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Existence of density function for the running maximum of SDEs driven by nontruncated pure-jump Lévy processes

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Abstract The existence of density function of the running maximum of a stochastic differential equation (SDE) driven by a Brownian motion and a nontruncated pure-jump process is verified. This is proved by the existence of density function of the running maximum of the Wiener–Poisson functionals resulting from Bismut's approach to the Malliavin calculus for jump processes.

Keywords Running maximum, density functions, Malliavin calculus, stochastic differential equations, Lévy processes

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1 Introduction

We consider a solution of the following one-dimensional SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma_1 \mathrm{d}W_t + \sigma_2 \mathrm{d}L_t,\ \ X_0 = x \in \mathbb{R}, \tag{1.1}$$

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for $t \geq 0$, where σ_1 and σ_2 are constants, $b : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative is bounded, $W = \{W_t\}_{t\in[0,T]}$ is a standard Brownian motion and $L = \{L_t\}_{t\in[0,T]}$ is a Lévy process with the Lévy triplet $(0, 0, \nu)$. The infinitesimal generator A of L is defined by

$$Af(x) := \int_{\mathbb{R}\setminus\{0\}} \big\{f(x+y) - f(x) - \mathbb{1}_{\{|y|<1\}} y f'(x)\big\}\nu(\mathrm{d}y)$$

for any $f \in C_b^2(\mathbb{R})$ and $x \in \mathbb{R}$. See, e.g., equation (3.18) in [1]. The Lévy measure ν satisfies assumptions (2.1) and (2.2). We assume that W and L are independent. In considering SDE (1.1), we introduce the following SDE for each $n \in \mathbb{N}$:

$$\mathrm{d}X_t^{(n)} = b\big(X_t^{(n)}\big)\mathrm{d}t + \sigma_1 \mathrm{d}W_t + \sigma_2 \mathrm{d}L_t^{(n)}, \quad X_0^{(n)} = x \in \mathbb{R}, \tag{1.2}$$

for $t \in [0, T]$, where $L^{(n)} = \{L_t^{(n)}\}_{t\in[0,T]}$ is a truncated pure-jump process L with jump sizes larger than n. Let $X^* := \{X_t^*\}_{t\in[0,T]}$ and $X^{(*,n)} := \{X_t^{(*,n)}\}_{t\in[0,T]}$, defined as

$$X_t^* = \sup_{t\in[0,t]} X_s, \quad X_t^{(*,n)} = \sup_{s\in[0,t]} X_s^{(n)} \text{ for each } t \in [0, T] \text{ and } n \in \mathbb{N},$$

be the running maximums of the solutions X and $X^{(n)}$ to SDE (1.1) and (1.2), respectively. It is well known that when b is Lipschitz continuous, SDEs (1.1) and (1.2) have a unique solution (e.g., see [12]). The purpose of this paper is to show that the distribution of X_t^* is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ for all $t > 0$. The running maximum process has received widespread attention in recent years as an interesting objectboth practically and theoretically (cf. [6, 3]). The following results for the special cases of the law of X^* are known. The density function of the maximum of Brownian motion (i.e. $x = b = \sigma_2 = 0$ and $\sigma_1 = 1$) is well known. See, e.g., [7]. The law of the maximum of Lévy motion (i.e. $x = b = \sigma_1 = 0$ and $\sigma_2 = 1$) is also well known. See, for example, [4, 10].

The following prior studies are based on the simultaneous dealing of Brownian motion and truncation Lévy processes. If L is a compound Poisson process, Coutin et al. [5] consider a joint density of (X_t^*, X_t). Song and Zhang [15] study the existence of distributional density of X_t and the weak continuity in the first variable of the distributional density under full Hörmander's conditions. This proof is given by showing the statement for $X_t^{(1)}$. Song and Xie [14] show the existence of density functions for the running maximum $X_t^{(*,1)}$ of a Lévy–Itô diffusion. They claimed that if b is Lipschitz continuous in Lemma 4.3 of [14], they can prove the existence of the density function of $X_t^{(*,1)}$. However, we cannot follow them because the product of weakly convergent sequences does not necessarily converge to the product of their limits. These [15, 14] are proved similarly if jump size n is a finite value. However, to the best of our knowledge, the results of the nontruncated Lévy process are not known. This is since Bismut's approach to the Malliavin calculus for jump processes in [2] can simply calculate the concrete form only for finite jumps by using Proposition 2.11 of [15]. In this paper, we show the existence of a density function for X^* using the proof method of [14] and the fact that the Malliavin calculus for L can be defined by the limit of that of $L^{(n)}$.

