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Gamma mixed fractional Lévy Ornstein–Uhlenbeck process

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Abstract In this article, a non-Gaussian long memory process is constructed by the aggregation of independent copies of a fractional Lévy Ornstein–Uhlenbeck process with random coefficients. Several properties and a limit theorem are studied for this new process. Finally, some simulations of the limit process are shown.

Keywords Fractional Lévy process, Ornstein–Uhlenbeck process, non-Gaussian process, random coefficients
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1 Introduction

An Ornstein–Uhlenbeck (OU) process is a diffusion process introduced by the physicists Leonard Salomon Ornstein and George Eugene Uhlenbeck [27] to describe the

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stochastic behavior of the velocity of a particle undergoing Brownian motion. The OU process $X = \{X(t),\ t \geq 0\}$ is the solution of the Langevin equation

$$dX(t) = \alpha X(t)dt + \sigma dB(t), \quad t \geq 0, \tag{1}$$

where $X(0) = x \in \mathbb{R}$, $B = \{B(t), t \geq 0\}$ is a Brownian motion and α, σ are constants. This process is stationary, Gaussian and Markovian; in fact, it is the only stochastic process which has all these three properties. This process is used for modeling in many different fields such as physics, biology and finance among others (see [1, 6, 14, 27, 23, 28] and references therein) and it has been widely generalized.

Different extensions of the Ornstein–Uhlenbeck processes have been obtained replacing the Brownian motion in (1) by more general noise processes; for example, Lévy OU [2], fractional OU [7], subfractional OU [22] or Hermite OU processes [16]. These are introduced by solution of the Langevin equation with driving noise given by a Lévy process, fractional Brownian motion, subfractional Brownian motion, Hermite process, respectively.

In the study of long-range dependence, Igoli and Terdik [13] defined a generalization of OU process with this property. This process is called Gamma-mixed Ornstein–Uhlenbeck process and it is built via aggregation of a sequence of Ornstein–Uhlenbeck processes with random coefficients. Let us be more precise, given a sequence $(X_k)_{k\in\mathbb{N}}$ of stochastic processes such that for each $k \geq 1$, the process X_k is the solution of the Langevin equation

$$dX_k(t) = \alpha_k X(t)dt + dB(t), \tag{2}$$

where B is a Brownian motion with time parameter $t \in \mathbb{R}$ and $(-\alpha_k)_{k\in\mathbb{N}}$ are independent random variables (also independent of B) with Gamma distribution $\Gamma(1-h, \lambda)$ with $h \in (0, 1)$ and $\lambda > 0$. The aggregated process is given by

$$Y_n(t) = \frac{1}{n}\sum_{k=1}^{n} X_k(t),$$

and it converges, as $n \to \infty$, to a stochastic process Y which is a stationary Gaussian process, semimartingale, asymptotically self-similar and it has long-range dependence. This limit process is the so-called Gamma-mixed Ornstein–Uhlenbeck process. In a similar way, in [9] and [10] the authors studied the generalized cases where B in (2) is a fractional Brownian motion and Hermite process, respectively; they define the fractional Ornstein–Uhlenbeck process mixed with a Gamma distribution and the Hermite Ornstein–Uhlenbeck process mixed with a Gamma distribution, both processes exhibit long range dependence, the first one is a Gaussian process but the second one is not Gaussian.

The aim of this paper is to define and study some properties of the Gamma mixed fractional Lévy Ornstein–Uhlenbeck process obtained as the limit of the aggregated OU processes with random coefficients of Gamma distribution and driven by fractional Lévy process.

The fractional Lévy process (fLp) was defined in [18] as a generalization of the moving average representation of a fractional Brownian motion given by Mandelbrot and Van Ness [17] replacing the Brownian motion in this integral representation by a Lévy process with zero mean, finite variance and without Gaussian part. FLp is almost surely Hölderian, has stationary increments and long range dependence, but unlike fractional Brownian motion, this process is neither Gaussian nor self-similar process. The authors of [11] introduced the fractional Lévy Ornstein–Uhlenbeck process (fLOUp) as the unique stationary pathwise solution of the Langevin equation driven by a fLp and prove that its increments exhibit long range dependence. Recently, many authors have studied fLp and the fractional Lévy Ornstein–Uhlenbeck process on theoretical and applicable levels, see, for example, [4, 3, 12, 15, 24, 26, 29] and the references therein.

This paper is organized as follows. In Section 2 we give a brief introduction to the fLp and the stochastic calculus related to this. Fractional Lévy Ornstein–Uhlenbeck process with random coefficient is introduced in Section 3. In Section 4, we define the aggregated process of fLOUp with random coefficients and study its limit process, which we will call Gamma mixed fractional Lévy Ornstein–Uhlenbeck process. Finally, in Section 5 we present some simulations of the paths of the Gamma mixed fractional Lévy Ornstein–Uhlenbeck process.

2 Preliminaries

In this section, we briefly recall some relevant aspects of the fractional Lévy process (fLp), its main properties and stochastic integrals with respect to this fLp. This process will be used in the remainder of the paper. We work on a complete probability space $(\Omega_L, \mathcal{F}_L, \mathbb{P}_L)$ and we denote by $\mathbb{E}_L$ the expectation in this space.

2.1 Fractional Lévy process

The fractional Lévy process $L^d = \{L^d_t,\ t \in \mathbb{R}\}$, with $d \in (0, 1/2)$, is a non-Gaussian process defined as follows (see [18]):

$$L^d_t = \int_{\mathbb{R}} f^{(d)}_t(s)dL(s), \quad t \in \mathbb{R}, \tag{3}$$

where the kernel function $f^{(d)}_t$ is given by

$$f^{(d)}_t(s) = \frac{1}{\Gamma(d+1)}[(t-s)^d_+ - (-s)^d_+], \quad s \in \mathbb{R}, \tag{4}$$

and $L = \{L(t),\ t \in \mathbb{R}\}$ is a zero-mean two-sided Lévy process with $\mathbb{E}_L[(L(1))^2] < \infty$ and without Brownian component, i.e.

$$L(t) = L^{(1)}(t)1_{\{t\geq 0\}} - L^{(2)}(-t)1_{\{t\leq 0\}},$$

where $L^{(1)}$ and $L^{(2)}$ are two independent copies of the same one-sided Lévy process.

The following Lemma (see [18, 15]) establishes that the fLp is well defined in the $L^2(\Omega_L)$-sense and gives its characteristic function.

