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Generalized BSDEs driven by RCLL martingales with stochastic monotone coefficients

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Abstract A solution is given to generalized backward stochastic differential equations driven by a real-valued RCLL martingale on an arbitrary filtered probability space. The existence and uniqueness of a solution are proved via the Yosida approximation method when the generators are only stochastic monotone with respect to the y-variable and stochastic Lipschitz with respect to the z-variable, with different linear growth conditions.

Keywords Generalized BSDEs with jumps, RCLL martingale, stochastic monotone coefficient, stochastic Lipschitz coefficient, Yosida approximation

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Introduction

Backward stochastic differential equations (BSDEs, for short) were initially investigated as adjoint processes under the maximum stochastic control principle by Bismut [3]. Pardoux and Peng [13] obtained the first result dealing with nonlinear BSDEs. They showed the existence and uniqueness of the solution in a Brownian setting under specific conditions, most notably the Lipschitz continuity of the generator via the martingale representation theorem. The case of nonlinear discontinuous BSDEs

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has been carried out by Tang and Li [18] and Situ [17] in a more general filtration generated by a Brownian motion and an independent Poisson random measure.

In other context, Pardoux and Zhang [15] provided a probabilistic representation for a solution of a system of parabolic and elliptic semilinear partial differential equations (PDEs, for short) with the Neumann boundary condition using a new class of BSDEs which involves the integral with respect to a continuous increasing process interpreted as the local time of a diffusion process on the boundary. This kind of BSDEs is called generalized BSDEs (GBSDEs, for short). A solution of this equation, associated with a terminal value ξ and generators or drivers $f(\omega, t, y, z)$ and $g(\omega, t, y)$, is a couple of stochastic processes $(Y_t, Z_t)_{t\leq T}$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (1)$$

where $(B_t)_{t\leq T}$ is a standard Brownian motion and the process $(Y_t, Z_t)_{t\leq T}$ is adapted to the natural filtration of $(B_t)_{t\leq T}$.

Following this work, Pardoux [12] has considered the case of GBSDEs with jumps where the discontinuity stems from the Poisson random measure. The same problem has been treated by El Otmani [6] in the case of GBSDEs driven by the Lévy process, where the author provides the link between those equations and a class of PDEs with the Neumann boundary condition.

In this paper, we are interested in exploring, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a complete, quasi-left continuous, right continuous filtration $(\mathcal{F}_t)_{t\leq T}$, a generalized backward stochastic differential equation of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)d\langle M\rangle_s + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad (2)$$

driven by the square integrable martingale $(M_t)_{t\leq T}$, where $\langle M\rangle$ denotes the predictable projection of the quadratic variation $[M]$, T is a fixed time horizon and $\mathcal{F}_T = \mathcal{F}$.

As opposed to (1), the solution is now a triplet $(Y_t, Z_t, N_t)_{t\leq T}$, where $(Z_t)_{t\leq T}$ is predictable and $(N_t)_{t\leq T}$ is a square integrable martingale orthogonal to M, fulfilling a few integrability requirements that we will specify more explicitly in the following section. We do not impose any constraints on the filtration $(\mathcal{F}_t)_{t\leq T}$ other than the usual conditions and the quasi-left continuity.

In comparison to existing literature, our work reexamines and generalizes papers by Barles et al. [1], Bender and Kohlmann [2], El Karoui and Huang [5], Carbone et al. [4], Nie and Rutkowski [10] for classical BSDEs, the book by [14] and the aforementioned works [6, 12, 15, 17, 18] since: on the one hand, we allow for a more general filtration instead of dealing with the one generated by a Brownian motion or a Poisson random measure; on the other hand, the generators f and g are stochastic monotone with respect to the y-variable, and the coefficient f is stochastic Lipschitz in the z-variable – this includes all Lipschitz and monotonic constraints placed on drivers that have been published previously. Moreover, under some suitable integrability assumptions on the terminal value ξ and linear growth condition on

$f(t, y, 0)$ and $g(t, y)$, we provide existence and uniqueness results in an appropriate $\mathbb{L}^2$ space using Yosida's approximation method. We note that Yosida's approximation technique has been employed by several authors in this context. Notably, Hu [7] proved the existence and uniqueness results for a class of equations known as forward-backward stochastic differential equations, defined over an arbitrarily prescribed time duration, under specific monotonicity conditions on the coefficients (for other related works on FBSDEs, readers can refer to [8]).

The main difficulties of our problem lies in the fact that, firstly, the GBSDE (2) is considered on an arbitrary filtered probability space, which allows for a more general semimartingale setting, and secondly, the drivers of our GBSDE satisfy weaker assumptions than the ones considered in the literature. To the best of our knowledge, there is no existence and uniqueness result for GBSDEs driven by a right continuous with left limits (RCLL, for short) martingale in a general arbitrary probability space with stochastic monotone coefficients. Finally, note that our proofs can be easily extended to the multidimensional case.

The article is structured as follows. The notations, assumptions, definitions and other properties are covered in Section 1. Section 2 is devoted to giving a priori estimates of the solutions of the GBSDE (2) under stochastic monotonicity of the drivers f and g in the y-variable and the stochastic Lipschitz condition of f with respect to z, which specifically produces the solution's uniqueness. Finally, in Section 3 we establish the article's main result. Namely, we prove the existence of the solution using the Yosida approximation method by dividing the proof into several steps in which we deal first with a generator f independent of z, then approximate the resulting equation using a family of GBSDE with a deterministic Lipschitz constant; using this, the general results may be obtained via a fixed point argument in an appropriate Banach space.

1 Setting of the problem and assumptions

Let $T > 0$ be a fixed time and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is quasi-left continuous and satisfies the usual conditions of right continuity and completeness. The initial σ-field $\mathcal{F}_0$ is assumed to be trivial and $\mathcal{F}_T = \mathcal{F}$. The equality $X = Y$ between any two processes $(X_t)_t$ and $(Y_t)_t$ must be understood in the indistinguishably sens i.e. $\mathbb{P}(\omega : X_t(\omega) = Y_t(\omega), \forall t \in [0, T]) = 1$. For a given RCLL process $(Y_t)_t$, $Y_{t-} = \lim_{s\nearrow t} Y_s$ is the left limits of Y at t, we set $Y_{0-} = Y_0$ by convention. $Y_- = (Y_{t-})_t$ the left limited process, $\Delta Y_t = Y_t - Y_{t-}$ the jump of Y at time t and $\mathbb{E}^{\mathcal{F}_t}[Y] := \mathbb{E}[Y \mid \mathcal{F}_t]$. Next, for given two locally square integrable $\mathbb{F}$-martingales M and N, we denote by $\langle M, N\rangle$ the predictable $\mathbb{F}$-dual projection of the quadratic co-variation process $[M, N]$, by M^c the continuous part of M. Finally, the integral $\int_s^t Y_u dX_u$ is interpreted as $\int_{]s,t]} Y_u dX_u$ where $]s, t] = \{u \in [0, T] : s < u \leq t\}$.

We assume given an $\mathbb{R}$-valued, square-integrable $\mathbb{F}$-martingale $M := (M)_{t\leq T}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Since the filtration $\mathbb{F}$ is right-continuous, there exists a modification of M with RCLL paths, hence, we may assume throughout this paper that M is an RCLL process.

Now, we present the conditions imposed on the data of the generalized BSDE (2) that enable us to prove the main result of the paper.

The basic assumptions on the data (ξ, f, g, A).

Measurability of the data and trajectory properties of the process $(A_t)_{t\leq T}$.

- The process $(A_t)_{t\leq T}$ is $\mathcal{F}_t$-progressively measurable continuous increasing such that $A_0 = 0$;
- ξ is an $\mathcal{F}_T$-measurable random variable;
- $\forall y, z \in \mathbb{R}$, the processes $f(\cdot,\cdot,y,z) : \Omega \times [0,T] \to \mathbb{R}$ and $g(\cdot,\cdot,y) : \Omega \times [0,T] \to \mathbb{R}$ are $\mathcal{F}_t$-progressively measurable.

Stochastic monotonicity of f *and* g *in* y. There exists two $\mathcal{F}_t$-progressively measurable processes $\alpha : \Omega \times [0,T] \to \mathbb{R}$ and $\beta : \Omega \times [0,T] \to \mathbb{R}^-$ such that:

(i) for all $y, y', z \in \mathbb{R}$, $d\mathbb{P} \otimes d\langle M\rangle_t$-a.e.,

$$\big(y - y'\big)\big(f(t,y,z) - f\big(t,y',z\big)\big) \leq \alpha_t \big|y - y'\big|^2;$$

(ii) for all $y, y' \in \mathbb{R}$, $d\mathbb{P} \otimes dA_t$-a.e.,

$$\big(y - y'\big)\big(g(t,y) - g\big(t,y'\big)\big) \leq \beta_t \big|y - y'\big|^2.$$

Stochastic Lipschitz condition on f *in* z. There exists an $\mathcal{F}_t$-progressively measurable process $\gamma : \Omega \times [0,T] \to \mathbb{R}^+$ such that

(iii) for all $y, z, z' \in \mathbb{R}$, $d\mathbb{P} \otimes d\langle M\rangle_t$-a.e.,

$$\big|f(t,y,z) - f\big(t,y,z'\big)\big| \leq \gamma_t \big|z - z'\big|.$$

Linear growth of f *and* g. For some constant $\kappa > 0$ and some $[1,\infty)$-valued adapted processes $\{\varphi_t, \psi_t; 0 \leq t \leq T\}$ and all $(t,y) \in [0,T] \times \mathbb{R}$, we have

(iv) $|f(t,y,0)| \leq \varphi_t + \kappa|y|$ and $|g(t,y)| \leq \psi_t + \kappa|y|$.

