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Stochastic Lotka–Volterra mutualism model with jumps

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Abstract The existence and uniqueness of the global positive solution are proved for the system of stochastic differential equations describing a two-species Lotka–Volterra mutualism model disturbed by white noise, centered and noncentered Poisson noises. For the considered system, sufficient conditions of stochastic ultimate boundedness, stochastic permanence, nonpersistence and strong persistence in the mean are obtained.

Keywords Stochastic Lotka–Volterra mutualism model, global solution, stochastic ultimate boundedness, stochastic permanence, nonpersistence, strong persistence in the mean

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1 Introduction

In nature we can find many examples where the interaction of two or more species is to the advantage of all. These population systems are described by mutualism models. The simplest two-spices Lotka–Volterra mutualism model and its properties are

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presented in J.D. Murray [1]. A deterministic nonautonomous two-species Lotka–Volterra mutualism model is described by the system

$$dx_i(t) = x_i(t)\big(r_i(t) - a_{ii}(t)x_i(t) + a_{ij}(t)x_j(t)\big)dt,\ i, j = 1, 2,\ i \neq j,$$

where $x_i(t), i = 1, 2$, denote population densities of each species at time t, $r_i(t) > 0$, $i = 1, 2$, denote the intrinsic growth rates of species $x_i(t), i = 1, 2$. The carrying capacities of species $x_i(t)$ at time t are $r_i(t)/a_{ii}(t) > 0$, $i = 1, 2$, and coefficients $a_{ij}(t) > 0$, $i, j = 1, 2, i \neq j$, describe the influence of the j-th population upon the i-th population at time t.

In the real world population systems are often subject to environmental noise. Therefore, it is natural to describe such systems by the systems of stochastic differential equations. In the paper by Peiyan Xia et al. [2], the authors consider the stochastic nonautonomous two-species Lotka–Volterra mutualism model of the form

$$dx_i(t) = x_i(t)\big[\big(r_i(t) - a_{ii}(t)x_i(t) + a_{ij}(t)x_j(t)\big)dt + \sigma_i(t)dw_i(t)\big], \tag{1}$$

$i, j = 1, 2, i \neq j$, where $r_i(t), a_{ij}(t), \sigma_i(t), i = 1, 2$, are all positive, continuous and bounded functions on $[0, +\infty)$, and $w_1(t)$, $w_2(t)$ are mutually independent Wiener processes. The authors show that the stochastic system (1) has a unique global (no explosion in a finite time) solution for any positive initial value and that the p-th moment of the solution is bounded. The sufficient conditions for stochastic permanence, persistence in the mean, nonpersistence and global attractivity of the system (1) are obtained.

In the paper by L. Shaikhet and A. Korobeinikov [3], the authors studied the asymptotic properties of Lotka–Volterra competition and mutualism models driven by autonomous system of stochastic differential equations

$$\begin{aligned} dx(t) &= a_1x(t)\big(1 - b_{11}x(t) - b_{12}y(t)\big)dt + \sigma_1x(t)dw_1(t),\\ dy(t) &= a_2y(t)\big(1 - b_{21}x(t) - b_{22}y(t)\big)dt + \sigma_2y(t)dw_2(t).\end{aligned}$$

Here, a_1 and a_2 are per capita rates of growth of populations $x(t)$ and $y(t)$, respectively, b_{11}, b_{12}, b_{21} and b_{22} reflect the intraspecific competition ($b_{11} > 0$ and $b_{22} > 0$) and interspecies interaction (b_{12} and b_{21}). For competing or symbiotic species, $a_1 > 0$, $a_2 > 0$. For competing populations $b_{12} > 0$, $b_{21} > 0$, and for symbiotic populations $b_{12} < 0$, $b_{21} < 0$. The authors showed that solutions to the considered system with positive initial conditions converge to a certain compact region in the model phase space and oscillate around this region thereafter. So, the solutions of the considered stochastic system are bounded and the system is persistent. The necessary condition of the species extinction is obtained, sufficient conditions of the extinction of competing species, for which $b_{12} > 0$, $b_{21} > 0$ are derived, but sufficient conditions of the extinction of symbiotic species, for which $b_{12} < 0$, $b_{21} < 0$, were not obtained in this paper.

If we want to take into account abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. in the considered models, we must introduce Poisson noises into the population models for describing such discontinuous systems. In the paper by Y. Gao and X. Zhang [4], the authors considered a two-dimensional autonomous stochastic Lotka–Volterra mutualistic parasite–host system with "white" noise and “small” jumps, corresponding to the centered Poisson measure

$$dx_i(t) = x_i(t)\big[\big((-1)^{i-1}r_i - a_{ii}x_i(t) + a_{ij}x_j(t)\big)dt + \sigma_i dw_i(t)\big] + \int_{\mathbb{Z}} \gamma_i(z)x_i\big(t^-\big)\tilde{N}(dt, dz),\ i, j = 1, 2,\ i \neq j,$$

