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Almost everywhere continuity of conditional expectations

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Abstract A necessary and sufficient condition on a sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ of σ-subalgebras which assures convergence almost everywhere of conditional expectations for functions in L^∞ is given. It is proven that for $f \in L^\infty(\mathscr{A})$

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{a.e.} \mathsf{E}(f|\mathscr{A}_{\mu a.e.}).$$

Keywords Conditional expectations, probability

1 Introduction

Ever since the appearance of the Martingales Convergence Theorem, there have been attempts to find criteria for a larger group of sequences of σ-subalgebras that imply the convergence of conditional expectations. Several attempts have been made in this regard (see [8] for an excellent summary), but few have dealt with conditions that assure convergence a.e.

In this paper we will give necessary and sufficient conditions for almost-everywhere convergence of $\mathsf{E}(f|\mathscr{A}_n)$ for $f \in L^\infty(\mathscr{A})$. This result is new. In a sense it closes the question of finding the necessary and sufficient conditions on a sequence $\{\mathscr{A}_n\}$ of σ-subalgebras which assure a.e. convergence of conditional expectations.

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Conditional expectations convergence theorems have many applications in probability theory, data science, economics, finance, statistics and give alternative proofs of theorems such as Kolmogorov large-numbers, Levy generalized Borel–Cantelli, Radon–Nikodym. For potential applications, see [6] and [5].We would like to remark that the existence of convergence a.e. provides the possibility of using tools not necessarily available in a given situation.

Let $\mathscr{A}$ be a σ-algebra on a set $\mathbb{X}$ with a probability measure μ. Given a sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ of σ-subalgebras of $\mathscr{A}$, it is widely known that we have convergence a.e. in the case when the sequence is monotone. However, this is not the case in more general cases. For example, in the set-theoretical approach of Fetter [4], in which she estabilished very natural conditions for convergence of sigma algebras, one has convergence in L^p but not neccesarily convergence a.e. [1].

In [2], we defined two σ-subalgebras $\mathscr{A}_\mu$ and $\mathscr{A}_\perp$ which satisfy

$$\underline{\mathscr{A}} \subset \mathscr{A}_\mu \subset \mathscr{A}_\perp \subset \overline{\mathscr{A}},$$

where $\underline{\mathscr{A}} = \bigvee_{m=1}^\infty \bigcap_{n=m}^\infty \mathscr{A}_n$ and $\overline{\mathscr{A}} = \bigcap_{m=1}^\infty \bigvee_{n=m}^\infty \mathscr{A}_n$ are the inferior and the superior limits of the σ-subalgebras $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$. With this setup we proved that we have convergence in L^p if and only if $\mathscr{A}_\mu = \mathscr{A}_\perp$.

The quest of this paper was to see if we could establish somewhat similar conditions on the σ-subalgebras in order to have convergence a.e. It is clear that in view of [1] one needs stronger conditions and has to deal with sets that will not necessarily be σ-subalgebras. To do so, we strengthen the conditions and define a set $\mathscr{A}_{\mu a.e.}$, but loose the assumption that it is a σ-subalgebra, although in the case when it is such, for $f \in L^\infty(\mathscr{A}_{\mu a.e.})$ we proved that

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{a.e.} \mathsf{E}(f|\mathscr{A}_{\mu a.e.}) = f.$$

To deal with the other part of the problem, we first characterize the conditions through a set $W^{\perp a.e.}$ to have, for $f \in L^\infty$, convergence a.e. to zero if and only if $f \in W^{\perp a.e.}$. Finally we define a family of σ-subalgebras $\mathfrak{D}$ and establish what conditions are necessary and sufficient for convergence a.e.

It has been pointed out by one of the reviewers that in the literature the results are usually referred to L^p while our results use L^∞. We believe that the results will hold in L^p but the proofs have been elusive.

2 Previous work

As usual, we will use the notation $\mathsf{E}(f|\mathscr{A})$ for the conditional expectation of f given the σ-algebra $\mathscr{A}$. A^c will stand for $\mathbb{X} \setminus A$, χ_A for the characteristic function of a set A, $A \triangle B$ for the symmetric difference of the sets A and B, and $A = B\ a.e$ for $\mu(A \triangle B) = 0$. All the σ-subalgebras that we deal with are considered to be complete. Some well-known results on a.e.-convergence and L^p-convergence ($1 \le p < \infty$) are presented in [7, page 124, section IV, 3.2].

Theorem 2.1 (Martingales [7]). *If $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ is monotone increasing sequence of σ-subalgebras of $\mathscr{A}$, that is, $\mathscr{A}_n \subset \mathscr{A}_{n+1}$ for any $n \in \mathbb{N}$, then*

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow[L^p]{a.e.} \mathsf{E}\left(f \,\middle|\, \bigvee_{n=1}^{\infty} \mathscr{A}_n\right)$$

for every $f \in L^p(\mathscr{A})$, where $\bigvee_{n=1}^{\infty} \mathscr{A}_n$ stands for the minimum σ-algebra that contains $\bigcup_{n=1}^{\infty} \mathscr{A}_n$. (Or if $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ is monotone decreasing, that is, $\mathscr{A}_n \supset \mathscr{A}_{n+1}$, then $\mathsf{E}(f|\mathscr{A}_n) \xrightarrow[L^p]{a.e.} \mathsf{E}(f\,|\bigcap_{n=1}^{\infty} \mathscr{A}_n))$.

In [3], Boylan introduced a Hausdorff metric in the space of σ-algebras. It gives us a relationship between Cauchy sequences of σ-subalgebras and L^p-convergence of conditional expectations. Using the notation $A \triangle B = (A \setminus B) \cup (B \setminus A)$, Boylan's Theorem 4 reads as follows.

Theorem 2.2 (Boylan, Equiconvergence). *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space of σ-algebras with the Hausdorff metric, that is,*

$$d(\mathscr{A}_n, \mathscr{A}_m) = \sup_{A\in\mathscr{A}_n}\Big(\inf_{B\in\mathscr{A}_m} \mu(A \triangle B)\Big) + \sup_{B\in\mathscr{A}_m}\Big(\inf_{A\in\mathscr{A}_n} \mu(A \triangle B)\Big).$$

The there is a σ-subalgebra $\mathscr{D}$ such that

$$\lim_{n\to\infty} d(\mathscr{A}_n, \mathscr{D}) = 0,$$

and

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{L^p} \mathsf{E}(f|\mathscr{D}),$$

for every $f \in L^p(\mathscr{A})$ with $1 \le p < \infty$.

Another approach was given by Fetter [4]. She proved that if the lim sup of a sequence of σ-algebras coincides with the lim inf, then we have convergence in L^p. Indeed, Theorem 4 of [4] reads as follows.

Theorem 2.3 (Fetter). *If $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ is such that $\underline{\mathscr{A}} = \overline{\mathscr{A}}$, where*

$$\underline{\mathscr{A}} = \bigvee_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathscr{A}_n \quad \textit{and} \quad \overline{\mathscr{A}} = \bigcap_{m=1}^{\infty} \bigvee_{n=m}^{\infty} \mathscr{A}_n,$$

then

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow[L^p]{} \mathsf{E}(f|\overline{\mathscr{A}}),$$

for every $f \in L^p(\mathscr{A})$, $1 \le p < \infty$.

Since the condition of the above theorem is fulfilled for monotone sequences of σ-algebras, Fetter's result implies that of the monotone convergence theorem in the case of L^p. However, we point out that in [1] it was proved that the condition $\underline{\mathscr{A}} = \overline{\mathscr{A}}$ does not imply convergence almost everywhere.