Lemma 1. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided Lévy process without a Brownian component such that $\mathbb{E}_L[L(1)] = 0$ and $\mathbb{E}_L[(L(1))^2] < \infty$. For $t \in \mathbb{R}$, let $f_t \in L^2(\Omega)$. Then the integral $S(t) := \int_{\mathbb{R}} f_t(u)dL(u)$ exists in the $L^2(\Omega_L)$ sense and $\mathbb{E}_L[S(t)] = 0$. Furthermore, $S(t)$ satisfies the isometry*

$$\mathbb{E}_L[(S(t))^2] = \mathbb{E}_L[(L(1))^2]\|f_t(\cdot)\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R},$$

the covariance function of process S is given by

$$\tilde{\Gamma}(s,t) = cov(S(s), S(t)) = \mathbb{E}_L[(L(1))^2]\int_{\mathbb{R}} f_t(u)f_s(u)du, \quad s,t \in \mathbb{R},$$

and the characteristic function of $S(t_1), \dots, S(t_m)$ for $t_1 < \cdots < t_m$ and $m \in \mathbb{N}$ is given by

$$\mathbb{E}_L\left[\exp\left(i\sum_{j=1}^{m}\theta_j S(t_j)\right)\right] = \exp\left(\int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m}\theta_j f_{t_j}(s)\right)ds\right),$$

for $\theta_j \in \mathbb{R}$, $j = 1, \dots, m$, where

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx), \quad u \in \mathbb{R},$$

and ν is the Lévy measure of L.

From (3) we can see that the covariance function of L^d is given by

$$\mathbb{E}_L(L_t^d L_s^d) = \frac{1}{2}V_d^2\left(|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right), \quad t,s \in \mathbb{R}, \tag{5}$$

where $V_d^2 = \frac{\mathbb{E}(L_1^2)}{2\Gamma(2d+2)\sin(\pi(d+1/2))}$. Up to a scaling constant, this is the covariance of a fractional Brownian motion.

The fractional Lévy process L^d defined by (3) has the following properties (see [18] for their proofs).

- For any $\beta \in (0,d)$, the sample paths of L^d are a.s. β- Hölder continuous.
- L^d is a process with stationary increments and symmetric, i.e. $\{L_{-t}^d\}_{t\in\mathbb{R}} \overset{(d)}{=} \{-L_t^d\}_{t\in\mathbb{R}}$.
- L^d cannot be self-similar. However, L^d is asymptotically self-similar with parameter $0 < d < 0.5$, i.e.

$$\lim_{c\to\infty}\left\{\frac{L_{ct}^d}{c^d}\right\}_{t\in\mathbb{R}} \overset{(d)}{=} \left\{B_t^d\right\}_{t\in\mathbb{R}},$$

where the equality is in the sense of finite-dimensional distributions and $B = \{B_t^d\}_{t\in\mathbb{R}}$ is a fractional Brownian motion of index d.

- For $h > 0$, the covariance between two increments $L^d_{t+h} - L^d_t$ and $L^d_{s+h} - L^d_s$, where $s + h \leq t$ and $t - s = nh$ is

$$\begin{aligned}\delta_d(n) &= V_d^2 h^{2d+1}\left[(n+1)^{2d+1} + (n-1)^{2d+1} - 2n^{2d+1}\right]\\ &= V_d^2 d(2d+1)h^{2d+1}n^{2d-1} + O(n^{2d-2}), \quad n \to \infty. \end{aligned} \tag{6}$$

- The increments of L^d exhibit long memory in the sense that for $d > 0$, we have

$$\sum_{n=1}^{\infty} \delta_d(n) = \infty.$$

In the following, we recall two results from the reference [18] concerning to stochastic integrals with respect to fractional Lévy process.

Let $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ be the spaces of integrable and square integrable real functions, respectively, and H the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm $\|g\|_H^2 = \mathbb{E}_L[(L(1))^2]\int_{\mathbb{R}}(I_-^d g)^2(u)du$, where $(I_-^d g)$ denotes the right-sided Riemann–Liouville fractional integral defined by

$$(I_-^d g)(x) = \frac{1}{\Gamma(d)}\int_x^{\infty} g(t)(t-x)^{d-1}dt.$$

Lemma 2. *If $g \in H$, then*

$$\int_{\mathbb{R}} g(s)dL_s^d = \int_{\mathbb{R}} (I_-^d g)(s)dL_s,$$

where the equality is in the L^2 sense.

The next second-order property of the stochastic integral with respect to fLp will be a key tool in this article.

Lemma 3. *If $|f|, |g| \in H$, then*

$$\mathbb{E}_L\left[\int_{\mathbb{R}} f(s)dL_s^d \int_{\mathbb{R}} g(s)dL_s^d\right] = C_d \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)|t-s|^{2d-1}dsdt,$$

where

$$C_d = \frac{\Gamma(1-2d)E[L_1^2]}{\Gamma(d)\Gamma(1-d)}. \tag{7}$$

The following result from Ref. [11] establishes the solution of the Langevin equation driven by a fractional Lévy process.

Theorem 1. *Let L^d be an fLp, $d \in (0, 1/2)$ and $\lambda > 0$. Then the unique stationary pathwise solution of the Langevin equation*

$$dX^{d,\lambda}(t) = \lambda X^{d,\lambda}(t)dt + dL_t^d$$

is given a.s. by

$$X^{d,\lambda}(t) = \int_{-\infty}^{t} e^{\lambda(t-u)}dL^d(u), \quad t \in \mathbb{R}. \tag{8}$$

This process is called the fractional Lévy Ornstein–Uhlenbeck processes (fLOUp).

3 Ornstein–Uhlenbeck process with random coefficient

In this section, we study the fractional Lévy Ornstein–Uhlenbeck processes (fLOUp) with random coefficients. First, we establish the existence of the solution, and then some properties of the process are shown.

We consider the fractional Lévy Ornstein–Uhlenbeck process $V^d = \{V^d(t), t \in \mathbb{R}\}$ given as the solution of

$$dV^d(t) = \lambda V^d(t)dt + dL^d(t), \quad t \in \mathbb{R}, \tag{9}$$

where L^d is an fLp with $d \in (0, 1/2)$ and defined on $(\Omega_L, \mathcal{F}_L, \mathbb{P}_L)$; and the coefficient λ is a random variable defined on the probability space $(\Omega_\lambda, \mathcal{F}_\lambda, \mathbb{P}_\lambda)$ and independent of L^d. We assume that $-\lambda$ follows a Gamma distribution with parameters $1 - h$ and α, i.e. $-\lambda \sim \Gamma(1 - h, \alpha)$ with $h \in (0, 1)$ and $\alpha > 0$.