where $x_i(t^-)$, $i = 1, 2$, are the left limits of $x_i(t)$, $i = 1, 2$, $w_i(t)$, $i = 1, 2$, are mutually independent standard one-dimensional Wiener processes, $\tilde{N}(t, A) = N(t, A) - t\lambda(A)$, A is a Borel set in $\mathbb{R}$, $\lambda(\mathbb{Z}) < +\infty$, $N(t, A)$ is the Poisson measure, which is independent of $w_i(t)$, $i = 1, 2$, $r_i > 0$, $a_{ij} > 0$, $\sigma_i > 0$, $i, j = 1, 2$. The sufficient conditions for extinction and persistent in the mean of species $x_i(t)$, $i = 1, 2$, are obtained. Then, the authors established the sufficient criteria for stability in distribution of the considered parasite–host system.

In the paper by J. Bao et al. [5], the authors studied the stochastic competitive multi-species Lotka–Volterra model under the action of the one-dimensional “white” noise and jumps, generated by a centered Poisson measure. The authors proved that the model admits a unique global positive solution, which has a uniformly finite p-th moment with $p > 0$. Stochastic ultimate boundedness, existence of invariant measure and long-term behaviors of solutions are discussed.

The paper by Q. Liu et al. [6] is devoted to the study of two-species mutualism model driven by the system of the autonomous stochastic differential equations

$$dx_i(t) = x_i(t)\left[\left(r_i - \frac{b_i x_i(t)}{K_i + \kappa_i x_j(t)} - \varepsilon_i x_i(t)\right)dt + \alpha_{ii}dw_{ii}(t) + \alpha_{ij}x_i(t)dw_{ij}(t)\right] + \int_{\mathbb{Z}} \gamma_i(z)x_i\big(t^-\big)\tilde{N}(dt, dz),\ i, j = 1, 2,\ i \neq j,$$

where $w_{ij}(t)$, $i, j = 1, 2$, are mutually independent standard one-dimensional Wiener processes, independent of $N(t, A)$. It is shown that the positive solution of the considered system is stochastically ultimate bounded. Then authors establish sufficient and necessary conditions for the stochastic permanence and extinction of the system.

The impact of centered and noncentered Poisson noises to the stochastic nonautonomous mutualism model is studied in the paper by Olg. Borysenko and O. Borysenko [7]. The authors considered the system of nonautonomous stochastic differential equations

$$dx_i(t) = x_i(t)\left[\frac{a_{i1}(t) + a_{i2}(t)x_{3-i}(t)}{1 + x_{3-i}(t)} - c_i(t)x_i(t)\right]dt + \sigma_i(t)x_i(t)dw_i(t) + \int_{\mathbb{R}} \gamma_i(t, z)x_i(t)\tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \delta_i(t, z)x_i(t)\nu_2(dt, dz),$$
$$x_i(0) = x_{i0} > 0,\ i = 1, 2,$$

where $w_i(t)$, $i = 1, 2$, are mutually independent standard one-dimensional Wiener processes, $\tilde{\nu}_1(t, A) = \nu_1(t, A) - t\Pi_1(A)$, $\nu_i(t, A)$, $i = 1, 2$, are mutually independent

Poisson measures, which are independent of $w_i(t)$, $i=1,2$, $E[\nu_i(t,A)]=t\Pi_i(A)$, $i=1,2$, $\Pi_i(A)$, $i=1,2$, are a finite measures on the Borel sets A in $\mathbb{R}$. The existence and uniqueness of the global positive solution to the considered system is proved. The authors obtain sufficient conditions of stochastic ultimate boundedness, stochastic permanence, nonpersistence in the mean, strong persistence in the mean and extinction of the solution to the considered system.

In this paper, we consider the nonautonomous stochastic mutualism model with jumps generated by centered and noncentered Poisson measures. So, the novelty of considered model is following: we investigate the nonautonomous stochastic Lotka–Volterra model and we take into account not only "small" jumps, corresponding to the centered Poisson measure but also the "large" jumps, corresponding to the noncentered Poisson measure. This model is driven by the system of nonautonomous stochastic differential equations

$$\begin{aligned} dx_i(t) &= x_i(t)\big[\big(r_i(t)-a_{ii}(t)x_i(t)+a_{ij}(t)x_j(t)\big)dt+\sigma_i(t)dw_i(t)\big] \\ &\quad+\int_{\mathbb{R}}\gamma_i(t,z)x_i\big(t^-\big)\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\delta_i(t,z)x_i\big(t^-\big)\nu_2(dt,dz), \\ &\qquad x_i(0)=x_{i0}>0,\ i,j=1,2,\ i\neq j, \end{aligned} \tag{2}$$

where $x_i(t^-)$, $i=1,2$, are the left limits of $x_i(t)$, $i=1,2$, $w_i(t)$, $i=1,2$, are mutually independent standard one-dimensional Wiener processes, $\tilde{\nu}_1(t,A)=\nu_1(t,A)-t\Pi_1(A)$, $\nu_i(t,A)$, $i=1,2$, are mutually independent Poisson measures, which are independent of $w_i(t)$, $i=1,2$, $E[\nu_i(t,A)]=t\Pi_i(A)$, $i=1,2$, $\Pi_i(A)$, $i=1,2$, are finite measures on the Borel sets A in $\mathbb{R}$.