Given a sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ of σ-algebras, two relevant σ-algebras were defined by Alonso and Brambila in [2]:

$$\mathscr{A}_\mu = \left\{A \in \mathscr{A} \ : \ \exists \{A_n\}_{n\in\mathbb{N}},\, A_n \in \mathscr{A}_n,\ \lim_{n\to\infty} \mu(A_n \triangle A) = 0\right\}, \tag{2.1}$$

and if we consider the set:

$$W = \left\{g \in L^2(\mathscr{A}) \ : \ \exists n_k \text{ and, } A_{n_k} \in \mathscr{A}_{n_k}, \text{ with } \chi_{A_{n_k}} \overset{\text{weakly}}{\longrightarrow} g\right\}. \tag{2.2}$$

$\mathscr{A}_\perp$ was defined as the minimal complete σ-algebra such that g is $\mathscr{A}_\perp$ measurable for all $g \in W$. The importance of these σ-algebras is clear due to the following results, see Lemmas 1.3, 2.4 and Proposition 3.3 in [2].

Lemma 2.4. *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-subalgebras. Then $\mathsf{E}(f|\mathscr{A}_n) \overset{L^2}{\longrightarrow} \mathsf{E}(f|\mathscr{A}_\mu) = f$ for every $f \in L^2(\mathscr{A}_\mu)$.*

Lemma 2.5. *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-subalgebras and $f \in L^2(\mathscr{A}_\perp)^\perp$. Then $\mathsf{E}(f|\mathscr{A}_n) \longrightarrow 0$ in $L^2(\mathscr{A})$ norm.*

Theorem 2.6. *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-subalgebras and p such that $1 \le p < \infty$. Then $\mathscr{A}_\mu = \mathscr{A}_\perp$ if and only if for every $f \in L^p(\mathscr{A})$, $\mathsf{E}(f|\mathscr{A}_n)$ converges in $L^p(\mathscr{A})$. Furthermore, if $\mathscr{A}_\infty = \mathscr{A}_\mu = \mathscr{A}_\perp$, then*

$$\mathsf{E}(f|\mathscr{A}_n) \overset{L^p(\mathscr{A})}{\longrightarrow} \mathsf{E}(f|\mathscr{A}_\infty).$$

Now, since these σ-algebras satisfy the relationship

$$\underline{\mathscr{A}} \subseteq \mathscr{A}_\mu \subseteq \mathscr{A}_\perp \subseteq \overline{\mathscr{A}}, \tag{2.3}$$

we have that if $\underline{A} = \overline{A}$, then Fetter's theorem becomes an immediate corollary.

Finally we point out that none of the theorems but Theorem 2.1 deal with the problem of convergence a.e.

3 A useful lemma

In [2], the following concept was introduced: a sequence of σ-subalgebras $\{\mathscr{A}_n\}$ μ-approaches a σ-subalgebra $\mathscr{D}$, if for each $D \in \mathscr{D}$ there are $A_n \in \mathscr{A}_n$ such that $\mu(A_n \triangle D) \to 0$. It was established that when a sequence of σ-subalgebras $\{\mathscr{A}_n\}$ μ-approaches a σ-subalgebra $\mathscr{D}$, and only in that case, we have, for $f \in L^p(\mathscr{D})$ $(1 \le p < \infty)$, that

$$\mathsf{E}(f|\mathscr{A}_n) \overset{L^p}{\longrightarrow} \mathsf{E}(f|\mathscr{D}) = f.$$

It is clear that in order to have convergence a.e. we need a stronger concept for sets in $\{\mathscr{A}_n\}$ to express their approaching sets in $\mathscr{D}$.

We begin by establishing the following lemma.

Lemma 3.1. *Let $(\mathbb{X}, \mathscr{A}, \mu)$ be a probability space with σ-algebra $\mathscr{A}$ and measure μ. If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of elements in $\mathscr{A}$ and $A \in \mathscr{A}$, the following statements are equivalent:*

i) $\chi_{A_n} \xrightarrow[a.e.]{} \chi_A$.

ii) $A = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n$ *a.e., that is,* $\mu(A \triangle \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} A_n) = \mu(A \triangle \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n) = 0$.

iii) $\lim_{N\to\infty} \mu(\bigcup_{n>N} (A_n \triangle A)) = 0$.

Proof. Notice that $\chi_{A_n} \to \chi_A$ $a.e.$ implies that for almost all $x \in \mathbb{X}$, there is an $N_x \in \mathbb{N}$ such that, if $n > N_x$, then

$$\left|\chi_{A_n}(x) - \chi_A(x)\right| < \frac{1}{2}. \tag{3.1}$$

To prove $i) \to ii)$, we first take a look at the elements of A. Since $|\chi_{A_n}(x) - \chi_A(x)| = |\chi_{A_n^c}(x) - \chi_{A^c}(x)|$, we have that, for $n > N_x$ and almost all $x \in A$, $\chi_{A_n^c}(x) < 1/2$, and so $x \in A_n$. Therefore, $A \subset \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n$ $a.e.$

Using the same argument for A^c, we get $A^c \subset \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n^c$ $a.e.$ Thus $A \supset \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n$ $a.e.$

Since

$$\bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n \subset \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n,$$

we have

$$\left(\bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n\right) \subset A \subset \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n \subset \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n \ a.e.$$

Therefore $i \to ii)$.

We will prove now that $iii) \to i)$. Let

$$M = \bigcap_{N=1}^{\infty} \bigcup_{n>N} (A_n \triangle A).$$

Then, by hypothesis, $\mu(M) = 0$. Now, as $M^c = \bigcup_{N=1}^{\infty} \bigcap_{n>N} (A_n \triangle A)^c$, for almost all $x \in M^c$ there is an $N \in \mathbb{N}$ such that

$$x \in \bigcap_{n>N} (A_n \triangle A)^c,$$

and hence $x \in (A_n \triangle A)^c$ for all $n > N$. That is,

$$0 = \chi_{A_n \triangle A}(x) = \left|\chi_A(x) - \chi_{A_n}(x)\right|, \quad \text{for } n > N.$$

Finally, to prove that $ii)$ implies $iii)$, we notice that

$$\mu\left(\bigcup_{n>N} (A_n \triangle A)\right) = \mu\left(\left(\bigcup_{n>N} A_n\right) \cap A^c\right) + \mu\left(\left(\bigcup_{n>N} A_n^c\right) \cap A\right).$$

Since by hypothesis

$$0 = \mu\Big(A \triangle \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n\Big) \geq \mu\Big(A \setminus \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n\Big) = \mu\Big(A \cap \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n^c\Big)$$
$$= \lim_{N\to\infty} \mu\Big(A \cap \bigcup_{n>N} A_n^c\Big),$$

and

$$0 = \mu\Big(A \triangle \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n\Big) \geq \mu\Big(\bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n \setminus A\Big) = \lim_{N\to\infty} \mu\Big(\bigcup_{n>N} A_n \cap A^c\Big),$$

we get

$$\lim_{N\to\infty} \mu\Big(\bigcup_{n>N} (A_n \triangle A)\Big) = 0. \qquad \square$$

4 Necessary and sufficient conditions for almost everywhere convergence

4.1 On almost everywhere convergence of characteristic functions

Definition 4.1. Let $\mathscr{B}$ be a σ-subalgebra of $\mathscr{A}$. The seminorm $\|\cdot\|_{\mathscr{B}}$ for $f \in L^{\infty}(d\mu)$ is defined as

$$\|f\|_{\mathscr{B}} = \big\|\mathsf{E}(f|\mathscr{B})\big\|_{\infty}.$$

The relationship between the measure of a set and the above norm will be shown in the appendix. We prove there that for $A \in \mathscr{A}$, $\|\chi_A\|_{\mathscr{B}} = \sup_{\substack{B \in \mathscr{B} \\ \mu(B)>0}} \frac{\mu(A\cap B)}{\mu(B)}$.