Integrability condition. Let $(V_t)_{t\leq T}$ and $(Q_t)_{t\leq T}$ be the two $\mathcal{F}_t$-progressively measurable continuous increasing stochastic processes defined by

$$V_t := \int_0^t \big(|\alpha_s| + \alpha_s^2 + \gamma_s^2\big) d\langle M\rangle_s + \int_0^t \beta_s^2 dA_s, \quad Q_t := \langle M\rangle_t + A_t.$$

Let us set

$$\phi_t^{\lambda,\theta,\mu} := \lambda V_t + \theta\langle M\rangle_t + \mu A_t \quad \text{and} \quad \Phi^{\lambda,\theta,\mu} := e^{\phi^{\lambda,\theta,\mu}} \quad \text{for } \lambda, \theta, \mu > 0.$$

We assume that, for any $\lambda, \theta, \mu > 0$,

(v) $\mathbb{E}[\Phi_T^{\lambda,\theta,\mu}|\xi|^2] < \infty$;

(vi) $\mathbb{E}[\int_0^T \Phi_t^{\lambda,\theta,\mu}\varphi_t^2 d\langle M\rangle_t + \int_0^T \Phi_t^{\lambda,\theta,\mu}\psi_t^2 dA_t] < \infty$.

Remark 1. Note that, since $\{\varphi_t, \psi_t;\ 0 \leq t \leq T\}$ are $[1,\infty)$-valued processes,

$$\mathbb{E}\left[\int_0^T \Phi_t^{\lambda,\theta,\mu} dQ_t\right] < \infty.$$

Continuity condition on f and g. For all $y, y', z, z' \in \mathbb{R}$:

(vii) $d\mathbb{P} \otimes d\langle M\rangle_t$-a.e., the mapping $y \mapsto f(t, y, z) : \mathbb{R} \to \mathbb{R}$ is continuous;

(viii) $d\mathbb{P} \otimes dA_t$-a.e., the mapping $y \mapsto g(t, y) : \mathbb{R} \to \mathbb{R}$ is continuous.

In the rest of this paper the previous assumptions will be denoted by **(H-M)**.

Spaces. For $\lambda, \theta, \mu \geq 0$, we introduce the following spaces to describe the parameters and the solution of the equation (2).

- $\mathcal{H}^2$: The space of $\mathbb{R}$-valued $\mathcal{F}_t$-predictable processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|^2_{\mathcal{H}^2} = \mathbb{E}\left[\int_0^T |Z_s|^2 d\langle M\rangle_s\right] < \infty.$$

- $\mathcal{M}^2$: The space of one-dimensional square-integrable $\mathbb{F}$-martingale $(N_t)_{t\leq T}$ orthogonal to M such that

$$\|N\|^2_{\mathcal{M}^2} = \mathbb{E}\left[\int_0^T d[N]_s\right] < \infty.$$

- $\mathcal{S}^2_{\lambda,\theta,\mu}$: The space of one-dimensional $\mathcal{F}_t$-adapted RCLL processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathcal{S}^2_{\lambda,\theta,\mu}} = \mathbb{E}\left[\sup_{0\leq t\leq T} \Phi_t^{\lambda,\theta,\mu}|Y_t|^2\right] < \infty.$$

- $\mathcal{C}^{2,V}_{\lambda,\theta,\mu}$: The space of one-dimensional $\mathcal{F}_t$-adapted RCLL processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathcal{C}^{2,V}_{\lambda,\theta,\mu}} = \mathbb{E}\left[\int_0^T \Phi_s^{\lambda,\theta,\mu}|Y_s|^2 dV_s\right] < \infty.$$

- $\mathcal{H}^2_{\lambda,\theta,\mu}$: The space of $\mathbb{R}$-valued $\mathcal{F}_t$-predictable processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|^2_{\mathcal{H}^2_{\lambda,\theta,\mu}} = \mathbb{E}\left[\int_0^T \Phi_s^{\lambda,\theta,\mu}|Z_s|^2 d\langle M\rangle_s\right] < \infty.$$

- $\mathcal{M}^2_{\lambda,\theta,\mu}$: The subspace of $\mathcal{M}^2$ such that

$$\|N\|^2_{\mathcal{M}^2_{\lambda,\theta,\mu}} = \mathbb{E}\left[\int_0^T \Phi_s^{\lambda,\theta,\mu} d[N]_s\right] < \infty.$$

- $\mathfrak{D}^2_{\lambda,\theta,\mu} = (\mathcal{S}^2_{\lambda,\theta,\mu} \cap \mathcal{C}^{2,V}_{\lambda,\theta,\mu}) \times \mathcal{H}^2_{\lambda,\theta,\mu} \times \mathcal{M}^2_{\lambda,\theta,\mu}$ and $\mathfrak{D}^2 := \mathfrak{D}^2_{0,0,0}$.

We adopt the following definition of a solution to the GBSDE (2).

Definition 1. A solution to GBSDE associated with parameters (ξ, f, g, A) is a triplet of processes $(Y_t, Z_t, N_t)_{t\leq T}$ which satisfy (2) and belongs to $\mathfrak{D}^2_{\lambda,\theta,\mu}$.

First, we give some remarks that will be used subsequently.

Remark 2. Recall that

(i) For any process Z that belongs to $\mathcal{H}^2$, the process $(\int_0^{\cdot} Z_s dM_s)^2 - \int_0^{\cdot} |Z_s|^2 d[M]_s$ is an $\mathbb{F}$-martingale (Theorem 27 in [16, p. 71]) and we have

$$\mathbb{E}^{\mathcal{F}_t}\left[\left(\int_t^T Z_s dM_s\right)^2\right] = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T |Z_s|^2 d[M]_s\right] = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T |Z_s|^2 d\langle M\rangle_s\right]. \tag{3}$$

(ii) We point out that, since $([M, N] - \langle M, N\rangle)$ is a martingale (see Proposition 4.50-b in [9, p. 53]), if Z is an element of $\mathcal{H}^2_{\lambda,\theta,\mu}$, we have: $\forall t \in [0, T]$

$$\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \Phi_s^{\lambda,\theta,\mu} Z_s d[M, N]_s\right] = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \Phi_s^{\lambda,\theta,\mu} Z_s d\langle M, N\rangle_s\right], \tag{4}$$

and this last term is equal to zero if N is orthogonal to M.

(iii) The jump part of the process $[Y]$ is described by

$$\sum_{0<s\leq\cdot} (\Delta Y_s)^2 = \sum_{0<s\leq\cdot} |Z_s|^2 (\Delta M_s)^2 + 2\sum_{0<s\leq\cdot} Z_s \Delta M_s \Delta N_s + \sum_{0<s\leq\cdot} (\Delta N_s)^2, \tag{5}$$

and the path-by-path continuous part of $t \mapsto [Y]_t$ is given by

$$[Y]^c = \int_0^{\cdot} |Z_s|^2 d\langle M^c\rangle_s + 2\int_0^{\cdot} Z_s d\langle M^c, N^c\rangle_s + \int_0^{\cdot} d\langle N^c\rangle_s. \tag{6}$$

(For such a path-wise decomposition for the quadratic variation, the reader is referred to [16, p. 70].)

2 A priori estimates and uniqueness

Let (ξ, f, g, A) and (ξ', f', g', A') be two sets of data, each satisfying the above assumption **(H-M)**. Let (Y, Z, N) (resp. (Y', Z', N')) denote a solution of the generalized BSDE (2) with data (ξ, f, g, A) (resp. (ξ', f', g', A')) in the sense of Definition 1. The following result is useful for further applications.

Proposition 1. *Define $\bar{\mathfrak{R}} = \mathfrak{R} - \mathfrak{R}'$ for $\mathfrak{R} = Y, Z, N, A$ and ξ. Then, for any $\lambda > 2$ and θ, $\mu > 0$, there exists a constant $\mathfrak{C} = \mathfrak{C}(\lambda, \theta, \mu)$ such that, for all $s \in [0, T]$,*

$$\mathbb{E}^{\mathcal{F}_s}\left[\sup_{s\leq t\leq T} \bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2\right] + \mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}|\bar{Y}_r|^2 dV_r\right]$$
$$+ \mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}|\bar{Y}_r|^2 d\langle M\rangle_r\right] + \mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}|\bar{Y}_r|^2 d\|\bar{A}\|_r\right]$$
$$+ \mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}|\bar{Z}_r|^2 d\langle M\rangle_r\right] + \mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu} d[\bar{N}]_r\right]$$

$$\leq \mathfrak{C}\left\{\mathbb{E}^{\mathcal{F}_s}\left[\bar{\Phi}_T^{\lambda,\theta,\mu}|\bar{\xi}|^2+\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}\big|f\big(r,Y_r',Z_r'\big)-f'\big(r,Y_r',Z_r'\big)\big|^2 d\langle M\rangle_r\right]\right.$$
$$\left.+\mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}\Big(\big|g\big(r,Y_r'\big)\big|^2 d\|\bar{A}\|_r+\big|g\big(r,Y_r'\big)-g'\big(s,Y_r'\big)\big|^2 dA_r'\Big)\right]\right\}.$$

Here $\|\bar{A}\|$ *denote the total variation of the process* $\bar{A}=A-A'$, *and* $\bar{\Phi}^{\lambda,\theta,\mu}:=e^{\bar{\phi}^{\lambda,\theta,\mu}}$ *where* $\bar{\phi}_t^{\lambda,\theta,\mu}=\lambda V_t+\theta\langle M\rangle_t+\mu(\|\bar{A}\|_t+A_t')$.

