Modern Stochastics: Theory and Applications 11 (2) (2024) 133–148
https://doi.org/10.15559/23-VMSTA241

Arithmetic properties of multiplicative integer-valued perturbed random walks

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Received: 4 October 2023, Revised: 12 December 2023, Accepted: 12 December 2023, Published online: 4 January 2024

Abstract Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be independent identically distributed $\mathbb{N}^2$-valued random vectors with arbitrarily dependent components. The sequence $(\Theta_k)_{k\in\mathbb{N}}$ defined by $\Theta_k = \Pi_{k-1} \cdot \eta_k$, where $\Pi_0 = 1$ and $\Pi_k = \xi_1 \cdot \ldots \cdot \xi_k$ for $k \in \mathbb{N}$, is called a multiplicative perturbed random walk. Arithmetic properties of the random sets $\{\Pi_1, \Pi_2, \ldots, \Pi_k\} \subset \mathbb{N}$ and $\{\Theta_1, \Theta_2, \ldots, \Theta_k\} \subset \mathbb{N}$, $k \in \mathbb{N}$, are studied. In particular, distributional limit theorems for their prime counts and for the least common multiple are derived.

Keywords Least common multiple, multiplicative perturbed random walk, prime counts
2010 MSC 11A05, 60F05, 11K65

1 Introduction

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be independent copies of an $\mathbb{N}^2$-valued random vector (ξ, η) with arbitrarily dependent components. Denote by $(\Pi_k)_{k\in\mathbb{N}_0}$ (as usual, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$) the standard multiplicative random walk defined by

$$\Pi_0 := 1, \quad \Pi_k = \xi_1 \cdot \xi_2 \cdots \xi_k, \quad k \in \mathbb{N}.$$

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A *multiplicative perturbed random walk* is the sequence $(\Theta_k)_{k\in\mathbb{N}}$ given by

$$\Theta_k := \Pi_{k-1} \cdot \eta_k, \quad k \in \mathbb{N}.$$

Note that if $\mathbb{P}\{\eta = \xi\} = 1$, then $\Pi_k = \Theta_k$ for all $k \in \mathbb{N}$. If $\mathbb{P}\{\xi = 1\} = 1$, then $(\Theta_k)_{k\in\mathbb{N}}$ is just a sequence of independent copies of a random variable η. In this article we investigate some arithmetic properties of the random sets $(\Pi_k)_{k\in\mathbb{N}}$ and $(\Theta_k)_{k\in\mathbb{N}}$.

To set the scene, we introduce first some necessary notation. Let $\mathcal{P}$ denote the set of prime numbers. For an integer $n \in \mathbb{N}$ and $p \in \mathcal{P}$, let $\lambda_p(n)$ denote the multiplicity of prime p in the prime decomposition of n, that is,

$$n = \prod_{p\in\mathcal{P}} p^{\lambda_p(n)}.$$

For every $p \in \mathcal{P}$, the function $\lambda_p : \mathbb{N} \mapsto \mathbb{N}_0$ is totally additive in the sense that

$$\lambda_p(mn) = \lambda_p(m) + \lambda_p(n), \quad p \in \mathcal{P}, \quad m, n \in \mathbb{N}.$$

The set of functions $(\lambda_p)_{p\in\mathcal{P}}$ is a basic brick from which many other arithmetic functions can be constructed. For example, with GCD (A) and LCM (A) denoting the greatest common divisor and the least common multiple of a set $A \subset \mathbb{N}$, respectively, we have

$$\text{GCD}\,(A) = \prod_{p\in\mathcal{P}} p^{\min_{n\in A} \lambda_p(n)} \quad \text{and} \quad \text{LCM}\,(A) = \prod_{p\in\mathcal{P}} p^{\max_{n\in A} \lambda_p(n)}.$$

The listed arithmetic functions applied either to $A = \{\Pi_1, \dots, \Pi_n\}$ or $A = \{\Theta_1, \dots, \Theta_n\}$ are the main objects of investigation in the present paper. From the additivity of λ_p we infer

$$S_k(p) := \lambda_p(\Pi_k) = \sum_{j=1}^{k} \lambda_p(\xi_j), \quad p \in \mathcal{P}, \quad k \in \mathbb{N}_0, \tag{1}$$

and

$$T_k(p) := \lambda_p(\Theta_k) = \sum_{j=1}^{k-1} \lambda_p(\xi_j) + \lambda_p(\eta_k), \quad p \in \mathcal{P}, \quad k \in \mathbb{N}. \tag{2}$$

Fix any $p \in \mathcal{P}$. Formulae (1) and (2) demonstrate that $S(p) := (S_k(p))_{k\in\mathbb{N}_0}$ is a standard additive random walk with the generic step $\lambda_p(\xi)$, whereas the sequence $T(p) := (T_k(p))_{k\in\mathbb{N}}$ is a particular instance of an *additive perturbed random walk*, see [6], generated by the pair $(\lambda_p(\xi), \lambda_p(\eta))$.

2 Main results

2.1 Distributional properties of the prime counts $(\lambda_p(\xi), \lambda_p(\eta))$

As is suggested by (1) and (2) the first step in the analysis of $S(p)$ and $T(p)$ should be the derivation of the joint distribution $(\lambda_p(\xi), \lambda_p(\eta))_{p\in\mathcal{P}}$. The next lemma confirms that the finite-dimensional distributions of the infinite vector $(\lambda_p(\xi), \lambda_p(\eta))_{p\in\mathcal{P}}$, are expressible via the probability mass function of (ξ, η). However, the obtained formulae are not easy to handle except some special cases. For $i, j \in \mathbb{N}$, put

$$u_i := \mathbb{P}\{\xi = i\}, \quad v_j := \mathbb{P}\{\eta = j\}, \quad w_{i,j} := \mathbb{P}\{\xi = i, \eta = j\}.$$

Lemma 1. *Fix $p \in \mathcal{P}$ and nonnegative integers $(k_q)_{q\in\mathcal{P},q\le p}$ and $(\ell_q)_{q\in\mathcal{P},q\le p}$. Then*

$$\mathbb{P}\big\{\lambda_q(\xi) \ge k_q, \lambda_q(\eta) \ge \ell_q, q \in \mathcal{P}, q \le p\big\} = \sum_{i,j=1}^{\infty} w_{Ki,Lj},$$

where $K := \prod_{q\le p,q\in\mathcal{P}} q^{k_q}$ and $L := \prod_{q\le p,q\in\mathcal{P}} q^{\ell_q}$.

Proof. This follows from

$$\begin{aligned}
&\mathbb{P}\big\{\lambda_q(\xi) \ge k_q, \lambda_q(\eta) \ge \ell_q, q \in \mathcal{P}, q \le p\big\} \\
&= \mathbb{P}\Bigg\{\prod_{q\le p,q\in\mathcal{P}} q^{k_q} \text{ divides } \xi, \prod_{q\le p,q\in\mathcal{P}} q^{\ell_q} \text{ divides } \eta\Bigg\} = \sum_{i,j=1}^{\infty} w_{Ki,Lj}.
\end{aligned}$$

Obviously, if ξ and η are independent, then

$$\sum_{i,j=1}^{\infty} w_{Ki,Lj} = \Bigg(\sum_{i=1}^{\infty} u_{Ki}\Bigg)\Bigg(\sum_{j=1}^{\infty} v_{Lj}\Bigg). \qquad \square$$

We proceed with the series of examples.