The structure of this paper is as follows. In Section 2, we introduce the notations employed throughout this paper and present our main theorem. Section 3 revisits Bismut's approach to the Malliavin calculus with jumps. In Section 4, we discuss the results of Song and Xie [14] and extend their results. Section 5 is dedicated to applying the outcomes derived in the preceding section to our stochastic differential equations. Our primary contribution, Theorem 2.1, is proven in Section 6. Lastly, Section A offers several lemmas essential for the proof of the our main results.

2 Notations and a result

Let $L = \{L_t\}_{0\le t\le T}$ and $L^{(n)} = \{L_t^{(n)}\}_{t\in[0,T]}$ be a pure-jump process and the one truncated by $[-n, n] \setminus \{0\}$, respectively. The jump size of L and $L^{(n)}$ at time t is defined by $\Delta L_t = L_t - L_{t-}$ and $\Delta L_t^{(n)} := L_t^{(n)} - L_{t-}^{(n)}$ for any $t > 0$ and $\Delta L_0 := 0$ and $\Delta L_0^{(n)} := 0$. The Poisson random measures associated with L and $L^{(n)}$ on $\mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$ are denoted by $N(t, F) = \sum_{0\le s\le t} \mathbb{1}_F(\Delta L_s)$ and $N^{(n)}(t, G) = \sum_{0\le s\le t} \mathbb{1}_F(\Delta L_s^{(n)})$ for $t \in [0, T]$ and $F \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, respectively. The Lévy measures of L and $L^{(n)}$ on $\mathcal{B}(\mathbb{R} \setminus \{0\})$ are defined as $\nu(\mathrm{d}z) = c(z)\mathrm{d}z$ and $c(z)\mathbb{1}_{\{|z|\le n\}}(z)\mathrm{d}z$, where the positive function c satisfies the following requirements: there exist some constants $\beta > 1$ and $C > 0$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \left(1 \wedge |z|^2\right)\nu(\mathrm{d}z) < \infty, \quad \lim_{n\to\infty} \int_{\mathbb{R}\setminus\{0\}} |z|^p \mathbb{1}_{\{|z|>n\}}(z)\nu(\mathrm{d}z) = 0 \text{ for any } p \in (1, \beta), \tag{2.1}$$

$$\sup_{n\in\mathbb{N}} \left| \int_{1\le|z|\le n} z\nu(\mathrm{d}z) \right| < \infty \text{ and } \int_{0+} \nu(\mathrm{d}z) = \infty, \quad \left|\frac{c'(z)}{c(z)}\right| \le C\left(1 \vee \frac{1}{|z|}\right) \text{ for any } z \ne 0. \tag{2.2}$$

The compensated Poisson random measures of L and $L^{(n)}$ are defined as $\widetilde{N}$ and $\widetilde{N}^{(n)}$, respectively.

Example 2.1. If L is a symmetric α-stable process with $\alpha \in (1, 2)$, then for any $z \ne 0$

$$c(z) = \frac{c_\alpha}{|z|^{1+\alpha}}, \text{ where } c_\alpha = \pi^{-1}\Gamma(\alpha + 1) \sin\left(\frac{\alpha\pi}{2}\right),$$

so that a Lévy measure ν satisfies assumptions (2.1) and (2.2) for any $p \in (1, \alpha)$.

Our results are described below.

Theorem 2.1. *Assume that $b : \mathbb{R} \to \mathbb{R}$ is once differentiable and its derivative is bounded, and that a Lévy measure ν of L satisfies (2.1) and (2.2). Let $\{X_t\}_{t\in[0,T]}$ be the solution to equation (1.1). If $\sigma_1^2 + \sigma_2^2 \ne 0$, then for any $T > 0$ the law of X_T^* is absolutely continuous with respect to the Lebesgue measure.*

We will prove this result in Section 6. To prepare for that proof, we introduce the Malliavin calculus.