Proof. It suffices to prove the result in the case where $\|\bar{A}\|_T+A_T'$ is a bounded random variable, and then apply Fatou's Lemma. From the Itô formula (Theorem 33 in [16, p. 81]), we can write

$$\begin{aligned}
&\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2+\lambda\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2 dV_s+\theta\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2 d\langle M\rangle_s\\
&\quad+\mu\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2\big(d\|\bar{A}\|_s+dA_s'\big)+\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Z}_s|^2 d\big\langle M^c\big\rangle_s\\
&\quad+2\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\bar{Z}_s d\big\langle M^c,\bar{N}^c\big\rangle_s+\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}d\big\langle \bar{N}^c\big\rangle_s+\sum_{t<s\leq T}\bar{\Phi}_s^{\lambda,\theta,\mu}(\Delta\bar{Y}_s)^2\\
&=\bar{\Phi}_T^{\lambda,\theta,\mu}|\bar{\xi}|^2+2\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\bar{Y}_s\big(f(s,Y_s,Z_s)-f'\big(s,Y_s',Z_s'\big)\big)d\langle M\rangle_s\\
&\quad+2\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\bar{Y}_s\big(g(s,Y_s)dA_s-g'\big(s,Y_s'\big)dA_s'\big)\\
&\quad-2\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\bar{Y}_{s-}\bar{Z}_s dM_s-2\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\bar{Y}_{s-}d\bar{N}_s.
\end{aligned}\tag{7}$$

Note that from the integrability condition satisfied by the solution of GBSDE (2), we deduce that the two terms in the last line of (7) are uniformly integrable $\mathbb{F}$-martingales. Next, Assumptions (**H-M**)-((i)-(ii)-(iii)) imply in the sense of signed measures on $[0,T]$:

$$\begin{aligned}
&2\bar{Y}_s\big(f(s,Y_s,Z_s)-f'\big(s,Y_s',Z_s'\big)\big)d\langle M\rangle_s\\
&\quad\leq\left(\frac{\theta}{2}+2\big(\alpha_s+\gamma_s^2\big)\right)|\bar{Y}_s|^2 d\langle M\rangle_s+\frac{1}{2}|\bar{Z}|_s^2 d\langle M\rangle_s\\
&\qquad+\frac{2}{\theta}\big|f\big(s,Y_s',Z_s'\big)-f'\big(s,Y_s',Z_s'\big)\big|^2 d\langle M\rangle_s,
\end{aligned}$$

and

$$\begin{aligned}
&2\bar{Y}_s\big(g(s,Y_s)dA_s-g'\big(s,Y_s'\big)dA_s'\big)\\
&\quad=2\bar{Y}_s\big(g(s,Y_s)-g\big(s,Y_s'\big)\big)dA_s+2\bar{Y}_s g\big(s,Y_s'\big)\big(dA_s-dA_s'\big)\\
&\qquad+2\bar{Y}_s\big(g\big(s,Y_s'\big)-g'\big(s,Y_s'\big)\big)dA_s'\\
&\quad\leq 2\beta_s|\bar{Y}_s|^2 dA_s+\frac{\mu}{2}|\bar{Y}_s|^2\big(d\|\bar{A}\|_s+dA_s'\big)+\frac{4}{\mu}\big|g\big(s,Y_s'\big)\big|^2 d\|\bar{A}\|_s
\end{aligned}$$

$$+\frac{4}{\mu}\left|g\left(s, Y_s'\right) - g'\left(s, Y_s'\right)\right|^2 dA_s'.$$

After that, taking this into account with (3), (4), (5), and (6), and choosing $\lambda > 2$, we deduce, taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of (7), that there exists a constant $C_{\lambda,\theta,\mu}$ such that for any $t \in [0, T]$,

$$\begin{aligned}
&\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2 + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2 dV_s\right] + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2 d\langle M\rangle_s\right] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Y}_s|^2 dA_s\right] + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}|\bar{Z}_s|^2 d\langle M\rangle_s\right] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu} d[\bar{N}]_s\right] \\
&\leq C_{\lambda,\theta,\mu}\left\{\mathbb{E}^{\mathcal{F}_t}\left[\bar{\Phi}_T^{\lambda,\theta,\mu}|\bar{\xi}|^2\right]\right. \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\left|f\left(s, Y_s', Z_s'\right) - f'\left(s, Y_s', Z_s'\right)\right|^2 d\langle M\rangle_s\right] \\
&\quad \left.+ \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \bar{\Phi}_s^{\lambda,\theta,\mu}\left(\left|g\left(s, Y_s'\right)\right|^2 d\|\bar{A}\|_s + \left|g\left(s, Y_s'\right) - g'\left(s, Y_s'\right)\right|^2 dA_s'\right)\right]\right\}. \qquad (8)
\end{aligned}$$

On the other hand, starting from (7) and utilizing the estimates related to the generators, we obtain, for each $s \in [0, T]$ and all $s \leq t \leq T$,

$$\begin{aligned}
&\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2 \\
&\quad \leq \bar{\Phi}_T^{\lambda,\theta,\mu}|\bar{\xi}|^2 + \frac{2}{\theta}\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\left|f\left(r, Y_r', Z_r'\right) - f'\left(r, Y_r', Z_r'\right)\right|^2 d\langle M\rangle_r \\
&\qquad + \frac{4}{\mu}\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\left(\left|g\left(r, Y_r'\right)\right|^2 d\|\bar{A}\|_r + \left|g\left(r, Y_r'\right) - g'\left(r, Y_r'\right)\right|^2 dA_r'\right) \\
&\qquad + 2\sup_{s\leq t\leq T}\left|\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\bar{Y}_{r-}\bar{Z}_r dM_r\right| + 2\sup_{s\leq t\leq T}\left|\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\bar{Y}_{r-} d\bar{N}_r\right|. \qquad (9)
\end{aligned}$$

Using the B-D-G inequality (Theorem 48 in [16, p. 193]), we obtain

$$\begin{aligned}
&2\mathbb{E}^{\mathcal{F}_s}\left[\sup_{s\leq t\leq T}\left|\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\bar{Y}_{r-}\bar{Z}_r dM_r\right|\right] \\
&\quad \leq 2c\mathbb{E}^{\mathcal{F}_s}\left[\left|\int_s^T \bar{\Phi}_r^{2\lambda,2\theta,2\mu}|\bar{Y}_{r-}||\bar{Z}_r|^2 d[M]_r\right|^{\frac{1}{2}}\right] \\
&\quad \leq \frac{1}{4}\mathbb{E}^{\mathcal{F}_s}\left[\sup_{s\leq t\leq T}\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2\right] + 4c^2\mathbb{E}^{\mathcal{F}_s}\left[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}|\bar{Z}_r|^2 d\langle M\rangle_r\right]. \qquad (10)
\end{aligned}$$

Similarly, we get

$$2\mathbb{E}^{\mathcal{F}_s}\left[\sup_{s\leq t\leq T}\left|\int_t^T \bar{\Phi}_r^{\lambda,\theta,\mu}\bar{Y}_{r-} d\bar{N}_r\right|\right]$$

$$\leq \frac{1}{4}\mathbb{E}^{\mathcal{F}_s}\Big[\sup_{s\leq t\leq T}\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2\Big] + 4c^2\mathbb{E}^{\mathcal{F}_s}\Big[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}d[\bar{N}]_r\Big]. \tag{11}$$

Now coming back to (9), taking the supremum of the process $(\bar{\Phi}_t^{\lambda,\theta,\mu}|\bar{Y}_t|^2)_{t\in[0,T]}$ over the time interval $[s,T]$, using estimates (8) (with $t=s$), (10), and (11), and taking conditional expectation with respect to $\mathcal{F}_s$, we deduce that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s}\Big[\sup_{s\leq t\leq T}\bar{\Phi}_t^{\lambda,\theta,\mu}|Y_t|^2\Big]\\
&\quad\leq C_{\lambda,\theta,\mu}\Big\{\mathbb{E}^{\mathcal{F}_s}\big[\bar{\Phi}_T^{\lambda,\theta,\mu}|\bar{\xi}|^2\big]\\
&\qquad+\mathbb{E}^{\mathcal{F}_s}\Big[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}\big|f\big(r,Y'_r,Z'_r\big)-f'\big(r,Y'_r,Z'_r\big)\big|^2 d\langle M\rangle_r\Big]\\
&\qquad+\mathbb{E}^{\mathcal{F}_s}\Big[\int_s^T \bar{\Phi}_r^{\lambda,\theta,\mu}\Big(\big|g\big(r,Y'_r\big)\big|^2 d\|\bar{A}\|_r+\big|g\big(r,Y'_r\big)-g'\big(r,Y'_r\big)\big|^2 dA'_r\Big)\Big]\Big\}.
\end{aligned}$$

The proof is complete. □

Corollary 1. *Under assumption (**H-M**), there exists at most one $\mathcal{F}_t$-progressively measurable process $\{(Y_t,Z_t,N_t), 0\leq t\leq T\}$ which belongs to $\mathfrak{D}^2_{\lambda,\theta,\mu}$ for $(\lambda,\theta,\mu)\in(2,\infty)\times(0,\infty)^2$ and solves equation* (2).