Example 1. For $\alpha > 1$, let $\mathbb{P}\{\xi = k\} = (\zeta(\alpha))^{-1}k^{-\alpha}$, $k \in \mathbb{N}$, where ζ is the Riemann zeta-function. For $k \in \mathbb{N}$, $p_1, \dots, p_k \in \mathcal{P}$ and $j_1, \dots, j_k \in \mathbb{N}_0$ we have

$$\begin{aligned}
&\mathbb{P}\big\{\lambda_{p_1}(\xi) \ge j_1, \dots, \lambda_{p_k}(\xi) \ge j_k\big\} = \mathbb{P}\big\{p_1^{j_1} \cdots p_k^{j_k} \text{ divides } \xi\big\} \\
&= \sum_{i=1}^{\infty} \mathbb{P}\big\{\xi = \big(p_1^{j_1} \cdots p_k^{j_k}\big)i\big\} = \big(p_1^{j_1} \cdots p_k^{j_k}\big)^{-\alpha} = p_1^{-\alpha j_1} \cdots p_k^{-\alpha j_k}.
\end{aligned}$$

Thus, $(\lambda_p(\xi))_{p\in\mathcal{P}}$ are mutually independent and $\lambda_p(\xi)$ has a geometric distribution on $\mathbb{N}_0$ with parameter $p^{-\alpha}$, for every fixed $p \in \mathcal{P}$.

Example 2. For $\beta \in (0, 1)$, let $\mathbb{P}\{\xi = k\} = \beta^{k-1}(1 - \beta)$, $k \in \mathbb{N}$. Then

$$\mathbb{P}\big\{\lambda_p(\xi) \ge k\big\} = \frac{1-\beta}{\beta}\sum_{j=1}^{\infty} \beta^{p^k j} = \frac{(1-\beta)(\beta^{p^k-1})}{1-\beta^{p^k}}, \quad k \in \mathbb{N}_0.$$

Example 3. Let $\mathrm{Poi}(\lambda)$ be a random variable with the Poisson distribution with parameter λ and put

$$\mathbb{P}\{\xi = k\} = \mathbb{P}\big\{\mathrm{Poi}(\lambda) = k|\mathrm{Poi}(\lambda) \ge 1\big\} = \big(e^{\lambda} - 1\big)^{-1}\lambda^k/k!, \quad k \in \mathbb{N}.$$

Then

$$\mathbb{P}\{\lambda_p(\xi) \geq k\} = (e^{\lambda} - 1)^{-1} \sum_{j=1}^{\infty} \lambda^{p^k j}/(p^k j)!$$
$$= \left({}_0F_{p^k}\left(; \frac{1}{p^k}, \frac{2}{p^k}, \dots, \frac{p^k - 1}{p^k}; \left(\frac{\lambda}{p^k}\right)^{p^k}\right) - 1\right), \tag{3}$$

where ${}_0F_{p^k}$ is the generalized hypergeometric function, see Chapter 16 in [10].

In all examples above, the distribution of $\lambda_p(\xi)$ for every fixed $p \in \mathcal{P}$ is extremely light-tailed. It is not that difficult to construct 'weird' distributions where all $\lambda_p(\xi)$ have infinite expectations.

Example 4. Let $(g_p)_{p\in\mathcal{P}}$ be any probability distribution supported by $\mathcal{P}$, $g_p > 0$, and $(t_k)_{k\in\mathbb{N}_0}$ any probability distribution on $\mathbb{N}$ such that $\sum_{k=1}^{\infty} kt_k = \infty$ and $t_k > 0$. Define a probability distribution $\mathfrak{h}$ on $\mathcal{Q} := \bigcup_{p\in\mathcal{P}}\{p, p^2, \dots\}$ by

$$\mathfrak{h}(\{p^k\}) = g_p t_k, \quad p \in \mathcal{P}, \quad k \in \mathbb{N}.$$

If ξ is a random variable with distribution $\mathfrak{h}$, then

$$\mathbb{P}\{\lambda_p(\xi) \geq k\} = g_p \sum_{j=k}^{\infty} t_j, \quad k \in \mathbb{N}, \quad p \in \mathcal{P},$$

which implies $\mathbb{E}[\lambda_p(\xi)] = g_p \sum_{k=1}^{\infty} kt_k = \infty$, $p \in \mathcal{P}$.

This example can be modified by taking $g := \sum_{p\in\mathcal{P}} g_p < 1$ and charging all points of $\mathbb{N} \setminus \mathcal{Q}$ (this set contains 1 and all integers having at least two different prime factors) with arbitrary positive masses of the total weight $1 - g$. The obtained probability distribution charges all points of $\mathbb{N}$ and still possesses the property that all λ_p's have infinite expectations.

Let X be a random variable taking values in $\mathbb{N}$. Since

$$\log X = \sum_{p\in\mathcal{P}} \lambda_p(X) \log p,$$

we conclude that $\mathbb{E}[(\lambda_p(X))^k] < \infty$, for all $p \in \mathcal{P}$, whenever $\mathbb{E}[\log^k X] < \infty$, $k \in \mathbb{N}$. It is also clear that the converse implication is false in general. However, when $k = 1$ the inequality $\mathbb{E}[\log X] < \infty$ is in fact equivalent to $\sum_{p\in\mathcal{P}} \mathbb{E}[\lambda_p(X)] \log p < \infty$. As we have seen in the above examples, checking that $\mathbb{E}[(\lambda_p(X))^k] < \infty$ might be a much more difficult task than proving a stronger assumption $\mathbb{E}[\log^k X] < \infty$. Thus, we shall mostly work under moment conditions on $\log \xi$ and $\log \eta$.

Our standing assumption throughout the article is

$$\mu_\xi := \mathbb{E}[\log \xi] < \infty, \tag{4}$$

which, by the above reasoning, implies $\mathbb{E}[\lambda_p(\xi)] < \infty$, $p \in \mathcal{P}$.

2.2 Limit theorems for $S(p)$ and $T(p)$

From Donsker's invariance principle we immediately obtain the following proposition. Let $D := D([0,\infty),\mathbb{R})$ be the Skorokhod space endowed with the standard J_1-topology.