As far as we know, there are no papers devoted to the dynamical properties of the stochastic mutualism model (2), even in the case of a centered Poisson noise.

In the following we will use the notations $X(t)=(x_1(t),x_2(t))$, $X_0=(x_{10},x_{20})$, $|X(t)|=\sqrt{x_1^2(t)+x_2^2(t)}$, $\mathbb{R}_+^2=\{X\in\mathbb{R}^2:\ x_1>0,x_2>0\}$,

$$\begin{aligned} \beta_i(t) &= \frac{\sigma_i^2(t)}{2}+\int_{\mathbb{R}}\big[\gamma_i(t,z)-\ln\big(1+\gamma_i(t,z)\big)\big]\Pi_1(dz) \\ &\quad-\int_{\mathbb{R}}\ln\big(1+\delta_i(t,z)\big)]\Pi_2(dz), \end{aligned} \tag{3}$$

$i=1,2$. For the bounded, continuous function $f(t)$, $t\in[0,+\infty)$, let us denote

$$f^{\sup}=\sup_{t\geq 0}f(t),\quad f^{\inf}=\inf_{t\geq 0}f(t).$$

We prove that system (2) has a unique, positive, global solution for any positive initial value and this solution is stochastically ultimate bounded. The sufficient conditions for stochastic permanence, nonpersistence and strong persistence in the mean of the system are derived.

The rest of this paper is organized as follows. In Section 2, we prove the existence of the unique global positive solution to the system (2). In Section 3, we prove the stochastic ultimate boundedness of the solution to the system (2). In Section 4, we obtain conditions under which the solution to the system (2) is stochastically permanent and strong persistence in the mean. In Section 5 the sufficient conditions for nonpersistence of the system (2) are obtained.

2 Existence of a global solution

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $w_i(t)$, $i = 1, 2$, $t \ge 0$, are mutually independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, \mathrm{P})$, and $\nu_i(t, A)$, $i = 1, 2$, are mutually independent Poisson measures defined on $(\Omega, \mathcal{F}, \mathrm{P})$ independent of $w_i(t)$, $i = 1, 2$. Here $\mathrm{E}[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(\cdot)$, $i = 1, 2$, are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\ge 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s, A), s \le t, i = 1, 2\}$.

We need the following assumption.

Assumption 1. It is assumed, that $a_{ij}(t) > 0$, $i, j = 1, 2$, $r_i(t) > 0$, $\sigma_i(t)$, $i = 1, 2$, are bounded, continuous on t functions, $\gamma_i(t, z)$, $\delta_i(t, z)$, $i = 1, 2$, are continuous on t functions, $\ln(1 + \gamma_i(t, z))$, $\ln(1 + \delta_i(t, z))$, $i = 1, 2$, are bounded, $\Pi_i(\mathbb{R}) < \infty$, $i = 1, 2$ and $a_{12}^{\sup} a_{21}^{\sup} < a_{11}^{\inf} a_{22}^{\inf}$.

In what follows we will assume that Assumption 1 holds.

Theorem 1. *There exists a unique global solution $X(t)$ to the system (2) for any initial value $X(0) = X_0 > 0$, and $\mathrm{P}\{X(t) \in \mathbb{R}_+^2\} = 1$.*

Proof. The idea of proof is taken from [2]. The coefficients of the system (2) are locally Lipschitz continuous. Therefore, for any initial value X_0 there exists a unique local solution $X(t) = (x_1(t), x_2(t))$ on $[0, \tau_e)$, where $\sup_{t<\tau_e} |X(t)| = +\infty$ (cf. Theorem 6, p. 246, [8]). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0]$, $i = 1, 2$. For any $n \ge n_0$ we define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : X(t) \notin \left(\frac{1}{n}, n\right) \times \left(\frac{1}{n}, n\right)\right\}.$$

Clearly, τ_n is increasing as $n \to +\infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \le \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in \mathbb{R}_+^2$ a.s. for all $t \in [0, +\infty)$. So, we need to show that $\tau_\infty = \infty$ a.s. If this statement is false, there are constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $\mathrm{P}\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \ge n_0$ such that

$$\mathrm{P}\{\tau_n < T\} > \varepsilon, \quad \forall n \ge n_1. \tag{4}$$

For the nonnegative function

$$V(X) = \sum_{i=1}^{2} b_i(x_i - 1 - \ln x_i),\ \ b_1 = a_{21}^{\sup}, b_2 = a_{12}^{\sup}, x_i > 0, i = 1, 2,$$

by the Itô formula, system (2), and definition of the stochastic integral with respect to the noncentered Poisson measure $\nu_2(dt, dz)$ we derive