Corollary 2. *Let assumption (**H-M**) be satisfied. Then, for any $\lambda>2$ and $\theta,\mu>0$, there exists a constants $\mathfrak{C}$, which depends only on (λ,θ,μ) and the constant of the B-D-G inequality, such that whenever (Y,Z,N) satisfies* (2)*, we have*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq s\leq T}\Phi_s^{\lambda,\theta,\mu}|Y_s|^2\Big]+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}|Y_s|^2 dV_s\Big]+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}|Y_s|^2 dQ_s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}|Z_s|^2 d\langle M\rangle_s\Big]+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}d[N]_s\Big]\\
&\leq\mathfrak{C}\Big\{\mathbb{E}[\Phi_T^{\lambda,\theta,\mu}|\xi|^2]+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}|\varphi_s|^2 d\langle M\rangle_s\Big]+\mathbb{E}\Big[\int_0^T\Phi_s^{\lambda,\theta,\mu}|\psi_s|^2 dA_s\Big]\Big\}.
\end{aligned} \tag{12}$$

Proof. The result is a direct consequence of Proposition 1, (**H-M**)-(iv) and the fact that $(Y',Z',N')=(0,0,0)$ is the unique solution of (2) with data $(\xi',f',g',A')=(0,0,0,0)$. □

3 Existence theorem for the generalized BSDE (2)

We are going to show that equation (2) has a solution using the Yosida approximation method. Roughly speaking, we first assume that the driver f of the GBSDE (2) is independent of the z variable, i.e. $\mathbb{P}$-a.s., $f(t,y,z)=\mathfrak{f}(t,y)$, for any t, y and z. Then we approximate the coefficients $(\mathfrak{f}(t,y)+\alpha_t y)_{t\leq T}$ and $(g(t,y)+\beta_t y)_{t\leq T}$ by a family of Lipschitz mappings F_ϵ and G_ϵ indexed by some $\epsilon\in]0,1]$, which allows us to construct a certain form of equation that converges toward the solution of the

GBSDE in this special case. However, the method uses a contraction argument to keep the result in the general framework.

Assume for the moment that the driver f does not depend on z, i.e. $\mathbb{P}$-a.s., $f(t, y, z) = \mathfrak{f}(t, y)$, for any t, y and z. The following result establishes the existence of the solution of the GBSDE (2) linked to $(\xi, \mathfrak{f}, g, A)$.

Theorem 1. *Under (**H-M**), there exists a unique $\mathcal{F}_t$-progressively measurable process $(Y_t, Z_t, N_t)_{t\leq T}$ with values in $\mathbb{R}^3$ which satisfies (12) and*

$$Y_t = \xi + \int_t^T \mathfrak{f}(s, Y_s)d\langle M\rangle_s + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad (13)$$

for all $t \in [0, T]$, $\mathbb{P}$-a.s.

Proof. To prove that the solution exists, we first rewrite the equation as follows: $\mathbb{P}$-a.s. $0 \leq t \leq T$,

$$\begin{aligned} Y_t = \xi &+ \int_t^T \big\{F(s, Y_s) + \alpha_s Y_s\big\}d\langle M\rangle_s + \int_t^T \big\{G(s, Y_s) + \beta_s Y_s\big\}dA_s \\ &- \int_t^T Z_s dM_s - \int_t^T dN_s, \quad 0 \leq t \leq T, \end{aligned} \quad (14)$$

where

$$F(s, y) := \mathfrak{f}(s, y) - \alpha_s y, \quad \text{and} \quad G(s, y) := g(s, y) - \beta_s y.$$

Clearly, the new drivers F and G satisfy the following monotonicity property:

(H-M$_0$)
- $(y - y')(F(s, y) - F(s, y')) \leq 0,$
- $(y - y')(G(s, y) - G(s, y')) \leq 0.$

The proof is performed in 4 steps.

Step 1: Yosida approximation. Note that the new drivers F and G satisfy Assumption **(H-M)** except **(H-M)**-(i)-(ii) is transformed to **(H-M$_0$)**. It follows (see Annex B in [14, p. 524]) that for every $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}$ and $\epsilon > 0$, there exists unique $J_\epsilon^F = J_\epsilon^F(\omega, t, y)$ and $J_\epsilon^G = J_\epsilon^G(\omega, t, y) \in \mathbb{R}$ such that

$$J_\epsilon^F - \epsilon F\big(\omega, t, J_\epsilon^F\big) = y, \quad J_\epsilon^G - \epsilon G\big(\omega, t, J_\epsilon^G\big) = y.$$

The Yosida approximation of $\mathfrak{f}$ and g is defined respectively by $F_\epsilon = F_\epsilon(\omega, t, y)$ and $G_\epsilon = G_\epsilon(\omega, t, y) \in \mathbb{R}$ such that

$$\begin{aligned} F_\epsilon(\omega, t, y) &:= \frac{1}{\epsilon}\big(J_\epsilon^F(\omega, t, y) - y\big) = F(t, y + \epsilon F_\epsilon), \\ G_\epsilon(\omega, t, y) &:= \frac{1}{\epsilon}\big(J_\epsilon^G(\omega, t, y) - y\big) = G(t, y + \epsilon G_\epsilon). \end{aligned} \quad (15)$$

Note that (F_ϵ, G_ϵ) is the unique couple satisfying the system (15).

From Annex B, Proposition 6.7 in [14], one can recall that:

(Y1) for every $y \in \mathbb{R}$, the processes $F_\epsilon(\cdot,\cdot,y)$, $G_\epsilon(\cdot,\cdot,y) : \Omega \times [0,T] \to \mathbb{R}$ are $\mathcal{F}_t$-progressively measurable;

(Y2) $\forall \epsilon, \delta > 0, \forall t \in [0,T], \forall y, y' \in \mathbb{R}$, $\mathbb{P}$-a.s.

(i) $|J_\epsilon^F(t,y) - J_\epsilon^F(t,y')| + |J_\epsilon^G(t,y) - J_\epsilon^G(t,y')| \leq |y-y'|$,

(ii) $(y-y')(F_\epsilon(t,y) - F_\epsilon(t,y')) \leq 0$,

(iii) $(y-y')(G_\epsilon(t,y) - G_\epsilon(t,y')) \leq 0$,

(iv) $|F_\epsilon(t,y) - F_\epsilon(t,y')| + |G_\epsilon(t,y) - G_\epsilon(t,y')| \leq \frac{2}{\epsilon}|y-y'|$,

(v) $|F_\epsilon(t,y)| \leq |F(t,y)|$ and $\lim_{\epsilon\to 0} F_\epsilon(t,y) = F(t,y)$, $|G_\epsilon(t,y)| \leq |G(t,y)|$ and $\lim_{\epsilon\to 0} G_\epsilon(t,y) = G(t,y)$,

(vi) $(y-y')(F_\epsilon(t,y) - F_\delta(t,y')) \leq (\epsilon+\delta) F_\epsilon(t,y) F_\delta(t,y')$, $(y-y')(G_\epsilon(t,y) - G_\delta(t,y')) \leq (\epsilon+\delta) G_\epsilon(t,y) G_\delta(t,y')$.

Step 2: Approximating equation and uniform boundedness. Let $0 < \epsilon \leq 1$. The *Yosida approximating equation* of the generalized BSDE (14) is given by

$$Y_t^\epsilon = \xi + \int_t^T \left\{F_\epsilon\big(s, Y_s^\epsilon\big) + \alpha_s Y_s^\epsilon\right\} d\langle M\rangle_s + \int_t^T \left\{G_\epsilon\big(s, Y_s^\epsilon\big) + \beta_s Y_s^\epsilon\right\} dA_s - \int_t^T Z_s^\epsilon dM_s - \int_t^T dN_s^\epsilon, \quad 0 \leq t \leq T. \tag{16}$$

The GBSDE (16) has a unique solution $(Y_t^\epsilon, Z_t^\epsilon, N_t^\epsilon)_{t\leq T} \in \mathfrak{D}^2$ (see Appendix A).

Let $\chi_t^{\lambda,\theta,\mu} = e^{\lambda \hat{V}_t^\star + \theta\langle M\rangle_t + \mu A_t}$, where $\hat{V}_t^\star := \int_0^t (|\alpha_s| + \alpha_s^2) d\langle M\rangle_s + \int_0^t \beta_s^2 dA_s = V_t - \int_0^t \gamma_s^2 d\langle M\rangle_s$.