Proposition 1. *Assume that* $\mathbb{E}[\log^2 \xi] \in (0,\infty)$. *Then,*

$$\left(\left(\frac{S_{\lfloor ut\rfloor}(p) - ut\mathbb{E}[\lambda_p(\xi)]}{\sqrt{t}}\right)_{u\geq 0}\right)_{p\in\mathcal{P}} \Longrightarrow \left(\left(W_p(u)\right)_{u\geq 0}\right)_{p\in\mathcal{P}}, \quad t\to\infty,$$

on the product space $D^{\mathbb{N}}$, *where, for all* $n \in \mathbb{N}$ *and all* $p_1 < p_2 < \cdots < p_n$, $p_i \in \mathcal{P}$, $i \leq n$, $((W_{p_1}(u))_{u\geq 0},\ldots,(W_{p_n}(u))_{u\geq 0})$ *is an* n*-dimensional centered Wiener process with covariance matrix* $C = \|C_{i,j}\|_{1\leq i,j\leq n}$ *given by* $C_{i,j} = C_{j,i} = \mathrm{Cov}\,(\lambda_{p_i}(\xi), \lambda_{p_j}(\xi))$.

According to the proof of Proposition 1.3.13 in [6], see pp. 28–29 therein, the following holds true for the perturbed random walks $T(p)$, $p\in\mathcal{P}$.

Proposition 2. *Assume that* $\mathbb{E}[\log^2 \xi] \in (0,\infty)$ *and*

$$\lim_{t\to\infty} t^2\mathbb{P}\big\{\lambda_p(\eta) \geq t\big\} = 0, \quad p\in\mathcal{P}. \tag{5}$$

Then,

$$\left(\left(\frac{T_{\lfloor ut\rfloor}(p) - ut\mathbb{E}[\lambda_p(\xi)]}{\sqrt{t}}\right)_{u\geq 0}\right)_{p\in\mathcal{P}} \Longrightarrow \left(\left(W_p(u)\right)_{u\geq 0}\right)_{p\in\mathcal{P}}, \quad t\to\infty,$$

on the product space $D^{\mathbb{N}}$.

Remark 1. Since $\mathbb{P}\{\lambda_p(\eta)\log p \geq t\} \leq \mathbb{P}\{\log\eta \geq t\}$, the condition

$$\lim_{t\to\infty} t^2\mathbb{P}\{\log\eta \geq t\} = 0 \tag{6}$$

is clearly sufficient for (5).

From the continuous mapping theorem under the assumptions of Proposition 2 we infer

$$\left(\left(\frac{\max_{1\leq k\leq\lfloor ut\rfloor}(T_k(p) - k\mathbb{E}[\lambda_p(\xi)])}{\sqrt{t}}\right)_{u\geq 0}\right)_{p\in\mathcal{P}} \Longrightarrow \left(\left(\sup_{0\leq v\leq u} W_p(v)\right)_{u\geq 0}\right)_{p\in\mathcal{P}}, \quad t\to\infty, \tag{7}$$

see Proposition 1.3.13 in [6].

Formula (7), for a fixed $p\in\mathcal{P}$, belongs to the realm of limit theorems for the maximum of a single additive perturbed random walk. This circle of problems is well-understood, see Section 1.3.3 in [6] and [7], in the situation when the underlying additive standard random walk is *centered* and attracted to a stable Lévy process. In our setting the perturbed random walks $(T_k(p))_{k\in\mathbb{N}}$ and $(T_k(q))_{k\in\mathbb{N}}$ are dependent whenever $p,q\in\mathcal{P}$, $p\neq q$, which make derivation of the joint limit theorems harder and leads to various asymptotic regimes.

Note that (5) implies $\mathbb{E}[\lambda_p(\eta)] < \infty$ and (6) implies $\mathbb{E}[\log \eta] < \infty$. Theorem 5 below tells us that under such moment conditions and assuming also $\mathbb{E}[\log^2 \xi] < \infty$ the maxima $\max_{1\le k\le n} T_k(p)$, $p \in \mathcal{P}$, of *noncentered* perturbed random walks $T(p)$ have the same behavior as $S_n(p)$, $p \in \mathcal{P}$ as $n \to \infty$.

Theorem 5. *Assume that* $\mathbb{E}[\log^2 \xi] < \infty$ *and* $\mathbb{E}[\lambda_p(\eta)] < \infty$, $p \in \mathcal{P}$. *Suppose further that*

$$\mathbb{P}\{\xi \text{ is divisible by } p\} = \mathbb{P}\big\{\lambda_p(\xi) > 0\big\} > 0, \quad p \in \mathcal{P}. \tag{8}$$

Then, as $t \to \infty$,

$$\left(\left(\frac{\max_{1\le k\le \lfloor tu\rfloor} T_k(p) - \mathbb{E}[\lambda_p(\xi)]tu}{t^{1/2}}\right)_{u\ge 0}\right)_{p\in\mathcal{P}} \overset{\text{f.d.d.}}{\longrightarrow} \big(\big(W_p(u)\big)_{u\ge 0}\big)_{p\in\mathcal{P}}. \tag{9}$$

Moreover, if also (5) *holds for all* $p \in \mathcal{P}$, *then* (9) *holds on the product space* $D^{\mathbb{N}}$.

Remark 2. If (8) holds only for some $\mathcal{P}_0 \subseteq \mathcal{P}$, then (9) holds with $\mathcal{P}_0$ instead of $\mathcal{P}$.

In the next result we shall assume that η dominates ξ in a sense that the asymptotic behavior of $\max_{1\le k\le n} T_k(p)$ is regulated by the perturbations $(\lambda_p(\eta_k))_{k\le n}$ for all $p \in \mathcal{P}_0$, where $\mathcal{P}_0$ is a finite subset of prime numbers and those p's dominate all other primes.

Theorem 6. *Assume* (4). *Suppose further that there exists a finite set* $\mathcal{P}_0 \subseteq \mathcal{P}$, $d := |\mathcal{P}_0|$, *such that the distributional tail of* $(\lambda_p(\eta))_{p\in\mathcal{P}_0}$ *is regularly varying at infinity in the following sense. For some positive function* $(a(t))_{t>0}$ *and a measure* ν *satisfying* $\nu(\{x \in \mathbb{R}^d : \|x\| \ge r\}) = c \cdot r^{-\alpha}$, $c > 0$, $\alpha \in (0, 1)$, *it holds*

$$t\mathbb{P}\big\{\big(a(t)\big)^{-1}\big(\lambda_p(\eta)\big)_{p\in\mathcal{P}_0} \in \cdot\big\} \overset{\text{v}}{\longrightarrow} \nu(\cdot), \quad t \to \infty, \tag{10}$$

on the space of locally finite measures on $(0, \infty]^d$ *endowed with the vague topology. Then*

$$\left(\left(\frac{\max_{1\le k\le \lfloor tu\rfloor} T_k(p)}{a(t)}\right)_{u\ge 0}\right)_{p\in\mathcal{P}_0} \overset{\text{f.d.d.}}{\longrightarrow} \big(\big(M_p(u)\big)_{u\ge 0}\big)_{p\in\mathcal{P}_0}, \quad t \to \infty, \tag{11}$$

where $((M_p(u))_{u\ge 0})_{p\in\mathcal{P}_0}$ *is a multivariate extreme process defined by*

$$\big(M_p(u)\big)_{p\in\mathcal{P}_0} = \sup_{k:\, t_k \le u} y_k, \quad u \ge 0. \tag{12}$$

Here the pairs (t_k, y_k) *are the atoms of a Poisson point process on* $[0, \infty) \times (0, \infty]^d$ *with the intensity measure* $\mathbb{LEB} \otimes \nu$ *and the supremum is taken coordinatewise. Moreover, suppose that* $\mathbb{E}[\lambda_p(\eta)] < \infty$, *for* $p \in \mathcal{P} \setminus \mathcal{P}_0$. *Then*

$$\left(\left(\frac{\max_{1\le k\le \lfloor tu\rfloor} T_k(p)}{a(t)}\right)_{u\ge 0}\right)_{p\in\mathcal{P}\setminus\mathcal{P}_0} \overset{\text{f.d.d.}}{\longrightarrow} 0, \quad t \to \infty. \tag{13}$$

We shall deduce Theorems 5 and 6 in Section 3 by proving general limit results for coupled perturbed random walks.

