the formula

$$\Sigma^0 = (f'(\vartheta))^{-1}\widetilde{\Sigma}\left(f'(\vartheta)\right)^{-\top},$$

where $\widetilde{\Sigma}$ is defined in Theorem 2 and

$$f'(\vartheta) = \begin{pmatrix} 2\kappa^2 h^{2H_1}\log h & 2\sigma^2 h^{2H_2}\log h & h^{2H_1} & h^{2H_2} \\ 2\kappa^2(2h)^{2H_1}\log(2h) & 2\sigma^2(2h)^{2H_2}\log(2h) & (2h)^{2H_1} & (2h)^{2H_2} \\ 2\kappa^2(4h)^{2H_1}\log(4h) & 2\sigma^2(4h)^{2H_2}\log(4h) & (4h)^{2H_1} & (4h)^{2H_2} \\ 2\kappa^2(8h)^{2H_1}\log(8h) & 2\sigma^2(8h)^{2H_2}\log(8h) & (8h)^{2H_1} & (8h)^{2H_2} \end{pmatrix}.$$

4   Simulation study

In this section, we present results of estimators performance in numerical simulations. For each generated trajectory, we will estimate asymptotic covariance matrices defined in Theorem 2 and Theorem 3 by using values of estimators (9)–(12). For each set of parameters with $0 < H_1 < H_2 < 1$, we generate 1000 trajectories of the process X and calculate the empirical means, empirical standard deviations of the estimates, and percentage of iterations with constant d_N defined in (13) being equal to zero. Additionally, for $0 < H_1 < H_2 < \frac{3}{4}$, we calculate the average approximate theoretical standard deviation (ASD) – the square root of the estimated asymptotic variance divided by N ($\sqrt{(\hat{\Sigma}_N^0)_{ii}/N}$) – and the coverage probability (CP) for $\alpha = 5\%$, based on the estimator of the asymptotic covariance matrix. We are computing these statistics only for $H_1 < H_2 < \frac{3}{4}$, excluding interval $(\frac{3}{4}, 1)$ where the series (17) will not converge as per Lemma 3. Therefore, the asymptotic covariance matrix and corresponding statistics (ASD and CP) will be undefined. Moreover, when any of the estimates $\widehat{H}_N$ will be out of boundaries $(0, \frac{3}{4})$, or the calculated constant d_N defined in (13) is zero (e.g., in cases where discriminant D defined in the proof of Theorem 1 by (30) is nonpositive), then ASD and CP will also be considered undefined.

To numerically approximate limiting covariances, series alike (17) are divided into the sum of two convergent series

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) = \sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) + \sum_{i=0}^{+\infty} \tilde{\rho}(i+1-\alpha)\tilde{\rho}(i+1-\beta),$$

where $\alpha, \beta \in \mathbb{Z}$.

Then, series $\sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$ are approximated with precision $\delta = 10^{-3}$ by evaluating the sum $S_m = \sum_{i=0}^{m} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$, where m is the smallest number such that $|S_m - S_{m-1}| < 10^{-3}$. Obtained approximations will be used to compute the asymptotic covariance matrix defined in Theorem 2.

For $\widehat{H}_N^{(1)}$, $\widehat{H}_N^{(2)}$, $\hat{\kappa}_N^2$, $\hat{\sigma}_N^2$, $\hat{\Sigma}_N^0$ and fixed time step $h = 1$ we vary the horizon $T = h2^n$ for $n \in \{8, 10, 12, 14, 16, 18, 20\}$. For all simulations, the values $\sigma = 1$ and $\kappa = 1$ were used.

Results of parameters estimations provided in Tables 1–4 indicate that constructed estimators perform better, the higher the difference is between H_1 and H_2. That impact is especially significant for estimators of variances κ^2 and σ^2 since the denominator in (11) and (12) tends to 0 when the difference between two Hurst parameters tends to zero. Also, the smaller this difference is, the higher the percentage of iterations with constant d_N being equal to zero, which is presented in Table 5. Additionally, we also observe that constructed estimators perform better for higher values of H_1 and H_2. This also might be caused by the difference between H_1 and H_2 since most of the values in numerators and denominators of constructed estimators are highly dependent on the corresponding difference, as was shown in the proof of Theorem 1. And higher values of H_1 and H_2 might result in a more accurate estimate of the difference $\widehat{H}_N^{(2)} - \widehat{H}_N^{(1)}$ in terms of the ratio between the estimated value and its deviation, and therefore – a lower percentage of iterations with the estimated difference being negative.

Table 1. The estimator $\widehat{H}_N^{(1)}$ with $\sigma^2 = 1, \kappa^2 = 1$ ($h = 1$)

| H_1 | H_2 | | N | | | | | | |
			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	Mean	0.0722	0.0434	−0.0230	−0.1009	−0.0844	0.0482	0.0842
		S.dev.	0.5041	0.4727	0.5063	0.5160	0.4459	0.1629	0.0791
		ASD	0.7648	0.7039	1.1629	0.3871	0.2011	0.1239	0.0812
		CP%	100.00	100.00	100.00	92.18	75.61	75.58	87.91
0.1	0.5	Mean	0.0184	−0.0455	−0.0009	0.0674	0.0953	0.0988	0.0991
		S.dev.	0.5618	0.5945	0.3733	0.1710	0.0824	0.0414	0.0205
		ASD	1.1951	0.5894	0.2696	0.1457	0.0829	0.0439	0.0222
		CP%	100.00	100.00	99.30	89.82	93.76	96.85	96.50
0.1	0.7	Mean	−0.0408	−0.0071	0.0773	0.0945	0.1002	0.0998	0.1005
		S.dev.	0.5931	0.3828	0.1513	0.0715	0.0342	0.0179	0.0090
		ASD	0.7428	0.3223	0.1516	0.0742	0.0375	0.0188	0.0094
		CP%	100.00	100.00	100.00	99.30	97.09	95.90	95.40
0.1	0.9	Mean	0.0131	0.0725	0.0886	0.0956	0.0972	0.0986	0.0993
		S.dev.	0.3842	0.1554	0.0707	0.0350	0.0187	0.0113	0.0075
0.3	0.5	Mean	0.2096	0.1797	0.1161	0.0665	0.1731	0.2658	0.2901
		S.dev.	0.5181	0.5738	0.4782	0.5254	0.3445	0.1209	0.0564
		ASD	1.1101	1.0150	0.7507	0.4497	0.2337	0.1124	0.0570
		CP%	100.00	100.00	100.00	100.00	92.01	87.11	90.80
0.3	0.7	Mean	0.1748	0.1524	0.2330	0.2764	0.2931	0.2989	0.3001
		S.dev.	0.5138	0.4661	0.2630	0.1165	0.0563	0.0263	0.0137
		ASD	1.4395	0.5732	0.2682	0.1248	0.0566	0.0277	0.0138
		CP%	100.00	100.00	100.00	100.00	98.32	96.30	95.60
0.3	0.9	Mean	0.1863	0.2636	0.2863	0.2934	0.2969	0.2983	0.2988
		S.dev.	0.4268	0.1859	0.0824	0.0423	0.0211	0.0124	0.0078
0.5	0.7	Mean	0.4213	0.3300	0.2774	0.3367	0.4563	0.4859	0.4974
		S.dev.	0.6302	0.6258	0.5085	0.3908	0.1461	0.0763	0.0367
		ASD	2.0105	1.2367	0.8386	0.4502	0.1964	0.0830	0.0366
		CP%	100.00	100.00	100.00	100.00	99.84	99.09	98.09
0.5	0.9	Mean	0.3188	0.3701	0.4577	0.4878	0.4943	0.4975	0.4986
		S.dev.	0.5170	0.4018	0.1477	0.0627	0.0299	0.0162	0.0095
0.7	0.9	Mean	0.5487	0.4918	0.5172	0.5854	0.6706	0.6898	0.6957
		S.dev.	0.6543	0.6735	0.4569	0.2908	0.0914	0.0396	0.0193

We observe that the higher the difference is between the true values of H_1 and H_2, the closer the ASD is to the empirical standard deviation of the estimator and the closer the coverage probability to the theoretical 95%. At the same time, results similar to those described above are observed in the performance of our estimates for asymptotic variance of (4) constructed in Theorem 2, which is shown in Table 6 by CP estimates. This table also shows that for higher values of H_1 and H_2, the accuracy of (4) is the lower, the bigger the increment step (e.g., ϕ_N is less accurate than ζ_N, which is less accurate than η_N, which is less accurate than ξ_N). Unfortunately, the percentage of iterations with undefined asymptotic covariance matrix $\widehat{\Sigma}_N^0$ is high but decreases for bigger samples, which is shown in Table 7, due to estimated $\widehat{H}_N^{(1)}$, $\widehat{H}_N^{(2)}$ violating the conditions of Theorem 3. Also, the percentage of iterations with the undefined $\widehat{\Sigma}_N^0$ is the lower, the higher the difference between true H_1 and H_2.