3 Bismut's approach to the Malliavin calculus with jumps

This section provides a brief overview of Bismut's approach in the context of Malliavin calculus for jump processes (cf. [2, 14, 15], etc.). Consider an open set $\Gamma \subset \mathbb{R}^d$ containing the origin. We define

$$\Gamma_0 := \Gamma \setminus \{0\}, \quad \varrho(z) := 1 \vee \mathbf{d}\big(z, \Gamma_0^c\big)^{-1}, \tag{3.1}$$

where $\mathbf{d}(z, \Gamma_0^c)$ is the distance of z to the complement of Γ_0. Let Ω denote the canonical space consisting of all pairs $\omega = (w, \mu)$, where

- $w : [0, 1] \to \mathbb{R}^d$ is a continuous function satisfying $w(0) = 0$;
- μ represents an integer-valued measure on $[0, 1] \times \Gamma_0$ such that $\mu(A) < +\infty$ for any compact subset $A \subset [0, 1] \times \Gamma_0$.

Let us define the canonical process on Ω by setting for $\omega = (w, \mu)$:

$$W_t(\omega) := w(t), \quad N(\omega; \mathrm{d}t, \mathrm{d}z) := \mu(\omega; \mathrm{d}t, \mathrm{d}z) := \mu(\mathrm{d}t, \mathrm{d}z).$$

We consider the smallest right-continuous filtration $(\mathscr{F}_t)_{t\in[0,1]}$ on Ω ensuring that both W and N are optional processes. Throughout our discussion, we set $\mathscr{F} := \mathscr{F}_1$. The space $(\Omega, \mathscr{F})$ is equipped with a unique probability measure $\mathbb{P}$ satisfying the following conditions:

- W is a standard d-dimensional Brownian motion;
- N is a Poisson random measure with intensity $\mathrm{d}t\,\nu(\mathrm{d}z)$, where $\nu(\mathrm{d}z) = \kappa(z)\mathrm{d}z$ with

$$\kappa \in C^1\big(\Gamma_0; (0, \infty)\big), \int_{\Gamma_0} \big(1 \wedge |z|^2\big)\kappa(z)\mathrm{d}z < +\infty, \ |\nabla \log \kappa(z)| \le C\varrho(z), \tag{3.2}$$

  where $\varrho(z)$ is defined by equation (3.1);
- W and N are independent.

We denote the compensated Poisson random measure N by

$$\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t.$$

Let $p \ge 1$, $i \in \{1, 2\}$, and let k be an integer. We introduce the following spaces for subsequent discussions.

- We denote $L^p(\Omega)$ as the space of all $\mathscr{F}$-measurable random variables for which the norm represented by

$$\|F\|_p := \mathbb{E}\big[|F|^p\big]^{\frac{1}{p}}$$

  is finite.

- Let $\mathbb{L}_p^i$ be the space of all predictable processes $\xi : \Omega \times [0,1] \times \Gamma_0 \to \mathbb{R}^k$ with finite norm

$$\|\xi\|_{\mathbb{L}_p^i} := \mathbb{E}\left[\left(\int_0^1 \int_{\Gamma_0} |\xi(s,z)|^i \nu(\mathrm{d}z)\mathrm{d}s\right)^{\frac{p}{i}}\right]^{\frac{1}{p}} + \mathbb{E}\left[\int_0^1 \int_{\Gamma_0} |\xi(s,z)|^p \nu(\mathrm{d}z)\mathrm{d}s\right]^{\frac{1}{p}} < \infty.$$

- We introduce $\mathbb{H}_p$ as the set of all measurable adapted processes $h : \Omega \times [0,1] \to \mathbb{R}^d$ that possess a finite norm defined by

$$\|h\|_{\mathbb{H}_p} := \mathbb{E}\left[\left(\int_0^1 |h(s)|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

- Consider the space $\mathbb{V}_p$ of all predictable processes $\mathbf{v} : \Omega \times [0,1] \times \Gamma_0 \to \mathbb{R}^d$ that satisfy the finite norm condition

$$\|\mathbf{v}\|_{\mathbb{V}_p} := \|\nabla_z \mathbf{v}\|_{\mathbb{L}_p^1} + \|\mathbf{v}\varrho\|_{\mathbb{L}_p^1} < \infty,$$

  where $\varrho(z)$ is defined by equation (3.1). For later discussions, we will use the notations

$$\mathbb{H}_{\infty-} := \bigcap_{p \geq 1} \mathbb{H}_p, \quad \mathbb{V}_{\infty-} := \bigcap_{p \geq 1} \mathbb{V}_p.$$

- The space $\mathbb{H}_0$ encompasses all bounded, measurable, and adapted processes $h : \Omega \times [0,1] \to \mathbb{R}^d$.