2.3 *Limit theorems for the* LCM

The results from the previous section will be applied below to the analysis of

$$\mathfrak{P}_n := \mathrm{LCM}\big(\{\Pi_1, \Pi_2, \dots, \Pi_n\}\big) \quad \text{and} \quad \mathfrak{T}_n := \mathrm{LCM}\big(\{\Theta_1, \Theta_2, \dots, \Theta_n\}\big).$$

A moment's reflection shows that the analysis of $\mathfrak{P}_n$ is trivial. Indeed, by definition, Π_{n-1} divides Π_n and thereupon $\mathfrak{P}_n = \Pi_n$ for $n \in \mathbb{N}$. Thus, assuming that $\sigma_\xi^2 := \mathrm{Var}\,(\log \xi) \in (0, \infty)$, an application of the Donsker functional limit theorem yields

$$\left(\frac{\log \mathfrak{P}_{\lfloor tu \rfloor} - \mu_\xi tu}{t^{1/2}}\right)_{u \geq 0} \implies \big(\sigma_\xi W(u)\big)_{u \geq 0}, \quad t \to \infty, \tag{14}$$

on the Skorokhod space D, where $(W(u))_{u \geq 0}$ is a standard Brownian motion and $\mu_\xi = \mathbb{E}[\log \xi]$ was defined in (4).

A simple structure of the sequence $(\mathfrak{P}_n)_{n\in\mathbb{N}}$ breaks down completely upon introducing the perturbations (η_k), which makes the analysis of $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ a much harder problem. As an illustration, consider the case $\xi = 1$ in which

$$\mathfrak{T}_n = \mathrm{LCM}\,(\eta_1, \dots, \eta_n).$$

Thus, the problem encompasses, as a particular case, the investigation of the LCM of an independent sample. This itself constitutes a highly nontrivial challenge. Note that

$$\log \mathfrak{T}_n = \log \prod_{p \in \mathcal{P}} p^{\max_{1 \leq k \leq n} (\lambda_p(\xi_1) + \dots + \lambda_p(\xi_{k-1}) + \lambda_p(\eta_k))} = \sum_{p \in \mathcal{P}} \max_{1 \leq k \leq n} T_k(p) \log p,$$

which shows that the asymptotics of $\mathfrak{T}_n$ is intimately connected with the behavior of $\max_{1 \leq k \leq n} T_k(p)$, $p \in \mathcal{P}$.

As one can guess from Theorem 5 in a 'typical' situation relation (14) holds with $\log \mathfrak{T}_{\lfloor tu \rfloor}$ replacing $\log \mathfrak{P}_{\lfloor tu \rfloor}$. The following heuristics suggest the right form of assumptions ensuring that perturbations $(\eta_k)_{k\in\mathbb{N}}$ have an asymptotically negligible impact on $\log \mathfrak{T}_n$. Take a prime $p \in \mathcal{P}$. Its contribution to $\log \mathfrak{T}_n$ (up to a factor $\log p$) is $\max_{1 \leq k \leq n} T_k(p)$. According to Theorem 5, this maximum is asymptotically the same as $S_n(p)$. However, as p gets large, the mean $\mathbb{E}[\lambda_p(\xi)]$ of the random walk $S_{n-1}(p)$ becomes small because of the identity

$$\sum_{p \in \mathcal{P}} \mathbb{E}\big[\lambda_p(\xi)\big] \log p = \mathbb{E}[\log \xi] < \infty.$$

Thus, for large $p \in \mathcal{P}$, the remainder $\max_{1 \leq k \leq n} T_k(p) - S_{n-1}(p)$ can, in principle, become larger than $S_{n-1}(p)$ itself if the tail of $\lambda_p(\eta)$ is sufficiently heavy. In order to rule out such a possibility, we introduce the deterministic sets

$$\mathcal{P}_1(n) := \big\{p \in \mathcal{P} : \mathbb{P}\big\{\lambda_p(\xi) > 0\big\} \geq n^{-1/2}\big\} \quad \text{and} \quad \mathcal{P}_2(n) := \mathcal{P} \setminus \mathcal{P}_1(n), \tag{15}$$

and bound the rate of growth of $\max_{1 \leq k \leq n} \lambda_p(\eta_k)$ for all $p \in \mathcal{P}_2(n)$. It is important to note that under the assumption (8) it holds

$$\min \mathcal{P}_2(n) = \min\big\{p \in \mathcal{P} : p \in \mathcal{P}_2(n)\big\}$$

$$= \min\{p \in \mathcal{P} : \mathbb{P}\{\lambda_p(\xi) > 0\} < n^{-1/2}\} \to \infty, \quad n \to \infty.$$

Therefore, if $\mathbb{E}[\log \xi] < \infty$ and (8) holds, then

$$\lim_{n\to\infty} \sum_{p\in\mathcal{P}_2(n)} \mathbb{E}\big[\lambda_p(\xi)\big] \log p = 0. \tag{16}$$

Theorem 7. *Assume* $\mathbb{E}[\log^2 \xi] < \infty$*,* $\mathbb{E}[\log \eta] < \infty$*,* (8) *and the following two conditions:*

$$\sum_{p\in\mathcal{P}} \mathbb{E}\Big[\big((\lambda_p(\eta) - \lambda_p(\xi))^+\big)^2\Big] \log p < \infty \tag{17}$$

and

$$\sum_{p\in\mathcal{P}_2(n)} \mathbb{E}\big[(\lambda_p(\eta) - \lambda_p(\xi))^+\big] \log p = o\big(n^{-1/2}\big), \quad n \to \infty. \tag{18}$$

Then

$$\left(\frac{\log \mathfrak{T}_{\lfloor tu\rfloor} - \mu_\xi tu}{t^{1/2}}\right)_{u\geq 0} \overset{\text{f.d.d.}}{\longrightarrow} \big(\sigma_\xi W(u)\big)_{u\geq 0}, \quad t \to \infty, \tag{19}$$

where $\mu_\xi = \mathbb{E}[\log \xi] < \infty$*,* $\sigma_\xi^2 = \operatorname{Var}[\log \xi]$ *and* $(W(u))_{u\geq 0}$ *is a standard Brownian motion.*