We denote by $\mathbb{P}$ the product probability measure on $\Omega = \Omega_L \times \Omega_\lambda$ and $\mathbb{E}_L$, $\mathbb{E}_\lambda$, $\mathbb{E}$ denote the expectation with respect to the probability measure $\mathbb{P}_L$, $\mathbb{P}_\lambda$ and $\mathbb{P}$, respectively.

From Theorem 1, the SDE given by (9) has the explicit solution

$$V^d(t) = \int_{-\infty}^{t} e^{\lambda(t-u)} dL^d(u), \quad t \in \mathbb{R}, \tag{10}$$

where the initial condition is given by

$$V^d(0) = \int_{-\infty}^{0} e^{-\lambda u} dL^d(u).$$

By Lemma 2, we can get

$$V^d(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_{-\infty}^{t} e^{\lambda(t-v)} (v-u)_+^{d-1} dv dL(u), \quad t \in \mathbb{R}.$$

We will prove that for every $\omega_\lambda \in \Omega_\lambda$, the process V^d is well defined in $L^2(\Omega_L)$. In fact, by Lemma 3 we have that for $C_d = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \mathbb{E}_L[(L(1))^2]$

$$\begin{aligned}
\mathbb{E}_L[(V^d(t))^2] &= C_d \int_{-\infty}^{t} \int_{-\infty}^{t} e^{\lambda(t-u)} e^{\lambda(t-v)} |u-v|^{2d-1} du dv \\
&= C_d \int_{0}^{\infty} \int_{0}^{\infty} e^{\lambda(u+v)} |u-v|^{2d-1} du dv \\
&= 2C_d \int_{0}^{\infty} \int_{0}^{u} e^{\lambda(u+v)} (u-v)^{2d-1} dv du \\
&= 2C_d \int_{0}^{\infty} e^{2\lambda u} \int_{0}^{u} e^{-\lambda v} v^{2d-1} dv du \\
&= 2C_d \int_{0}^{\infty} e^{-\lambda v} v^{2d-1} \int_{v}^{\infty} e^{2\lambda u} du dv \\
&= -\frac{C_d}{\lambda} \int_{0}^{\infty} e^{\lambda v} v^{2d-1} dv = \frac{C_d}{(-\lambda)^{2d+1}} \Gamma(2d).
\end{aligned} \tag{11}$$

Remark 1. For $c \neq 0$, by (11) we have that $\mathbb{E}_L[(V^d(ct))^2] = \frac{C_d}{(-\lambda)^{2d+1}}\Gamma(2d)$. Hence the processes V^d is not self-similar.

Lemma 4. *The process V^d (for fixed $\omega_\lambda \in \Omega_\lambda$) is stationary, i.e. for $b > 0$ and $t_1 < \cdots < t_n$, with $n \in \mathbb{N}$*

$$(V^d(t_1 + b), \dots, V^d(t_n + b)) \overset{(d)}{=} (V^d(t_1), \dots, V^d(t_n)),$$

where $\overset{(d)}{=}$ means equality in the sense of finite-dimensional distributions.

Proof. For $b > 0$, $u_1, \dots, u_n$ and $-\infty < t_1 < \cdots < t_n$, $n \in \mathbb{R}$, by the stationarity of the increments of L^d we get

$$\begin{aligned}\sum_{i=1}^n u_i V^d(t_i + b) &= \sum_{i=1}^n u_i \int_{-\infty}^{t_i+b} e^{\lambda(t_i+b-u)} dL^d(u) \\ &\overset{(d)}{=} \sum_{i=1}^n u_i \int_{-\infty}^{t_i} e^{\lambda(t_i-u)} dL^d(u) \\ &= \sum_{i=1}^n u_i V^d(t_i).\end{aligned}$$

□

We will provide a spectral representation of the process V^d, given by (10), based on the results from [19, 21] already used in Ref. [20] to construct a long memory process based on a CARMA process driven by a Lévy processes.

Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two sided square integrable Lévy process with $\mathbb{E}_L[L(1)] = 0$ and $\mathbb{E}_L[(L(1))^2] = \Sigma_L$, then there exists a random orthogonal measure Φ_L with spectral measure F_L such that $\mathbb{E}_L[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ,

$$F_L(dt) = \frac{\Sigma_L}{2\pi} dt.$$

Also, the random measure Φ_L is uniquely determined by

$$\Phi_L([a, b)) = \int_{\mathbb{R}} \frac{e^{-i\alpha a} - e^{-i\alpha b}}{2\pi i\alpha} L(d\alpha), \tag{12}$$

for all $-\infty < a < b < \infty$. Moreover,

$$L(t) = \int_{\mathbb{R}} \frac{e^{i\alpha t} - 1}{i\alpha} \Phi_L(d\alpha), \quad t \in \mathbb{R}.$$

Hence, for any function $f \in L^2(\mathbb{C})$,

$$\int_{\mathbb{R}} f(\alpha)\Phi_L(d\alpha) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\alpha t} f(\alpha) d\alpha L(dt) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) L(dt), \tag{13}$$

$$\int_{\mathbb{R}} \hat{f}(t) L(dt) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\alpha t} \hat{f}(t) dt \Phi_L(d\alpha) = \sqrt{2\pi} \int_{\mathbb{R}} f(\alpha)\Phi_L(d\alpha), \tag{14}$$

where

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\alpha t} f(\alpha) d\alpha \quad \text{and} \quad f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\alpha t} \hat{f}(t) d\alpha.$$

Lemma 5. *Let V^d be an fLpOU with arandom coefficient, given by (10). Then the spectral representation of V^d is*

$$V^d(t) = \int_{\mathbb{R}} e^{it\alpha} \frac{1}{i\alpha - \lambda} \Phi_M(d\alpha), \quad t \in \mathbb{R}, \tag{15}$$

where

$$\Phi_M([a,b]) = \int_{\mathbb{R}} \frac{e^{-ias} - e^{-ibs}}{2\pi i s} dL^d(s).$$

Proof. By Theorem 2.8 in [19], we know that

$$L^d(t) = \int_{\mathbb{R}} (e^{i\alpha t} - 1)(i\alpha)^{-(d+1)} \Phi_L(d\alpha), \quad t \in \mathbb{R}.$$

Furthermore, following the proof of Theorem 2.8, Remark 2.9 and equality (2.31) in the same reference, we can obtain

$$\begin{aligned} V^d(t) &= \int_{\mathbb{R}} e^{\lambda(t-u)} 1_{(-\infty,t)}(u) dL^d(u) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\alpha u} e^{\lambda(t-u)} 1_{(-\infty,t)}(u) du \Phi_M(d\alpha) \\ &= \int_{\mathbb{R}} e^{it\alpha} \frac{1}{i\alpha - \lambda} \Phi_M(d\alpha). \end{aligned}$$

Thus, the result is achieved. □

Corollary 1. *With almost no major effort, we can obtain, by Remark 2.9 in [19], that*

$$V^d(t) = \int_{\mathbb{R}} e^{it\alpha} \frac{1}{i\alpha - \lambda} (i\alpha)^{-d} \Phi_L(d\alpha), \quad t \in \mathbb{R}.$$

4 Gamma mixed fractional Lévy Ornstein–Uhlenbeck process

In this section, first, we will study a process defined by the aggregation of independent fLOUp with random coefficient λ such that $-\lambda \sim \Gamma(1-h, \alpha)$ (see Section 3). Then we study its limit process, which we call Gamma mixed fractional Lévy Ornstein–Uhlenbeck process. Some properties of this limit process are given, namely, we give the characteristic function of the finite-dimensional distributions of the process, we determine its covariance, stationarity and long memory property; and finally, we analyze the asymptotic behavior with respect to the parameter α, and prove that this process will tend to a fractional Lévy process when α goes to ∞.