Table 2. The estimator $\widehat{H}_N^{(2)}$ with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

| H_1 | H_2 | | N | | | | | | |
			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	Mean	0.2296	0.2845	0.3834	0.5157	0.4571	0.3634	0.3182
		S.dev.	0.5205	0.6239	0.5643	0.5944	0.3578	0.1543	0.0594
		ASD	2.2888	2.0419	1.4188	0.8483	0.4334	0.1937	0.0791
		CP%	100.00	100.00	100.00	100.00	100.00	100.00	99.88
0.1	0.5	Mean	0.5437	0.6554	0.6072	0.5302	0.5093	0.5024	0.5003
		S.dev.	0.6078	0.5234	0.2668	0.0985	0.0411	0.0193	0.0094
		ASD	0.9381	0.5332	0.2632	0.1158	0.0481	0.0218	0.0107
		CP%	100.00	100.00	100.00	100.00	99.66	98.37	96.80
0.1	0.7	Mean	0.7912	0.7303	0.7069	0.7018	0.7006	0.7000	0.7001
		S.dev.	0.4123	0.1341	0.0554	0.0254	0.0126	0.0066	0.0032
		ASD	0.3193	0.1257	0.0557	0.0265	0.0139	0.0064	0.0032
		CP%	100.00	100.00	100.00	99.77	95.98	94.50	94.70
0.1	0.9	Mean	0.8818	0.8825	0.8882	0.8930	0.8959	0.8979	0.8991
		S.dev.	0.1599	0.0601	0.0381	0.0272	0.0201	0.0151	0.0121
0.3	0.5	Mean	0.4179	0.4999	0.5838	0.6819	0.5997	0.5329	0.5097
		S.dev.	0.6199	0.6409	0.5791	0.4784	0.2349	0.0921	0.0418
		ASD	0.9471	0.7822	0.6198	0.3870	0.1737	0.0861	0.0412
		CP%	100.00	100.00	100.00	100.00	97.81	90.36	90.90
0.3	0.7	Mean	0.7636	0.8383	0.7481	0.7094	0.7023	0.7007	0.7004
		S.dev.	0.5831	0.4365	0.1412	0.0574	0.0271	0.0128	0.0066
		ASD	0.4241	0.1793	0.0914	0.0466	0.0256	0.0130	0.0065
		CP%	100.00	100.00	99.38	90.74	93.60	95.30	94.00
0.3	0.9	Mean	0.9267	0.8926	0.8898	0.8936	0.8954	0.8975	0.8981
		S.dev.	0.2765	0.0827	0.0440	0.0301	0.0213	0.0166	0.0130
0.5	0.7	Mean	0.5823	0.6867	0.8179	0.8324	0.7494	0.7181	0.7056
		S.dev.	0.6097	0.6288	0.5302	0.3556	0.1229	0.0581	0.0265
		ASD	0.5033	0.2277	0.1997	0.1055	0.0672	0.0422	0.0245
		CP%	100.00	100.00	99.54	82.13	78.62	85.38	92.58
0.5	0.9	Mean	0.8978	0.9350	0.8980	0.8936	0.8939	0.8965	0.8978
		S.dev.	0.5267	0.2147	0.0793	0.0413	0.0274	0.0198	0.0150
0.7	0.9	Mean	0.7878	0.8862	0.9672	0.9309	0.9021	0.8963	0.8965
		S.dev.	0.6431	0.5628	0.4037	0.1334	0.0578	0.0316	0.0213

Additionally, by conducting provided simulations, we observed that the closer parameters H_1 and H_2 to 0.75, the lower the convergence rate in the numerical approximation of the limiting covariance. To study this effect, we have conducted simulations for the series

$$\sum_{i=0}^{+\infty} \tilde{\rho}(i)^2. \tag{16}$$

We estimate (16) by finding the smallest number $m \in \mathbb{N}$ such that for $S_m = \sum_{i=0}^{m} \tilde{\rho}(i)^2$ the inequality $|S_m - S_{m-1}| \leq \delta$ holds for some predefined δ.

This estimation was conducted for $\delta \in \{10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}\}$ for $\sigma = 1$, $\kappa = 1$, $H_1 = 0.1$ and $H_2 \in \{0.3, 0.5, 0.7\}$. Corresponding estimation results are presented in Table 8. As one can see, in order to estimate S_m with the precision $\delta = 10^{-9}$ for $H_2 = 0.3$, only a few hundred iterations are needed, while for $H_2 = 0.7$, it requires dozens of millions of iterations. Moreover, lowering the

Table 3. The estimator $\hat{\kappa}_N^2$ with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

H_1	H_2		N						
			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	Mean	NaN	NaN	NaN	Inf	1.0591	1.0732	1.0460
		S.dev.	NaN	NaN	NaN	NaN	0.7876	0.6474	0.4604
		ASD	4.3622	4.9405	5.0730	2.8475	1.5443	0.9266	0.6165
		CP%	100.00	100.00	93.08	80.66	70.73	74.27	89.53
0.1	0.5	Mean	NaN	Inf	1.1104	1.0594	1.0340	1.0098	1.0006
		S.dev.	NaN	NaN	0.6161	0.4314	0.2528	0.1333	0.0672
		ASD	4.4678	2.4422	1.0822	0.5453	0.2928	0.1484	0.0743
		CP%	100.00	100.00	99.30	92.92	96.15	97.97	96.20
0.1	0.7	Mean	Inf	1.0400	1.0183	1.0054	1.0035	1.0004	1.0010
		S.dev.	NaN	0.4440	0.2538	0.1260	0.0623	0.0330	0.0164
		ASD	1.8115	0.6744	0.2911	0.1395	0.0693	0.0346	0.0173
		CP%	100.00	100.00	100.00	100.00	97.79	95.70	95.70
0.1	0.9	Mean	1.0054	0.9872	0.9878	0.9922	0.9950	0.9972	0.9987
		S.dev.	0.3366	0.1663	0.0823	0.0437	0.0264	0.0177	0.0132
0.3	0.5	Mean	NaN	NaN	Inf	1.0451	1.0379	1.0466	1.0232
		S.dev.	NaN	NaN	NaN	0.8444	0.7341	0.5658	0.3768
		ASD	4.8428	4.2202	3.2605	2.1396	1.0325	0.6043	0.3862
		CP%	100.00	100.00	100.00	96.51	83.23	77.25	84.00
0.3	0.7	Mean	NaN	1.1084	1.0696	1.0255	1.0106	1.0026	1.0017
		S.dev.	NaN	NaN	0.5169	0.3305	0.1797	0.0877	0.0455
		ASD	3.5137	1.3194	0.6535	0.3286	0.1770	0.0917	0.0461
		CP%	100.00	100.00	0.9979	0.9074	0.9496	0.9570	0.9460
0.3	0.9	Mean	Inf	0.9970	0.9867	0.9887	0.9923	0.9955	0.9968
		S.dev.	NaN	0.2908	0.1463	0.0802	0.0436	0.0285	0.0201
0.5	0.7	Mean	NaN	NaN	Inf	1.0557	1.0674	1.0502	1.0270
		S.dev.	NaN	NaN	NaN	0.7737	0.6163	0.4567	0.2703
		ASD	4.6051	2.2181	2.1464	1.2319	0.7646	0.4713	0.2722
		CP%	100.00	100.00	92.17	77.05	78.14	86.29	93.96
0.5	0.9	Mean	NaN	1.0008	0.9750	0.9800	0.9814	0.9905	0.9940
		S.dev.	NaN	0.5558	0.3666	0.1997	0.1057	0.0621	0.0402
0.7	0.9	Mean	NaN	NaN	$-$Inf	0.9524	0.9568	0.9572	0.9690
		S.dev.	NaN	NaN	NaN	0.6566	0.4605	0.2777	0.1643

precision from 10^{-4} to 10^{-3} has almost no impact on the estimated value S_m for low H_2, but for high H_2, it could result in around three times difference.

5 Proofs

5.1 Auxiliary properties of covariance functions

The following statement provides some important properties of sequences $\tilde{\rho}(i)$ and $\rho(i, H)$, $i \in \mathbb{Z}$, defined by (14) and (15), respectively.

Lemma 3.

1. *For any* $H \in (0, 1)$, $\rho(i, H) \sim H(2H - 1)|i|^{2H-2}$ *as* $|i| \to \infty$.

2. *For any* $H \in (0, \frac{3}{4})$, $\sum_{i=-\infty}^{\infty} \rho^2(i, H) < \infty$.

Table 4. The estimator $\hat{\sigma}_N^2$ with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