$$V\big(X(T\wedge\tau_n)\big) = V(X_0) + \int_0^{T\wedge\tau_n} L\big(x_1(t), x_2(t), t\big)dt$$
$$+\sum_{i=1}^{2} b_i\Bigg\{\int_0^{T\wedge\tau_n}\big(x_i(t)-1\big)\sigma_i(t)dw_i(t) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}}\big[\gamma_i(t,z)x_i\big(t^-\big)$$
$$-\ln\big(1+\gamma_i(t,z)\big)\big]\tilde{\nu}_1(dt,dz)$$
$$+\int_0^{T\wedge\tau_n}\int_{\mathbb{R}}\big[\delta_i(t,z)x_i\big(t^-\big)-\ln\big(1+\delta_i(t,z)\big)\big]\tilde{\nu}_2(dt,dz)\Bigg\}, \tag{5}$$

where

$$L(x_1,x_2,t) = \sum_{i=1}^{2} b_i\bigg[(x_i-1)\big(r_i(t)-a_{ii}(t)x_i\big)+\beta_i(t)+x_i\int_{\mathbb{R}}\delta_i(t,z)\Pi_2(dz)\bigg]$$
$$+b_1(x_1-1)a_{12}(t)x_2+b_2(x_2-1)a_{21}(t)x_1.$$

Using the inequality $x_1x_2 \le \varepsilon x_1^2 + x_2^2/(4\varepsilon)$, $\varepsilon > 0$, we derive the estimate

$$L(x_1,x_2,t) \le x_1^2\big(2a_{21}^{\sup}a_{12}^{\sup}\varepsilon - a_{21}^{\sup}a_{11}^{\inf}\big)$$
$$+x_1\big[\big(r_1^{\sup}+a_{11}^{\sup}+\tilde{\delta}_1^{\sup}\Pi_2(\mathbb{R})\big)a_{21}^{\sup}-a_{12}^{\sup}a_{21}^{\inf}\big]-a_{21}^{\sup}\big(r_1^{\inf}-\beta_1^{\sup}\big)$$
$$+x_2^2\bigg(\frac{a_{21}^{\sup}a_{12}^{\sup}}{2\varepsilon}-a_{12}^{\sup}a_{22}^{\inf}\bigg)+x_2\Big[\big(r_2^{\sup}+a_{22}^{\sup}+\tilde{\delta}_2^{\sup}\Pi_2(\mathbb{R})\big)a_{12}^{\sup}$$
$$-a_{21}^{\sup}a_{12}^{\inf}\Big]-a_{12}^{\sup}\big(r_2^{\inf}-\beta_2^{\sup}\big),\ \ \tilde{\delta}_i^{\sup} = \sup_{t\ge 0, z\in\mathbb{R}}\delta_i(t,z),\ \ i=1,2.$$

From the condition $a_{12}^{\sup}a_{21}^{\sup} < a_{11}^{\inf}a_{22}^{\inf}$ we can choose ε such that

$$\frac{a_{21}^{\sup}}{2a_{22}^{\inf}} < \varepsilon < \frac{a_{11}^{\inf}}{2a_{12}^{\sup}}.$$

Therefore, we obtain

$$2a_{21}^{\sup}a_{12}^{\sup}\varepsilon - a_{21}^{\sup}a_{11}^{\inf} < 0,\ \ \frac{a_{21}^{\sup}a_{12}^{\sup}}{2\varepsilon} - a_{12}^{\sup}a_{22}^{\inf} < 0,$$

and under conditions of the theorem, there is a constant $K > 0$, such that $L(x_1, x_2, t) \le K$. Hence, from (5) we have

$$V\big(X(T\wedge\tau_n)\big) \le V(X_0) + K(T\wedge\tau_n) + \sum_{i=1}^{2} b_i\Bigg\{\int_0^{T\wedge\tau_n}\big(x_i(t)-1\big)\sigma_i(t)dw_i(t)$$

$$+\int_0^{T\wedge\tau_n}\int_{\mathbb{R}}\big[\gamma_i(t,z)x_i\big(t^-\big)-\ln\big(1+\gamma_i(t,z)\big)\big]\tilde{\nu}_1(dt,dz)$$
$$\left.+\int_0^{T\wedge\tau_n}\int_{\mathbb{R}}\big[\delta_i(t,z)x_i\big(t^-\big)-\ln\big(1+\delta_i(t,z)\big)\big]\tilde{\nu}_2(dt,dz)\right\}.$$

Whence taking expectations, we have

$$\mathrm{E}\big[V\big(X(T\wedge\tau_n)\big)\big]\le V(X_0)+KT. \tag{6}$$

Set $\Omega_n=\{\omega\in\Omega:\tau_n\le T\}$ for $n\ge n_1$. Then by (4), $\mathrm{P}(\Omega_n)=\mathrm{P}\{\tau_n\le t\}>\varepsilon$, $\forall n\ge n_1$. Note that for every $\omega\in\Omega_n$ there is some i such that $x_i(\tau_n,\omega)$ equals either n or $1/n$. So,

$$V\big(X(\tau_n)\big)\ge\min\{b_1,b_2\}\min\left\{n-1-\ln n,\frac{1}{n}-1+\ln n\right\}.$$