By applying the Itô formula, we have, for all $t \in [0,T]$ and each $\epsilon \in]0,1]$,

$$\begin{aligned}
&\chi_t^{\lambda,\theta,\mu}\big|Y_t^\epsilon\big|^2 + \lambda \int_t^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\hat{V}_s^\star + \theta \int_t^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s \\
&\quad + \mu \int_t^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 dA_s + \int_t^T \chi_s^{\lambda,\theta,\mu}\big|Z_s^\epsilon\big|^2 d\big\langle M^c\big\rangle_s \\
&\quad + 2\int_t^T \chi_s^{\lambda,\theta,\mu} Z_s^\epsilon d\big\langle M^c, \big(N^\epsilon\big)^c\big\rangle_s + \int_t^T \chi_s^{\lambda,\theta,\mu} d\big\langle \big(N^\epsilon\big)^c\big\rangle_s \\
&\quad + \sum_{t<s\leq T} \chi_s^{\lambda,\theta,\mu}\big(\Delta Y_s^\epsilon\big)^2 \\
&= \chi_T^{\lambda,\theta,\mu}|\xi|^2 + 2\int_t^T \chi_s^{\lambda,\theta,\mu} Y_s^\epsilon \big(F_\epsilon\big(s, Y_s^\epsilon\big) + \alpha_s Y_s^\epsilon\big) d\langle M\rangle_s \\
&\quad + 2\int_t^T \chi_s^{\lambda,\theta,\mu} Y_s^\epsilon \big(G_\epsilon\big(s, Y_s^\epsilon\big) + \beta_s Y_s^\epsilon\big) dA_s - 2\int_t^T \chi_s^{\lambda,\theta,\mu} Y_{s-}^\epsilon Z_s^\epsilon dM_s \\
&\quad - 2\int_t^T \chi_s^{\lambda,\theta,\mu} Y_{s-}^\epsilon dN_s^\epsilon.
\end{aligned}$$

From proprieties **(Y2)**-(ii)-(iii)-(iv) and **(H-M)**-(iv), we obtain, for $\delta_1, \delta_2 > 0$,

$$\begin{aligned}
&2Y_s^\epsilon\big(F_\epsilon\big(s, Y_s^\epsilon\big) + \alpha_s Y_s^\epsilon\big)d\langle M\rangle_s \\
&\quad \le 2\big|Y_s^\epsilon\big|\big|F(s,0)\big|d\langle M\rangle_s + 2|\alpha_s|\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s \\
&\quad \le 2|\alpha_s|\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s + \delta_1\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s + \frac{1}{\delta_1}\varphi_s^2 d\langle M\rangle_s,
\end{aligned}$$

and

$$2Y_s^\epsilon\big(G_\epsilon\big(s, Y_s^\epsilon\big) + \beta_s Y_s^\epsilon\big)dA_s \le \delta_2\big|Y_s^\epsilon\big|^2 dA_s + \frac{1}{\delta_2}\psi_s^2 dA_s.$$

Next, choosing $\lambda > 2$, $\delta_1 < \theta$ and $\delta_2 < \mu$, we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\chi_t^{\lambda,\theta,\mu}\big|Y_t^\epsilon\big|^2\Big] + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\hat{V}_s^\star\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s\Big] + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 dA_s\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Z_s^\epsilon\big|^2 d\langle M\rangle_s\Big] + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu} d\big[N^\epsilon\big]_s\Big] \\
&\le \mathfrak{C}\Big\{\mathbb{E}\big[\chi_T^{\lambda,\theta,\mu}|\xi|^2\big] + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}|\varphi_s|^2 d\langle M\rangle_s\Big] + \mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}|\psi_s|^2 dA_s\Big]\Big\}.
\end{aligned} \tag{17}$$

Using Assumption **(H-M)**-(iv) and property **(Y2)**-(v), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|F_\epsilon\big(s, Y_s^\epsilon\big)\big|^2 d\langle M\rangle_s\Big] \\
&\le 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|\mathfrak{f}\big(s, Y_s^\epsilon\big)\big|^2 d\langle M\rangle_s\Big] + 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\alpha_s^2\big|Y_s^\epsilon\big|^2 d\langle M\rangle_s\Big] \\
&\le 4\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big(\varphi_s^2 + \kappa^2\big|Y_s^\epsilon\big|^2\big)d\langle M\rangle_s\Big] + 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\hat{V}_s^\star\Big].
\end{aligned} \tag{18}$$

Similarly, we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|G_\epsilon\big(s, Y_s^\epsilon\big)\big|^2 dA_s\Big] \\
&\le 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|g\big(s, Y_s^\epsilon\big)\big|^2 d\langle M\rangle_s\Big] + 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\beta_s^2\big|Y_s^\epsilon\big|^2 dA_s\Big] \\
&\le 4\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big(\psi_s^2 + \kappa^2\big|Y_s^\epsilon\big|^2\big)dA_s\Big] + 2\mathbb{E}\Big[\int_0^T \chi_s^{\lambda,\theta,\mu}\big|Y_s^\epsilon\big|^2 d\hat{V}_s^\star\Big].
\end{aligned} \tag{19}$$

Step 3: $(Y^\epsilon, Z^\epsilon, N^\epsilon)_{0<\epsilon\le 1}$ *is a Cauchy sequence in* $\mathfrak{D}^2$. Let $0 < \epsilon, \delta \le 1$. From (16), we have

$$Y_t^{\epsilon,\delta} = \int_t^T dK_s^{\epsilon,\delta} - \int_t^T Z_s^{\epsilon,\delta} dM_s - \int_t^T dN_s^{\epsilon,\delta}, \quad 0 \le t \le T,$$

where $Y_t^{\epsilon,\delta} := Y_t^{\epsilon} - Y_t^{\delta}$, $Y_T^{\epsilon,\delta} = 0$, $Z_t^{\epsilon,\delta} := Z_t^{\epsilon} - Z_t^{\delta}$, $dN_t^{\epsilon,\delta} := dN_t^{\epsilon} - dN_t^{\delta}$, and

$$K_t^{\epsilon,\delta} := \int_0^t \big(F_\epsilon(s, Y_s^\epsilon) + \alpha_s Y_s^\epsilon - F_\delta(s, Y_s^\delta) - \alpha_s Y_s^\delta\big) d\langle M\rangle_s + \int_0^t \big(G_\epsilon(s, Y_s^\epsilon) + \beta_s Y_s^\epsilon - G_\delta(s, Y_s^\delta) - \beta_s Y_s^\delta\big) dA_s.$$

In order to apply the Itô formula to the process $\chi_t^{\lambda,\theta,\mu}|Y_t^{\epsilon,\delta}|^2$, we should first estimate the term $Y_s^{\epsilon,\delta} dK_s^{\epsilon,\delta}$. But from **(Y2)**-(vii), we have

$$Y_s^{\epsilon,\delta} dK_s^{\epsilon,\delta} \leq (\epsilon+\delta)\big\{F_\epsilon(s, Y_s^\epsilon) F_\delta(s, Y_s^\delta) d\langle M\rangle_s + G_\epsilon(s, Y_s^\epsilon) G_\delta(s, Y_s^\delta) dA_s\big\} + |\alpha_s| \big|Y_s^\epsilon - Y_s^\delta\big|^2 d\langle M\rangle_s.$$

Using the Kunita–Watanabe inequality (see Corollary on p. 70 in [16]) in conjunction with inequalities (17) and (18), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \chi_s^{\lambda,\theta,\mu} \big|F_\epsilon(s, Y_s^\epsilon) F_\delta(s, Y_s^\delta)\big| d\langle M\rangle_s\right] \\
&\leq \left(\mathbb{E}\int_0^T \chi_s^{\lambda,\theta,\mu} \big|F_\epsilon(s, Y_s^\epsilon)\big|^2 d\langle M\rangle_s\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \chi_s^{\lambda,\theta,\mu} \big|F_\delta(s, Y_s^\delta)\big|^2 d\langle M\rangle_s\right)^{\frac{1}{2}} \\
&\leq \mathfrak{C}_{\lambda,\theta,\mu}.
\end{aligned} \tag{20}$$

In the same way, using (19) and the Hölder inequality, we obtain

$$\mathbb{E}\left[\int_0^T \big|G_\epsilon(s, Y_s^\epsilon) G_\delta(s, Y_s^\delta)\big| dA_s\right] \leq \mathfrak{C}_{\lambda,\theta,\mu}. \tag{21}$$

Combining (20) and (21) yields, for any $t \in [0, T]$,

$$\mathbb{E}\left[\int_t^T \big|Y_s^{\epsilon,\delta}\big| d\big\|K^{\epsilon,\delta}\big\|_s\right] \leq \mathfrak{C}^\star_{\lambda,\theta,\mu}(\epsilon+\delta) + \mathbb{E}\left[\int_t^T \chi_s^{\lambda,\theta,\mu} \big|Y_s^{\epsilon,\delta}\big|^2 d\hat{V}_s^\star\right].$$