Definition 4.2. We say that $A \in \mathscr{A}$ is uniformly covered by the sequence of σ-subalgebras $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ if there is a sequence $\{A_n \in \mathscr{A}_n\}_{n\in\mathbb{N}}$ such that

i) $A = \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} A_n \quad a.e.$
ii) $\|\chi_{A\setminus A_n}\|_{\mathscr{A}_n} \to 0$ as $n \to \infty$.

The next lemma shows that actually we can relax the condition *ii*) a little bit.

Lemma 4.3. *A is uniformly covered by $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ if and only if for any $r > 0$ there is a sequence $\{A_n^r\}_{n\in\mathbb{N}}$, $A_n^r \in \mathscr{A}_n$, and $M_r \in \mathbb{N}$ such that*

i') $A = \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n^r = \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n^r \quad a.e.$
ii') $\|\chi_{A\setminus A_n^r}\|_{\mathscr{A}_n} < r$ *if* $n > M_r$.

Proof. By Lemma 3.1 the condition $i)$ means that

$$\lim_{N\to\infty} \mu\Big(\bigcup_{n>N} \big(A \triangle A_n^r\big)\Big) = 0.$$

Thus, for $r = 1$ let $N_1 > M_1$ with

$$\mu\Big(\bigcup_{n>N_1}\big(A \triangle A_n^1\big)\Big) < 1.$$

In general, for $k \in \mathbb{N}$, let $r_k = 2^{-k}$ and $\tilde{N}_k$ be such that

$$\mu\Big(\bigcup_{n>\tilde{N}_k}\big(A \triangle A_n^{r_k}\big)\Big) < r_k,$$

and M_{r_k} be such that

$$\|\chi_{A\setminus A_n^{r_k}}\|_{\mathscr{A}_n} < r_k \quad \text{for } n > M_{r_k}.$$

Let N_k be a strictly increasing sequence such that $N_k \geq \max\{\tilde{N}_k, M_{r_k}\}$, and let us define $A_n \in \mathscr{A}_n$ as

$$A_n = A_n^{r_k} \quad \text{if } N_k \leq n < N_{k+1}.$$

Then

$$\|\chi_{A\setminus A_n}\|_{\mathscr{A}_n} = \|\chi_{A\setminus A_n^{r_k}}\|_{\mathscr{A}_n} < \frac{1}{2^k},$$

thus $\lim_{n\to\infty}\|\chi_{A\setminus A_n}\|_{\mathscr{A}_n} = 0$. Finally,

$$\begin{aligned}\mu\Big(\bigcup_{n\geq N_{k'}}(A \triangle A_n)\Big) &= \mu\Big(\bigcup_{k=k'}^{\infty}\bigcup_{N_k\leq n<N_{k+1}}(A \triangle A_n)\Big)\\ &\leq \sum_{k=k'}^{\infty}\mu\Big(\bigcup_{N_k\leq n<N_{k+1}} A \triangle A_n^{r_k}\Big) < \sum_{k=k'}^{\infty}\frac{1}{2^k} = \frac{1}{2^{k'-1}}.\end{aligned}$$

Now, since $\mu(\bigcup_{n\geq N} A \triangle A_n)$ is monotone, we have $\lim_{N\to\infty}\mu(\bigcup_{n\geq N} A \triangle A_n) = 0$. □

We will say that a sequence of sets $\{A_n \in \mathscr{A}_n\}$ uniformly covers a set $A \in \mathscr{A}$ if conditions $i)$ and $ii)$ of Definition 4.2 are satisfied.

It is easily seen that, if $\{A_n \in \mathscr{A}_n\}$ uniformly covers a set $A \in \mathscr{A}$ and $\{A'_n \in \mathscr{A}_n\}$ is such that $A_n \subset A'_n$ for all $n \in \mathbb{N}$ and $\chi_{A'_n} \longrightarrow \chi_A$ *a.e.*, then it uniformly covers A.

Lemma 4.4. *If A and B are sets in $\mathscr{A}$ uniformly covered by $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$, then so are $A \bigcup B$ and $A \bigcap B$.*

Proof. Let $A_n \in \mathscr{A}_n$ and $B_n \in \mathscr{A}_n$ sequences of sets that uniformly cover A and B, respectively. By property $i)$ of the definition of uniform covering we have that $\chi_{A_n} \xrightarrow{a.e.} \chi_A$ and $\chi_{B_n} \xrightarrow{a.e.} \chi_B$. So $\chi_{A_n \bigcap B_n} = \chi_{A_n}\chi_{B_n} \xrightarrow{a.e.} \chi_A\chi_B = \chi_{A\bigcap B}$ and $\chi_{A_n\cup B_n} \xrightarrow{a.e.} \chi_{A\cup B}$.

Since $(\chi_A\chi_B - \chi_{A_n}\chi_{B_n})_+ \leq (\chi_A(\chi_B - \chi_{B_n})_+ + (\chi_{B_n}(\chi_A - \chi_{A_n})_+$, we have that

$$\mathsf{E}(\chi_{A\cap B\setminus(A_n\cap B_n)}|\mathscr{A}_n) \leq \mathsf{E}(\chi_{A\cap A_n^c}|\mathscr{A}_n) + \mathsf{E}(\chi_{B\cap B_n^c}|\mathscr{A}_n) \xrightarrow{L^\infty} 0.$$

A similar argument can be used for the case of the union of two sets. □

Lemma 4.5. *If A is uniformly covered by $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$, then*

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}(\chi_A|\mathscr{A}_n)(x) \le \chi_A(x) \ a.e.$$

Proof. Let $\{A_n\}$ be a sequence with the properties of Definition 4.2. Then

$$\begin{aligned}\mathsf{E}(\chi_A|\mathscr{A}_n) &= \mathsf{E}(\chi_A\chi_{A_n} + \chi_A\chi_{A_n^c}|\mathscr{A}_n)\\ &= \chi_{A_n}\mathsf{E}(\chi_A|\mathscr{A}_n) + \mathsf{E}(\chi_{A\setminus A_n}|\mathscr{A}_n)\\ &\le \chi_{A_n} + \|\chi_{A\setminus A_n}\|_{\mathscr{A}_n} \longrightarrow \chi_A \text{ a.e.}\end{aligned}$$ □

In view of the proof, we can actually relax somewhat the condition of uniformly covering to get a similar result.

Lemma 4.6. *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-subalgebras. If $A \in \mathscr{A}$ is such that there is a sequence $\{A_n \in \mathscr{A}_n\}_{n\in\mathbb{N}}$ satisfying*

i) $A = \bigcup_{N=1}^{\infty}\bigcap_{n\ge N} A_n = \bigcap_{N=1}^{\infty}\bigcup_{n\ge N} A_n$ *a.e.,*
ii) $\mathsf{E}(\chi_{A\setminus A_n}|\mathscr{A}_n) \to 0$ *a.e. as* $n \to \infty$,

then

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}(\chi_A|\mathscr{A}_n)(x) \le \chi_A(x) \ a.e.$$

Proof. The proof is exactly the same as that of the above lemma. □

Notice that if A is uniformly covered by $\{\mathscr{A}_n\}$ the conditions of Lemma 4.6 are satisfied.

The interesting case occurs when both A and A^c are uniformly covered by $\{\mathscr{A}_n\}$ or satisfy the conditions of the above lemma.