- The space $\mathbb{V}_0$ is constituted of all predictable processes $\mathbf{v} : \Omega \times [0,1] \times \Gamma_0 \to \mathbb{R}^d$ satisfying the following conditions:

  (i) both $\mathbf{v}$ and $\nabla_z \mathbf{v}$ are bounded;

  (ii) there exists a compact subset $U \subset \Gamma_0$ such that

$$\mathbf{v}(t,z) = 0, \quad \forall z \in U.$$

Let $C_p^\infty(\mathbb{R}^m)$ denote the set of smooth functions on $\mathbb{R}^m$ for which all derivatives exhibit at most polynomial growth. Define the collection of Wiener–Poisson functionals on Ω given by

$$F(\omega) = f\big(W(h_1), \ldots, W(h_{m_1}), N(g_1), \ldots, N(g_{m_2})\big),$$

where f belongs to $C_p^\infty(\mathbb{R}^{m_1+m_2})$, $h_1, \ldots, h_{m_1}$ are elements of $\mathbb{H}_0$, and $g_1, \ldots, g_{m_2}$ are in $\mathbb{V}_0$, with all of them being nonrandom and real-valued. Additionally, for each j in the range $1 \leq j \leq m_1$ and each k in the range $1 \leq k \leq m_2$, we define

$$W(h_j) := \int_0^1 \langle h_j(s), \mathrm{d}W_s \rangle_{\mathbb{R}^d} \text{ and } N(g_k) := \int_0^1 \int_{\Gamma_0} g_k(s,z) N(\mathrm{d}s, \mathrm{d}z).$$

Given any $p > 1$ and $\Theta = (h, \mathbf{v}) \in \mathbb{H}_p \times \mathbb{V}_p$, we denote

$$D_\Theta F := \sum_{i=1}^{m_1} (\partial_i f)(\cdot) \int_0^1 \langle h(s), h_i \rangle_{\mathbb{R}^d} \mathrm{d}s \\ + \sum_{j=1}^{m_2} (\partial_{j+m_1} f)(\cdot) \int_0^1 \int_{\Gamma_0} \langle \mathbf{v}(s,z), \nabla_z g_j \rangle_{\mathbb{R}^d} N(\mathrm{d}s, \mathrm{d}z),$$

where "$(\cdot)$" represents the collection $W(h_1), \ldots, W(h_{m_1}), N(g_1), \ldots, N(g_{m_2})$.

Definition 3.1. Given $p > 1$ and $\Theta = (h, \mathbf{v}) \in \mathbb{H}_p \times \mathbb{V}_p$, we introduce the first-order Sobolev space $\mathbb{W}_\Theta^{1,p}$ as the completion of $\mathscr{F}C_p$ in $L^p(\Omega)$ with respect to the norm

$$\|F\|_{\Theta;1,p} := \|F\|_{L^p} + \|D_\Theta F\|_{L^p}.$$

It is well known that the Banach space $\mathbb{W}_\Theta^{1,p}$ possesses weak compactness, a crucial property for the proof of Theorem 2.1 (see Lemma 2.3 in [14]). We next present the results obtained by applying the Malliavin calculus developed above to the running maximum processes.