Remark 3. If $\mathbb{E}[\log^2 \eta] < \infty$, then (17) holds true. Indeed, since we assume $\mathbb{E}[\log^2 \xi] < \infty$,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{p\in\mathcal{P}} \big((\lambda_p(\eta) - \lambda_p(\xi))^+\big)^2 \log p\Big] &\leq \mathbb{E}\Big[\sum_{p\in\mathcal{P}} \big(\lambda_p^2(\eta) + \lambda_p^2(\xi)\big) \log p\Big] \\
&\leq \frac{1}{\log 2} \mathbb{E}\Big[\Big(\sum_{p\in\mathcal{P}} \lambda_p(\eta) \log p\Big)^2\Big] + \mathbb{E}\Big[\Big(\sum_{p\in\mathcal{P}} \lambda_p(\xi) \log p\Big)^2\Big] \\
&= \frac{1}{\log 2}\big(\mathbb{E}\big[\log^2 \eta\big] + \mathbb{E}\big[\log^2 \xi\big]\big) < \infty.
\end{aligned}$$

The condition (18) can be replaced by a stronger one which only involves the distribution of η, namely

$$\sum_{p\in\mathcal{P}_2(n)} \mathbb{E}\big[\lambda_p(\eta)\big] \log p = o\big(n^{-1/2}\big), \quad n \to \infty. \tag{20}$$

Taking into account (16) and the fact that $\mathbb{E}[\log \eta] < \infty$, the assumption (20) is nothing else but a condition of the speed of convergence of the series

$$\sum_{p\in\mathcal{P}} \mathbb{E}\big[\lambda_p(\eta)\big] \log p = \mathbb{E}[\log \eta].$$

Example 8. In the settings of Example 1, let ξ and η be arbitrarily dependent with

$$\mathbb{P}\{\xi = k\} = \frac{1}{\zeta(\alpha)k^\alpha}, \quad \mathbb{P}\{\eta = k\} = \frac{1}{\zeta(\beta)k^\beta}, \quad k \in \mathbb{N},$$

for some $\alpha, \beta > 1$. Note that $\mathbb{E}[\log^2 \xi] < \infty$ and $\mathbb{E}[\log^2 \eta] < \infty$. Direct calculations show that

$$\mathcal{P}_1(n) = \{p \in \mathcal{P} : p^{-\alpha} \geq n^{-1/2}\} = \{p \in \mathcal{P} : p \leq n^{1/(2\alpha)}\},$$
$$\mathcal{P}_2(n) = \{p \in \mathcal{P} : p > n^{1/(2\alpha)}\}.$$

From the chain of relations

$$\mathbb{E}[\lambda_p(\eta)] = \sum_{j\geq 1} \mathbb{P}\{\lambda_p(\eta) \geq j\} = \sum_{j\geq 1} p^{-\beta j} = \frac{p^{-\beta}}{1 - p^{-\beta}} \leq 2p^{-\beta},$$

and using the notation $\pi(x)$ for the number of primes smaller than x, we obtain

$$\sum_{p\in\mathcal{P}_2(n)} \mathbb{E}[\lambda_p(\eta)] \log p \leq 2 \sum_{p\in\mathcal{P},\, p>n^{1/(2\alpha)}} \frac{\log p}{p^\beta} = 2\int_{(n^{1/(2\alpha)},\,\infty)} \frac{\log x}{x^\beta} \mathrm{d}\pi(x)$$
$$\sim 2\int_{n^{1/(2\alpha)}}^{\infty} \frac{\log x}{x^\beta} \frac{\mathrm{d}x}{\log x} = \frac{2n^{(1-\beta)/(2\alpha)}}{\beta - 1}, \quad n \to \infty.$$

Here the asymptotic equivalence follows from the prime number theorem and integration by parts, see, for example Eq. (16) in [3]. Thus, (20) holds if

$$\frac{1}{2} + \frac{1-\beta}{2\alpha} < 0 \iff \alpha + 1 < \beta.$$

In the settings of Theorem 6 the situation is much simpler in a sense that almost no extra assumptions are needed to derive a limit theorem for $\mathfrak{T}_n$.

Theorem 9. *Under the same assumptions as in Theorem 6 and assuming additionally that*

$$\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0} \mathbb{E}[\lambda_p(\eta)] \log p < \infty, \tag{21}$$

it holds

$$\left(\frac{\log \mathfrak{T}_{\lfloor tu \rfloor}}{a(t)}\right)_{u\geq 0} \overset{\text{f.d.d.}}{\longrightarrow} \left(\sum_{p\in\mathcal{P}_0} M_p(u) \log p\right)_{u\geq 0}, \quad t \to \infty. \tag{22}$$

Note that in Theorem 9 it is allowed to take $\xi = 1$, which yields the following limit theorem for the LCM of an independent integer-valued random variables.

Corollary 1. *Under the same assumptions on η as in Theorem 6, it holds*

$$\left(\frac{\log \mathrm{LCM}\,(\eta_1, \eta_2, \ldots, \eta_{\lfloor tu \rfloor})}{a(t)}\right)_{u\geq 0} \overset{\text{f.d.d.}}{\longrightarrow} \left(\sum_{p\in\mathcal{P}_0} M_p(u) \log p\right)_{u\geq 0}, \quad t \to \infty.$$

Remark 4. The results presented in Theorems 7 and 9 constitute a contribution to a popular topic in probabilistic number theory, namely, the asymptotic analysis of the LCM of various random sets. For random sets comprised of independent random variables uniformly distributed on $\{1, 2, \ldots, n\}$ this problem has been addressed in [2–5, 9]. Some models with a more sophisticated dependence structure have been studied [1] and [8].

3 Limit theorems for coupled perturbed random walks

Theorems 5 and 6 will be derived from general limit theorems for the maxima of arbitrary additive perturbed random walks indexed by some parameters ranging in a countable set in the situation when the underlying additive standard random walks are positively divergent and attracted to a Brownian motion.

Let $\mathcal{A}$ be a countable or finite set of real numbers and

$$\big((X_1(r), Y_1(r))\big)_{r\in\mathcal{A}}, \quad \big((X_2(r), Y_2(r))\big)_{r\in\mathcal{A}}, \dots$$

be independent copies of an $\mathbb{R}^{2\times|\mathcal{A}|}$ random vector $(X(r), Y(r))_{r\in\mathcal{A}}$ with arbitrarily dependent components. For each $r \in \mathcal{A}$, the sequence $(S^*_k(r))_{k\in\mathbb{N}_0}$ given by

$$S^*_0(r) := 0, \quad S^*_k(r) := X_1(r) + \dots + X_k(r), \quad k \in \mathbb{N},$$

is an additive standard random walk. For each $r \in \mathcal{A}$, the sequence $(T^*_k(r))_{k\in\mathbb{N}}$ defined by

$$T^*_k(r) := S^*_{k-1}(r) + Y_k(r), \quad k \in \mathbb{N},$$

is an additive perturbed random walk. The sequence $((T^*_k(r))_{k\in\mathbb{N}})_{r\in\mathcal{A}}$ is a collection of (generally) dependent additive perturbed random walks.