Lemma 4.7. *If $A \in \mathscr{A}$ is such that A and A^c are uniformly covered by $\{\mathscr{A}_n\}$ (or satisfy the conditions of Lemma 4.6), then*

$$\mathsf{E}(\chi_A|\mathscr{A}_n) \xrightarrow{a.e.} \chi_A.$$

Proof.

$$\begin{aligned}\overline{\lim_{n\to\infty}}\mathsf{E}(\chi_{A^c}|\mathscr{A}_n) &= \overline{\lim_{n\to\infty}}\,\mathsf{E}(1-\chi_A|\mathscr{A}_n) = \overline{\lim_{n\to\infty}}\big(1-\mathsf{E}(\chi_A|\mathscr{A}_n)\big)\\ &= 1 - \underline{\lim_{n\to\infty}}\,\mathsf{E}(\chi_A|\mathscr{A}_n).\end{aligned}$$

Since A^c is uniformly covered,

$$\chi_{A^c} = 1-\chi_A \ge \overline{\lim_{n\to\infty}}\mathsf{E}(\chi_{A^c}|\mathscr{A}_n) = 1 - \underline{\lim_{n\to\infty}}\,\mathsf{E}(\chi_A|\mathscr{A}_n),$$

and so

$$\underline{\lim_{n\to\infty}}\,\mathsf{E}(\chi_A|\mathscr{A}_n) \ge \chi_A \ a.e.$$

Finally, as A is uniformly covered,

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}(\chi_A|\mathscr{A}_n) \le \chi_A \le \underline{\lim_{n\to\infty}}\, \mathsf{E}(\chi_A|\mathscr{A}_n) \; a.e.,$$

and then

$$\mathsf{E}(\chi_A|\mathscr{A}_n) \xrightarrow{a.e.} \chi_A. \qquad \square$$

Now we state the necessary lemma for a.e. convergence of characteristic functions.

Lemma 4.8. *Let $A \in \mathscr{A}$ and $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be σ-subalgebras such that*

$$\mathsf{E}(\chi_A|\mathscr{A}_n) \xrightarrow{a.e.} \chi_A.$$

Then A and A^c are uniformly covered by $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$.

Proof. Let $0 < r < 1$. Define $A_n \in \mathscr{A}_n$ as

$$A_n = \left\{x \in \mathbb{X} \;:\; \mathsf{E}(\chi_A|\mathscr{A}_n)(x) \ge r\right\}.$$

Since $\mathsf{E}(\chi_A|\mathscr{A}_n) \overset{a.e.}{\to} \chi_A$, we have that $\chi_{A_n} \xrightarrow{a.e.} \chi_A$. Indeed, for almost all $x \in A$ there is an $N_x \in \mathbb{N}$ such that, if $n > N_x$, then $\mathsf{E}(\chi_A|\mathscr{A}_n) > r$. Thus $x \in A_n$, and so $\chi_{A_n}(x) = \chi_A(x)$. We can proceed similarly for A^c.

Therefore,

$$A = \bigcap_{N=1}^{\infty} \bigcup_{n>N} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n>N} A_n.$$

It is also clear that

$$\begin{aligned} 0 \le \mathsf{E}(\chi_{A\setminus A_n}|\mathscr{A}_n) &= \mathsf{E}(\chi_A \chi_{A_n^c}|\mathscr{A}_n) \\ &= \chi_{A_n^c}\mathsf{E}(\chi_A|\mathscr{A}_n) < \chi_{A_n^c} r \le r. \end{aligned}$$

Thus, $\|\mathsf{E}(\chi_{A\setminus A_n}|\mathscr{A}_n)\|_\infty \le r$, and by Lemma 4.3 A is uniformly covered.

Finally, since $\mathsf{E}(\chi_A|\mathscr{A}_n) \xrightarrow{a.e.} \chi_A$ implies that $\mathsf{E}(\chi_{A^c}|\mathscr{A}_n) \xrightarrow{a.e.} \chi_{A^c}$, we have that A^c is also uniformly covered. $\square$

Combining Lemma 4.7 and Lemma 4.8 we get the following theorem.

Theorem 4.9. *Let $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-algebras and let $A \in \mathscr{A}$. Then*

$$\mathsf{E}(\chi_A|\mathscr{A}_n) \xrightarrow{a.e.} \chi_A,$$

if and only if A and A^c are uniformly covered by $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$.

In view of the above theorem it is clear that it is convenient to establish the following definition.

Definition 4.10. Given a sequence of σ-subalgebras $\{\mathscr{A}_n\}$, define $\mathscr{A}_{\mu.a.e.}$ as

$$\mathscr{A}_{\mu.a.e.} = \left\{A \in \mathscr{A} : A \text{ and } A^c \text{ are uniformly covered by } \{\mathscr{A}_n\}_{n\in\mathbb{N}}\right\}.$$

Notice the following lemma.

Lemma 4.11. *$\mathscr{A}_{\mu.a.e.}$ is an algebra.*

Proof. Taking $A_n = \mathbb{X}$ or $A_n = \phi$ for all $n \in \mathbb{N}$ it is clear that $\mathbb{X}$ and ϕ are in $\mathscr{A}_{\mu.a.e.}$. Lemma 4.4 shows that finite union and finite intersection of uniformly covered sets are uniformly covered. Finally, by definition if A is in $\mathscr{A}_{\mu.a.e.}$, then A^c is also there. $\square$

Notice that if A is in $\bigcap_{n>M} \mathscr{A}_n$ for a given M, A is uniformly covered, since we can take $A_n = A$ for $n \geq M$. Since the intersection is a σ-subalgebra, $\bigcap_{n>M} \mathscr{A}_n \subset \mathscr{A}_{\mu.a.e.}$. Thus, in the case when $\mathscr{A}_{\mu.a.e.}$ is a σ-subalgebra, we have that $\underline{\mathscr{A}} \subset \mathscr{A}_{\mu.a.e.}$ and thus the chain

$$\underline{\mathscr{A}} \subset \mathscr{A}_{\mu.a.e.} \subset \mathscr{A}_{\mu} \subset \mathscr{A}_{\perp} \subset \overline{\mathscr{A}}.$$

Lemma 4.12. *If $\mathscr{A}_{\mu.a.e.}$ is σ-subalgebra, then:*

i) If $f \in L^{\infty}(\mathscr{A}_{\mu.a.e.})$, then

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{a.e.} \mathsf{E}(f|\mathcal{A}_{\mu.a.e.}) = f.$$

ii) If a σ-subalgebra $\mathscr{B}$ is such that, for all $f \in L^{\infty}(\mathscr{A}_{\mu.a.e.})$,

$$\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{a.e.} \mathsf{E}(f|\mathscr{B}),$$

then $\mathscr{B} \subset \mathscr{A}_{\mu.a.e.}$.