4.1 Aggregated fractional Lévy Ornstein–Uhlenbeck process

Consider a sequence of fLOUp with random coefficients $V_k^d = (V_k^d(t), t \in \mathbb{R})$, $k \geq 1$, given by

$$V_k^d(t) = \int_{-\infty}^{t} e^{\lambda_k(t-u)} dL^d(u), \quad t \in \mathbb{R}, \quad k \geq 1, \tag{16}$$

where L^d is an fLp with $d \in (0, 1/2)$ defined on $(\Omega_L, \mathcal{F}_L, \mathbb{P}_L)$. The random variables λ_k are assumed independent and identically distributed defined on the probability space $(\Omega_\lambda, \mathcal{F}_\lambda, \mathbb{P}_\lambda)$, and for $k \geq 1$ we assume that $-\lambda_k$ follows the Gamma distribution with parameters $1-h$ and α, i.e. $-\lambda_k \sim \Gamma(1-h, \alpha)$ with $h \in (0, 1)$ and $\alpha > 0$. Furthermore, we also assume that the random variables $(\lambda_k)_{k\geq 1}$ are independent of L^d.

The m-aggregated fractional Lévy Ornstein–Uhlenbeck processs $Z_m^d = (Z_m^d(t), t \in \mathbb{R})$ is defined by

$$Z_m^d(t) = \frac{1}{m}\sum_{k=1}^{m} V_k^d(t), \tag{17}$$

for $m \geq 1$ and $d \in (0, 1/2)$.

Lemma 6. *For all $m \geq 1$, the m-aggregated fractional Lévy Ornstein–Uhlenbeck processs Z_m^d is stationary and*

$$\mathbb{E}_L[(Z_m^d(t))^2] = \frac{C_d}{m^2} 2\Gamma(2d) \sum_{k=1}^{m}\sum_{j=1}^{m} \frac{1}{-(\lambda_k + \lambda_j)(-\lambda_k)^{2d}},$$

where C_d is given by (7).

Proof. The stationarity follows by Lemma 4.

From Lemma 3 and the change of variables $\tilde{u} = t - u$ and $\tilde{v} = t - v$, we can see

$$\begin{aligned}
&\mathbb{E}_L[(Z_m^d(t))^2] \\
&= \frac{C_d}{m^2}\int_{-\infty}^{t}\int_{-\infty}^{t}\sum_{k=1}^{m}\sum_{j=1}^{m} e^{\lambda_k(t-u)}e^{\lambda_j(t-v)}|u-v|^{2d-1}dvdu \\
&= \frac{C_d}{m^2}\sum_{k=1}^{m}\sum_{j=1}^{m}\int_{0}^{\infty}\int_{0}^{\infty} e^{\lambda_k u+\lambda_j v}|u-v|^{2d-1}dvdu \\
&= \frac{C_d}{m^2}\sum_{k=1}^{m}\sum_{j=1}^{m}\bigg(\int_{0}^{\infty}\int_{0}^{u} e^{\lambda_k u+\lambda_j v}(u-v)^{2d-1}dvdu \\
&\quad + \int_{0}^{\infty}\int_{u}^{\infty} e^{\lambda_k u+\lambda_j v}(v-u)^{2d-1}dvdu\bigg) \\
&\mathbb{E}_L[(Z_m^d(t))^2] \\
&= \frac{C_d}{m^2}\sum_{k=1}^{m}\sum_{j=1}^{m}\bigg(\int_{0}^{\infty} e^{(\lambda_k+\lambda_j)u}\int_{0}^{u} e^{-\lambda_j v}v^{2d-1}dvdu \\
&\quad + \int_{0}^{\infty} e^{(\lambda_k+\lambda_j)u}\int_{0}^{\infty} e^{\lambda_j v}v^{2d-1}dvdu\bigg) \\
&= \frac{C_d}{m^2}\sum_{k=1}^{m}\sum_{j=1}^{m}\bigg(\int_{0}^{\infty} e^{\lambda_j v}v^{2d-1}\int_{v}^{\infty} e^{(\lambda_k+\lambda_j)u}dudv \\
&\quad + \int_{0}^{\infty} e^{\lambda_j v}v^{2d-1}\int_{0}^{\infty} e^{(\lambda_k+\lambda_j)u}dudv\bigg)
\end{aligned}$$

$$= \frac{C_d}{m^2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{1}{-(\lambda_k+\lambda_j)}\left(\int_0^\infty e^{\lambda_k v}v^{2d-1}dv + \int_0^\infty e^{\lambda_j v}v^{2d-1}dv\right)$$

$$= \frac{C_d}{m^2}\Gamma(2d)\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{1}{-(\lambda_k+\lambda_j)}\left(\frac{1}{(-\lambda_k)^{2d}} + \frac{1}{(-\lambda_j)^{2d}}\right)$$

$$= \frac{C_d}{m^2}2\Gamma(2d)\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{1}{-(\lambda_k+\lambda_j)(-\lambda_k)^{2d}}. \qquad \square$$

4.2 Limit of *m*-aggregated fractional Lévy Ornstein–Uhlenbeck process

By (10) we can see that Z_m^d can be written as

$$Z_m^d(t) = \int_{-\infty}^{t}\frac{1}{m}\sum_{k=1}^{m}e^{\lambda_k(t-u)}dL^d(u)$$
$$= \int_{-\infty}^{t} f_m(t-u)dL^d(u). \tag{18}$$

Moreover, by the law of large numbers we get

$$\frac{1}{m}\sum_{k=1}^{m}e^{\lambda_k(t-u)} \to \mathbb{E}_\lambda\left[e^{\lambda_1(t-u)}\right] = \left(\frac{\alpha}{\alpha+t-u}\right)^{1-h}, \tag{19}$$

as $m \to \infty$, where the convergence is $\mathbb{P}_\lambda$-almost surely.