H_1	H_2		N						
			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	Mean	NaN	NaN	NaN	Inf	0.9764	0.9499	0.9781
		S.dev.	NaN	NaN	NaN	NaN	0.7882	0.6480	0.4606
		ASD	4.3572	4.9387	5.0724	2.8472	1.5442	0.9265	0.6165
		CP%	100.00	100.00	93.08	80.25	70.95	74.27	89.53
0.1	0.5	Mean	NaN	$-$Inf	0.8916	0.9408	0.9656	0.9902	0.9995
		S.dev.	NaN	NaN	0.6168	0.4320	0.2527	0.1334	0.0672
		ASD	4.4647	2.4412	1.0818	0.5452	0.2928	0.1484	0.0748
		CP%	100.00	100.00	99.54	93.07	96.15	97.87	96.40
0.1	0.7	Mean	$-$Inf	0.9609	0.9795	0.9941	0.9959	0.9994	0.9989
		S.dev.	NaN	0.4474	0.2513	0.1245	0.0617	0.0326	0.0162
		ASD	1.8071	0.6716	0.2897	0.1388	0.0690	0.0344	0.0172
		CP%	100.00	100.00	100.00	100.00	97.49	95.90	96.20
0.1	0.9	Mean	1.0170	1.0253	1.0273	1.0157	1.0099	1.0064	1.0067
		S.dev.	0.5639	0.3708	0.2810	0.2096	0.1452	0.1039	0.0784
0.3	0.5	Mean	NaN	NaN	$-$Inf	0.9554	0.9623	0.9536	0.9769
		S.dev.	NaN	NaN	NaN	0.8435	0.7340	0.5662	0.3768
		ASD	4.8407	4.2199	3.2605	2.1396	1.0325	0.6043	0.3862
		CP%	100.00	100.00	100.00	96.80	83.23	77.03	84.00
0.3	0.7	Mean	NaN	$-$Inf	0.9327	0.9886	0.9978	0.9984	0.9982
		S.dev.	NaN	NaN	0.5156	0.3295	0.1787	0.0872	0.0453
		ASD	3.5101	1.3180	0.6527	0.3283	0.1768	0.0915	0.0460
		CP%	100.00	100.00	100.00	90.61	95.07	95.50	94.80
0.3	0.9	Mean	$-$Inf	1.0177	1.0233	1.0235	1.0092	1.0092	1.0044
		S.dev.	NaN	0.4173	0.2704	0.1891	0.1256	0.1035	0.0803
0.5	0.7	Mean	NaN	NaN	$-$Inf	0.9450	0.9323	0.9497	0.9730
		S.dev.	NaN	NaN	NaN	0.7735	0.6158	0.4568	0.2702
		ASD	4.5998	2.2161	2.1461	1.2318	0.7645	0.4712	0.2721
		CP%	100.00	100.00	91.71	77.05	78.30	86.42	94.06
0.5	0.9	Mean	NaN	1.0025	1.0274	1.0277	1.0211	1.0122	1.0075
		S.dev.	NaN	0.6212	0.4051	0.2374	0.1476	0.1002	0.0726
0.7	0.9	Mean	NaN	NaN	Inf	1.0399	1.0375	1.0414	1.0288
		S.dev.	NaN	NaN	NaN	0.6359	0.4416	0.2525	0.1337

Table 5. The percentage (%) of iterations with constant d_N being equal to zero with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

H_1	H_2	N						
		2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	2.9	1.3	0.3	0	0	0	0
0.1	0.5	1.1	0.1	0	0	0	0	0
0.1	0.7	0.2	0	0	0	0	0	0
0.1	0.9	0	0	0	0	0	0	0
0.3	0.5	3.8	1.5	0.1	0	0	0	0
0.3	0.7	0.9	0	0	0	0	0	0
0.3	0.9	0.1	0	0	0	0	0	0
0.5	0.7	3.5	2.2	0.1	0	0	0	0
0.5	0.9	0.6	0	0	0	0	0	0
0.7	0.9	4.3	1.1	0.4	0	0	0	0

Table 6. The coverage probability (%) for estimators $(\xi_N, \eta_N, \zeta_N, \phi_N)^\top$ with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

H_1	H_2		N						
			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	ξ_N	94.29	92.19	95.38	94.24	94.68	92.25	93.14
		η_N	100.0	98.44	99.23	99.59	98.67	99.27	98.95
		ζ_N	100.0	100.0	100.0	100.0	100.0	100.0	99.88
		ϕ_N	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.1	0.5	ξ_N	95.06	94.55	94.20	94.25	93.54	94.41	95.10
		η_N	98.77	97.52	97.45	99.12	97.73	98.27	98.50
		ζ_N	97.53	100.0	98.84	99.41	98.75	99.19	99.20
		ϕ_N	100.0	100.0	99.54	99.26	99.55	99.19	99.40
0.1	0.7	ξ_N	95.83	95.11	96.33	95.24	94.48	94.00	94.40
		η_N	93.75	95.49	97.10	94.66	94.68	94.10	93.70
		ζ_N	90.62	95.11	93.82	91.88	91.77	90.00	91.10
		ϕ_N	88.54	92.48	91.31	89.91	90.26	87.50	88.60
0.3	0.5	ξ_N	95.00	95.28	95.21	95.64	94.36	95.77	94.80
		η_N	95.00	96.23	97.01	96.51	97.34	97.29	96.50
		ζ_N	96.67	98.11	98.20	97.67	99.06	98.70	97.80
		ϕ_N	100.0	99.06	97.01	98.84	98.90	98.81	97.90
0.3	0.7	ξ_N	96.15	94.21	94.24	92.70	93.07	94.00	93.70
		η_N	98.72	93.82	94.65	92.18	91.71	91.80	91.40
		ζ_N	96.15	93.05	91.98	90.61	90.35	89.40	89.00
		ϕ_N	93.59	93.44	89.51	87.22	88.88	86.90	85.70
0.5	0.7	ξ_N	89.36	96.15	95.85	91.06	94.18	92.56	94.27
		η_N	87.23	97.69	92.17	87.68	91.04	89.16	90.88
		ζ_N	89.36	96.92	87.56	84.30	88.52	86.81	87.91
		ϕ_N	91.49	97.69	85.25	82.37	86.95	84.60	86.00

Table 7. The percentage (%) of iterations with undefined asymptotic covariance matrix $\hat{\Sigma}_N^0$ with $\sigma^2 = 1$, $\kappa^2 = 1$ ($h = 1$)

H_1	H_2	N						
		2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	0.3	96.50	93.60	87.00	75.70	54.90	31.60	14.00
0.1	0.5	91.90	79.80	56.90	32.20	11.80	1.60	0
0.1	0.7	90.40	73.40	48.20	13.80	0.40	0	0
0.3	0.5	94.00	89.40	83.30	65.60	36.20	7.70	0
0.3	0.7	92.20	74.10	51.40	23.30	4.70	0	0
0.5	0.7	95.30	87.00	78.30	58.60	36.40	23.40	5.70

Table 8. The convergence of the series (16) with $\sigma^2 = 1$, $\kappa^2 = 1$ and $H_1 = 0.1$ ($h = 1$)

H_2		N						
		10^{-3}	10^{-4}	10^{-5}	10^{-6}	2^{-7}	10^{-8}	10^{-9}
0.3	m	4	8	16	35	76	168	377
	S_m	4.45362	4.45427	4.45445	4.45451	4.45452	4.45452	4.45453
0.5	m	3	4	7	12	22	42	78
	S_m	4.18198	4.18203	4.18206	4.18207	4.18208	4.18208	4.18208
0.7	m	$5.9 \cdot 10^6$	$4.9 \cdot 10^7$	$7.2 \cdot 10^7$	$7.2 \cdot 10^7$	$7.2 \cdot 10^7$	$7.2 \cdot 10^7$	$7.2 \cdot 10^7$
	S_m	9981.86	35818.4	45796.6	45796.6	45796.6	45796.6	45796.6

3. For any $0 < H_1 < H_2 < \frac{3}{4}$ and for all $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) < \infty. \tag{17}$$

Proof. First two statements are well known, see, e.g., [22]. In order to prove the third one, we observe that

$$\sum_{i=-\infty}^{\infty} \tilde{\rho}^2(i) \leq 2\kappa^4 h^{4H_1} \sum_{i=-\infty}^{\infty} \rho^2(i, H_1) + 2\sigma^4 h^{4H_2} \sum_{i=-\infty}^{\infty} \rho^2(i, H_2) < \infty,$$

by the second statement. Then, using the Cauchy–Schwarz inequality, we get for all $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) \leq \sqrt{\sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i+\alpha) \cdot \sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i+\beta)} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i) < \infty.$$
$\square$

5.2   Proofs of ergodic results

Proof of Lemma 1. Since B^{H_1} and B^{H_2} are independent centered Gaussian processes with stationary increments, we see that $\{\Delta X_k, k \geq 0\}$ is a stationary Gaussian sequence with $\mathsf{E}\,\Delta X_k = 0$. Therefore, in order to establish the ergodicity of ΔX_k, it suffices to prove that its autocovariance function $\tilde{\rho}(k)$ vanishes at infinity. In turn, this fact follows immediately from the first statement of Lemma 3:

$$\tilde{\rho}(k) = \kappa^2 h^{2H_1} \rho(k, H_1) + \sigma^2 h^{2H_2} \rho(k, H_2)$$
$$\sim \kappa^2 h^{2H_1} H_1 (2H_1 - 1)k^{(2H_1 - 2)} + \sigma^2 h^{2H_2} H_2 (2H_2 - 1)k^{(2H_2 - 2)} \to 0,$$

as $k \to \infty$.
$\square$

Proof of Lemma 2. (*i*) Taking $g(x_0) = x_0^2$ in (3), we obtain that

$$\xi_N \to \mathsf{E}\,X_h^2 = \mathsf{E}\left(\kappa B_h^{H_1} + \sigma B_h^{H_2}\right)^2 = \kappa^2 \mathsf{E}(B_h^{H_1})^2 + \sigma^2 \mathsf{E}(B_h^{H_2})^2$$
$$= \kappa^2 h^{2H_1} + \sigma^2 h^{2H_2}.$$

(*ii*) Applying $g(x_0, x_1) = (x_0 + x_1)^2$ to (3), we arrive at

$$\eta_N \to \mathsf{E}\,X_{2h}^2 = \mathsf{E}\left(\kappa B_{2h}^{H_1} + \sigma B_{2h}^{H_2}\right)^2 = \kappa^2 \mathsf{E}(B_{2h}^{H_1})^2 + \sigma^2 \mathsf{E}(B_{2h}^{H_2})^2$$
$$= \kappa^2 h^{2H_1} 2^{2H_1} + \sigma^2 h^{2H_2} 2^{2H_2}.$$

Statements (*iii*) and (*iv*) can be proved similarly; we need to choose $g(x_0, x_1, x_2, x_3) = (x_0 + x_1 + x_2 + x_3)^2$ and $g(x_0, \dots, x_7) = (\sum_{i=0}^{7} x_i)^2$ for (3), respectively. $\square$