4 Regularity of the running maximum processes

In this section we discuss the results of Song and Xie [14] and their extensions. Let $X^{(n)} = \{X_s^{(n)}\}_{s \geq 0}$ be a right continuous real-valued process. For any fixed $T > 0$ and $n \in \mathbb{N}$, in the following we shall write

$$X_T^{(*,n)} := \sup_{s \in [0,T]} X_s^{(n)}, \quad X_T^* := \sup_{s \in [0,T]} X_s.$$

Lemma 4.1. *Let $X^{(n)} = \{X_s^{(n)}\}_{s \geq 0}$ and $X = \{X_s\}_{s \geq 0}$ be a right continuous process for each $n \in \mathbb{N}$. Suppose that for some $p > 1$ and $\Theta = (h, \mathbf{v}) \in \mathbb{H}_{\infty-} \times \mathbb{V}_{\infty-}$:*

1. $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_s^{(*,n)}|^p] < \infty$*, and for any $s \in [0, T]$, $X_s^{(n)} \in \mathbb{W}_\Theta^{1,p}$, and*
$$\sup_{n \in \mathbb{N}} \mathbb{E}\Big[\sup_{s \in [0,T]} \big|D_\Theta X_s^{(n)}\big|^p\Big] < \infty;$$
2. *the process $\{D_\Theta X_s^{(n)}\}_{s \in [0,T]}$ possesses a right continuous version for each $n \in \mathbb{N}$;*
3. $\lim_{n \to \infty} \mathbb{E}[\sup_{s \in [0,T]} |X_s^{(n)} - X_s|^p] = 0.$

Then $X_T^* \in \mathbb{W}_\theta^{1,p}$ and the sequence $\{D_\Theta X_s^{(n)}\}_{s \in [0,T]}$ converges to $D_\Theta X = \{D_\Theta X_s\}_{s \in [0,T]}$ in the weak topology of $L^p(\Omega \times [0,T])$. Moreover, if this $D_\Theta X$ has a right continuous version and

$$\mathbb{P}\big(D_\Theta X_t \neq 0 \text{ on } \{t \in (0,T] : X_t = X_T^*\}\big) = 1, \tag{4.1}$$

then the law of X_T^* is absolutely continuous with respect to the Lebesgue measure.

Proof. It can be seen that $X_T^{(*,n)} \in \mathbb{W}_\theta^{1,p}$ follows from Proposition 3.1 in [14] for each $n \in \mathbb{N}$. From Lemma 2.3 in [14], we obtain $X_T^* \in \mathbb{W}_\theta^{1,p}$ and

$$\lim_{n\to\infty} D_\Theta X_\cdot^{(n)} = D_\Theta X_\cdot \text{ weakly in } L^p\big(\Omega \times [0,T]\big).$$

In exactly the same way as in Theorem 3.2 in [14], the following equality follows if $D_\Theta X$ has a right continuous path almost surely:

$$\begin{aligned}
1 &= \mathbb{P}\big(\big\{\exists t \in [0,T] \text{ such that } D_\Theta X_t \neq D_\Theta X_T^* \text{ and } X_t = X_T^*\big\}^c\big)\\
&= \mathbb{P}\big(D_\Theta X_t = D_\Theta X_T^* \text{ on } \big\{t \in [0,T] : X_t = X_t^*\big\}\big)\\
&\leq \mathbb{P}\big(D_\Theta X_t = D_\Theta X_T^* \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big)\\
&= 1.
\end{aligned}$$

Subsequently, we prove that X_T^* has a density function if (4.1) holds. In addition, by the closability of D_Θ (see Theorem 2.6 in [14]), we obtain

$$\mathbb{P}\big(\mathbb{1}_A\big(D_\Theta X_T^*\big) D_\Theta X_T^* = 0\big) = 1,$$

for any $A \in \mathcal{B}(\mathbb{R})$ with $\mathrm{Leb}(A) = 0$. With these facts and (4.1) we obtain the following equation:

$$\begin{aligned}
1 = \mathbb{P}\big(&\big\{\mathbb{1}_A\big(D_\Theta X_T^*\big) D_\Theta X_T^* = 0\big\} \cap \big\{D_\Theta X_t = D_\Theta X_T^* \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\\
&\cap \big\{D_\Theta X_t \neq 0 \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\big)\\
= \mathbb{P}\big(&\big\{\mathbb{1}_A\big(D_\Theta X_T^*\big) D_\Theta X_t = 0 \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\\
&\cap \big\{D_\Theta X_t = D_\Theta X_T^* \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\\
&\cap \big\{D_\Theta X_t \neq 0 \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\big)\\
= \mathbb{P}\big(&\big\{\mathbb{1}_A\big(D_\Theta X_T^*\big) = 0\big\} \cap \big\{D_\Theta X_t = D_\Theta X_T^* \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\\
&\cap \big\{D_\Theta X_t \neq 0 \text{ on } \big\{t \in (0,T] : X_t = X_t^*\big\}\big\}\big)\\
\leq \mathbb{P}\big(&\mathbb{1}_A\big(D_\Theta X_T^*\big) = 0\big)\\
= 1.&
\end{aligned}$$

Therefore, this lemma is completed. □

Now we know the relationship between the Malliavin calculus of the running maximum processes and the existence of the density function. Next, we note the results of applying of the Malliavin calculus to the SDE (1.1).