Proof. Let $\epsilon > 0$. As $f \in L^{\infty}(\mathscr{A}_{\mu.a.e.})$, there is a simple $\mathscr{A}_{\mu.a.e.}$-measurable function g, such that $\|f - g\|_{\infty} < \epsilon/2$. It is clear that Lemma 4.7 implies that $\mathsf{E}(g|\mathscr{A}_n) \xrightarrow{a.e.} g$. Thus,

$$\begin{aligned}\left|\mathsf{E}(f|\mathscr{A}_n) - f\right|(x) &\leq \left|\mathsf{E}(f|\mathscr{A}_n) - \mathsf{E}(g|\mathscr{A}_n)\right|(x) + \left|\mathsf{E}(g|\mathscr{A}_n) - g\right|(x) + |g - f|(x)\\ &\leq \left|\mathsf{E}(f - g|\mathscr{A}_n)\right|(x) + \left|\mathsf{E}(g|\mathscr{A}_n) - g\right|(x) + \|g - f\|_{\infty}\\ &\leq \epsilon + \left|\mathsf{E}(g|\mathscr{A}_n) - g\right|(x).\end{aligned}$$

So $\overline{\lim}_{n\to\infty}|\mathsf{E}(f|\mathscr{A}_n) - f|(x) \leq \epsilon$ $a.e.$ for every ϵ. Therefore, $\lim_{n\to\infty} \mathsf{E}(f|\mathscr{A}_n) = f$ $a.e.$

To prove *ii*), let $A \in \mathscr{B}$. By hypothesis $\mathsf{E}(\chi_A|\mathscr{A}_n) \longrightarrow \chi_A$ $a.e.$

For $0 < \epsilon < 1$ define $A_n = \{x \ : \ \mathsf{E}(\chi_A|\mathscr{A}_n) \geq \epsilon\} \in \mathscr{A}_n$. Since for almost all $x \in A$, $\mathsf{E}(\chi_A|\mathscr{A}_n)(x) \underset{n\to\infty}{\longrightarrow} 1$, $x \in A_n$ for n big enough. So $\chi_A(x)\chi_{A_n}(x) \longrightarrow \chi_A(x)$ a.e. The case for A^c is similar. In this case, for almost all $x \notin A$, $\mathsf{E}(\chi_A|\mathscr{A}_n)(x) \underset{n\to\infty}{\longrightarrow} 0$. Hence $x \notin A_n$ for n big enough. Thus $\chi_{A^c}(x)\chi_{A_n}(x) \longrightarrow 0$ a.e. We have then $\chi_{A_n}(x) \longrightarrow \chi_A(x)$ a.e.

We also have

$$\|A \setminus A_n\|_{\mathscr{A}_n} = \left\|\mathsf{E}(\chi_A \chi_{A_n^c}|\mathscr{A}_n)\right\|_{\infty} = \left\|\chi_{A_n^c}\mathsf{E}(\chi_A|\mathscr{A}_n)\right\|_{\infty} < \epsilon\|\chi_{A_n^c}\|_{\infty} < \epsilon.$$

And so by Lemma 4.3 A is uniformly covered by $\{\mathscr{A}_n\}$. $\square$

How do the above results look in the case of $\mathscr{A}_n$ being monotone? In the case of monotone decreasing sequence of σ-subalgebras it is clear that $\bigvee_{n=N}^{\infty} \mathscr{A}_n = \mathscr{A}_N$ and that for any N, $\bigcap_{n=N}^{\infty} \mathscr{A}_n = \bigcap_{n=1}^{\infty} \mathscr{A}_n$. Therefore,

$$\bigcap_{n=1}^{\infty} \mathscr{A}_n = \underline{\mathscr{A}} \subset \overline{\mathscr{A}} \subset \bigcap_{n=1}^{\infty} \mathscr{A}_n.$$

Thus $\underline{\mathscr{A}} = \overline{\mathscr{A}}$. So if $A \in \underline{\mathscr{A}}$, the sequence of sets $A_n = A \in \mathscr{A}_n$ trivially uniformly covers A. The same can be said for A^c. Since both sequences trivially fulfill the second condition of Definition 4.2, we have that $\mathscr{A}_{\mu.a.e.} = \mathscr{A}_n$.

4.2 *Necessary and sufficient conditions for a.e. convergence to zero*

In [2], we defined a σ-subalgebra $\mathscr{A}_\perp$ by considering the set W,

$$W = \left\{ g \in L^2(\mathscr{A}) \ : \ \exists A_{n_k} \in \mathscr{A}_{n_k} \ \text{ with } \ \chi_{A_{n_k}} \overset{L^2\text{-weakly}}{\longrightarrow} g \right\}.$$

$\mathscr{A}_\perp$ was defined as the minimal σ-algebra generated by W. We are going to do something similar. Notice the obvious fact that if $A_n \in \mathscr{A}_n$, $\chi_{A_n} = \mathsf{E}(\chi_{A_n}|\mathscr{A}_n)$. We are going to study when the conditional expectations of a sequence of σ-subalgebras converge a.e. to zero. To do this, consider first the set defined, given a sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ σ-subalgebras and $N \in \mathbb{N}$, as follows:

$$C_N = \Bigg\{ h \in L^1(\mu) \ : h = \sum_{k\geq N} \mathsf{E}(\chi_{B_k}|\mathscr{A}_k), \ B_k \in \mathscr{A}, \\ \text{disjoint, and such that there are only finite many of such } B_k's \Bigg\}.$$

Notice that, since we defined $h \in C_N$ as a finite sum, h is in L^2 although its norm could be large. That is, if $\{h_N\}_{N\in\mathbb{N}}$ is a sequence, then the norms $\|h_N\|$ are not necessarily bounded. However this is not the case in L^1, since we have

$$\|h\|_1 = \int \sum_{n\geq N} \mathsf{E}(\chi_{B_n}|\mathscr{A}_n) d\mu = \sum_{n\geq N} \mu(B_n) = \mu\Bigg(\bigcup_{n\geq N} B_n \Bigg) \leq 1.$$

Now we give the following definition.

Definition 4.13.

$$W^{\perp a.e.} = \left\{ f \in L^\infty(\mu) : \text{for every subsequence } \{h_{N_k}\}, \ h_{N_k} \in C_{N_k}, \ \langle f, h_{N_k} \rangle \underset{k\to\infty}{\longrightarrow} 0 \right\}.$$

We have the following result.

Theorem 4.14. *If $f \in L^\infty(\mu)$, then*

$$\mathsf{E}(f|\mathscr{A}_n) \overset{a.e.}{\longrightarrow} 0, \tag{4.1}$$

if and only if $f \in W^{\perp a.e.}$.

Proof. $\Rightarrow$) Let $f \in L^\infty(\mu)$. Without loss of generality we can assume that $\|f\|_\infty \leq 1$. First, notice that, if $h \in C_N$, we have

$$\langle h, f\rangle = \sum_{n\geq N} \big\langle \mathsf{E}(\chi_{B_n}|\mathscr{A}_n), f\big\rangle = \sum_{n\geq N} \big\langle \chi_{B_n}, \mathsf{E}(f|\mathscr{A}_n)\big\rangle.$$

Let $\epsilon > 0$. Since we are supposing that $\mathsf{E}(f|\mathscr{A}_n) \xrightarrow{a.e.} 0$, Egoroff's theorem implies that there is a set M_ϵ and $N_1 \in \mathbb{N}$ such that, if $n \geq N_1$, then

$$\mu\big(M_\epsilon^c\big) < \frac{\epsilon}{2} \quad \text{and} \quad \big\|\chi_{M_\epsilon} \mathsf{E}(f|\mathscr{A}_n)\big\|_\infty < \frac{\epsilon}{2}.$$

Thus, if $N > N_1$ and h_N is in C_N,

$$\begin{aligned}
\big|\langle h_N, f\rangle\big| &= \Big|\sum_{n\geq N} \big\langle \chi_{B_n}, \mathsf{E}(f|\mathscr{A}_n)\big\rangle\Big| \\
&\leq \sum_{n\geq N} \Big(\big|\big\langle \chi_{B_n}, \chi_{M_\epsilon}\mathsf{E}(f|\mathscr{A}_n)\big\rangle\big| + \big|\big\langle \chi_{B_n}\chi_{M_\epsilon^c}, \mathsf{E}(f|\mathscr{A}_n)\big\rangle\big|\Big) \\
&\leq \frac{\epsilon}{2}\sum_{n\geq N} \langle \chi_{B_n}, \chi_{M_\epsilon}\rangle + \sum_{n\geq N} \big\|\mathsf{E}(f|\mathscr{A}_n)\big\|_\infty \|\chi_{B_n}\chi_{M_\epsilon^c}\|_1 \\
&\leq \frac{\epsilon}{2}\mu(M_\epsilon) + \|f\|_\infty \mu\big(M_\epsilon^c\big) \leq \epsilon,
\end{aligned}$$

therefore, $\langle h_N, f\rangle \underset{N\to\infty}{\longrightarrow} 0$.