5.3 Construction of estimators. Proof of Theorem 1

Let us briefly explain, how to solve the estimating equation (7). Denote $u = \kappa^2 h^{2H_1}$, $v = \sigma^2 h^{2H_2}$, $x = 2^{2H_1}$, $y = 2^{2H_2}$, $\xi = \xi_N$, $\eta = \eta_N$, $\zeta = \zeta_N$, $\phi = \phi_N$. Then (7) takes the form

$$\xi = u + v, \tag{18}$$

$$\eta = ux + vy, \tag{19}$$

$$\zeta = ux^2 + vy^2, \tag{20}$$

$$\phi = ux^3 + vy^3. \tag{21}$$

Using equations (18) and (19) we can express the variables u and v in terms of x and y as follows:

$$v = \frac{\eta - \xi x}{y - x}, \quad u = \frac{\eta - \xi y}{x - y}. \tag{22}$$

Combining (22) with (20) we get

$$\zeta = x^2 \cdot \frac{\eta - \xi y}{x - y} + y^2 \cdot \frac{\eta - \xi x}{y - x} = \frac{1}{y - x}\left(-\eta x^2 + \xi y x^2 + \eta y^2 - \xi x y^2\right)$$

$$= \frac{1}{y - x}\left(-\xi(y - x)xy + \eta(y^2 - x^2)\right) = -\xi xy + \eta(y + x). \tag{23}$$

Similarly, from (21) and (22) we derive

$$\phi = x^3 \cdot \frac{\eta - \xi y}{x - y} + y^3 \cdot \frac{\eta - \xi x}{y - x} = \frac{1}{y - x}\left(-\xi(y^2 - x^2)xy + \eta(y^3 - x^3)\right)$$

$$= -\xi(y + x)xy + \eta(y^2 + xy + x^2). \tag{24}$$

Multiplying (23) by $x + y$ results in

$$\zeta(x + y) = -\xi(x + y)xy + \eta(y^2 + 2xy + x^2). \tag{25}$$

Finally, by subtracting (24) from (25) we get $\zeta(x + y) - \phi = \eta xy$, whence

$$x = \frac{\phi - \zeta y}{\zeta - \eta y}. \tag{26}$$

At the same time, we can express x via y using (23):

$$x = \frac{\zeta - \eta y}{\eta - \xi y}. \tag{27}$$

From equations (26) and (27) we obtain

$$\frac{\phi - \zeta y}{\zeta - \eta y} = \frac{\zeta - \eta y}{\eta - \xi y},$$

whence we get the following quadratic equation for y

$$y^2\left(\eta^2 - \xi\zeta\right) - y\left(\eta\zeta - \xi\phi\right) + \left(\zeta^2 - \eta\phi\right) = 0. \tag{28}$$

It is easy to see that x satisfies the same quadratic equation, in view of symmetry. Therefore, x and y ($y > x$) are two roots of (28), which are given by

$$\frac{\eta\zeta - \xi\phi \pm \sqrt{D}}{2(\eta^2 - \xi\zeta)}, \tag{29}$$

where

$$\begin{aligned} D &= (\eta\zeta - \xi\phi)^2 - 4(\eta^2 - \xi\zeta)(\zeta^2 - \eta\phi) \\ &= \xi^2\phi^2 - 6\xi\eta\zeta\phi - 3\eta^2\zeta^2 + 4\eta^3\phi + 4\xi\zeta^3. \end{aligned} \tag{30}$$

It is not hard to derive the following relations from equations (18)–(21):

$$\eta\zeta - \xi\phi = (ux + vy)(ux^2 + vy^2) - (u + v)(ux^3 + vy^3) = -uv(y - x)^2(y + x);$$
$$\eta^2 - \xi\zeta = (ux + vy)^2 - (u + v)(ux^2 + vy^2) = -uv(y - x)^2;$$
$$\zeta^2 - \eta\phi = (ux^2 + vy^2)^2 - (ux + vy)(ux^3 + vy^3) = -uvxy(y - x)^2.$$

Thus

$$\begin{aligned} D &= (\eta\zeta - \xi\phi)^2 - 4(\eta^2 - \xi\zeta)(\zeta^2 - \eta\phi) \\ &= u^2v^2(y - x)^4(y + x)^2 - 4u^2v^2xy(y - x)^4 = u^2v^2(y - x)^6 > 0. \end{aligned}$$

Therefore, the discriminant is well defined and the corresponding equation has two different roots. From equation (29)

$$\frac{\eta\zeta - \xi\phi \pm \sqrt{D}}{2(\eta^2 - \xi\zeta)} = \frac{-uv(y - x)^2(y + x) \pm uv(y - x)^3}{-2uv(y - x)^2} = \frac{-1}{2} \cdot (-y - x \pm (y - x)),$$

which provides

$$x = \frac{\eta\zeta - \xi\phi + \sqrt{D}}{2(\eta^2 - \xi\zeta)}, \quad y = \frac{\eta\zeta - \xi\phi - \sqrt{D}}{2(\eta^2 - \xi\zeta)}.$$

Recall that $x = 2^{2H_1}$ and $y = 2^{2H_2}$, so we get

$$H_1 = \frac{1}{2}\log_2\left(\frac{\eta\zeta - \xi\phi + \sqrt{D}}{2(\eta^2 - \xi\zeta)}\right), \tag{31}$$

$$H_2 = \frac{1}{2}\log_2\left(\frac{\eta\zeta - \xi\phi - \sqrt{D}}{2(\eta^2 - \xi\zeta)}\right). \tag{32}$$

Also, from equation (22) we obtain

$$v = \frac{\eta - \xi x}{y - x} = \frac{2\eta^3 - 2\xi\eta\zeta - \xi\eta\zeta + \xi^2\phi - \xi\sqrt{D}}{-2\sqrt{D}} = \frac{\xi\sqrt{D} - 2\eta^3 + 3\xi\eta\zeta - \xi^2\phi}{2\sqrt{D}}.$$

Taking into account that $\sigma^2 h^{2H_2} = v$ and $a^{\log b} = b^{\log a}$ we get

$$\sigma^2 = \left(\frac{2(\eta^2 - \xi\zeta)}{\eta\zeta - \xi\phi - \sqrt{D}}\right)^{\log_2 h}\left(\frac{\xi\sqrt{D} - 2\eta^3 + 3\xi\eta\zeta - \xi^2\phi}{2\sqrt{D}}\right). \tag{33}$$

And similarly, we get

$$\kappa^2 = \left(\frac{2(\eta^2 - \xi\zeta)}{\eta\zeta - \xi\phi + \sqrt{D}} \right)^{\log_2 h} \left(\frac{\xi\sqrt{D} + 2\eta^3 - 3\xi\eta\zeta + \xi^2\phi}{2\sqrt{D}} \right). \qquad (34)$$

Equations (31)–(34) are providing us with inverse function $q = f^{-1}$, where f is defined by (5), such that for $f(\vartheta) = \tau_0$ we get $q(\tau_0) = \vartheta$. Therefore, by substituting strongly consistent estimator τ_N instead of τ_0 into equations (31)–(34) we obtain strongly consistent estimator $\hat{\vartheta}_N = (\widehat{H}_N^{(1)}, \widehat{H}_N^{(2)}, \hat{\kappa}_N^2, \hat{\sigma}_N^2)^\top$ for $\vartheta = (H_1, H_2, \kappa^2, \sigma^2)^\top$ defined by (9)–(12). $\qquad\square$

5.4 Proof of Theorem 2

The proof consists of two parts: in the first part we compute the asymptotic covariance matrix $\widetilde{\Sigma}$, while the second part contains the proof of asymptotic normality.

Part 1: Identification of the asymptotic covariance matrix.

Firstly, we will find the explicit form of covariance matrix $\widetilde{\Sigma}$ by evaluating convergence limits as $N \to \infty$ for the following variances and covariances: $N \mathbf{Var}(\xi_N)$, $N \mathbf{Var}(\eta_N)$, $N \mathbf{Var}(\zeta_N)$, $N \mathbf{Var}(\phi_N)$, $N \mathbf{cov}(\xi_N, \eta_N)$, $N \mathbf{cov}(\xi_N, \zeta_N)$, $N \mathbf{cov}(\xi_N, \phi_N)$, $N \mathbf{cov}(\eta_N, \zeta_N)$, $N \mathbf{cov}(\eta_N, \phi_N)$, $N \mathbf{cov}(\zeta_N, \phi_N)$.

1.1 Evaluation of convergence limit for $N \mathbf{Var}(\xi_N)$. Using Isserlis' theorem,[1] and stationarity of the sequence $\{\Delta X_k\}$, we can write

$$\mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2 \right) = 2 \, \mathbf{cov}\left((\Delta X_k), (\Delta X_j) \right)^2 = 2\tilde{\rho}(k-j)^2.$$

Therefore

$$N \mathbf{Var}(\xi_N) = \frac{1}{N} \mathbf{Var}\left(\sum_{k=0}^{N-1} (\Delta X_k)^2 \right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2 \right)$$

$$= \frac{2}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j)^2.$$

Further, by rearranging sums, we get

$$N \mathbf{Var}(\xi_N) = \frac{2}{N} \sum_{k=0}^{N-1} \sum_{i=k-N+1}^{k} \tilde{\rho}(i)^2 = \frac{2}{N} \sum_{i=-N+1}^{0} \sum_{k=0}^{N-1+i} \tilde{\rho}(i)^2 + \frac{2}{N} \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} \tilde{\rho}(i)^2$$

$$= 2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N} \right) \tilde{\rho}(i)^2 \to 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)^2, \quad \text{as } N \to \infty. \qquad (35)$$

---

[1]Isserlis' theorem [12]: if (X_1, X_2, X_3, X_4) is a zero-mean multivariate normal random vector, then $\mathsf{E}(X_1 X_2 X_3 X_4) = \mathsf{E}\, X_1 X_2 \, \mathsf{E}\, X_3 X_4 + \mathsf{E}\, X_1 X_3 \, \mathsf{E}\, X_2 X_4 + \mathsf{E}\, X_1 X_4 \, \mathsf{E}\, X_2 X_3$. In particular, $\mathbf{cov}(X_1^2, X_2^2) = 2\, \mathbf{cov}(X_1, X_2)^2$.

where the passage to the limit can be justified by the dominated convergence theorem due to Lemma 1.