It then follows from (6) that

$$\begin{aligned}V(X_0)+KT&\ge\mathrm{E}\big[\mathbf{1}_{\Omega_n}V\big(X(\tau_n)\big)\big]\\&\ge\varepsilon\min\{b_1,b_2\}\min\left\{n-1-\ln n,\frac{1}{n}-1+\ln n\right\},\end{aligned}$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n. Letting $n\to\infty$ leads to the contradiction $\infty>V(X_0)+KT=\infty$. This completes the proof of the theorem. □

3 Stochastically ultimate boundedness

Definition 1 ([9])**.** The solution $X(t)$ to the system (2) is said to be stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there is a positive constant $\chi=\chi(\varepsilon)>0$, such that for any initial value $X_0\in\mathbb{R}^2_+$, the solution to the system (2) has the property that

$$\limsup_{t\to\infty}\mathrm{P}\big\{|X(t)|>\chi\big\}<\varepsilon.$$

Theorem 2. *The solution $X(t)$ to the system* (2) *is stochastically ultimately bounded for any initial value $X_0\in\mathbb{R}^2_+$.*

Proof. Let τ_n be the stopping time defined in Theorem 1. Applying the Itô formula to the process $V(t,x_i(t))=e^tx_i^p(t)$, $i=1,2$, $p>0$, we obtain for $i,j=1,2$, $i\ne j$,

$$V\big(t\wedge\tau_n,x_i(t\wedge\tau_n)\big)=x_{i0}^p+\int_0^{t\wedge\tau_n}e^sx_i^p(s)\Big\{1+p\big[r_i(s)-a_{ii}(s)x_i(s)$$
$$+a_{ij}(s)x_j(s)\big]+\frac{p(p-1)\sigma_i^2(s)}{2}+\int_{\mathbb{R}}\big[\big(1+\gamma_i(s,z)\big)^p-1-p\gamma_i(s,z)\big]\Pi_1(dz)$$

$$+\int_{\mathbb{R}}\big[\big(1+\delta_i(s,z)\big)^p-1\big]\Pi_2(dz)\Big\}ds+\int_0^{t\wedge\tau_n} pe^s x_i^p(s)\sigma_i(s)dw_i(s)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p\big(s^-\big)\big[\big(1+\gamma_i(s,z)\big)^p-1\big]\tilde{\nu}_1(ds,dz)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p\big(s^-\big)\big[\big(1+\delta_i(s,z)\big)^p-1\big]\tilde{\nu}_2(ds,dz). \tag{7}$$

Under Assumption 1 there are constants $K_i(p)>0$, $i=1,2$, such that

$$e^s x_i^p\Big\{1+p\big[r_i(s)-a_{ii}(s)x_i+a_{ij}(s)x_j\big]+\frac{p(p-1)\sigma_i^2(s)}{2}+$$
$$+\int_{\mathbb{R}}\big[\big(1+\gamma_i(s,z)\big)^p-1-p\gamma_i(s,z)\big]\Pi_1(dz)+\int_{\mathbb{R}}\big[\big(1+\delta_i(s,z)\big)^p-1\big]\Pi_2(dz)\Big\}$$
$$\le e^s\big(x_i^p K_i(p)-pa_{ii}^{\inf}x_i^{p+1}+pa_{ij}^{\sup}x_i^p x_j\big). \tag{8}$$

From (7) and (8) for the process $G(t,x_1(t),x_2(t))=c_1V(t,x_1(t))+c_2V(t,x_2(t))$, where $c_i>0$, $i=1,2$, some constants, which we will define later, we have

$$G\big(t\wedge\tau_n,x_1(t\wedge\tau_n),x_2(t\wedge\tau_n)\big)\le c_1x_{10}^p+c_2x_{20}^p+\int_0^{t\wedge\tau_n} e^s\Big[c_1\Big(x_1^p(s)K_1(p)$$
$$-pa_{11}^{\inf}x_1^{p+1}(s)+pa_{12}^{\sup}x_1^p(s)x_2(s)\Big)+c_2\Big(x_2^p(s)K_2(p)-pa_{22}^{\inf}x_2^{p+1}(s)$$
$$+pa_{21}^{\sup}x_1(s)x_2^p(s)\Big)\Big]ds+\sum_{i=1}^{2}c_i\Big\{\int_0^{t\wedge\tau_n} pe^s x_i^p(s)\sigma_i(s)dw_i(s)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p\big(s^-\big)\big[\big(1+\gamma_i(s,z)\big)^p-1\big]\tilde{\nu}_1(ds,dz)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p\big(s^-\big)\big[\big(1+\delta_i(s,z)\big)^p-1\big]\tilde{\nu}_2(ds,dz)\Big\}=c_1x_{10}^p+c_2x_{20}^p$$
$$+\int_0^{t\wedge\tau_n} e^s L\big(x_1(s),x_2(s)\big)ds+\sum_{i=1}^{2}c_iS_i^{stoch}, \tag{9}$$