5 Applying of Malliavin calculus to SDEs

In this section, to find an equation satisfied by $D_\Theta X$ for X in equation (1.1), we check an equation satisfied by $D_\Theta X^{(n)}$ for $X^{(n)}$ in equation (1.2). The following lemma is shown in the same way as for Lemma 4.3 in [14].

Lemma 5.1 ([14], Lemma 4.3). *Assume that $b : \mathbb{R} \to \mathbb{R}$ is once differentiable and its derivative is bounded. Then for any $\Theta = (h, \mathbf{v}) \in \mathbb{H}_{\infty-} \times \mathbb{V}_{\infty-}$ and $t \in [0, T]$, $X_t^{(n)} \in \mathbb{W}_\Theta^{1,2}$ and*

$$D_\Theta X_t^{(n)} = \int_0^t b'\big(X_s^{(n)}\big) D_\Theta X_s^{(n)} \mathrm{d}s + \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{0<|z|\le n} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z).$$

The following lemma defines $D_\Theta X$ and confirms that it satisfies (5.1) below.

Lemma 5.2. *Assume the same assumptions as in Lemma 5.1. Then for some $p \in (1, \beta)$, for some $n \in \mathbb{N}$, for any $q \in (1, \beta)$ and for any $\Theta = (h, \mathbf{v}) \in \mathbb{H}_{\infty-} \times \mathbb{V}_{\infty-}$, where*

$$\lim_{n\to\infty} \int_0^T \int_{|z|>n} \big|\mathbf{v}(s,z)\big|^{\frac{q}{\beta}} \mathrm{d}s\,\nu(\mathrm{d}z) = 0 \ \text{ and } \ \int_0^T \int_{|z|>n} \big|\mathbf{v}(s,z)\big|^{q} \mathrm{d}s\,\nu(\mathrm{d}z) < \infty,$$

$X_t^* \in \mathbb{W}_\Theta^{1,p}$ for any $t \in [0, T]$, and

$$D_\Theta X_t = \int_0^t b'(X_s) D_\Theta X_s \mathrm{d}s + \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{|z|>0} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z). \tag{5.1}$$

Then this $D_\Theta X$ is the limit of weak $L^p(\Omega \times [0, T])$ convergence of the sequence $D_\Theta X^{(n)}$, and this sequence is strongly $L^p(\Omega \times [0, T])$ convergent in practice.

Proof. It can be seen immediately from Lemma 5.1 that Assumptions 1 and 2 of Lemma 4.1 are satisfied. Note that the L^p integrability of $D_\Theta X^{(n)}$ in Assumption 1 shall be checked later. See Lemma A.5 for the fact that Assumption 3 is satisfied. By using Lemma 2.3 in [14], Lemma A.5 and the closability of D_Θ (cf. Lemma 2.7 in [15]), we have

$$\lim_{n\to\infty} D_\Theta X_\cdot^{(n)} = D_\Theta X_\cdot \ \text{ weakly in } \ L^p\big(\Omega \times [0, T]\big).$$

We verify that this $D_\Theta X = \{D_\Theta X_t\}_{t\in[0,T]}$ satisfies equation (5.1). We set $\{Y_t\}_{t\in[0,T]}$ as a solution of

$$\begin{aligned} Y_t &= \int_0^t b'(X_s) Y_s \mathrm{d}s + C_t, \quad \text{where} \\ C_t &= \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{|z|>0} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z), \\ C_t^{(n)} &= \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{0<|z|\le n} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

We prove

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|D_\Theta X_t^{(n)} - Y_t\big|^p\Big] = 0. \tag{5.2}$$