1.2. Evaluation of convergence limit for $N \operatorname{\textbf{cov}}(\xi_N, \eta_N)$. Write

$$N \operatorname{\textbf{cov}}(\xi_N, \eta_N) = \frac{1}{N} \operatorname{\textbf{cov}}\left(\sum_{k=0}^{N-1}(\Delta X_k)^2, \sum_{j=0}^{N-1}(\Delta X_j + \Delta X_{j+1})^2\right)$$

$$= \frac{1}{N} \operatorname{\textbf{cov}}\left(\sum_{k=0}^{N-1}(\Delta X_k)^2, \sum_{j=0}^{N-1}\sum_{a,b=0}^{1}\Delta X_{j+a}\Delta X_{j+b}\right)$$

$$= \frac{1}{N} \sum_{a,b=0}^{1}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \operatorname{\textbf{cov}}\left((\Delta X_k)^2, \Delta X_{j+a}\Delta X_{j+b}\right).$$

By Isserlis' theorem,

$$\operatorname{\textbf{cov}}\left((\Delta X_k)^2, \Delta X_{j+a}\Delta X_{j+b}\right)$$
$$= \operatorname{\textsf{E}}(\Delta X_k)^2\Delta X_{j+a}\Delta X_{j+b} - \operatorname{\textsf{E}}(\Delta X_k)^2 \operatorname{\textsf{E}}\Delta X_{j+a}\Delta X_{j+b}$$
$$= \operatorname{\textsf{E}}\Delta X_k\Delta X_{j+a}\operatorname{\textsf{E}}\Delta X_k\Delta X_{j+b} + \operatorname{\textsf{E}}\Delta X_k\Delta X_{j+b}\operatorname{\textsf{E}}\Delta X_k\Delta X_{j+a}$$
$$= 2\operatorname{\textbf{cov}}(\Delta X_k, \Delta X_{j+a})\operatorname{\textbf{cov}}(\Delta X_k, \Delta X_{j+b}) = 2\tilde{\rho}(j + a - k)\tilde{\rho}(j + b - k).$$

Therefore arguing as in (35), we obtain

$$N \operatorname{\textbf{cov}}(\xi_N, \eta_N) = \sum_{a,b=0}^{1}\frac{2}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde{\rho}(j + a - k)\tilde{\rho}(j + b - k)$$

$$= \sum_{a,b=0}^{1}\frac{2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde{\rho}(i)\tilde{\rho}(i + b - a)$$

$$= 2\sum_{a,b=0}^{1}\sum_{i=-N+1}^{N-1}\left(1 - \frac{|i|}{N}\right)\tilde{\rho}(i)\tilde{\rho}(i + b - a)$$

$$\to 2\sum_{a,b=0}^{1}\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)\tilde{\rho}(i + b - a)$$

$$= \sum_{i=-\infty}^{\infty}\tilde{\rho}(i)(4\tilde{\rho}(i) + 4\tilde{\rho}(i + 1)),$$

as $N \to \infty$, where the last series converges according to Lemma 3.

1.3. Evaluation of convergence limit for $N \operatorname{\textbf{Var}}(\eta_N)$. We have

$$N \operatorname{\textbf{Var}}(\eta_N) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\operatorname{\textbf{cov}}\left(\sum_{a,b=0}^{1}\Delta X_{k+a}\Delta X_{k+b}, \sum_{c,d=0}^{1}\Delta X_{j+c}\Delta X_{j+d}\right)$$

$$= \sum_{a,b,c,d=0}^{1} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}(\Delta X_{k+a} \Delta X_{k+b}, \Delta X_{j+c} \Delta X_{j+d}).$$

By Isserlis' theorem,

$$
\begin{aligned}
\mathbf{cov}&(\Delta X_{k+a} \Delta X_{k+b}, \Delta X_{j+c} \Delta X_{j+d}) \\
&= \mathsf{E}\, \Delta X_{k+a} \Delta X_{k+b} \Delta X_{j+c} \Delta X_{j+d} - \mathsf{E}\, \Delta X_{k+a} \Delta X_{k+b} \,\mathsf{E}\, \Delta X_{j+c} \Delta X_{j+d} \\
&= \mathsf{E}\, \Delta X_{k+a} \Delta X_{k+b} \,\mathsf{E}\, \Delta X_{j+c} \Delta X_{j+d} + \mathsf{E}\, \Delta X_{k+a} \Delta X_{j+c} \,\mathsf{E}\, \Delta X_{k+b} \Delta X_{j+d} \\
&\quad + \mathsf{E}\, \Delta X_{k+a} \Delta X_{j+d} \,\mathsf{E}\, \Delta X_{j+c} \Delta X_{k+b} - \mathsf{E}\, \Delta X_{k+a} \Delta X_{k+b} \,\mathsf{E}\, \Delta X_{j+c} \Delta X_{j+d} \\
&= \mathsf{E}\, \Delta X_{k+a} \Delta X_{j+c} \,\mathsf{E}\, \Delta X_{k+b} \Delta X_{j+d} + \mathsf{E}\, \Delta X_{k+a} \Delta X_{j+d} \,\mathsf{E}\, \Delta X_{j+c} \Delta X_{k+b} \\
&= \mathbf{cov}(\Delta k + a, \Delta X_{j+c}) \,\mathbf{cov}(\Delta X_{k+b}, \Delta X_{j+d}) \\
&\quad + \mathbf{cov}(\Delta X_{k+a}, \Delta X_{j+d}) \,\mathbf{cov}(\Delta X_{k+b}, \Delta X_{j+c}) \\
&= \tilde{\rho}(k - j + c - a)\tilde{\rho}(k - j + d - b) + \tilde{\rho}(k - j + c - b)\tilde{\rho}(k - j + d - a) \\
&= \sum_{\gamma=0}^{1} \tilde{\rho}(k - j + c - a + \gamma(a - b))\tilde{\rho}(k - j + d - b + \gamma(b - a)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
N\,\mathbf{Var}(\eta_N) &= \sum_{a,b,c,d,\gamma=0}^{1} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k - j + c - a + \gamma(a - b)) \\
&\quad \times \tilde{\rho}(k - j + d - b + \gamma(b - a)) \\
&= \sum_{a,b,c,d,\gamma=0}^{1} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i + c - a + \gamma(a - b)) \\
&\quad \times \tilde{\rho}(i + d - b + \gamma(b - a)) \\
&\to \sum_{a,b,c,d,\gamma=0}^{1} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i + c - a + \gamma(a - b))\tilde{\rho}(i + d - b + \gamma(b - a)) \\
&= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{a,b,c,d,\gamma=0}^{1} \tilde{\rho}(i + d - c + a - b + 2\gamma(b - a)),
\end{aligned}
$$

as $N \to \infty$. After simplifications, we arrive at

$$\lim_{N\to\infty} N\,\mathbf{Var}(\eta_N) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(12\tilde{\rho}(i) + 16\tilde{\rho}(i + 1) + 4\tilde{\rho}(i + 2)\Big).$$

1.4. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N, \zeta_N)$.
We rewrite $N\,\mathbf{cov}(\xi_N, \zeta_N)$ in the form

$$N\,\mathbf{cov}(\xi_N, \zeta_N) = \frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1} (\Delta X_k)^2, \sum_{j=0}^{N-1} \sum_{a,b=0}^{3} \Delta X_{j+a} \Delta X_{j+b}\right)$$

$$= \frac{1}{N} \sum_{a,b=0}^{3} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left((\Delta X_k)^2, \Delta X_{j+a} \Delta X_{j+b}\right).$$

By Isserlis' theorem,

$$
\begin{aligned}
N \, \mathbf{cov}(\xi_N, \zeta_N) &= \sum_{a,b=0}^{3} \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \tilde{\rho}(j+a-k)\tilde{\rho}(j+b-k) \\
&= \sum_{a,b=0}^{3} \frac{2}{N} \sum_{j=0}^{N-1} \sum_{i=j-N+1}^{j} \tilde{\rho}(i)\tilde{\rho}(i+b-a) \\
&= 2 \sum_{a,b=0}^{3} \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i)\tilde{\rho}(i+b-a) \\
&\to 2 \sum_{a,b=0}^{3} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+b-a) \\
&= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\left(8\tilde{\rho}(i) + 12\tilde{\rho}(i+1) + 8\tilde{\rho}(i+2) + 4\tilde{\rho}(i+3)\right),
\end{aligned}
$$

as $N \to \infty$, where the last series converges according to Lemma 3.