Due to the previous result, a natural candidate to be the limit of the m-aggregated fractional Lévy Ornstein–Uhlenbeck process is the process $Z^d = (Z^d(t), t \in \mathbb{R})$ given by

$$Z^d(t) := \int_{-\infty}^{t}\left(\frac{\alpha}{\alpha+t-u}\right)^{1-h}dL^d(u)$$
$$= \int_{-\infty}^{t} g(t-u)dL^d(u), \tag{20}$$

where $g(t) = (\frac{\alpha}{\alpha+t})^{1-h}$. We refer to this process as the Gamma mixed fractional Lévy Ornstein–Uhlenbeck process.

We can see that the process Z^d belongs to $L^2(\Omega_L)$ if $0 < h < 1/2 - d$. Actually, by Lemma 3 and applying consecutively the change of variables $\tilde{u} = t-u$, $\tilde{v} = t-v$, $x = u/\alpha$, $y = v/\alpha$, $r = 1/(1+x)$, $s = 1/(1+y)$, and $\hat{u} = r/s$, we obtain

$$\mathbb{E}_L\left[(Z^d(t))^2\right] = C_d\int_{-\infty}^{t}\int_{-\infty}^{t} g(t-u)g(t-v)|u-v|^{2d-1}dvdu$$
$$= C_d\int_0^\infty\int_0^\infty g(u)g(v)|u-v|^{2d-1}dvdu$$
$$= C_d\int_0^\infty\int_0^\infty\left(1+\frac{u}{\alpha}\right)^{h-1}\left(1+\frac{v}{\alpha}\right)^{h-1}|u-v|^{2d-1}dvdu$$

$$
\begin{aligned}
&= C_d \alpha^{2d+1} \int_0^\infty \int_0^\infty (1+x)^{h-1} (1+y)^{h-1} |x-y|^{2d-1} dy dx \\
&= C_d \alpha^{2d+1} \int_0^1 \int_0^1 r^{1-h} s^{1-h} (rs)^{-2} \left| \frac{1}{r} - \frac{1}{s} \right|^{2d-1} dr ds \\
&= 2C_d \alpha^{2d+1} \int_0^1 s^{-1-h} \int_0^s r^{-h-2d} \left(1 - \frac{r}{s}\right)^{2d-1} dr ds \\
&= 2C_d \alpha^{2d+1} \int_0^1 s^{-2(h+d)} \int_0^1 \hat{u}^{-h-2d} (1-\hat{u})^{2d-1} d\hat{u} ds \\
&= \frac{2C_d \alpha^{2d+1}}{1-2(h+d)} B(1-h-2d, 2d) = C_{\alpha,h,d}.
\end{aligned}
$$

Clearly, the condition $0 < h < 1/2 - d$ emerges from the last line.

Now, we present the main result of this section related to the limit of the aggregated fractional Lévy Ornstein–Uhlenbeck process. This result is analogous to that obtained in Theorem 3 in [10] for the m-aggregated fractional Ornstein–Uhlenbeck processes.

Theorem 2. *Let Z_m^d and Z^d be defined by (17) and (20), respectively. Assume that $0 < h < 1/2 - d$. Then $\mathbb{P}_\lambda$-a.s., for every $t \in \mathbb{R}$,*

$$
Z_m^d(t) \longrightarrow Z^d(t) \quad in\ L^2(\Omega_L), \tag{21}
$$

and for $a, b \in \mathbb{R}$,

$$
Z_m^d \longrightarrow Z^d \quad weakly\ \ in\ \ C[a,b]\ \ under\ \mathbb{P}_L, \tag{22}
$$

as $m \to \infty$.

Proof. To establish weak convergence, we prove, first, the convergence of finite-dimensional distributions of Z_m^d to those of Z^d, and second, that the sequence $\{Z_m^d\}$ is tight.

By Lemma 3, (18) and (20), we have

$$
\begin{aligned}
&\mathbb{E}_L[(Z_m^d(t) - Z^d(t))^2] \\
&= \mathbb{E}_L \left[\left(\int_{-\infty}^t \left[\frac{1}{m} \sum_{k=1}^m e^{\lambda_k (t-u)} - \left(\frac{\alpha}{\alpha + t - u} \right)^h \right] dL^d(u) \right)^2 \right] \\
&= \mathbb{E}_L \left[\left(\int_{-\infty}^t [f_m(t-u) - g(t-u)] \, dL^d(u) \right)^2 \right] \\
&= C_d \int_0^\infty \int_0^\infty (f_m(u) - g(u))(f_m(v) - g(v)))|u-v|^{2d-1} du dv.
\end{aligned}
$$

Now, following the lines of the proof of Theorem 3 (first part) in [10] and the fact that $0 < h < 1/2 - d$, we can obtain that $\mathbb{P}_\lambda$-a.s.

$$
\lim_{m \to \infty} \mathbb{E}_L[(Z_m^d(t) - Z^d(t))^2] = 0.
$$

Hence we have the $\mathbb{P}_\lambda$-a.s. convergence of the sequence $(Z_m^d)_{m\geq 1}$ in $L^2(\Omega_L)$ for each $t \in \mathbb{R}$, which in turn implies the $\mathbb{P}_\lambda$-a.s. convergence of the finite-dimensional distributions.

It remains to prove the tightness. Due to Theorem 12.3 in [5], it is sufficient to show that $\mathbb{E}_L[(Z_m^d(t) - Z_m^d(s))^2] \leq C(t-s)^{1+\rho}$ for $a \leq s < t \leq b$, where $\rho > 0$ and C may depend upon parameters.