By using an inequality $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$ for any $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \big|D_\Theta X_t^{(n)} - Y_t\big|^p\Big] \\
&\le 2^{p-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \big\{b'\big(X_s^{(n)}\big)D_\Theta X_s^{(n)} - b'(X_s)Y_s\big\}\mathrm{d}s\Big|^p\Big] \\
&\quad + 2^{p-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|C_t^{(n)} - C_t\big|^p\Big] \\
&\le 2^{2(p-1)}\int_0^T \mathbb{E}\big[\big|b'\big(X_s^{(n)}\big) - b'(X_s)\big|^p\big|D_\Theta X_s^{(n)}\big|^p\big]\mathrm{d}s \\
&\quad + 2^{2(p-1)}\|b'\|_\infty^p \int_0^T \mathbb{E}\big[\big|D_\Theta X_s^{(n)} - Y_s\big|^p\big]\mathrm{d}s + 2^{p-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|C_t^{(n)} - C_t\big|^p\Big].
\end{aligned}$$

The last and last second inequalities in the last chain follow from Jensen's inequality and Fubini's theorem. By using Gronwall's inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \big|D_\Theta X_t^{(n)} - Y_t\big|^p\Big] \\
&\le 2^{2(p-1)}\exp\big(2^{2(p-1)}\|b'\|_\infty^p\big)\int_0^T \mathbb{E}\big[\big|b'\big(X_s^{(n)}\big) - b'(X_s)\big|^p\big|D_\Theta X_s^{(n)}\big|^p\big]\mathrm{d}s \\
&\quad + 2^{2(p-1)}\exp\big(2^{2(p-1)}\|b'\|_\infty^p\big)\mathbb{E}\Big[\sup_{t\in[0,T]}\big|C_t^{(n)} - C_t\big|^p\Big].
\end{aligned}$$

We show

$$\lim_{n\to\infty}\int_0^T \mathbb{E}\big[\big|b'\big(X_s^{(n)}\big) - b'(X_s)\big|^p\big|D_\Theta X_s^{(n)}\big|^p\big]\mathrm{d}s = 0, \tag{5.3}$$

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|C_t^{(n)} - C_t\big|^p\Big] = 0. \tag{5.4}$$

See Lemma A.6 for proof of (5.4). Here we show equation (5.3). Notice that $p \in (1, \beta)$, there exists $q > 1$ such that $pq < \beta$ because of the denseness of rational numbers. By using the Hölder inequality, we have

$$\begin{aligned}
&\int_0^T \mathbb{E}\big[\big|b'\big(X_s^{(n)}\big) - b'(X_s)\big|^p\big|D_\Theta X_s^{(n)}\big|^p\big]\mathrm{d}s \\
&\le \int_0^T \mathbb{E}\big[\big|b'\big(X_s^{(n)}\big) - b'(X_s)\big|^{\frac{pq}{q-1}}\big]^{\frac{q-1}{q}}\mathbb{E}\big[\big|D_\Theta X_s^{(n)}\big|^{pq}\big]^{\frac{1}{q}}\mathrm{d}s \\
&\le T\mathbb{E}\Big[\sup_{t\in[0,T]}\big|b'\big(X_t^{(n)}\big) - b'(X_t)\big|^{\frac{pq}{q-1}}\Big]^{\frac{q-1}{q}}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|D_\Theta X_t^{(n)}\big|^{pq}\Big]^{\frac{1}{q}}.
\end{aligned}$$

Due to an inequality $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$ for any $a, b \in \mathbb{R}$ and $p \ge 1$ and Jensen's inequality, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|D_\Theta X_t^{(n)}\big|^{pq}\Big]$$

$$\le 2^{pq-1}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t b'\big(X_s^{(n)}\big)D_\Theta X_s^{(n)}\mathrm{d}s\Big|^{pq}\Big] + 2^{2(pq-1)}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t h(s)\mathrm{d}s\Big|^{pq}\Big]$$
$$+ 2^{2(pq-1)}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_{0<|z|\le n}\mathbf{v}(s,z)N(\mathrm{d}s,\mathrm{d}z)\Big|^{pq}\Big]$$
$$\le 2^{pq-1}\|b'\|_\infty^{pq}\int_0^T\mathbb{E}\Big[\sup_{u\in[0,s]}\big|D_\Theta X_u^{(n)}\big|^{pq}\Big]\mathrm{d}s$$
$$+ 2^{2(pq-1)}\Big(\mathbb{E}\Big[\int_0^T\big|h(s)\big|^{pq}\mathrm{d}s\Big] + \mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_{0<|z|\le n}\mathbf{v}(s,z)N(\mathrm{d}s,\mathrm{d}z)\Big|^{pq}\Big]\Big).$$