    1.5. Evaluation of convergence limit for $N \, \mathbf{cov}(\eta_N, \zeta_N)$. We have:

$$
\begin{aligned}
N \, \mathbf{cov}(\eta_N, \zeta_N) &= \frac{1}{N} \mathbf{cov}\left(\sum_{k=0}^{N-1} \sum_{a,b=0}^{1} \Delta X_{j+a} \Delta X_{j+b}, \sum_{j=0}^{N-1} \sum_{c,d=0}^{3} \Delta X_{j+c} \Delta X_{j+d}\right) \\
&= \frac{1}{N} \sum_{a,b=0}^{1} \sum_{c,d=0}^{3} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left(\Delta X_{j+a} \Delta X_{j+b}, \Delta X_{j+c} \Delta X_{j+d}\right).
\end{aligned}
$$

By Isserlis' theorem,

$$
\begin{aligned}
N \, \mathbf{cov}(\eta_N, \zeta_N) &= \sum_{a,b,\gamma=0}^{1} \sum_{c,d,=0}^{3} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j+c-a+\gamma(a-b)) \\
&\qquad \times \tilde{\rho}(k-j+d-b+\gamma(b-a)) \\
&= \sum_{a,b,\gamma=0}^{1} \sum_{c,d,=0}^{3} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i+c-a+\gamma(a-b)) \\
&\qquad \times \tilde{\rho}(i+d-b+\gamma(b-a)) \\
&\to \sum_{a,b,\gamma=0}^{1} \sum_{c,d,=0}^{3} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i+c-a+\gamma(a-b)) \\
&\qquad \times \tilde{\rho}(i+d-b+\gamma(b-a))
\end{aligned}
$$

$$= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{a,b,\gamma=0}^{1} \sum_{c,d,=0}^{3} \tilde{\rho}(i + d - c + a - b + 2\gamma(b - a)),$$

as $N \to \infty$. After simplifications, we arrive at

$$\lim_{N\to\infty} N\,\mathbf{cov}(\eta_N, \zeta_N) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(28\tilde{\rho}(i) + 48\tilde{\rho}(i + 1)$$

$$+ 32\tilde{\rho}(i + 2) + 16\tilde{\rho}(i + 3) + 4\tilde{\rho}(i + 4)\Big).$$

1.6. Evaluation of convergence limit for $N\,\mathbf{Var}(\zeta_N)$. We have:

$$N\,\mathbf{Var}(\zeta_N)) = \frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1}\sum_{a,b=0}^{3} \Delta X_{j+a}\Delta X_{j+b}, \sum_{j=0}^{N-1}\sum_{c,d=0}^{3} \Delta X_{j+c}\Delta X_{j+d}\right)$$

$$= \frac{1}{N}\sum_{a,b,c,d=0}^{3}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \mathbf{cov}\left(\Delta X_{j+a}\Delta X_{j+b}, \Delta X_{j+c}\Delta X_{j+d}\right).$$

By Isserlis' theorem,

$$N\,\mathbf{Var}(\zeta_N) = \sum_{\gamma=0}^{1}\sum_{a,b,c,d=0}^{3} \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \tilde{\rho}(k - j + c - a + \gamma(a - b))$$

$$\times \tilde{\rho}(k - j + d - b + \gamma(b - a))$$

$$= \sum_{\gamma=0}^{1}\sum_{a,b,c,d=0}^{3}\sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right)\tilde{\rho}(i + c - a + \gamma(a - b))$$

$$\times \tilde{\rho}(i + d - b + \gamma(b - a))$$

$$\to \sum_{\gamma=0}^{1}\sum_{a,b,c,d=0}^{3}\sum_{i=-\infty}^{\infty} \tilde{\rho}(i + c - a + \gamma(a - b))$$

$$\times \tilde{\rho}(i + d - b + \gamma(b - a))$$

$$= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{\gamma=0}^{1}\sum_{a,b,c,d=0}^{3} \tilde{\rho}(i + d - c + a - b + 2\gamma(b - a)),$$

as $N \to \infty$. After simplifications, we arrive at

$$\lim_{N\to\infty} N\,\mathbf{Var}(\zeta_N)) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(88\tilde{\rho}(i) + 160\tilde{\rho}(i + 1)$$

$$+ 124\tilde{\rho}(i + 2) + 80\tilde{\rho}(i + 3) + 40\tilde{\rho}(i + 4) + 16\tilde{\rho}(i + 5) + 4\tilde{\rho}(i + 6)\Big).$$

1.7. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N, \phi_N)$. We have:

$$N\,\mathbf{cov}(\xi_N, \phi_N) = \frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1}(\Delta X_k)^2, \sum_{j=0}^{N-1}\sum_{a,b=0}^{7}\Delta X_{j+a}\Delta X_{j+b}\right)$$

$$= \frac{1}{N}\sum_{a,b=0}^{7}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}\left((\Delta X_k)^2, \Delta X_{j+a}\Delta X_{j+b}\right).$$

By Isserlis' theorem,

$$N\,\mathbf{cov}(\xi_N, \phi_N) = \sum_{a,b=0}^{7}\frac{2}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde{\rho}(j+a-k)\tilde{\rho}(j+b-k)$$

$$= \sum_{a,b=0}^{7}\frac{2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde{\rho}(i)\tilde{\rho}(i+b-a)$$

$$= 2\sum_{a,b=0}^{7}\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i)\tilde{\rho}(i+b-a)$$

$$\rightarrow 2\sum_{a,b=0}^{7}\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)\tilde{\rho}(i+b-a),$$

as $N \to \infty$. After simplifications, we arrive at

$$\lim_{N\to\infty} N\,\mathbf{cov}(\xi_N, \phi_N) = \sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(16\tilde{\rho}(i) + 28\tilde{\rho}(i+1) + 24\tilde{\rho}(i+2)$$

$$+ 20\tilde{\rho}(i+3) + 16\tilde{\rho}(i+4) + 12\tilde{\rho}(i+5) + 8\tilde{\rho}(i+6) + 4\tilde{\rho}(i+7)\Big).$$

1.8. Evaluation of convergence limit for $N\,\mathbf{cov}(\eta_N, \phi_N)$. We have:

$$N\,\mathbf{cov}(\eta_N, \phi_N) = \frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1}\sum_{a,b=0}^{1}\Delta X_{j+a}\Delta X_{j+b}, \sum_{j=0}^{N-1}\sum_{c,d=0}^{7}\Delta X_{j+c}\Delta X_{j+d}\right)$$

$$= \frac{1}{N}\sum_{a,b=0}^{1}\sum_{c,d=0}^{7}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}\left(\Delta X_{j+a}\Delta X_{j+b}, \Delta X_{j+c}\Delta X_{j+d}\right).$$

By Isserlis' theorem,

$$N\,\mathbf{cov}(\eta_N, \phi_N) = \sum_{a,b,\gamma=0}^{1}\sum_{c,d=0}^{7}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j+c-a+\gamma(a-b))$$

$$\times \tilde{\rho}(k-j+d-b+\gamma(b-a))$$

$$= \sum_{a,b,\gamma=0}^{1} \sum_{c,d=0}^{7} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i+c-a+\gamma(a-b))$$

$$\times \, \tilde{\rho}(i+d-b+\gamma(b-a))$$

$$\to \sum_{a,b,\gamma=0}^{1} \sum_{c,d=0}^{7} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i+c-a+\gamma(a-b))$$

$$\times \, \tilde{\rho}(i+d-b+\gamma(b-a))$$

$$= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{a,b,\gamma=0}^{1} \sum_{c,d=0}^{7} \tilde{\rho}(i+d-c+a-b+2\gamma(b-a)),$$

as $N \to \infty$. After simplifications, we arrive at

$$\lim_{N\to\infty} N \, \mathbf{cov}(\eta_N, \phi_N) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(60\tilde{\rho}(i) + 112\tilde{\rho}(i+1) + 96\tilde{\rho}(i+2)$$

$$+ 80\tilde{\rho}(i+3) + 64\tilde{\rho}(i+4) + 48\tilde{\rho}(i+5) + 32\tilde{\rho}(i+6) + 16\tilde{\rho}(i+7) + 4\tilde{\rho}(i+8)\Big).$$

1.9. Evaluation of convergence limit for $N \, \mathbf{cov}(\zeta_N, \phi_N)$. We have:

$$N \, \mathbf{cov}(\zeta_N, \phi_N) = \frac{1}{N} \, \mathbf{cov}\left(\sum_{k=0}^{N-1} \sum_{a,b=0}^{3} \Delta X_{j+a} \Delta X_{j+b}, \sum_{j=0}^{N-1} \sum_{c,d=0}^{7} \Delta X_{j+c} \Delta X_{j+d}\right)$$

$$= \frac{1}{N} \sum_{a,b=0}^{3} \sum_{c,d=0}^{7} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left(\Delta X_{j+a} \Delta X_{j+b}, \Delta X_{j+c} \Delta X_{j+d}\right).$$