Proposition 3. *Assume that, for each $r \in \mathcal{A}$, $\mu(r) := \mathbb{E}[X(r)] \in (0, \infty)$,* Var $[X(r)] \in [0, \infty)$ *and $\mathbb{E}[Y(r)] < \infty$. Then*

$$\left(\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} T^*_k(r) - \mu(r)tu}{t^{1/2}}\right)_{u\ge0}\right)_{r\in\mathcal{A}} \overset{\text{f.d.d.}}{\longrightarrow} \big((W_r(u))_{u\ge0}\big)_{r\in\mathcal{A}}, \quad t\to\infty, \tag{23}$$

where, for all $n \in \mathbb{N}$ and arbitrary $r_1 < r_2 < \dots < r_n$ with $r_i \in \mathcal{A}$, $i \le n$, $((W_{r_1}(u))_{u\ge0}, \dots, (W_{r_n}(u))_{u\ge0})$ is an n-dimensional centered Wiener process with covariance matrix $C = \|C_{i,j}\|_{1\le i,j\le n}$ with the entries $C_{i,j} = C_{j,i} =$ $\mathrm{Cov}(X(r_i), X(r_j))$.

Proof. We shall prove an equivalent statement that, as $t \to \infty$,

$$\left(\left(\frac{\max_{0\le k\le\lfloor tu\rfloor} T^*_{k+1}(r) - \mu(r)tu}{t^{1/2}}\right)_{u\ge0}\right)_{r\in\mathcal{A}} \overset{\text{f.d.d.}}{\longrightarrow} \big((W_r(u))_{u\ge0}\big)_{r\in\mathcal{A}},$$

which differs from (23) by a shift of the subscript k. By the multidimensional Donsker theorem,

$$\left(\left(\frac{S^*_{\lfloor tu\rfloor}(r) - \mu(r)tu}{t^{1/2}}\right)_{u\ge0}\right)_{r\in\mathcal{A}} \Longrightarrow \big((W_r(u))_{u\ge0}\big)_{r\in\mathcal{A}}, \quad t\to\infty, \tag{24}$$

in the product topology of $D^{\mathbb{N}}$. Fix any $r \in \mathcal{A}$ and write

$$\begin{aligned}\max_{0\le k\le\lfloor tu\rfloor} & T^*_{k+1}(r) - \mu(r)tu \\ &= \max_{0\le k\le\lfloor tu\rfloor}\big(S^*_k(r) - S^*_{\lfloor tu\rfloor}(r) + Y_{k+1}(r)\big) + S^*_{\lfloor tu\rfloor}(r) - \mu(r)tu. \end{aligned} \tag{25}$$

In view of (24) the proof is complete once we can show that

$$n^{-1/2}\Big(\max_{0\le k\le n}\big(S_k^*(r)-S_n^*(r)+Y_{k+1}(r)\big)\Big)\overset{\mathbb{P}}{\to}0,\quad n\to\infty. \tag{26}$$

Let $(X_0(r), Y_0(r))$ be a copy of $(X(r), Y(r))$ which is independent of the vector $(X_k(r), Y_k(r))_{k\in\mathbb{N}}$. Since the collection

$$\big((X_1(r), Y_1(r)),\dots,(X_{n+1}(r), Y_{n+1}(r))\big)$$

has the same distribution as

$$\big((X_n(r), Y_n(r)),\dots,(X_0(r), Y_0(r))\big),$$

the variable

$$\max_{0\le k\le n}\big(S_k^*(r)-S_n^*(r)+Y_{k+1}(r)\big)$$

has the same distribution as

$$\max\Big(Y_0(r),\ \max_{0\le k\le n-1}\big(-S_k^*(r)+Y_{k+1}(r)-X_{k+1}(r)\big)\Big).$$

By assumption, $\mathbb{E}(-S_1^*(r))\in(-\infty,0)$ and $\mathbb{E}(Y(r)-X(r))^+<\infty$. Hence, by Theorem 1.2.1 and Remark 1.2.3 in [6],

$$\lim_{k\to\infty}\big(-S_k^*(r)+Y_{k+1}(r)-X_{k+1}(r)\big)=-\infty\quad\text{a.s.}$$

As a consequence, the a.s. limit

$$\lim_{n\to\infty}\max(Y_0(r),\ \max_{0\le k\le n-1}\big(-S_k^*(r)+Y_{k+1}(r)-X_{k+1}(r)\big)$$
$$=\max(Y_0(r),\max_{k\ge 0}\big(-S_k^*(r)+Y_{k+1}(r)-X_{k+1}(r)\big)$$

is a.s. finite. This completes the proof of (26). □

Remark 5. Proposition 3 tells us that fluctuations of $\max_{1\le k\le\lfloor tu\rfloor}T_k^*(r)$ on the level of finite-dimensional distributions are driven by the Brownian fluctuations of $S_{\lfloor tu\rfloor}^*(r)$. According to formula (25), a functional version of this statement would be true if we could check that, for every fixed $T>0$,

$$t^{-1/2}\sup_{u\in[0,T]}\max_{0\le k\le\lfloor tu\rfloor}\big(S_k^*(r)-S_{\lfloor tu\rfloor}^*(r)+Y_{k+1}(r)\big)\overset{\mathbb{P}}{\to}0,\quad t\to\infty.$$

But the left-hand side is bounded from below by

$$t^{-1/2}\sup_{u\in[0,T]}Y_{\lfloor tu\rfloor+1}(r)=t^{-1/2}\max_{0\le k\le\lfloor Tt\rfloor+1}Y_k(r).$$

Under the sole assumption $\mathbb{E}[Y(r)]<\infty$ this maximum does not converge to zero in probability, as $t\to\infty$. Thus, under the standing assumptions of Proposition 3 the functional convergence does not hold.

Proof of Theorem 5. To deduce the finite-dimensional convergence (9) we apply Proposition 3 with $\mathcal{A} = \mathcal{P}$, $X(p) = \lambda_p(\xi)$ and $Y(p) = \lambda_p(\eta)$. The assumption (8) in conjunction with $\mathbb{E}[\log^2 \xi] < \infty$ implies that $\mathbb{E}[\lambda_p(\xi)] \in (0, \infty)$ and $\mathrm{Var}\,[\lambda_p(\xi)] \in [0, \infty)$, for all $p \in \mathcal{P}$.