Then, following an argument similar to the one used in Proposition 1 or in **Step 2** yields, for any $\epsilon, \delta \in]0, 1]$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T} \chi_t^{\lambda,\theta,\mu} \big|Y_t^{\epsilon,\delta}\big|^2\Big] + \mathbb{E}\left[\int_0^T \chi_s^{\lambda,\theta,\mu} \big|Y_s^{\epsilon,\delta}\big|^2 d\hat{V}_s^\star\right] \\
&\quad + \mathbb{E}\left[\int_0^T \chi_s^{\lambda,\theta,\mu} \big|Y_s^{\epsilon,\delta}\big|^2 dQ_s\right] + \mathbb{E}\left[\int_0^T \chi_s^{\lambda,\theta,\mu} \big|Z_s^{\epsilon,\delta}\big|^2 d\langle M\rangle_s\right] \\
&\quad + \mathbb{E}\left[\int_0^T \chi_s^{\lambda,\theta,\mu} d\big[N^{\epsilon,\delta}\big]_s\right] \\
&\leq \mathfrak{C}(\epsilon+\delta).
\end{aligned}$$

Thus $\{(Y_t^\epsilon, Z_t^\epsilon, N_t^\epsilon); 0 \leq t \leq T\}_{0<\epsilon\leq 1}$ is a Cauchy sequence in the Banach space $\mathfrak{D}^2_{\lambda,\theta,\mu}$. So there exists a triplet $(Y_t, Z_t, N_t)_{t\leq T} \in \mathfrak{D}^2_{\lambda,\theta,\mu}$ such that $(Y^\epsilon, Z^\epsilon, N^\epsilon) \xrightarrow[\epsilon\to 0^+]{\mathfrak{D}^2}$

(Y, Z, N). In particular, letting $\delta \to 0^+$, we get

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|Y_t^\epsilon - Y_t\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Y_s^\epsilon - Y_s\big|^2 d\langle M\rangle_s\Big] + \mathbb{E}\Big[\int_0^T \big|Y_s^\epsilon - Y_s\big|^2 dA_s\Big]$$
$$+\,\mathbb{E}\Big[\int_0^T \big|Z_s^\epsilon - Z_s\big|^2 d\langle M\rangle_s\Big] + \mathbb{E}\Big[\int_0^T d\big[N^\epsilon - N\big]_s\Big] \le \mathfrak{C}\epsilon.$$

Step 4: The limited process $((Y_t, Z_t, N_t)_{t\le T}$ satisfies the GBSDE (13). From the definition of the Yosida approximation, we can write

$$F_\epsilon\big(s, Y_s^\epsilon\big) + \alpha_s Y_s^\epsilon = \mathfrak{f}\big(s, Y_s^\epsilon + \epsilon F_\epsilon\big(s, Y_s^\epsilon\big)\big) - \epsilon\alpha_s F_\epsilon\big(s, Y_s^\epsilon\big),$$
$$G_\epsilon\big(s, Y_s^\epsilon\big) + \beta_s Y_s^\epsilon = g\big(s, Y_s^\epsilon + \epsilon G_\epsilon\big(s, Y_s^\epsilon\big)\big) - \epsilon\beta_s G_\epsilon\big(s, Y_s^\epsilon\big).$$

Due to the Kunita–Watanabe inequality, the uniform estimations (17) and (18) given in **Step 2**, the Cauchy–Schwarz inequality, the fact that $(M_t)_{t\le T}$ is a square integrable martingale and Assumption **(H-M)**-(vi), we have

$$\big(\epsilon\alpha_s F_\epsilon\big(s, Y_s^\epsilon\big), \epsilon F_\epsilon\big(s, Y_s^\epsilon\big)\big)_{\epsilon\in]0,1]} \in \mathbb{L}^{1,2}\big(\Omega\times[0,T]; \mathbb{P}(d\omega)\otimes d\langle M\rangle_s(\omega)\big).$$

For simplicity of notation, we denote $\mathbb{L}^q_{\mathbb{P}\otimes\langle M\rangle} := \mathbb{L}^q(\Omega\times[0,T]; \mathbb{P}(d\omega)\otimes d\langle M\rangle_s(\omega))$ for $q\in\{1,2\}$, where $d\mathbb{P}\otimes d\langle M\rangle$ is the positive measure on $(\Omega\times[0,T], \mathcal{F}\otimes\mathcal{B}([0,T]))$ defined for any $\mathcal{V}\in\mathcal{F}\otimes\mathcal{B}([0,T])$ by $d\mathbb{P}\otimes d\langle M\rangle(\mathcal{V}) := \mathbb{E}[\int_0^T \mathbb{1}_{\mathcal{V}}(\omega, s)d\langle M\rangle_s]$.

Also

$$\epsilon\alpha_s F_\epsilon\big(s, Y_s^\epsilon\big) \xrightarrow[\epsilon\to 0^+]{\mathbb{L}^1_{\mathbb{P}\otimes\langle M\rangle}} 0 \quad\text{and}\quad \epsilon F_\epsilon\big(s, Y_s^\epsilon\big) \xrightarrow[\epsilon\to 0^+]{\mathbb{L}^2_{\mathbb{P}\otimes\langle M\rangle}} 0.$$

Then, applying the partial reciprocal of the dominated convergence theorem in $\mathbb{L}^2_{\mathbb{P}\otimes\langle M\rangle}$, we deduce the existence of two subsequences $(\epsilon_k\alpha_s F_{\epsilon_k}(s, Y_s^{\epsilon_k}))_{k\in\mathbb{N}}$ and $(\epsilon_k F_{\epsilon_k}(s, Y_s^{\epsilon_k}))_{k\in\mathbb{N}}$ such that $\epsilon_k \to 0$ as $k\to\infty$, and

$$\epsilon_k\alpha_s(\omega)F_{\epsilon_k}\big(\omega, s, Y_s^{\epsilon_k}(\omega)\big) \xrightarrow[k\to\infty]{} 0, \qquad \epsilon_k F_{\epsilon_k}\big(\omega, s, Y_s^{\epsilon_k}(\omega)\big) \xrightarrow[k\to\infty]{} 0,$$

$d\mathbb{P}(\omega)\otimes d\langle M\rangle_t(\omega)$-a.e.

Using the continuity of the driver $\mathfrak{f}$ and the fact that $Y_t^{\epsilon_k}(\omega) \xrightarrow[k\to\infty]{} Y_t(\omega)$ as $k\to\infty$, $d\mathbb{P}(\omega)\otimes d\langle M\rangle_t(\omega)$-a.e., we infer by the Lebesgue dominated convergence

$$\mathbb{E}\Big[\int_t^T \big|\alpha_s Y_s^{\epsilon_k} - \alpha_s Y_s\big| d\langle M\rangle_s\Big] \xrightarrow[k\to\infty]{} 0,$$

and

$$\mathbb{E}\Big[\int_t^T \big|\mathfrak{f}\big(s, Y_s^{\epsilon_k} + \epsilon F_\epsilon\big(s, Y_s^{\epsilon_k}\big)\big) - \mathfrak{f}(s, Y_s)\big| d\langle M\rangle_s\Big] \xrightarrow[k\to\infty]{} 0.$$

A similar argument gives

$$\mathbb{E}\Big[\int_t^T \big|\beta_s Y_s^{\epsilon_k} - \beta_s Y_s\big|^2 dA_s + \int_t^T \big|g\big(s, Y_s^{\epsilon_k} + \epsilon G_{\epsilon_k}\big(s, Y_s^{\epsilon_k}\big)\big) - g(s, Y_s)\big|^2 dA_s\Big]$$
$$\xrightarrow[k\to\infty]{} 0.$$

Consequently, $F_{\epsilon_k}(\cdot, Y^{\epsilon_k}_.) + \alpha.Y^{\epsilon_k}_. \xrightarrow[k\to\infty]{\mathbb{L}^1_{\mathbb{P}\otimes\langle M\rangle}} \mathfrak{f}(\cdot, Y_.)$ and $G_{\epsilon_k}(\cdot, Y^{\epsilon_k}_.) + \beta.Y_. \xrightarrow[k\to\infty]{\mathbb{L}^1_{\mathbb{P}\otimes A}} g(\cdot, Y_.)$.

Then, passing to the limit term by term in $\mathbb{L}^1(\Omega)$ as $k \to \infty$ in the approximating equation (16) for a subsequence $\{(Y^{\epsilon_k}_t, Z^{\epsilon_k}_t, N^{\epsilon_k}_t); 0 \le t \le T\}_{k\in\mathbb{N}}$, using the orthogonal property (4) yields

$$Y_t = \xi + \int_t^T \mathfrak{f}(s, Y_s)d\langle M\rangle_s + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s.$$

Henceforth, $(Y_t, Z_t, N_t)_{t\le T} \in \mathfrak{D}^2$ is the unique solution of the generalized BSDE (13). Moreover, using the Fatou lemma, we clearly infer that (Y, Z, N) satisfies (12) and then $(Y, Z, N) \in \mathfrak{D}^2_{\lambda,\theta,\mu}$ for $\lambda > 2$ and $\theta, \mu > 0$. Which completes the proof of Theorem 1. □

We are now prepared to present the paper's main result.