where S_i^{stoch}, $i=1,2$, are the sums of corresponding stochastic integrals in (9). From the Young inequality we have

$$\begin{aligned} x_1^p x_2 &\le \theta_1 x_1^{p+1} + \frac{1}{p+1}\frac{1}{\theta_1^p}\left(\frac{p}{p+1}\right)^p x_2^{p+1}, \theta_1 = \frac{p}{p+1}\frac{a_{11}^{\inf}}{a_{12}^{\sup}},\\ x_1 x_2^p &\le \theta_2 x_2^{p+1} + \frac{1}{p+1}\frac{1}{\theta_2^p}\left(\frac{p}{p+1}\right)^p x_1^{p+1}, \theta_2 = \frac{p}{p+1}\frac{a_{22}^{\inf}}{a_{21}^{\sup}}. \end{aligned} \tag{10}$$

So, applying (10), we derive the estimate

$$\begin{aligned} L(x_1,x_2) \le \sum_{i=1}^{2} c_i K_i(p) x_i^p - x_1^{p+1}\Big[c_1 p\big(a_{11}^{\inf} - a_{12}^{\sup}\theta_1\big)\\ - c_2 a_{21}^{\sup}\frac{1}{\theta_2^p}\left(\frac{p}{p+1}\right)^{p+1}\Big] - x_2^{p+1}\Big[c_2 p\big(a_{22}^{\inf} - a_{21}^{\sup}\theta_2\big)\\ - c_1 a_{12}^{\sup}\frac{1}{\theta_1^p}\left(\frac{p}{p+1}\right)^{p+1}\Big]. \end{aligned}$$

Using the condition $a_{12}^{\sup} a_{21}^{\sup} < a_{11}^{\inf} a_{22}^{\inf}$, we can choose constants $c_i, i = 1, 2$, such that

$$\frac{a_{21}^{\sup}}{a_{11}^{\inf}}\left(\frac{a_{21}^{\sup}}{a_{22}^{\inf}}\right)^p < \frac{c_1}{c_2} < \frac{a_{22}^{\inf}}{a_{12}^{\sup}}\left(\frac{a_{11}^{\inf}}{a_{12}^{\sup}}\right)^p. \tag{11}$$

Due to (11) we have

$$\begin{aligned} c_1 p\big(a_{11}^{\inf} - a_{12}^{\sup}\theta_1\big) - c_2 a_{21}^{\sup}\frac{1}{\theta_2^p}\left(\frac{p}{p+1}\right)^{p+1} > 0,\\ c_2 p\big(a_{22}^{\inf} - a_{21}^{\sup}\theta_2\big) - c_1 a_{12}^{\sup}\frac{1}{\theta_1^p}\left(\frac{p}{p+1}\right)^{p+1} > 0. \end{aligned}$$

Therefore, there is a constant $C(p) > 0$ such that $L(x_1, x_2) \le C(p)$. So, taking the expectation in (9), we obtain

$$\mathrm{E}\big[G\big(t\wedge\tau_n, x_1(t\wedge\tau_n), x_2(t\wedge\tau_n)\big)\big] \le c_1 x_{10}^p + c_2 x_{20}^p + C(p)e^t.$$

Letting $n \to \infty$ leads to the estimate

$$e^t \mathrm{E}\big[c_1 x_1^p(t) + c_2 x_2^p(t)\big] \le c_1 x_{10}^p + c_2 x_{20}^p + C(p)e^t.$$

So, we have

$$\limsup_{t\to\infty} \mathrm{E}\big[c_1 x_1^p(t) + c_2 x_2^p(t)\big] \le C(p). \tag{12}$$

For $X = (x_1, x_2) \in \mathbb{R}_+^2$ we have $|X|^p \le \frac{2^{p/2}}{\min(c_1,c_2)}(c_1 x_1^p + c_2 x_2^p)$, therefore, from (12) $\limsup_{t\to\infty} E[|X(t)|^p] \le M(p) = \frac{2^{p/2}}{\min(c_1,c_2)} C(p)$. Let $\chi > (M(p)/\varepsilon)^{1/p}$, $p > 0$, $\forall\varepsilon \in (0, 1)$. Then applying the Chebyshev inequality yields

$$\limsup_{t\to\infty} \mathrm{P}\big\{|X(t)| > \chi\big\} \le \frac{1}{\chi^p}\limsup_{t\to\infty} \mathrm{E}\big[|X(t)|^p\big] \le \frac{M(p)}{\chi^p} < \varepsilon.$$

The proof is completed. □

4 Stochastic permanence and strong persistence in the mean

Definition 2 ([10])**.** The solution $X(t)$ to the system (2) is said to be stochastically permanent if for any $\varepsilon > 0$, there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that for $i = 1, 2$

$$\liminf_{t\to\infty} \mathrm{P}\big\{x_i(t) \le H\big\} \ge 1-\varepsilon, \quad \liminf_{t\to\infty} \mathrm{P}\big\{x_i(t) \ge h\big\} \ge 1-\varepsilon,$$

for any inial value $X_0 \in \mathbb{R}^2_+$.