Suppose that (5) holds true for all $p \in \mathcal{P}$. Fix $p \in \mathcal{P}$, $t > 0$, and note that by the subadditivity of the supremum and the fact that $(S_k(p))_{k\in\mathbb{N}_0}$ is nondecreasing we have

$$S_{\lfloor tu\rfloor-1}(p) \le \max_{1\le k\le \lfloor tu\rfloor} T_k(p) \le S_{\lfloor tu\rfloor-1}(p) + \max_{1\le k\le \lfloor tu\rfloor} \lambda_p(\eta_k), \quad u \ge 0. \tag{27}$$

Assumption (5) implies that, for every fixed $T > 0$,

$$t^{-1/2} \sup_{u\in[0,T]} \max_{1\le k\le \lfloor tu\rfloor} \lambda_p(\eta_k) = t^{-1/2} \max_{1\le k\le \lfloor tT\rfloor} \lambda_p(\eta_k) \overset{\mathbb{P}}{\to} 0, \quad t \to \infty.$$

By Proposition 1 and taking into account (27) this means that (9) holds true on the product space $D^{\mathbb{N}}$. □

Proposition 4. *Assume $\mathbb{E}[X(r)] < \infty$, $r \in \mathcal{A}$. Assume further that there exists a finite set $\mathcal{A}_0 \subseteq \mathcal{A}$, $d := |\mathcal{A}_0|$, such that the distributional tail of $(Y(r))_{r\in\mathcal{A}_0}$ is regularly varying at infinity in the following sense. For some positive function $(a(t))_{t>0}$ and a measure ν satisfying $\nu(\{x \in \mathbb{R}^d : \|x\| \ge r\}) = c \cdot r^{-\alpha}$, $c > 0$, $\alpha \in (0, 1)$, it holds*

$$t\mathbb{P}\big\{\big(a(t)\big)^{-1}\big(Y(r)\big)_{r\in\mathcal{A}_0} \in \cdot\big\} \overset{\mathrm{v}}{\longrightarrow} \nu(\cdot), \quad t \to \infty, \tag{28}$$

on the space of locally finite measures on $(0, \infty]^d$ endowed with the vague topology. Then

$$\left(\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} T_k^*(r)}{a(t)}\right)_{u\ge0}\right)_{r\in\mathcal{A}_0} \overset{\text{f.d.d.}}{\longrightarrow} \big(\big(M_r(u)\big)_{u\ge0}\big)_{r\in\mathcal{A}_0}, \quad t \to \infty, \tag{29}$$

where $((M_r(u))_{u\ge0})_{r\in\mathcal{A}_0}$ is defined as in (12). If $\mathbb{E}[|Y(r)|] < \infty$, for $r \in \mathcal{A} \setminus \mathcal{A}_0$, then also

$$\left(\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} T_k^*(r)}{a(t)}\right)_{u\ge0}\right)_{r\in\mathcal{A}\setminus\mathcal{A}_0} \overset{\text{f.d.d.}}{\longrightarrow} 0, \quad t \to \infty. \tag{30}$$

Proof. According to Corollary 5.18 in [11]

$$\left(\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} Y_k(r)}{a(t)}\right)_{u\ge0}\right)_{r\in\mathcal{A}_0} \Longrightarrow \big(\big(M_r(u)\big)_{u\ge0}\big)_{r\in\mathcal{A}_0}, \quad t \to \infty,$$

in the product topology of $D^{\mathbb{N}}$. The function $(a(t))_{t\ge0}$ is regularly varying at infinity with index $1/\alpha > 1$. Thus, by the law of large numbers, for all $r \in \mathcal{A}$,

$$\left(\frac{\min_{1\le k\le\lfloor tu\rfloor} S_{k-1}^*(r)}{a(t)}\right)_{u\ge0} \overset{\text{f.d.d.}}{\longrightarrow} 0, \quad t \to \infty, \tag{31}$$

$$\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} S_{k-1}^*(r)}{a(t)}\right)_{u\ge0} \overset{\text{f.d.d.}}{\longrightarrow} 0, \quad t \to \infty, \tag{32}$$

and (29) follows from the inequalities

$$\min_{1\le k\le\lfloor tu\rfloor} S^*_{k-1}(r)+\max_{1\le k\le\lfloor tu\rfloor} Y_k(r)\le \max_{1\le k\le\lfloor tu\rfloor} T^*_k(r)$$
$$\le \max_{1\le k\le\lfloor tu\rfloor} S^*_{k-1}(r)+\max_{1\le k\le\lfloor tu\rfloor} Y_k(r).$$

In view of (31) and (32), to prove (30) it suffices to check that

$$\left(\left(\frac{\max_{1\le k\le\lfloor tu\rfloor} Y_k(r)}{a(t)}\right)_{u\ge 0}\right) \overset{\text{f.d.d.}}{\longrightarrow} 0,\quad t\to\infty,$$

for every fixed $r\in\mathcal{A}\setminus\mathcal{A}_0$. This, in turn, follows from

$$\frac{Y_n(r)}{n}\overset{\text{a.s.}}{\longrightarrow} 0,\quad n\to\infty,\quad r\in\mathcal{A}\setminus\mathcal{A}_0,$$

which is a consequence of the assumption $\mathbb{E}[|Y(r)|]<\infty$, $r\in\mathcal{A}\setminus\mathcal{A}_0$, and the Borel–Cantelli lemma. □

Proof of Theorem 6. Follows immediately from Proposition 4 applied with $\mathcal{A}=\mathcal{P}$, $X(p)=\lambda_p(\xi)$ and $Y(p)=\lambda_p(\eta)$. □

4 Proof of Theorem 7

We aim at proving that

$$\frac{\sum_{p\in\mathcal{P}}(\max_{1\le k\le n} T_k(p)-S_{n-1}(p))\log p}{\sqrt{n}}\overset{\mathbb{P}}{\longrightarrow} 0,\quad n\to\infty,\tag{33}$$

which together with the relation

$$\sum_{p\in\mathcal{P}} S_n(p)\log p=\log\Pi_n=\log\mathfrak{P}_n,\quad n\in\mathbb{N},$$

implies Theorem 7 by the Slutsky lemma and (14).

Let (ξ_0,η_0) be an independent copy of (ξ,η) which is also independent of $(\xi_n,\eta_n)_{n\in\mathbb{N}}$. By the same reasoning as we have used in the proof of (26) we obtain

$$\left(\max_{1\le k\le n} T_k(p)-S_{n-1}(p)\right)_{p\in\mathcal{P}}$$
$$\overset{d}{=}\left(\max\Big(\lambda_p(\eta_0),\max_{1\le k<n}\big(\lambda_p(\eta_k)-\lambda_p(\xi_k)-S_{k-1}(p)\big)\Big)\right)_{p\in\mathcal{P}}.\tag{34}$$

Taking into account

$$\sum_{p\in\mathcal{P}}\lambda_p(\eta_0)\log p=\log\eta_0,$$

we see that (33) is a consequence of

$$\frac{\sum_{p\in\mathcal{P}}\max_{1\le k<n}(\lambda_p(\eta_k)-\lambda_p(\xi_k)-S_{k-1}(p))^+\log p}{\sqrt{n}}\overset{\mathbb{P}}{\longrightarrow} 0,\quad n\to\infty.\tag{35}$$