By Isserlis' theorem,

$$N \, \mathbf{cov}(\zeta_N, \phi_N) = \sum_{\gamma=0}^{1} \sum_{a,b=0}^{3} \sum_{c,d=0}^{7} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j+c-a+\gamma(a-b))$$

$$\times \, \tilde{\rho}(k-j+d-b+\gamma(b-a))$$

$$= \sum_{\gamma=0}^{1} \sum_{a,b=0}^{3} \sum_{c,d=0}^{7} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i+c-a+\gamma(a-b))$$

$$\times \, \tilde{\rho}(i+d-b+\gamma(b-a))$$

$$\to \sum_{\gamma=0}^{1} \sum_{a,b=0}^{3} \sum_{c,d=0}^{7} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i+c-a+\gamma(a-b))$$

$$\times \, \tilde{\rho}(i+d-b+\gamma(b-a))$$

$$= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{\gamma=0}^{1} \sum_{a,b=0}^{3} \sum_{c,d=0}^{7} \tilde{\rho}(i+d-c+a-b+2\gamma(b-a)),$$

as $N \to \infty$. After simplifications, we arrive at

$$
\lim_{N \to \infty} N \, \mathbf{cov}(\zeta_N, \phi_N) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i) \Big(216 \tilde{\rho}(i) + 416 \tilde{\rho}(i+1) + 376 \tilde{\rho}(i+2)
$$
$$
+ 320 \tilde{\rho}(i+3) + 256 \tilde{\rho}(i+4) + 192 \tilde{\rho}(i+5) + 132 \tilde{\rho}(i+6)
$$
$$
+ 80 \tilde{\rho}(i+7) + 40 \tilde{\rho}(i+8) + 16 \tilde{\rho}(i+9) + 4 \tilde{\rho}(i+10) \Big).
$$

1.10. Evaluation of convergence limit for $N \, \mathbf{Var}(\phi_N)$. We have:

$$
N \, \mathbf{Var}(\phi_N) = \frac{1}{N} \, \mathbf{cov} \left(\sum_{k=0}^{N-1} \sum_{a,b=0}^{7} \Delta X_{j+a} \Delta X_{j+b}, \sum_{j=0}^{N-1} \sum_{c,d=0}^{7} \Delta X_{j+c} \Delta X_{j+d} \right)
$$
$$
= \frac{1}{N} \sum_{a,b,c,d=0}^{7} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov} \left(\Delta X_{j+a} \Delta X_{j+b}, \Delta X_{j+c} \Delta X_{j+d} \right).
$$

By Isserlis' theorem,

$$
N \, \mathbf{Var}(\phi_N) = \sum_{\gamma=0}^{1} \sum_{a,b,c,d=0}^{7} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k - j + c - a + \gamma(a-b))
$$
$$
\times \tilde{\rho}(k - j + d - b + \gamma(b-a))
$$
$$
= \sum_{\gamma=0}^{1} \sum_{a,b,c,d=0}^{7} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N} \right) \tilde{\rho}(i + c - a + \gamma(a-b))
$$
$$
\times \tilde{\rho}(i + d - b + \gamma(b-a))
$$
$$
\to \sum_{\gamma=0}^{1} \sum_{a,b,c,d=0}^{7} \sum_{i=-\infty}^{\infty} \tilde{\rho}(i + c - a + \gamma(a-b))
$$
$$
\times \tilde{\rho}(i + d - b + \gamma(b-a))
$$
$$
= \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) \sum_{\gamma=0}^{1} \sum_{a,b,c,d=0}^{7} \tilde{\rho}(i + d - c + a - b + 2\gamma(b-a)),
$$

as $N \to \infty$. After simplifications, we arrive at

$$
\lim_{N \to \infty} N \, \mathbf{Var}(\phi_N) = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i) \Big(688 \tilde{\rho}(i) + 1344 \tilde{\rho}(i+1) + 1260 \tilde{\rho}(i+2)
$$
$$
+ 1136 \tilde{\rho}(i+3) + 984 \tilde{\rho}(i+4) + 816 \tilde{\rho}(i+5) + 644 \tilde{\rho}(i+6) + 480 \tilde{\rho}(i+7)
$$
$$
+ 336 \tilde{\rho}(i+8) + 224 \tilde{\rho}(i+9) + 140 \tilde{\rho}(i+10) + 80 \tilde{\rho}(i+11) + 40 \tilde{\rho}(i+12)
$$
$$
+ 16 \tilde{\rho}(i+13) + 4 \tilde{\rho}(i+14) \Big).
$$

Part 2: Proof of asymptotic normality.

Let us define $Y_k = (Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}, Y_k^{(4)})^\top$ by

$$Y_k^{(1)} := \Delta X_k, \quad Y_k^{(2)} := \Delta X_{k+1}, \quad Y_k^{(3)} := \Delta X_{k+2} + \Delta X_{k+3}$$
$$Y_k^{(4)} := \Delta X_{k+4} + \Delta X_{k+5} + \Delta X_{k+6} + \Delta X_{k+7}. \tag{36}$$

Then

$$\xi_N = \frac{1}{N} \sum_{k=0}^{N-1} \left(Y_k^{(1)}\right)^2, \qquad \eta_N = \frac{1}{N} \sum_{k=0}^{N-1} \left(Y_k^{(1)} + Y_k^{(2)}\right)^2,$$

$$\zeta_N = \frac{1}{N} \sum_{k=0}^{N-1} \left(Y_k^{(1)} + Y_k^{(2)} + Y_k^{(3)}\right)^2,$$

$$\phi_N = \frac{1}{N} \sum_{k=0}^{N-1} \left(Y_k^{(1)} + Y_k^{(2)} + Y_k^{(3)} + Y_k^{(4)}\right)^2.$$

We shall prove the convergence of vector $(\xi_N, \eta_N, \zeta_N, \phi_N)^\top$ with the help of the Cramér–Wold device. Let the parameters $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$ be such that $\alpha^2 + \beta^2 + \gamma^2 + \lambda^2 \neq 0$. We introduce the function

$$f(y) = \alpha y_1^2 + \beta(y_1 + y_2)^2 + \gamma(y_1 + y_2 + y_3)^2 + \lambda(y_1 + y_2 + y_3 + y_4)^2,$$
$$y = (y_1, y_2, y_3)^\top \in \mathbb{R}^3,$$

so that

$$\alpha \xi_N + \beta \eta_N + \gamma \zeta_N + \lambda \phi_N = \frac{1}{N} \sum_{k=0}^{N-1} f(Y_k).$$

Thus, we need to prove that the sequence

$$\sqrt{N}\Big(\alpha(\xi_N - \mathsf{E}\,\xi_0) + \beta(\eta_N - \mathsf{E}\,\eta_0) + \gamma(\zeta_N - \mathsf{E}\,\zeta_0) + \lambda(\phi_N - \mathsf{E}\,\phi_0)\Big)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \Big(f(Y_k) - \mathsf{E}\,f(Y_k)\Big) \quad (37)$$

converges to a normal distribution. This fact can be established by application of the Breuer–Major theorem for stationary vectors [1, Theorem 4]. In order to apply this theorem, it suffices to verify the following condition:

$$\sum_{j\in\mathbb{Z}} \left|r^{(p,l)}(j)\right|^q < \infty, \quad \text{for all } p, l \in \{1, 2, 3\}, \tag{38}$$

where

$$r^{(p,l)}(k) := \mathbf{cov}\left(Y_1^{(p)}, Y_{1+k}^{(l)}\right), \quad k \in \mathbb{Z},$$

and q is the Hermite rank[2] of f with respect to Y_1.

It is not hard to see that this Hermite rank $q \geq 2$. Indeed, note that $f(Y_1)$ is a second-order polynomial of zero-mean normally distributed random variables $Y_1^{(1)}$, $Y_1^{(2)}$, $Y_1^{(3)}$, $Y_1^{(4)}$ in which only terms of the second order are present. Therefore, according to Isserlis' theorem, we will have that the expected value of the product of $f(Y_1)$ and any of $Y_1^{(t)}$, $t \in \{1, 2, 3\}$, will be equal to zero. All the obtained terms will have the form $\mathsf{E}\, G_1 G_2 G_3$, where $(G_1, G_2, G_3)^\top$ is a zero-mean multivariate normal random vector and according to Isserlis' theorem $\mathsf{E}\, G_1 G_2 G_3 = 0$. Therefore, the expected value of the product of $f(Y_1)$ and any first-order polynomial of $\{Y_1\}$ (a linear combination of $Y_1^{(1)}$, $Y_1^{(2)}$, $Y_1^{(3)}$, $Y_1^{(4)}$) will be equal to zero, and therefore the Hermite rank q of the function f with respect to Y_1 cannot be equal to 1.

Since $q \geq 2$, we easily see that in order to verify the condition (38), it suffices to prove that

$$\sum_{j \in \mathbb{Z}} \left| r^{(p,l)}(j) \right|^2 < \infty, \quad \text{for all } p, l \in \{1, 2, 3\}. \tag{39}$$