Theorem 2. *Under Assumption (**H-M**), there exists a unique $\mathcal{F}_t$-progressively measurable solution $(Y_t, Z_t, N_t)_{t\le T}$ of the GBSDE (2).*

Proof. All that is left is the proof of existence, which will be made by a fixed point reasoning. To this end, let $(\mathcal{Y}^i, \mathcal{Z}^i, \mathcal{N}^i) \in \mathfrak{D}^2_{\lambda,\theta,\mu}$ for $i = 1, 2$, and define $(Y^i, Z^i, N^i) = \Psi(\mathcal{Y}^i, \mathcal{Z}^i, \mathcal{N}^i)$ where

$$Y^i_t = \xi + \int_t^T f\big(s, Y^i_s, \mathcal{Z}^i_s\big)d\langle M\rangle_s + \int_t^T g\big(s, Y^i_s\big)dA_s - \int_t^T Z^i_s dM_s - \int_t^T dN^i_s.$$

Using the Itô formula, similarly to Proposition 1, taking the expectation and choosing $\lambda > 2$ and $\theta, \mu \ge 1$, it follows that

$$\begin{aligned}&\big\|\Psi\big(\mathcal{Y}^1, \mathcal{Z}^1, \mathcal{N}^1\big) - \Psi\big(\mathcal{Y}^2, \mathcal{Z}^2, \mathcal{N}^2\big)\big\|^2_{\lambda,\theta,\mu}\\ &\le \frac{1}{2}\big\|\big(\mathcal{Y}^1 - \mathcal{Y}^2, \mathcal{Z}^1 - \mathcal{Z}^2, \mathcal{N}^1 - \mathcal{N}^2\big)\big\|^2_{\lambda,\theta,\mu}.\end{aligned}$$

Hence, Ψ is a strict contraction on the Banach space $\mathfrak{D}^2_{\lambda,\theta,\mu}$ equipped with the norm $\|\cdot\|^2_{\lambda,\theta,\mu}$ provided that $\lambda > 2$ and $\theta, \mu \ge 1$, and by the Banach fixed point theorem we conclude that Ψ has a unique fixed point $(Y_t, Z_t, N_t)_{t\le T}$, which solves (2). □

A Appendix

In this section, we will prove that the GBSDE (16) has a unique solution in $\mathfrak{D}^2$.

The remark that follows proves that investigating the generalized BSDE (16) is equivalent to studying an analogous kind of equation with a particular Lipschitz generators.

Remark 3. Set $\varpi_t = \int_0^t \alpha_s d\langle M\rangle_s + \int_0^t \beta_s dA_s$. Obviously, $(\varpi_t)_{t\le T}$ is a continuous process with bounded variations over $[0, T]$. Let $(Y^\epsilon_t, Z^\epsilon_t, N^\epsilon_t)_{t\le T}$ be a solution of the GBSDE (16) associated with $(\xi, F_\epsilon(t,y)+\alpha_t y, G_\epsilon(t,y)+\beta_t y, A_t)_{t\le T}$, then the integration by part formula (Corollary 2 in [16, p. 68]) yields

$$e^{\varpi_t}Y_t^\epsilon = e^{\varpi_T}\xi + \int_t^T e^{\varpi_s}F_\epsilon\big(s,Y_s^\epsilon\big)d\langle M\rangle_s + \int_t^T e^{\varpi_s}G_\epsilon\big(s,Y_s^\epsilon\big)dA_s$$
$$-\int_t^T e^{\varpi_s}Z_s^\epsilon dM_s - \int_t^T e^{\varpi_s}dN_s^\epsilon, \quad 0\le t\le T.$$

Henceforth, the process $(Y_t^\epsilon, Z_t^\epsilon, N_t^\epsilon)_{t\le T}$ is a solution of the GBSDE (16) if and only if the process $(\bar{Y}_t^\epsilon, \bar{Z}_t^\epsilon, d\bar{N}_t^\epsilon)_{t\le T} := (e^{\varpi_t}Y_t^\epsilon, e^{\varpi_t}Z_t^\epsilon, e^{\varpi_t}dN_t^\epsilon)$ satisfies an analogous GBSDE, with ξ, $(F_\epsilon(t,y)+\alpha_t y)_{t\le T}$ and $(G_\epsilon(t,y)+\beta_t y)_{t\le T}$ replaced by

$$\bar{\xi} := e^{\varpi_T}\xi, \quad \bar{F}_\epsilon(s,y) := e^{\varpi_s}F_\epsilon\big(s,e^{-\varpi_s}y\big) \quad \text{and} \quad \bar{G}_\epsilon(s,y) := e^{\varpi_s}G_\epsilon\big(s,e^{-\varpi_s}y\big).$$

Hence, it suffices to show the existence and uniqueness result for the GBSDE

$$\bar{Y}_t^\epsilon = \bar{\xi} + \int_t^T \bar{F}_\epsilon\big(s,\bar{Y}_s^\epsilon\big)d\langle M\rangle_s + \int_t^T \bar{G}_\epsilon\big(s,\bar{Y}_s^\epsilon\big)dA_s - \int_t^T \bar{Z}_s^\epsilon dM_s - \int_t^T d\bar{N}_s^\epsilon.$$

Moreover, the drivers $\bar{F}_\epsilon$ and $\bar{G}_\epsilon$ verify the following properties, $\forall\epsilon>0$, $\forall t\in[0,T]$, $\forall y,y'\in\mathbb{R}$, $\mathbb{P}$-a.s.

- *Lipschitz property:* from **(Y2)**-(iv), we get

$$\big|\bar{F}_\epsilon(s,y)-\bar{F}_\epsilon\big(s,y'\big)\big| + \big|\bar{G}_\epsilon(s,y)-\bar{G}_\epsilon\big(s,y'\big)\big| \le \frac{2}{\epsilon}\big|y-y'\big|.$$

- Linear growth of $\bar{F}_\epsilon$ and $\bar{G}_\epsilon$: using **(Y2)**-(v), we obtain

$$\big|\bar{F}_\epsilon(s,0)\big| = e^{\varpi_s}\big|F_\epsilon(s,0)\big| \le e^{\varpi_s}\big|F(s,0)\big| \le e^{\varpi_s}\varphi_s \le e^{V_s}\varphi_s;$$
$$\big|\bar{G}_\epsilon(s,0)\big| = e^{\varpi_s}\big|G_\epsilon(s,0)\big| \le e^{\varpi_s}\big|G(s,0)\big| \le e^{\varpi_s}\psi_s \le e^{V_s}\psi_s.$$

Now consider the GBSDE

$$Y_t = \xi + \int_t^T h(s,Y_s)d\langle M\rangle_s + \int_t^T H(s,Y_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad (22)$$

where h and H are κ-Lipschitz and verifies the integrability condition

$$\mathbb{E}\bigg[\int_0^T e^{\theta Q_t}\big|h(t,0)\big|^2 d\langle M\rangle_t + \mathbb{E}\int_0^T e^{\theta Q_t}\big|H(t,0)\big|^2 dA_t\bigg] < \infty, \quad \theta>0.$$

We are now in position to state the main theorem of this section.

Theorem 3. *The GBSDE* (22) *has a unique* $\mathcal{F}_t$*-progressively measurable solution* $(Y_t,Z_t,N_t)_{t\le T}$ *which belongs to* $\mathfrak{D}^2$.

Proof. Uniqueness. Follows from Proposition 1.

Existence. The proof is divided into two stages: in the first part, we examine the case when the drivers h and H are independent of y, then we give the existence result that will be needed in the second part to deduce the general case via a fixed point argument.

Part 1: The maps h and H does not depend on y.

Let $(X_t, W_t, O_t)_{t\leq T}$ be in $\mathfrak{D}^2_{0,\theta,\theta}$ such that

$$\mathbb{E}\left[\int_0^T e^{\theta Q_s}|X_s|^2\big(d\langle M\rangle_s + dA_s\big)\right] < \infty.$$

Then we define the processes $(Y_t, Z_t, N_t)_{t\leq T}$ as follows. First,

$$\begin{aligned} Y_t &= \mathbb{E}^{\mathcal{F}_t}\left[\xi + \int_0^T h(s, X_s)d\langle M\rangle_s + \int_0^T H(s, X_s)dA_s\right] \\ &\quad - \int_0^t h(s, X_s)d\langle M\rangle_s - \int_0^t H(s, X_s)dA_s \\ &=: m_t - \int_0^t h\big(s, X_s\big)d\langle M\rangle_s - \int_0^t H(s, X_s)dA_s. \end{aligned}$$

Note that

$$\begin{aligned} &\mathbb{E}\left[\left|\xi + \int_0^T h(s, X_s)d\langle M\rangle_s + \int_0^T H(s, X_s)dA_s\right|^2\right] \\ &\leq 3\mathbb{E}\left[|\xi|^2 + \left|\int_0^T h(s, X_s)d\langle M\rangle_s\right|^2 + \left|\int_0^T H(s, X_s)dA_s\right|^2\right]. \end{aligned}$$

From the sharp bracket version of the Kunita–Watanabe inequality (see p. 148 in [16]) and **(H-M)**-(vi), we obtain

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T h(s, X_s)d\langle M\rangle_s\right|^2\right] \\ &\quad\leq \mathbb{E}\left[\int_0^T e^{-\theta\langle M\rangle_s}d\langle M\rangle_s \int_0^T e^{\theta\langle M\rangle_s}\big|h(s, X_s)\big|^2 d\langle M\rangle_s\right] \\ &\quad\leq \frac{2\kappa^2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta\langle M\rangle_s}|X_s|^2 d\langle M\rangle_s\right] + \frac{2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta\langle M\rangle_s}\big|h(s, 0)\big|^2 d\langle M\rangle_s\right] \\ &\quad< \infty. \end{aligned}$$