$\Leftarrow$) To prove the remaining part the theorem, suppose that $\mathsf{E}(f|\mathscr{A}_n)$ does not converge to zero a.e. Notice that, since we can take f or $-f$, without loss of generality, we can assume that there is an ϵ such that

$$\mu\Bigg(\bigcap_{N=1}^{\infty} \bigcup_{n\geq N} \big\{x \in \mathbb{X} \,:\, \mathsf{E}(f|\mathscr{A}_n)(x) \geq \epsilon\big\}\Bigg) > 0.$$

That is, there is an $r > 0$ such that for any N there is an M such that

$$\mu\Bigg(\bigcup_{N\leq n\leq M} \big\{x \in \mathbb{X} \,:\, \mathsf{E}(f|\mathscr{A}_n)(x) \geq \epsilon\big\}\Bigg) > r.$$

Let $A_n = \{x \in \mathbb{X} \,:\, \mathsf{E}(f|\mathscr{A}_n)(x) > \epsilon\}$ and let us define as usually the sequence $\{B_n\}$ as

$$B_N = A_N, \quad B_k = A_k \setminus \textstyle\bigcup_{j=N}^{k-1} B_j \quad \text{for } N < k < M. \tag{4.2}$$

$\{B_k\}$ is a disjoint family and

$$\bigcup_{N\leq n<M} B_n = \bigcup_{N\leq n<M} A_n.$$

Let $h_N = \sum_{N\leq n<M} \mathsf{E}(\chi_{B_n}|\mathscr{A}_n)$. We have that

$$\langle h_N, f\rangle = \sum_{N\leq n<M} \big\langle \chi_{B_n}, \mathsf{E}(f|\mathscr{A}_n)\big\rangle > \epsilon \sum_{N\leq n<M} \mu(\chi_{B_n}) = \epsilon\mu\Bigg(\bigcup_{N\leq n<M} B_n\Bigg) > \epsilon r.$$

We have now constructed a sequence $\{h_N\}_{N\in\mathbb{N}}$ such that for all N, $\langle h_N, f\rangle > \epsilon r$. Therefore $f \notin W^{\perp a.e.}$. $\square$

In [2], we defined the orthogonal conditional expectation induced by a σ-subalgebra $\mathscr{B}$ as the operator $\mathsf{E}^{\perp}_{\mathscr{B}} = I - \mathsf{E}_{\mathscr{B}}$, which in $L^2(\mathscr{A})$ is the orthogonal projection $\mathsf{E}^{\perp}_{\mathscr{B}} : L^2(\mathscr{A}) \to L^2(\mathscr{B})^{\perp}$. Let $\mathfrak{D}$ be the following family of σ-subalgebras:

$$\mathfrak{D} = \left\{\mathscr{B} : \mathscr{B} \text{ is a } \sigma\text{-subalgebra of } \mathscr{A} \text{ and } \mathsf{E}^{\perp}_{\mathscr{B}} f \in W^{\perp a.e.} \text{ for all } f \in L^{\infty}(\mathscr{A})\right\}.$$

It is clear that $\mathfrak{D}$ is not empty since it trivially contains $\mathscr{A}$.

The following proposition is an immediate property.

Proposition 4.15. *Let $\mathscr{C}$, $\mathscr{B}$ be two σ-subalgebras. If $\mathscr{C} \in \mathfrak{D}$ and $\mathscr{B} \supset \mathscr{C}$, then $\mathscr{B} \in \mathfrak{D}$*

Proof. Notice that in this case $\mathsf{E}^{\perp}_{\mathscr{B}} = \mathsf{E}^{\perp}_{\mathscr{C}}\mathsf{E}^{\perp}_{\mathscr{B}}$. So, as f in L^{∞} implies that $\mathsf{E}^{\perp}_{\mathscr{B}} f$ is also in L^{∞},

$$\mathsf{E}^{\perp}_{\mathscr{B}} f = \mathsf{E}^{\perp}_{\mathscr{C}}\big(\mathsf{E}^{\perp}_{\mathscr{B}} f\big) \in W^{\perp a.e.} \qquad \square$$

Definition 4.16. $\mathscr{A}_{\min}$ will be the minimal complete σ-subalgebra that contains the set

$$\tilde{W} = \Big\{g \in L^2(\mathscr{A}) : \text{there is a subsequence } \{h_{N_k} \in C_{N_k}\} \text{ such that } h_{N_k} \underset{k\to\infty}{\longrightarrow} g \text{ weakly in } L^2\Big\}.$$

Two properties of this σ-subalgebra are presented in the following lemma.

Lemma 4.17. $\mathscr{A}_{\min}$ *satisfies:*

i) $\mathscr{A}_{\perp} \subset \mathscr{A}_{\min} \subset \overline{\mathscr{A}}$,
ii) If $\mathscr{B} \in \mathfrak{D}$ then $\mathscr{A}_{\min} \subset \mathscr{B}$.

Proof. *i)* If $\chi_{A_{n_k}}$ is such that $\chi_{A_{n_k}} \longrightarrow g$ L^2-weakly, since $\chi_{A_{n_k}} \in C_{N_k}$, we have that $W \subset \tilde{W}$ and so $\mathscr{A}_{\perp} \subset \mathscr{A}_{\min}$. On the other hand, let $h_{N_k} \in C_{N_k}$ be such that $h_{N_k} \longrightarrow g$ weakly. Then h_{N_k} is $\bigvee_{n=N_k}^{\infty} \mathscr{A}_n$ measurable. That is, $h_{N_k} \in L^2(\bigvee_{n=N_k}^{\infty} \mathscr{A}_n)$ which is a closed subspace of $L^2(\mathscr{A})$. Thus, g is $\bigvee_{n=N_k}^{\infty} \mathscr{A}_n$ measurable. Since this is true for any N_k, g is $\bigcap_{m=1}^{\infty} \bigvee_{n=m}^{\infty} \mathscr{A}_n = \overline{\mathscr{A}}$ measurable.

ii) Let $g \in \tilde{W}$ and $h_{N_k} \in C_{N_k}$ be such that $h_{N_k} \xrightarrow{w} g$. Since $\mathscr{B}$ is in $\mathfrak{D}$ we have that, for all f in L^{∞},

$$\begin{aligned}\big\langle f, \mathsf{E}(g|\mathscr{B})\big\rangle &= \big\langle \mathsf{E}(f|\mathscr{B}), g\big\rangle = \lim_{k\to\infty}\big\langle \mathsf{E}(f|\mathscr{B}), h_{N_k}\big\rangle = \lim_{k\to\infty}\big\langle\big(I - \mathsf{E}^{\perp}_{\mathscr{B}}\big) f, h_{N_k}\big\rangle \\ &= \lim_{k\to\infty}\langle f, h_{N_k}\rangle - \lim_{k\to\infty}\big\langle \mathsf{E}^{\perp}_{\mathscr{B}} f, h_{N_k}\big\rangle = \langle f, g\rangle.\end{aligned}$$

Therefore g is equal to $\mathsf{E}(g|\mathscr{B})$ almost everywhere, and so is $\mathscr{B}$ measurable. Since by definition $\mathscr{A}_{\min}$ is the minimal σ-subalgebra that makes all such g measurable, we have $\mathscr{A}_{\min} \subset \mathscr{B}$. $\square$

Before we proceed we will study a special case.