Theorem 3. *If* $\min_{i=1,2}\inf_{t\ge 0}(r_i(t) - \beta_i(t)) > 0$*, where* $r_i(t)$ *and* $\beta_i(t)$ *are defined respectively in* (2) *and* (3)*, then the solution* $X(t)$ *to the system* (2) *with the initial condition* $X_0 \in \mathbb{R}^2_+$ *is stochastically permanent.*

Proof. Using the same arguments as in the corresponding part of the proof of Theorem 3 ([7]) and the condition $\min_{i=1,2}\inf_{t\ge 0}(r_i(t) - \beta_i(t)) > 0$, we can conclude that for sufficiently small $0 < \theta < 1$ and sufficiently small $\lambda = \lambda(\theta) > 0$ there exists a constant $K > 0$ such that

$$\begin{aligned} d\big[e^{\lambda t}\big(1+U_i(t)\big)^{\theta}\big] &\le K e^{\lambda t}\theta dt - \theta e^{\lambda t}\big(1+U_i(t)\big)^{\theta-1}U_i(t)\sigma_i(t)dw_i(t) \\ &\quad + e^{\lambda t}\int_{\mathbb{R}}\bigg[\bigg(1+\frac{U_i(t^-)}{1+\gamma_i(t,z)}\bigg)^{\theta} - \big(1+U_i\big(t^-\big)\big)^{\theta}\bigg]\tilde{\nu}_1(dt,dz) \\ &\quad + e^{\lambda t}\int_{\mathbb{R}}\bigg[\bigg(1+\frac{U_i(t^-)}{1+\delta_i(t,z)}\bigg)^{\theta} - \big(1+U_i\big(t^-\big)\big)^{\theta}\bigg]\tilde{\nu}_2(dt,dz), \end{aligned} \tag{13}$$

where $U_i(t) = 1/x_i(t)$, $i = 1, 2$. Let τ_n be the stopping time defined in Theorem 1. Then by integrating (13) and taking the expectation we have

$$\mathrm{E}\big[e^{\lambda(t\wedge\tau_n)}\big(1+U_i(t\wedge\tau_n)\big)^{\theta}\big] \le \bigg(1+\frac{1}{x_{i0}}\bigg)^{\theta} + \frac{\theta}{\lambda}K\big(e^{\lambda t}-1\big).$$

If $n \to \infty$, then we obtain the estimate

$$e^t\mathrm{E}\big[\big(1+U_i(t)\big)^{\theta}\big] \le \bigg(1+\frac{1}{x_{i0}}\bigg)^{\theta} + \frac{\theta}{\lambda}K\big(e^{\lambda t}-1\big). \tag{14}$$

From (14) we have

$$\begin{aligned} \limsup_{t\to\infty}\mathrm{E}\bigg[\bigg(\frac{1}{x_i(t)}\bigg)^{\theta}\bigg] &= \limsup_{t\to\infty}\mathrm{E}\big[U_i^{\theta}(t)\big] \\ &\le \limsup_{t\to\infty}\mathrm{E}\big[\big(1+U_i(t)\big)^{\theta}\big] \le \frac{\theta K}{\lambda}, i=1,2. \end{aligned} \tag{15}$$

From (12) and (15) by the Chebyshev inequality we can derive, that for arbitrary $\varepsilon \in (0, 1)$, there are positive constants $H = H(\varepsilon)$ and $h = h(\varepsilon)$ such that

$$\liminf_{t\to\infty}\mathrm{P}\big\{x_i(t) \le H\big\} \ge 1-\varepsilon, \quad \liminf_{t\to\infty}\mathrm{P}\big\{x_i(t) \ge h\big\} \ge 1-\varepsilon, i=1,2.$$

The proof is completed. □

Definition 3 ([11])**.** The solution $X(t) = (x_1(t), x_2(t))$, $t \geq 0$, to the system (2) is said to be strongly persistent in the mean if for every initial data $X_0 > 0$, we have $\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds > 0$ a.s., $i = 1, 2$.