Similarly, the Hölder inequality and **(H-M)**-(vi) imply

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T H(s, X_s)dA_s\right|^2\right] \\ &\quad\leq \frac{2\kappa^2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta A_s}|X_s|^2 dA_s\right] + \frac{2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta A_s}\big|H(s, 0)\big|^2 dA_s\right] < \infty. \end{aligned}$$

Thus, $(m_t)_{t\leq T}$ is a square-integrable $\mathbb{F}$-martingale. Further, from the fact that $\mathcal{F}_0$ is trivial and the predictable representation property of square-integrable martingales (see Remark 2.1 in [11, p. 323]), there exists a couple of processes $(Z_t, N_t)_{t\leq T} \in \mathcal{H}^2 \times \mathcal{M}^2$ such that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left[\xi + \int_0^T h(s, X_s)d\langle M\rangle_s + \int_0^T H(s, X_s)dA_s\right] \\
&\quad = \mathbb{E}\left[\xi + \int_0^T h(s, X_s)d\langle M\rangle_s + \int_0^T H(s, X_s)dA_s\right] + \int_0^t Z_s dM_s + \int_0^t dN_s.
\end{aligned}$$

In other words, the triplet $(Y_t, Z_t, N_t)_{t\leq T}$ is the unique solution of the GBSDE

$$Y_t = \xi + \int_t^T h(s, X_s)d\langle M\rangle_s + \int_t^T H(s, X_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s.$$

The fact that $(Y, Z, N) \in \mathfrak{D}^2_{0,\theta,\theta}$ for $\theta > 0$, follows from computations similar to those in the proof of Proposition 1.

Next, we endow the space $\mathfrak{D}^2_{0,\theta,\theta}$ with the norm

$$\|(Y, Z, N)\|_\theta = \left(\mathbb{E}\left[\int_0^T e^{\theta Q_s}\left\{|Y_s|^2 dQ_s + |Z_s|^2 d\langle M\rangle_s + d[N]_s\right\}\right]\right)^{\frac{1}{2}},$$

for some $\theta > 0$.

Part 2: General case.

Using the first part of the current proof, we may define a map Ψ from $\mathfrak{D}^2_{0,\theta,\theta}$ into itself as follows:

$$\begin{aligned}
&(Y, Z, N) = \Psi(X, W, O), \\
&Y_t = \xi + \int_t^T h(s, X_s)d\langle M\rangle_s + \int_t^T H(s, X_s)dA_s - \int_t^T Z_s dM_s - \int_t^T dN_s.
\end{aligned}$$

Let $\Phi(X', W', O') = (Y', Z', N')$ and $\bar{\Re} = \Re - \Re'$ for $\Re = Y, Z, N, X, W$ and O. In order to apply the Itô formula, we need the estimation

$$\begin{aligned}
2\bar{Y}_s\big(h(s, X_s) - h\big(s, X'_s\big)\big)d\langle M\rangle_s &\leq 2\kappa\big|\bar{Y}_s\big|\big|\bar{X}_s\big|d\langle M\rangle_s \\
&\leq 2\kappa^2|\bar{Y}_s|^2 d\langle M\rangle_s + \frac{1}{2}|\bar{X}_s|^2 d\langle M\rangle_s.
\end{aligned} \tag{23}$$

Following the same procedure, we get

$$2\bar{Y}_s\big(H(s, X_s) - H\big(s, X'_s\big)\big)dA_s \leq 2\kappa^2|\bar{Y}_s|^2 dA_s + \frac{1}{2}|\bar{X}_s|^2 dA_s. \tag{24}$$

Combining (23) and (24), we deduce

$$\begin{aligned}
&2\bar{Y}_s\big(h(s, X_s) - h\big(s, X'_s\big)\big)d\langle M\rangle_s + 2\bar{Y}_s\big(H(s, X_s) - H\big(s, X'_s\big)\big)dA_s \\
&\quad \leq 2\kappa^2|\bar{Y}_s|^2 dQ_s + \frac{1}{2}|\bar{X}_s|^2 dQ_s.
\end{aligned}$$

Using this with the Itô formula (7), we clearly infer for $\theta = 2\kappa^2 + 1$, that

$$\left\| \Psi(X, W, O) - \Psi\big(X', W', O'\big) \right\|_\theta^2 \leq \frac{1}{2} \left\| (X, W, O) - \big(X', W', O'\big) \right\|_\theta^2.$$

Hence Ψ is a strict contraction on the Banach space $\mathfrak{D}^2_{0,\theta,\theta}$, equipped with the norm $\| \cdot \|_\theta$ provided that $\theta = 2\kappa^2 + 1$, and by the Banach fixed point theorem we conclude that Ψ has a unique fixed point (Y, Z, N) which solves (22). This completes the proof of Theorem 3. □

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References

[1] Barles, G., Buckdahn, R., Pardoux, E.: Backward stochastic differential equations and integral-partial differential equations. Stoch. Int. J. Probab. Stoch. Process. **60**(1-2), 57–83 (1997). MR1436432. https://doi.org/10.1080/17442509708834099

[2] Bender, C., Kohlmann, M.: BSDEs with stochastic Lipschitz condition. Technical report, CoFE Discussion Paper (2000). https://www.econstor.eu/bitstream/10419/85163/1/dp00-08.pdf

[3] Bismut, J.M.: Conjugate convex functions in optimal stochastic control. J. Math. Anal. Appl. **44**(2), 384–404 (1973). MR0329726. https://doi.org/10.1016/0022-247X(73)90066-8

[4] Carbone, R., Ferrario, B., Santacroce, M.: Backward stochastic differential equations driven by càdlàg martingales. Theory Probab. Appl. **52**(2), 304–314 (2008). MR2742510. https://doi.org/10.1137/S0040585X97983055

[5] El Karoui, N., Huang, S.: A general result of existence and uniqueness of backward stochastic differential equations. In: Backward Stochastic Differential Equations (Paris, 1995–1996). Pitman Research Notes in Mathematics Series, vol. 364, pp. 27–38 (1997). MR1752673

[6] El Otmani, M.: Generalized BSDE driven by a Lévy process. J. Appl. Math. Stoch. Anal. **2006**, 85407 (2006). MR2253532. https://doi.org/10.1155/JAMSA/2006/85407

[7] Hu, Y.: On the solution of forward–backward sdes with monotone and continuous coefficients. Nonlinear Anal. **42**(1), 1–12 (2000). MR1769248. https://doi.org/10.1016/S0362-546X(98)00315-0

[8] Hu, Y., Peng, S.: Solution of forward-backward stochastic differential equations. Probab. Theory Relat. Fields **103**, 273–283 (1995). MR1355060. https://doi.org/10.1007/BF01204218

[9] Jacod, J., Shiryaev, A.: Limit Theorems for Stochastic Processes, vol. 288. Springer, Berlin (2013). MR0959133. https://doi.org/10.1007/978-3-662-02514-7

[10] Nie, T., Rutkowski, M.: Bsdes driven by multidimensional martingales and their applications to markets with funding costs. Theory Probab. Appl. **60**(4), 604–630 (2016). MR3583450. https://doi.org/10.1137/S0040585X97T987880

[11] Nie, T., Rutkowski, M.: Existence, uniqueness and strict comparison theorems for BSDEs driven by RCLL martingales. Probab. Uncertain. Quant. Risk **6**(4), 319–342 (2021). MR4399893. https://doi.org/10.3934/puqr.2021016

[12] Pardoux, E.: Generalized discontinuous backward stochastic differential equations. In: Backward Stochastic Differential Equations (Paris, 1995–1996). Pitman Research Notes in Mathematics Series, vol. 364, pp. 207–219 (1997). MR1752684. https://doi.org/10.1016/S0377-0427(97)00124-6

[13] Pardoux, E., Peng, S.: Adapted solution of a backward stochastic differential equation. Syst. Control Lett. **14**(1), 55–61 (1990). MR1037747. https://doi.org/10.1016/0167-6911(90)90082-6

[14] Pardoux, E., Râşcanu, A.: Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, vol. 69. Springer, Switzerland (2014). MR3308895. https://doi.org/10.1007/978-3-319-05714-9

[15] Pardoux, E., Zhang, S.: Generalized BSDEs and nonlinear Neumann boundary value problems. Probab. Theory Relat. Fields **110**, 535–558 (1998). MR1626963. https://doi.org/10.1007/s004400050158

[16] Protter, P.E.: Stochastic Integration and Differential Equations. Applications of Mathematics, 2nd edn. Springer, Berlin (2004). MR2020294

[17] Situ, R.: On solutions of backward stochastic differential equations with jumps and applications. Stoch. Process. Appl. **66**(2), 209–236 (1997). MR1440399. https://doi.org/10.1016/S0304-4149(96)00120-2

[18] Tang, S., Li, X.: Necessary conditions for optimal control of stochastic systems with random jumps. SIAM J. Control Optim. **32**(5), 1447–1475 (1994). MR1288257. https://doi.org/10.1137/S0363012992233858