From Lemma 6 and (18)

$$\begin{aligned} Z_m^d(t) - Z_m^d(s) &\overset{(d)}{=} Z_m^d(t-s) - Z_m^d(0) \\ &\overset{(d)}{=} \frac{1}{m}\sum_{k=1}^{m}(e^{\lambda_k(t-s)} - 1)\int_{-\infty}^{0} e^{-u\lambda_k} dL^d(u) \\ &\quad + \int_0^{t-s}\frac{1}{m}\sum_{k=1}^{m} e^{\lambda_k(t-s-u)} dL^d(u) \\ &= I_1 + I_2. \end{aligned}$$

Clearly, this implies

$$\mathbb{E}_L[(Z_m^d(t) - Z_m^d(s))^2] \leq 2\mathbb{E}_L[I_1^2] + 2\mathbb{E}_L[I_2^2].$$

We will estimate every term separately. In fact, by Lemma 3 we have

$$\begin{aligned} \mathbb{E}_L[I_1^2] &= \frac{1}{m^2}\sum_{k_1,k_2=1}^{m}(e^{\lambda_{k_1}(t-s)} - 1)(e^{\lambda_{k_2}(t-s)} - 1) \\ &\quad \times \mathbb{E}_L\left[\int_{-\infty}^{0} e^{-u\lambda_{k_1}} dL^d(u)\int_{-\infty}^{0} e^{-v\lambda_{k_1}} dL^d(v)\right] \\ &= \frac{1}{m^2}\sum_{k_1,k_2=1}^{m}(e^{\lambda_{k_1}(t-s)} - 1)(e^{\lambda_{k_2}(t-s)} - 1) \\ &\quad \times \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-u\lambda_{k_1}} e^{-v\lambda_{k_1}} |u-v|^{2d-1} du dv. \end{aligned}$$

By arguments similar to those of [9], the law of large number and the fact that $0 < h < 1/2 - d$,

$$\mathbb{E}_L[I_1^2] \leq C_{d,\lambda}(t-s)^2. \tag{23}$$

With respect to I_2, by Lemma 3 and making the change of variable $z = u - v$, we get

$$\begin{aligned} \mathbb{E}_L[I_2^2] &= \mathbb{E}_L\left[\int_0^{t-s}\frac{1}{m}\sum_{k=1}^{m} e^{\lambda_k(t-s-u)} dL^d(u) \cdot \int_0^{t-s}\frac{1}{m}\sum_{k=1}^{m} e^{\lambda_k(t-s-v)} dL^d(v)\right] \\ &= C_d \frac{1}{m^2}\sum_{k_1,k_2=1}^{m}\int_0^{t-s}\int_0^{t-s} e^{\lambda_{k_1}(t-s-u)} e^{\lambda_{k_2}(t-s-v)} |u-v|^{2d-1} du dv \end{aligned}$$

$$= C_d \frac{1}{m^2} \sum_{k_1,k_2=1}^{m} e^{\lambda_{k_1}(t-s)} e^{\lambda_{k_2}(t-s)} \int_0^{t-s} \int_0^{t-s} e^{-\lambda_{k_1}u-\lambda_{k_2}v} |u-v|^{2d-1} dv du$$

$$= 2C_d \frac{1}{m^2} \sum_{k_1,k_2=1}^{m} e^{\lambda_{k_1}(t-s)} e^{\lambda_{k_2}(t-s)} \int_0^{t-s} e^{-\lambda_{k_1}u} \int_0^{u} e^{-\lambda_{k_2}v} (u-v)^{2d-1} dv du$$

$$= 2C_d \frac{1}{m^2} \sum_{k_1,k_2=1}^{m} e^{\lambda_{k_1}(t-s)} e^{\lambda_{k_2}(t-s)} \int_0^{t-s} e^{-u(\lambda_{k_1}+\lambda_{k_2})} \int_0^{u} e^{\lambda_{k_2}z} z^{2d-1} dv du$$

$$\leq 2C_d \frac{1}{m^2} \sum_{k_1,k_2=1}^{m} e^{\lambda_{k_1}(t-s)} e^{\lambda_{k_2}(t-s)} \int_0^{t-s} e^{-\lambda_{k_1}u} u^{2d} du.$$

$$\leq 2C_d (t-s)^{1+2d} \frac{1}{m} \sum_{k_2=1}^{m} e^{\lambda_{k_2}(t-s)}.$$

Then, using the fact that $-\lambda_k \sim \Gamma(1-h, \alpha)$ with $h \in (0,1)$ and $\alpha > 0$, and $t > s$, we obtain

$$\mathbb{E}_L[I_2^2] \leq 2C_d (t-s)^{1+2d}. \tag{24}$$

Inequalities (23) and (24) imply

$$\mathbb{E}_L[(Z_m^d(t) - Z_m^d(s))^2] \leq C_{T,d,\lambda,\alpha,h} (t-s)^{1+2d}. \tag{25}$$

Hence, based on Kolmogorov's continuity theorem, for all $m \geq 1$, Z_m^d has a continuous modification, thereby fulfilling inequality (25) as well. Moreover, by (25) and Theorem 12.3 in [5], we obtain the tightness of the family $\{Z_m^d\}$.

By (21) and (25),

$$\mathbb{E}_L[(Z^d(t) - Z^d(s))^2] \leq C(t-s)^{1+2d},$$

then there is a continuous modification of Z^d. Finally, the weak convergence is obtained from Theorem 8.2 in [5]. □

4.3 Properties of $\mathbf{Z^d}$

Now we study some properties of the limit process Z^d.

Lemma 7. *Let Z^d be given by (20) with $h \in (0, 1/2-d)$. Then the characteristic function of $Z^d(t_1), Z^d(t_2), \ldots, Z^d(t_m)$ with $t_1 < t_2 < \cdots < t_m$ is given by*

$$\mathbb{E}_L\left[\exp\left(i \sum_{j=1}^{m} \theta_j Z^d(t_j)\right)\right] = \exp\left(\int_{\mathbb{R}} \psi \left[\sum_{j=1}^{m} \theta_j \tilde{f}_{t_j,h,\alpha}(s)\right] ds\right),$$

where

$$\tilde{f}_{t_j,h,\alpha}(s) = d \int_{-\infty}^{t_j} \left(\frac{\alpha}{\alpha + t_j - v}\right)^{1-h} (v-s)_+^{d-1} dv.$$

Proof. The result follows by Lemma 1 in Section 2 (also see Proposition 3.3 in [15]) and the fact that Z^d belongs to $L^2(\Omega_L)$ for $h \in (0, 1/2-d)$. □

Lemma 8. *Let Z^d be given by (20) with $h \in (0, 1/2 - d)$. Then Z^d is a stationary process with long memory.*

Proof. By (20), the stationarity of the increments of L^d and making the change of variable $\tilde{u} = u - b$, we can get, for every $b > 0$,

$$Z^d(t+b) = \int_{-\infty}^{t+b} \left(\frac{\alpha}{\alpha + t + b - u} \right)^{1-h} dL^d(u)$$
$$\stackrel{(d)}{=} \int_{-\infty}^{t} \left(\frac{\alpha}{\alpha + t - u} \right)^{1-h} dL^d(u) = Z^d(t).$$