In turn, (39) follows from Lemma 1, since using (36) and the definition of $\tilde{\rho}$, we can represent $r^{(p,l)}$ via $\tilde{\rho}$ as follows:

$$r^{(1,1)}(k) = r^{(2,2)}(k) = \tilde{\rho}(k), \quad r^{(1,2)}(k) = \tilde{\rho}(k+1), \quad r^{(2,1)}(k) = \tilde{\rho}(k-1),$$

$$r^{(1,3)}(k) = \tilde{\rho}(k) + \tilde{\rho}(k+2) + \tilde{\rho}(k+3), \quad r^{(3,1)}(k) = \tilde{\rho}(k-2) + \tilde{\rho}(k-3),$$

$$r^{(2,3)}(k) = +\tilde{\rho}(k+1) + \tilde{\rho}(k+2), \quad r^{(3,2)}(k) = \tilde{\rho}(k-1) + \tilde{\rho}(k-2),$$

$$r^{(3,3)}(k) = \tilde{\rho}(k+1) + 2\tilde{\rho}(k) + \tilde{\rho}(k-1),$$

$$r^{(1,4)}(k) = \tilde{\rho}(k+4) + \tilde{\rho}(k+5) + \tilde{\rho}(k+6) + \tilde{\rho}(k+7),$$

$$r^{(4,1)}(k) = \tilde{\rho}(k-4) + \tilde{\rho}(k-5) + \tilde{\rho}(k-6) + \tilde{\rho}(k-7),$$

$$r^{(2,4)}(k) = \tilde{\rho}(k+3) + \tilde{\rho}(k+4) + \tilde{\rho}(k+5) + \tilde{\rho}(k+6),$$

$$r^{(4,2)}(k) = \tilde{\rho}(k-3) + \tilde{\rho}(k-4) + \tilde{\rho}(k-5) + \tilde{\rho}(k-6),$$

$$r^{(3,4)}(k) = \tilde{\rho}(k+2) + 2\tilde{\rho}(k+2) + 2\tilde{\rho}(k+3) + 2\tilde{\rho}(k+4) + \tilde{\rho}(k+5),$$

$$r^{(4,3)}(k) = \tilde{\rho}(k-2) + 2\tilde{\rho}(k-2) + 2\tilde{\rho}(k-3) + 2\tilde{\rho}(k-4) + \tilde{\rho}(k-5),$$

$$r^{(4,4)}(k) = \tilde{\rho}(k-3) + 2\tilde{\rho}(k-2) + 3\tilde{\rho}(k-1) + 4\tilde{\rho}(k)$$
$$+ 3\tilde{\rho}(k+1) + 2\tilde{\rho}(k+2) + \tilde{\rho}(k+3).$$

Indeed, for example, the series

$$\sum_{k=-\infty}^{+\infty} \left(r^{(1,3)}(k) \right)^2 = \sum_{k=-\infty}^{+\infty} (\tilde{\rho}(k+2) + \tilde{\rho}(k+3))^2$$

---

[2] The function $f \colon \mathbb{R}^d \to \mathbb{R}$ is said to have Hermite rank equal to q with respect to a random vector X if (a) $\mathsf{E}[(f(X) - \mathsf{E}[f(X)]) p_m(X)] = 0$ for every polynomial p_m (on $\mathbb{R}^d$) of degree $m \leq q - 1$; and (b) there exists a polynomial p_q of degree q such that $\mathsf{E}[(f(X) - \mathsf{E}[f(X)]) p_m(X)] \neq 0$, see [23].

$$= \sum_{k=-\infty}^{+\infty} \tilde{\rho}(k+2)^2 + 2 \sum_{k=-\infty}^{+\infty} \tilde{\rho}(k+2)\tilde{\rho}(k+3) + \sum_{k=-\infty}^{+\infty} \tilde{\rho}(k+3)^2,$$

converges as a finite sum of series convergent according to Lemma 1. Similarly, other series in (39) are also finite.

Therefore, the assumptions of the Breuer–Major theorem for stationary vectors are satisfied, whence the desired weak converge of (37) to a zero-mean normal distribution follows. □

5.5 Proof of Theorem 3

In order to prove Theorem 3 we apply the delta method (see, e.g., [13, Theorem B.6]) for function g constructed in Theorem 1 and sequence τ_N that is asymptotically normal by Theorem 1, with the inverse function theorem. In order to apply the delta method we need to calculate the derivation matrix g' and prove that it is nonsingular. But considering the complicated structure of this function, we will calculate the derivation matrix f' for the function f defined in (5) and apply the inverse function theorem at the point ϑ since g is an inverse function of f, and thus $g' = (f')^{-1}$. Therefore, we will need to find matrix $f'(\vartheta)$ and to show that it is nonsingular.

We will start with evaluating partial derivatives of function f_1. Since

$$f_1(H_1, H_2, \kappa^2, \sigma^2) = \kappa^2 h^{2H_1} + \sigma^2 h^{2H_2},$$

we see that

$$\frac{\partial f_1}{\partial H_1} = 2\log h \kappa^2 h^{2H_1}, \qquad \frac{\partial f_1}{\partial H_2} = 2\log h \sigma^2 h^{2H_2},$$

$$\frac{\partial f_1}{\partial \kappa^2} = h^{2H_1}, \qquad \frac{\partial f_1}{\partial \sigma^2} = h^{2H_2}.$$

For the function

$$f_2(H_1, H_2, \kappa^2, \sigma^2) = \kappa^2 h^{2H_1} 2^{2H_1} + \sigma^2 h^{2H_2} 2^{2H_2}$$
$$= \kappa^2 e^{2H_1 \log(2h)} + \sigma^2 e^{2H_2 \log(2h)},$$

we get

$$\frac{\partial f_2}{\partial H_1} = 2\log(2h)\kappa^2 h^{2H_1} 2^{2H_1}, \qquad \frac{\partial f_2}{\partial H_2} = 2\log(2h)\sigma^2 h^{2H_2} 2^{2H_2},$$

$$\frac{\partial f_2}{\partial \kappa^2} = h^{2H_1} 2^{2H_1}, \qquad \frac{\partial f_2}{\partial \sigma^2} = h^{2H_2} 2^{2H_2}.$$

Similarly, one can evaluate partial derivatives for the functions f_3 and f_4. We arrive at the following derivative matrix:

$$f'(\vartheta) = \begin{pmatrix} 2\log(h)\kappa^2 h^{2H_1} & 2\log(h)\sigma^2 h^{2H_2} & h^{2H_1} & h^{2H_2} \\ 2\log(2h)\kappa^2(2h)^{2H_1} & 2\log(2h)\sigma^2(2h)^{2H_2} & (2h)^{2H_1} & (2h)^{2H_2} \\ 2\log(4h)\kappa^2(4h)^{2H_1} & 2\log(4h)\sigma^2(4h)^{2H_2} & (4h)^{2H_1} & (4h)^{2H_2} \\ 2\log(8h)\kappa^2(8h)^{2H_1} & 2\log(8h)\sigma^2(8h)^{2H_2} & (8h)^{2H_1} & (8h)^{2H_2} \end{pmatrix}.$$

Now, we need to compute its determinant, so that we could apply the delta method. We will have

$$\det(f'(\vartheta)) = 4\kappa^2\sigma^2 h^{4H_1+4H_2} \begin{vmatrix} \log(h) & \log(h) & 1 & 1 \\ \log(2h)2^{2H_1} & \log(2h)2^{2H_2} & 2^{2H_1} & 2^{2H_2} \\ \log(4h)4^{2H_1} & \log(4h)4^{2H_2} & 4^{2H_1} & 4^{2H_2} \\ \log(8h)8^{2H_1} & \log(8h)8^{2H_2} & 8^{2H_1} & 8^{2H_2} \end{vmatrix}$$

$$= 4\kappa^2\sigma^2 h^{4H_1+4H_2} \log 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 2^{2H_1} & 2^{2H_2} & 2^{2H_1} & 2^{2H_2} \\ 2\cdot2^{4H_1} & 2\cdot2^{4H_2} & 2^{4H_1} & 2^{4H_2} \\ 3\cdot2^{6H_1} & 3\cdot2^{6H_2} & 2^{6H_1} & 2^{6H_2} \end{vmatrix}$$

$$= 4\kappa^2\sigma^2 h^{4H_1+4H_2} \log 2 \left(\begin{vmatrix} 2^{2H_1} & 2^{2H_2} & 2^{2H_2} \\ 2\cdot2^{4H_1} & 2\cdot2^{4H_2} & 2^{4H_2} \\ 3\cdot2^{6H_1} & 3\cdot2^{6H_2} & 2^{6H_2} \end{vmatrix} \right.$$

$$\left. - \begin{vmatrix} 2^{2H_1} & 2^{2H_2} & 2^{2H_1} \\ 2\cdot2^{4H_1} & 2\cdot2^{4H_2} & 2^{4H_1} \\ 3\cdot2^{6H_1} & 3\cdot2^{6H_2} & 2^{6H_1} \end{vmatrix} \right).$$

Considering that

$$\begin{vmatrix} 2^{2H_1} & 2^{2H_2} & 2^{2H_2} \\ 2\cdot2^{4H_1} & 2\cdot2^{4H_2} & 2^{4H_2} \\ 3\cdot2^{6H_1} & 3\cdot2^{6H_2} & 2^{6H_2} \end{vmatrix} = -2^{2H_1}2^{10H_2} + 4\cdot2^{4H_1}2^{8H_2} - 3\cdot2^{6H_1}2^{6H_2},$$

and

$$\begin{vmatrix} 2^{2H_1} & 2^{2H_2} & 2^{2H_1} \\ 2\cdot2^{4H_1} & 2\cdot2^{4H_2} & 2^{4H_1} \\ 3\cdot2^{6H_1} & 3\cdot2^{6H_2} & 2^{6H_1} \end{vmatrix} = 2^{10H_1}2^{2H_2} + 3\cdot2^{6H_1}2^{6H_2} - 4\cdot2^{8H_1}2^{4H_2},$$

we get

$$\det(f'(\vartheta)) = -4\kappa^2\sigma^2 h^{6H_1+6H_2} \log(2) \left(2^{2H_2} - 2^{2H_1}\right)^4 \neq 0.$$

Therefore, the derivative matrix is nonsingular, and the corresponding derivative matrix of the inverse function $g'(\tau_0) = (f'(\vartheta))^{-1}$ is nonsingular by the inverse function theorem. Thus, the delta method can be applied, which implies the statement of the theorem and provides the formula for the asymptotic covariance matrix Σ^0 defined using the inverse derivative matrix $f'(\vartheta)$. □

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