Definition 4.18. We call the sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ 2-bounded if $\sup_{N\in\mathbb{N}}\|h_N\|_2 < \infty$ for any sequence $\{h_N \in C_N\}_{N\in\mathbb{N}}$.

The main property of these 2-bounded sequences of σ-subalgebras is that $\mathfrak{D}$ contains the minimal σ-subalgebra. Actually we have the following lemma:

Lemma 4.19. *If $\{\mathscr{A}_n\}$ is 2-bounded and $\mathscr{B}$ is a σ-subalgebra, then $\mathscr{B} \in \mathfrak{D}$ if and only if $\mathscr{A}_{\min} \subset \mathscr{B}$.*

Proof. In view of Lemma 4.17, we only need to prove the ($\Leftarrow$) part. Assume $\mathscr{A}_{\min} \subset \mathscr{B}$ and suppose that $\mathscr{A}_{\min}$ is not in $\mathfrak{D}$. This means that there is an $f \in L^\infty$ such that its orthogonal projection with respect to $\mathscr{A}_{\min}$ is not in $W^{\perp a.e.}$. That is, there is a subsequence $\{h_{N_k} \in C_{N_k}\}$ such that $\langle \mathsf{E}^{\perp}_{\mathscr{A}_{\min}} f, h_{N_k}\rangle$ does not converge to zero. Hence, there is an $\epsilon > 0$ and a sub-subsequence, which we still denote by $\{h_{N_k}\}$, such that $|\langle \mathsf{E}^{\perp}_{\mathscr{A}_{\min}} f, h_{N_k}\rangle| > \epsilon$. By hypothesis the sequence of σ-subalgebras is 2-bounded, so there is a sub-sub-subsequence that weakly converges to h in L^2. By definition, h is in $\tilde{W}$ and therefore is $\mathscr{A}_{\min}$ measurable. But this leads us to a contradiction, since $0 < \epsilon \leq |\langle \mathsf{E}^{\perp}_{\mathscr{A}_{\min}} f, h\rangle| = |\langle f, \mathsf{E}^{\perp}_{\mathscr{A}_{\min}} h\rangle| = 0$. □

Of course, the 2-boundeness condition is a very strong restriction. However, we would like to point out that any increasing or decreasing sequence of σ-subalgebras is 2-bounded. Indeed, let $\{\mathscr{A}_n\}$ be a decreasing sequence (the proof for the increasing case is similar) and let denote $\mathsf{E}(\cdot|\mathscr{A}_n) = \mathbb{P}_n\cdot$, for short, the L^2 orthogonal projection. Then,

$$\begin{aligned}
\langle h_N, h_N\rangle &= \sum_{k=N}^{M}\sum_{j=N}^{M}\big\langle \mathsf{E}(\chi_{B_k}|\mathscr{A}_k), \mathsf{E}(\chi_{B_j}|\mathscr{A}_j)\big\rangle = \sum_{k=N}^{M}\sum_{j=N}^{M}\langle \mathbb{P}_k\chi_{B_k}, \mathbb{P}_j\chi_{B_j}\rangle \\
&= \sum_{k=N}^{M}\sum_{N\leq j\leq k}\langle \mathbb{P}_k\chi_{B_k}, \mathbb{P}_j\chi_{B_j}\rangle + \sum_{k=N}^{M}\sum_{j=k+1}^{M}\langle \mathbb{P}_k\chi_{B_k}, \mathbb{P}_j\chi_{B_j}\rangle \\
&= \sum_{k=N}^{M}\sum_{N\leq j\leq k}\langle \chi_{B_k}, \mathbb{P}_j\chi_{B_j}\rangle + \sum_{k=N}^{M}\sum_{j=k+1}^{M}\langle \mathbb{P}_k\chi_{B_k}, \chi_{B_j}\rangle \\
&= \sum_{j=N}^{M}\sum_{k=j}^{M}\int \chi_{B_k}\mathbb{P}_j\chi_{B_j}\,d\mu + \sum_{k=N}^{M}\sum_{j=k+1}^{M}\int \mathbb{P}_k\chi_{B_k}\chi_{B_j}\,d\mu \\
&= \sum_{j=N}^{M}\int \chi_{\cup_{k=j}^{M} B_k}\mathbb{P}_j\chi_{B_j}\,d\mu + \sum_{k=N}^{M}\int \mathbb{P}_k\chi_{B_k}\chi_{\cup_{M\geq j>k} B_j}\,d\mu \\
&\leq \sum_{j=N}^{M}\int \mathbb{P}_j\chi_{B_j}\,d\mu + \sum_{k=N}^{M}\int \mathbb{P}_k\chi_{B_k}\,d\mu \leq \|h_N\|_1 + \|h_N\|_1 \leq 2.
\end{aligned}$$

We will see that the cases for which $\mathfrak{D}$ has a minimal σ-subalgebra play an important role. When this condition is fulfilled we will denote the minimal σ-subalgebra by $\mathscr{A}_{\perp a.e.}$.

Two immediate consequences follows.

Lemma 4.20. *If D has a minimal σ-subalgebra $\mathscr{A}_{\perp a.e.}$, then, for all f in $L^\infty(\mathscr{A})$,*

$$\mathsf{E}\big(\mathsf{E}^{\perp}_{\mathscr{A}_{\perp a.e.}} f|\mathscr{A}_n\big) \longrightarrow 0 \; a.e.$$

Lemma 4.21. *If the sequence of σ-subalgebras $\{\mathscr{A}_n\}$ is 2-bounded, then $\mathscr{A}_{\perp a.e.} = \mathscr{A}_{\min}$.*

4.3 Necessary and sufficient conditions for convergence a.e.

Theorem 4.22. *Let $\{\mathscr{A}_n\}$ be a sequence of σ-subalgebras. Then $\mathsf{E}(f|\mathscr{A}_n)$ converges a.e. for all $f \in L^\infty(\mathscr{A})$, if and only if the following conditions are satisfied:*

i) $\mathscr{A}_{\mu a.e.}$ *is a σ-subalgebra,*
ii) $\mathfrak{D}$ *has a minimal σ-subalgebra $\mathscr{A}_{\perp a.e.}$, and*
iii) $\mathscr{A}_{\mu a.e.} = \mathscr{A}_{\perp a.e.}$

Proof. $\Leftarrow$) Let the σ-subalgebra $\mathscr{B} = \mathscr{A}_{\mu a.e.} = \mathscr{A}_{\perp a.e.}$ Since for $f \in L^\infty(\mathscr{A})$, $\mathsf{E}(f|\mathscr{B})$ is in $L^\infty(\mathscr{A}_{\mu a.e.})$, Lemma 4.12 implies that $\mathsf{E}(\mathsf{E}(f|\mathscr{B})|\mathscr{A}_n) \longrightarrow \mathsf{E}(f|\mathscr{B})$ $a.e.$ On the other hand, Lemma 4.20 implies that $\mathsf{E}(\mathsf{E}^\perp_{\mathscr{B}} f|\mathscr{A}_n) \longrightarrow 0$ $a.e.$ Therefore,

$$\begin{aligned}\mathsf{E}(f|\mathscr{A}_n) &= \mathsf{E}\big(\mathsf{E}(f|\mathscr{B})|\mathscr{A}_n\big) + \mathsf{E}\big(f - \mathsf{E}(f|\mathscr{B})|\mathscr{A}_n\big)\\ &= \mathsf{E}\big(\mathsf{E}(f|\mathscr{B})|\mathscr{A}_n\big) + \mathsf{E}\big(\mathsf{E}^\perp_B f|\mathscr{A}_n\big) \longrightarrow \mathsf{E}(f|\mathscr{B}) \quad a.e.\end{aligned}$$