Theorem 4. *If* $\bar{p}_{i*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_i(s)ds > 0$, $i = 1, 2$, *where* $p_i(s) = r_i(s) - \beta_i(s)$, $i = 1, 2$, $r_i(t)$ *and* $\beta_i(t)$ *are defined respectively in* (2) *and* (3)*, then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds \geq \frac{\bar{p}_{i*}}{a_{ii}^{\sup}}.$$

Therefore, the solution $X(t)$ *to the system* (2) *with the initial condition* $X_0 \in \mathbb{R}_+^2$ *will be strongly persistent in the mean.*

Proof. For the system (2) by the Itô formula, we have for $i, j = 1, 2, i \neq j$,

$$\begin{aligned}\ln x_i(t) = \ln x_{i0} + \int_0^t p_i(s)ds - \int_0^t a_{ii}(s)x_i(s)ds + \int_0^t a_{ij}(s)x_j(s)ds \\ + \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln\big(1+\gamma_i(s,z)\big)\tilde{\nu}_1(ds,dz) \\ + \int_0^t\int_{\mathbb{R}} \ln\big(1+\delta_i(s,z)\big)\tilde{\nu}_2(ds,dz) \geq \ln x_{i0} + \int_0^t p_i(s)ds - a_{ii}^{\sup}\int_0^t x_i(s)ds + M_i(t),\end{aligned} \tag{16}$$

where the martingales

$$\begin{aligned}M_i(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln\big(1+\gamma_i(s,z)\big)\tilde{\nu}_1(ds,dz) \\ + \int_0^t\int_{\mathbb{R}} \ln\big(1+\delta_i(s,z)\big)\tilde{\nu}_2(ds,dz), i = 1, 2,\end{aligned} \tag{17}$$

have quadratic variation

$$\begin{aligned}\langle M_i, M_i\rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln^2\big(1+\gamma_i(s,z)\big)\Pi_1(dz)ds \\ + \int_0^t\int_{\mathbb{R}} \ln^2\big(1+\delta_i(s,z)\big)\Pi_2(dz)ds \leq Kt, i = 1, 2.\end{aligned}$$

The rest of the proof is the same as the proof of Theorem 6 in [7]. The proof is completed. □

5 Nonpersistence

Definition 4 ([2])**.** System (2) is said to be nonpersistent, if there are positive constants q_1, q_2 such that $\lim_{t\to\infty} x_1^{q_1}(t)x_2^{q_2}(t) = 0$ a.s.

Theorem 5. *If* $a_{22}^{\inf}\bar{p}_1^* + a_{12}^{\sup}\bar{p}_2^* < 0$, *or* $a_{11}^{\inf}\bar{p}_2^* + a_{21}^{\sup}\bar{p}_1^* < 0$, *where*

$$\bar{p}_i^* = \limsup_{t\to\infty}\frac{1}{t}\int_0^t p_i(s)ds,\;\; p_i(t) = r_i(t) - \beta_i(t),\;\; i = 1, 2,$$

$r_i(t)$ *and* $\beta_i(t)$ *are defined respectively in* (2) *and* (3)*, then the system* (2) *with the initial condition* $X_0 \in \mathbb{R}_+^2$ *will be nonpersistent.*

Proof. We prove the assertion of the theorem under condition $a_{22}^{\inf}\bar{p}_1^* + a_{12}^{\sup}\bar{p}_2^* < 0$. The proof of the theorem under condition $a_{11}^{\inf}\bar{p}_2^* + a_{21}^{\sup}\bar{p}_1^* < 0$ is similar. From the equality in (16), we have

$$\begin{aligned}
a_{22}^{\inf}\ln x_1(t) + a_{12}^{\sup}\ln x_2(t) &= a_{22}^{\inf}\ln x_{10} + a_{12}^{\sup}\ln x_{20} + a_{22}^{\inf}\int_0^t p_1(s)ds \\
&\quad + a_{12}^{\sup}\int_0^t p_2(s)ds - a_{22}^{\inf}\int_0^t a_{11}(s)x_1(s)ds + a_{12}^{\sup}\int_0^t a_{21}(s)x_1(s)ds \\
&\quad + a_{22}^{\inf}\int_0^t a_{12}(s)x_2(s)ds - a_{12}^{\sup}\int_0^t a_{22}(s)x_2(s)ds + a_{22}^{\inf}M_1(t) + a_{12}^{\sup}M_2(t) \\
&\le a_{22}^{\inf}\ln x_{10} + a_{12}^{\sup}\ln x_{20} + a_{22}^{\inf}\int_0^t p_1(s)ds + a_{12}^{\sup}\int_0^t p_2(s)ds \\
&\quad + a_{22}^{\inf}M_1(t) + a_{12}^{\sup}M_2(t),
\end{aligned} \tag{18}$$

where the martingales $M_i(t), i = 1, 2$, are defined in (17). Then the strong law of large numbers for local martingales ([12]) yields $\lim_{t\to\infty} M_i(t)/t = 0, i = 1, 2$, a.s. Therefore, from (18) we have

$$\limsup_{t\to\infty}\frac{1}{t}\big[a_{22}^{\inf}\ln x_1(t) + a_{12}^{\sup}\ln x_2(t)\big] \le a_{22}^{\inf}\bar{p}_1^* + a_{12}^{\sup}\bar{p}_2^* < 0 \quad \text{a.s.}$$

So, $\lim_{t\to\infty} x_1^{a_{22}^{\inf}}(t)x_2^{a_{12}^{\sup}}(t) = 0$ a.s. The proof is completed. □

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