Since, for every fixed $p \in \mathcal{P}$,

$$\max_{k\geq 1}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+ < \infty \quad \text{a.s.} \tag{36}$$

by assumption (8), it suffices to check that, for every fixed $\varepsilon > 0$,

$$\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big\{\sum_{p\in\mathcal{P},\, p>M}\max_{1\leq k<n}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\log p > \varepsilon\sqrt{n}\Big\}. \tag{37}$$

In order to check (37), we divide the sum into two disjoint parts with summations over $\mathcal{P}_1(n)$ and $\mathcal{P}_2(n)$. For the first sum, by Markov's inequality, we obtain

$$\begin{aligned}
&\mathbb{P}\Big\{\sum_{p\in\mathcal{P}_1(n),\, p>M}\max_{1\leq k<n}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\log p > \varepsilon\sqrt{n}/2\Big\}\\
&\quad\leq \frac{2}{\varepsilon\sqrt{n}}\sum_{p\in\mathcal{P}_1(n),\, p>M}\mathbb{E}\Big(\max_{1\leq k<n}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\Big)\log p\\
&\quad\leq \frac{2}{\varepsilon\sqrt{n}}\sum_{p\in\mathcal{P}_1(n),\, p>M}\log p\sum_{k\geq 1}\mathbb{E}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\\
&\quad= \frac{2}{\varepsilon\sqrt{n}}\sum_{p\in\mathcal{P}_1(n),\, p>M}\log p\sum_{j\geq 1}\mathbb{P}\big\{\lambda_p(\eta) - \lambda_p(\xi) = j\big\}\sum_{k\geq 1}\mathbb{E}\big(j - S_{k-1}(p)\big)^+\\
&\quad\leq \frac{2}{\varepsilon\sqrt{n}}\sum_{p\in\mathcal{P}_1(n),\, p>M}\log p\sum_{j\geq 1}j\mathbb{P}\big\{\lambda_p(\eta) - \lambda_p(\xi) = j\big\}\sum_{k\geq 0}\mathbb{P}\big\{S_k(p)\leq j\big\}\\
&\quad\leq \frac{2}{\varepsilon\sqrt{n}}\sum_{p\in\mathcal{P}_1(n),\, p>M}\log p\sum_{j\geq 1}j\mathbb{P}\big\{\lambda_p(\eta) - \lambda_p(\xi) = j\big\}\frac{2j}{\mathbb{E}[(\lambda_p(\xi)\wedge j)]},
\end{aligned}$$

where the last estimate is a consequence of Erickson's inequality for renewal functions, see Eq. (6.5) in [6]. Further, since for $p \in \mathcal{P}_1(n)$,

$$\mathbb{E}\big[\big(\lambda_p(\xi)\wedge j\big)\big] \geq \mathbb{P}\big\{\lambda_p(\xi)\geq 1\big\} = \mathbb{P}\big\{\lambda_p(\xi) > 0\big\} \geq n^{-1/2},$$

we obtain

$$\begin{aligned}
&\mathbb{P}\Big\{\sum_{p\in\mathcal{P}_1(n),\, p>M}\max_{1\leq k<n}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\log p > \varepsilon\sqrt{n}/2\Big\}\\
&\quad\leq \frac{4}{\varepsilon}\sum_{p\in\mathcal{P}_1(n),\, p>M}\log p\,\mathbb{E}\Big[\Big(\big(\lambda_p(\eta) - \lambda_p(\xi)\big)^+\Big)^2\Big]\\
&\quad\leq \frac{4}{\varepsilon}\sum_{p\in\mathcal{P},\, p>M}\log p\,\mathbb{E}\Big[\Big(\big(\lambda_p(\eta) - \lambda_p(\xi)\big)^+\Big)^2\Big].
\end{aligned}$$

The right-hand side converges to 0, as $M \to \infty$ by (17). For the sum over $\mathcal{P}_2(n)$ the derivation is simpler. By Markov's inequality

$$\mathbb{P}\Big\{\sum_{p\in\mathcal{P}_2(n),\, p>M}\max_{1\leq k<n}\big(\lambda_p(\eta_k) - \lambda_p(\xi_k) - S_{k-1}(p)\big)^+\log p > \varepsilon\sqrt{n}/2\Big\}$$

$$\leq \frac{2}{\varepsilon\sqrt{n}}\mathbb{E}\Big[\sum_{p\in\mathcal{P}_2(n),\,p>M}\max_{1\leq k<n}\big(\lambda_p(\eta_k)-\lambda_p(\xi_k)-S_{k-1}(p)\big)^+\log p\Big]$$

$$\leq \frac{2n}{\varepsilon\sqrt{n}}\mathbb{E}\Big[\sum_{p\in\mathcal{P}_2(n),\,p>M}\big(\lambda_p(\eta_k)-\lambda_p(\xi_k)\big)^+\log p\Big],$$

and the right-hand side tends to zero as $n\to\infty$ in view of (18). The proof is complete.

5 Proof of Theorem 9

From Theorem 6 with the aid of the continuous mapping theorem we conclude that

$$\left(\frac{\sum_{p\in\mathcal{P}_0}\max_{1\leq k\leq\lfloor tu\rfloor}T_k(p)\log p}{a(t)}\right)_{u\geq 0}\ \overset{\text{f.d.d.}}{\longrightarrow}\ \left(\sum_{p\in\mathcal{P}_0}M_p(u)\log p\right)_{u\geq 0},$$

as $t\to\infty$. It suffices to check

$$\left(\frac{\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\max_{1\leq k\leq\lfloor tu\rfloor}T_k(p)\log p}{a(t)}\right)_{u\geq 0}\ \overset{\text{f.d.d.}}{\longrightarrow}\ 0,\quad t\to\infty. \tag{38}$$

Since $(a(t))$ is regularly varying at infinity, (38) follows from

$$\frac{\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}[\max_{1\leq k\leq n}T_k(p)]\log p}{a(n)}\to 0,\quad n\to\infty, \tag{39}$$

by Markov's inequality. To check the latter, note that

$$\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}\Big[\max_{1\leq k\leq n}T_k(p)\Big]\log p\leq\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}\Big[S_{n-1}(p)+\max_{1\leq k\leq n}\lambda_p(\eta_k)\Big]\log p$$

$$\leq(n-1)\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}\big[\lambda_p(\xi)\big]\log p+n\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}\big[\lambda_p(\eta)\big]\log p$$

$$\leq(n-1)\mathbb{E}[\log\xi]+n\sum_{p\in\mathcal{P}\setminus\mathcal{P}_0}\mathbb{E}\big[\lambda_p(\eta)\big]\log p=O(n),\quad n\to\infty,$$

where we have used the inequality $\mathbb{E}[\log\xi]<\infty$ and the assumption (21). Using that $\alpha\in(0,1)$ and $(a(t))$ is regularly varying at infinity with index $1/\alpha$, we obtain (39).

Acknowledgement

We thank the anonymous referee for useful comments on the first version of the manuscript.

Funding

The research was supported by the National Research Foundation of Ukraine (project 2020.02/0014 "Asymptotic regimes of perturbed random walks: on the edge of modern and classical probability").

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