With respect to the long memory property we will study the nonsummability of the covariance. Since Z^d is a stationary process, we have

$$\mathbb{E}_L[Z^d(a)Z^d(t+a)] = \mathbb{E}_L[Z^d(0)Z^d(t)].$$

Then Lemma 3 implies

$$\begin{aligned}
\mathbb{E}_L[Z^d(0)Z^d(t)] &= C_d \int_{-\infty}^{0} \int_{-\infty}^{t} \left(\frac{\alpha}{\alpha + t - u} \right)^{1-h} \left(\frac{\alpha}{\alpha - v} \right)^{1-h} |u - v|^{2d-1} du dv \\
&= C_d \alpha^{2-2h} \int_{-\infty}^{0} \int_{-\infty}^{t} (\alpha + t - u)^{h-1} (\alpha - v)^{h-1} |u - v|^{2d-1} du dv \\
&= C_d \alpha^{2-2h} t^{2h+2d-1} \\
&\quad \times \int_{-\infty}^{0} \int_{-\infty}^{1} \left(\frac{\alpha}{t} + 1 - y \right)^{h-1} \left(\frac{\alpha}{t} - x \right)^{h-1} |x - y|^{2d-1} dx dy \\
&\sim C_d \alpha^{2-2h} t^{2h+2d-1} \\
&\quad \times \int_{-\infty}^{0} \int_{-\infty}^{1} (1 - y)^{h-1} (-x)^{h-1} |x - y|^{2d-1} dx dy \\
&= C_{d,\alpha,h} t^{2h+2d-1}.
\end{aligned}$$

Then the long memory property is obtain by just noticing that $h + d > 0$. □

Remark 2. Even if $d = 0$, the long memory property is satisfied if $h > 0$.

Remark 3. We can see that the process Z^d has the property

$$\frac{\rho(t)}{\rho(0)} \sim C_{\alpha,h,d} t^{2h+2d-1}, \quad \text{with } \rho(t) = \mathbb{E}_L[Z^d(0)Z^d(t)],$$

i.e. the process is almost self-similar (see [13, page 13] for details).

With respect to the behavior of the process Z^d with respect to the parameter α we have the following result.

Lemma 9. *Let $t \geq 0$ and $d \in (0, 1/2)$. Then the random variable*

$$Z^d(t) - Z^d(0)$$

converges in $L^2(\Omega_L)$ as $\alpha \to \infty$ to the random variable $L^d(t)$.

Proof. By (20) we obtain

$$
\begin{aligned}
&Z^d(t) - Z^d(0)\\
&= \int_{-\infty}^{t} \left(\frac{\alpha}{\alpha+t-u}\right)^{1-h} dL^d(u) - \int_{-\infty}^{0} \left(\frac{\alpha}{\alpha-u}\right)^{1-h} dL^d(u)\\
&= \int_{\mathbb{R}} \left[\left(\frac{\alpha}{\alpha+t-u}\right)^{1-h} 1_{(-\infty,t)}(u) - \left(\frac{\alpha}{\alpha-u}\right)^{1-h} 1_{(-\infty,0)}(u)\right] dL^d(u)\\
&= \int_{\mathbb{R}} f_{t,d}(u) dL^d(u).
\end{aligned}
$$

Clearly, $f_{t,d}$ converges to $1_{(0,t)}$ as $\alpha \to \infty$. Therefore, our candidate for a limit will be $L^d(t)$. In fact, by Lemma 3

$$
\begin{aligned}
&\mathbb{E}_L[(Z^d(t) - Z^d(0) - L^d(t))^2]\\
&= C_d \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\left(\frac{\alpha}{\alpha+t-u}\right)^{1-h} 1_{(-\infty,t)}(u) - \left(\frac{\alpha}{\alpha-u}\right)^{1-h} 1_{(-\infty,0)}(u) - 1_{(0,t)}(u)\right]\\
&\times \left[\left(\frac{\alpha}{\alpha+t-v}\right)^{1-h} 1_{(-\infty,t)}(v) - \left(\frac{\alpha}{\alpha-v}\right)^{1-h} 1_{(-\infty,0)}(v) - 1_{(0,t)}(v)\right]\\
&\times |u-v|^{2d-1} dudv.
\end{aligned}
$$

Finally, the result is obtained by means of the dominated convergence theorem. □

If $\alpha \to 0$, then we get the following result.

Lemma 10. *Let $t \geq 0$ and let us define $\tilde{Z}^d(t)$ by*

$$\tilde{Z}^d(t) = \alpha^{h-1} \int_0^t Z^d(s) ds.$$

Then, as $\alpha \to 0$, the random variable $\tilde{Z}^d(t)$ converges in $L^2(\Omega_L)$ to the random variable $Y^d(t)$ given by

$$Y^d(t) := \frac{1}{h} \int_{\mathbb{R}} [(t-u)_+^h - (-u)_+^h] dL^d(u). \tag{26}$$

Proof. By (20), we get

$$
\begin{aligned}
\tilde{Z}^d(t) &= \alpha^{h-1} \int_0^t \int_{-\infty}^{s} \left(\frac{\alpha}{\alpha+s-u}\right)^{1-h} dL^d(u) ds\\
&= \int_0^t \int_{-\infty}^{s} (\alpha+s-u)^{h-1} dL^d(u) ds\\
&= \int_{-\infty}^{t} \int_{v \vee 0}^{t} (\alpha+s-v)^{h-1} ds dL^d(v)\\
&= \frac{1}{h} \int_{-\infty}^{t} \left[(\alpha+t-v)^h - (\alpha+(v \vee 0)-v)^h\right] dL^d(v)
\end{aligned}
$$

$$= \frac{1}{h}\int_{-\infty}^{t} r_{\alpha,t,h}(v)dL^d(v).$$

We can see that $r_{\alpha,t,h}(v)$ converges to $(t-v)_+^h - (-v)_+^h$ for every v as $\alpha \to 0$. Therefore, the result is obtained by arguments similar to the proof of Lemma 9. □

Lemma 11. *Let $Y^d = (Y^d(t))_{t\geq 0}$ with $Y^d(t)$ be given by (26). Then Y^d is a stationary process with*

$$\mathbb{E}_L[(Y^d(t))^2] = t^{2h+2d+1}\mathbb{E}_L[(Y^d(1))^2].$$

Proof. By (26) and taking $b > 0$,

$$\begin{aligned} Y^d(t+b) - Y^d(b) &= \frac{1}{h}\int_{\mathbb{R}} [(t+b-u)_+^h - (b-u)_+^h]dL^d(u) \\ &\overset{(d)}{=} \frac{1}{h}\int_{\mathbb{R}} [(t-v)_+^h - (-v)_+^h]dL^d(v) = Y^d(t), \end{aligned}$$

here we have used the change of variable $v = u - b$ and the fact that L^d is a stationary increment process. With respect to the second part of the statement, we have that

$$\begin{aligned} &\mathbb{E}_L[(Y^d(t))^2] \\ &= \frac{1}{h^2}C_d \int_{\mathbb{R}}\int_{\mathbb{R}} ((t-u)_+^h - (-u)_+^h)((t-v)_+^h - (-v)_+^h)|u-v|^{2d-1}dvdu \\ &= t^{2h+2d+1}\frac{1}{h^2}C_d \\ &\quad\times \int_{\mathbb{R}}\int_{\mathbb{R}} ((1-u)_+^h - (-u)_+^h)((1-v)_+^h - (-v)_+^h)|u-v|^{2d-1}dvdu \\ &= t^{2h+2d+1}\mathbb{E}_L[(Y^d(1))^2]. \end{aligned}$$