$\Rightarrow$) Let us now assume that for all $f \in L^\infty(\mathscr{A})$, the conditional expectations $\mathsf{E}(f|\mathscr{A}_n)$ converge a.e. It is clear that the dominated convergence theorem implies that this convergence is also in L^2. It is also easily seen that if $\mathsf{E}(f|\mathscr{A}_n)$ converge for all f in L^∞ in the L^2 norm, it will also do so for all f in L^2. Indeed, given $\epsilon > 0$ take $\tilde{f}$ in L^∞ such that $\|f - \tilde{f}\|_2 < \epsilon/2$. Since $\{\mathsf{E}(\tilde{f}|\mathscr{A}_n)\}$ is a Cauchy sequence and $\|\mathsf{E}(f - \tilde{f}|\mathscr{A}_n)\|_2 \le \|f - \tilde{f}\|_2$, the sequence $\mathsf{E}(f|\mathscr{A}_n)$ is also Cauchy.

Notice that by Theorem 2.6 we have, as a first step, that

$$\mathscr{A}_\mu = \mathscr{A}_\perp \tag{4.3}$$

and

$$\mathsf{E}(f|\mathscr{A}_n) \overset{L^2(\mathscr{A})}{\longrightarrow} \mathsf{E}(f|\mathscr{A}_\mu) = \mathsf{E}(f|\mathscr{A}_\perp) \tag{4.4}$$

for every $f \in L^2(\mathscr{A})$.

To prove i), notice that Lemma 2.4 tells us that, if $A \in \mathscr{A}_\mu$, then $\mathsf{E}(\chi_A|\mathscr{A}_n) \longrightarrow \mathsf{E}(\chi_A|\mathscr{A}_\mu) = \chi_A$ in L^2. Since we assume convergence a.e. of the conditional expectations, the dominated convergence theorem implies that the convergence to the characteristic function is also a.e. We have then that $\mathscr{A}_{\mu a.e.}$ is a σ-subalgebra, since by Theorem 4.9 and the definition of $\mathscr{A}_{\mu a.e.}$ we have $\mathscr{A}_{\mu a.e.} = \mathscr{A}_\mu$.

We start the proof of the case ii) in a similar way. Since $\mathsf{E}^\perp_{\mathscr{A}_\perp} f$ clearly is in $L^2(\mathscr{A}_\perp)^\perp$, Lemma 2.5 implies that $\mathsf{E}(f|\mathscr{A}_n)$ converges to zero in L^2. By hypothesis this convergence is also a.e. Therefore, $\mathscr{A}_\perp$ is in $\mathfrak{D}$.

We are going to prove now that $\mathscr{A}_\perp$ is minimal. Let $\mathscr{B}$ be any element in $\mathfrak{D}$ and f in L^∞. By definition we have that $\mathsf{E}^\perp_{\mathscr{B}} f \in W^{\perp a.e.}$. That is, $\mathsf{E}(\mathsf{E}^\perp_B f|\mathscr{A}_n) \longrightarrow 0$ a.e.

But by (4.4)

$$\mathsf{E}\big(\mathsf{E}^\perp_{\mathscr{B}} f|\mathscr{A}_n\big) \overset{L^p(\mathscr{A})}{\longrightarrow} \mathsf{E}\big(\mathsf{E}^\perp_{\mathscr{B}} f|\mathscr{A}_\perp\big).$$

This means that $\mathsf{E}(\mathsf{E}^\perp_{\mathscr{B}} f|\mathscr{A}_\perp) = 0$ a.e. So $\mathsf{E}(f|\mathscr{A}_\perp) - \mathsf{E}(\mathsf{E}(f|\mathscr{B})|\mathscr{A}_\perp) = 0$. Since this is true for all f in L^∞, it is also true that, for all $g \in L^\infty$, $\mathsf{E}(g|\mathscr{A}_\perp) = \mathsf{E}(\mathsf{E}(g|\mathscr{A}_\perp)|\mathscr{B})$. Thus $\mathscr{A}_\perp \subset \mathscr{B}$, and hence $\mathscr{A}_\perp$ is minimal.

The property iii) follows easily since we have proven that in this case $\mathscr{A}_\mu = \mathscr{A}_{\mu a.e.}$, $\mathscr{A}_\perp = \mathscr{A}_{\perp a.e.}$ and by 4.3 above $\mathscr{A}_\mu = \mathscr{A}_\perp$. $\square$

Notice that in the case of a.e. convergence, $\mathfrak{D}$ has a minimal σ-subalgebra and it is $\mathscr{A}_\perp$. In view of Theorem 4.22, in this case we have also that $\mathscr{A}_{\min} = \mathscr{A}_\perp = \mathscr{A}_{\perp a.e.} = \mathscr{A}_{\mu a.e.} = \mathscr{A}_\mu$.

A Appendix

We will show the relation between the measure of a set and the $\mathscr{B}$ norm, which we claimed earlier.

Lemma A.1. *Let A be a measurable set in $\mathscr{A}$ and $\mathscr{B}$ be a σ-subalgebra of $\mathscr{A}$. Then*

$$\|\chi_A\|_{\mathscr{B}} = \sup_{\substack{B \in \mathscr{B} \\ \mu(B) > 0}} \frac{\mu(A \cap B)}{\mu(B)}.$$

Proof. Let $B \in \mathscr{B}$. We have

$$\begin{aligned}\mu(A \cap B) &= \int \chi_{A \cap B} d\mu = \int \chi_A \chi_B d\mu \\ &= \int \mathsf{E}(\chi_A|\mathscr{B})\chi_B d\mu \le \left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty \mu(B).\end{aligned}$$

Therefore, for any $B \in \mathscr{B}$ with $\mu(B) > 0$,

$$\frac{\mu(A \cap B)}{\mu(B)} \le \left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty,$$

and so the supremum is less or equal to $\|\mathsf{E}(\chi_A|\mathscr{B})\|_\infty$.

To prove the equality, notice that we can assume $\|\mathsf{E}(\chi_A|\mathscr{B})\|_\infty > 0$, since the case $\|\mathsf{E}(\chi_A|\mathscr{B})\|_\infty = 0$ is trivial. Now, take any $\varepsilon > 0$ such that

$$\left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty - \varepsilon > 0.$$

By definition the set

$$C = \left\{x \ : \ \mathsf{E}(\chi_A|\mathscr{B}) > \left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty - \varepsilon\right\}$$

is $\mathscr{B}$-measurable and

$$\int_C \mathsf{E}(\chi_A|\mathscr{B}) d\mu \ge \left(\left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty - \varepsilon\right)\mu(C).$$

As

$$\int_C \mathsf{E}(\chi_A|\mathscr{B}) d\mu = \int \chi_C \mathsf{E}(\chi_A|\mathscr{B}) d\mu = \int \chi_C \chi_A d\mu = \mu(C \cap A),$$

we have

$$\frac{\mu(C \cap A)}{\mu(C)} \ge \left\|\mathsf{E}(\chi_A|\mathscr{B})\right\|_\infty - \varepsilon.$$

□

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