□

Remark 4. Let us note that we can write

$$\tilde{Y}^d(t) = \int_{\mathbb{R}} m_t(u)dL^d(u),$$

where $m_t(u) = \frac{1}{h}[(t-u)_+^h - (-u)_+^h]$. Then Proposition 11 and Theorem 3.1 in [15] imply

$$Y^d(t) = \int_{\mathbb{R}} m_t(u)dL^d(u) = \int_{\mathbb{R}} (I_{-m}^g)(u)dL(s),$$

where (see Section 3 in the same reference for details)

$$(I_{-m}^g)(u) := \int_u^{\infty} m_t(v)g'(v-u)dv = \int_{\mathbb{R}} m_t(v)g'(v-u)dv$$

and $g(u) = (u)_+^d$. This implies that Y^d can be seen as a type of generalized fractional Lévy process (see [15] for details).

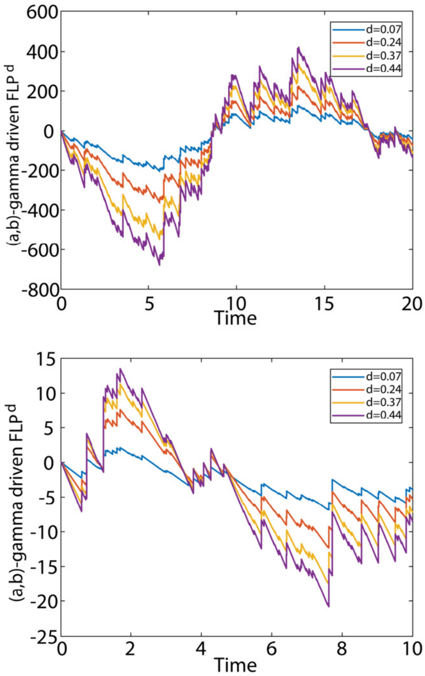

Fig. 1. Sample paths of a fractional Lévy process for different values of d. The approximation is made using the Riemann–Stieltjes approximation with the driving Lévy process being a stationary Gamma process with $a = 5$ and $b = 15$ ($a = 1$ and $b = 2$)

5 Simulations

In this section we are interested in obtaining some simulations related to the process Z^d. First, let us recall how we can simulate a fractional Lévy process.

In order to simulate sample paths of L^d we use the Riemann–Stieltjes approximation, that is, we approximate L^d in the following way (see [18] for details):

$$L^d(t) \approx \frac{1}{\Gamma(d+1)} \left(\sum_{k=-n^2}^{0} \left[\left(t - \frac{k}{n} \right)^d - \left(-\frac{k}{n} \right)^d \right] \left(L_{(k+1)/n} - L_{k/n} \right) \right.$$
$$\left. + \sum_{k=0}^{\lfloor nt \rfloor} \left(t - \frac{k}{n} \right)^d \left(L_{(k+1)/n} - L_{k/n} \right) \right).$$

Remark 5. An optimal form of simulating L^d is shown in [25]. Here, we have used the Riemann–Stieltjes approximation. This approximation can also be optimal if we take $a_n = n^{2-d/1-d}$. An advantage of this procedure is that a simulatation of increments of a Lévy process is relatively easy (see [8] for details about this simulation).

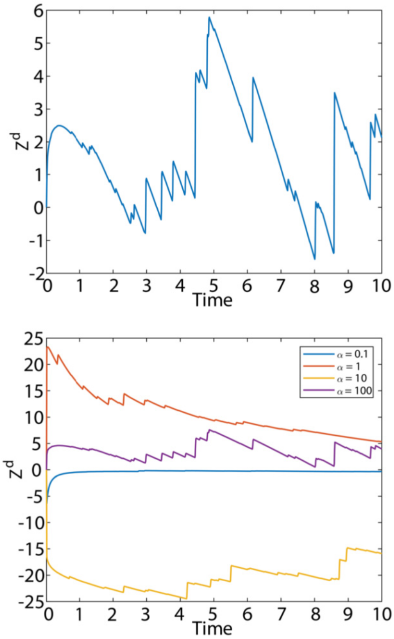

Fig. 2. Sample paths Z^d of the limit process for different α and $h = 0.12$ $(a = 1, b = 2)$

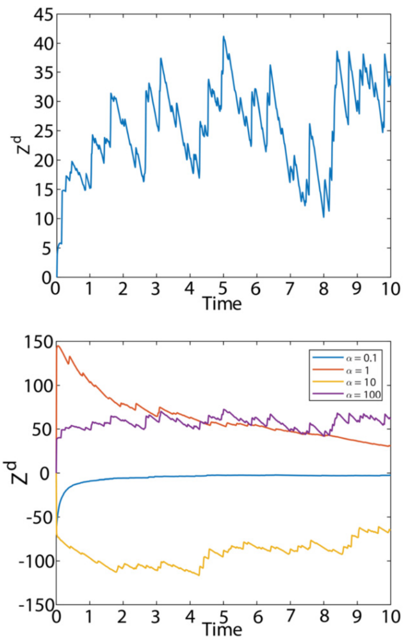

Fig. 3. Sample paths Z^d of the limit process for different α and $h = 0.12$ $(a = 5, b = 5)$

To simulate the process Z^d, we will use the Riemann type approximation

$$Z^{d,(n)}(t) = \alpha^{1-h} \sum_{k=-a_n}^{\lfloor nt \rfloor} \left(\alpha + t - \frac{k}{n}\right)^{h-1} \left(L^d\left(\frac{k+1}{n}\right) - L^d\left(\frac{k}{n}\right)\right), \quad t \in \mathbb{R},$$

where we take $a_n = n^2$ as in [18].

It can seen that the processes L^d and Z^d inherit some of the features of the underlying Lévy process (see [8] for details on the simulation of Lévy processes). Also, as expected, the paths of the process are more irregular; this is due the lack of the Gaussian part in the underlying Lévy proces in the definition of L^d and Z^d. Finally, the processes appear to be more regular for α small.

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