Gronwall's inequality implies

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|D_\Theta X_t^{(n)}\big|^{pq}\Big]$$
$$\le 2^{2(pq-1)}\Big(\mathbb{E}\Big[\int_0^T\big|h(s)\big|^{pq}\mathrm{d}s\Big]$$
$$+\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_{0<|z|\le n}\mathbf{v}(s,z)N(\mathrm{d}s,\mathrm{d}z)\Big|^{pq}\Big]\Big)e^{2^{pq-1}T\|b'\|_\infty^{pq}}.$$

The boundedness of the mean of sup with respect to time can be proved as in Lemma A.5 (ii). Due to assumptions on h and $\mathbf{v}$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|D_\Theta X_t^{(n)}\big|^{pq}\Big] < \infty.$$

This allows us to confirm the $L^p(\Omega\times[0,T])$ integrability of $D_\Theta X^{(n)}$ for assumption 1 in Lemma 4.1, and that $D_\Theta X$ can be defined as the limit of weak $L^p(\Omega\times[0,T])$ convergence of $D_\Theta X^{(n)}$. By Lemma A.5, boundedness of b' and continuous mapping theorem, we have

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|b'\big(X_t^{(n)}\big)-b'(X_t)\big|^{\frac{pq}{q-1}}\Big]^{\frac{q-1}{q}} = 0,$$

so that we obtain (5.3). Thus, by (5.2) and completeness of $L^p(\Omega\times[0,T])$, (5.1) follows. $\square$

From this proof, we see that $X^{(n)}$ and X satisfy the assumptions of Lemma 4.1. The proof of Theorem 2.1 is now ready to be presented.

6 Proof of Theorem 2.1

Proof. Applying the Itô formula to $e^{-\int_0^t b'(X_s)\mathrm{d}s}D_\Theta X_t$ (e.g., see [12], Corollary (Integration by Parts), P. 84), we obtain

$$e^{-\int_0^t b'(X_s)\mathrm{d}s}D_\Theta X_t = \int_{0+}^t e^{-\int_0^s b'(X_u)\mathrm{d}u}\circ\mathrm{d}D_\Theta X_{s-} + \int_{0+}^t D_\Theta X_{s-}\circ\mathrm{d}e^{-\int_0^s b'(X_u)\mathrm{d}u}$$

$$
\begin{aligned}
&= \int_0^t e^{-\int_0^s b'(X_u)\mathrm{d}u} \mathrm{d}D_\Theta X_{s-} + \int_0^t D_\Theta X_{s-} \mathrm{d}e^{-\int_0^s b'(X_u)\mathrm{d}u} \\
&= \int_0^t e^{-\int_0^s b'(X_u)\mathrm{d}u} \big(b'(X_s) D_\Theta X_s + \sigma_1 h(s)\big)\mathrm{d}s \\
&\quad + \int_0^t e^{-\int_0^s b'(X_u)\mathrm{d}u} \sigma_2 \int_{|z|>0} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t D_\Theta X_{s-} \left(-b'(X_s) \int_0^t e^{-\int_0^s b'(X_u)\mathrm{d}u} \right) \mathrm{d}s.
\end{aligned}
$$

For any $t > 0$, we set

$$
h(t) := \sigma_1 e^{-\int_0^t b'(X_s)\mathrm{d}s}, \quad \mathbf{v}(t,z) := \sigma_2 e^{-\int_0^t b'(X_s)\mathrm{d}s} \eta(z),
$$

$$
\eta(z) = \begin{cases} |z|^2, & |z| \le \frac{1}{4}, \\ 0, & |z| > \frac{1}{2}, \\ \text{smooth}, & \text{otherwise}. \end{cases} \tag{6.1}
$$

Since the function $\mathbf{v}$ is bounded, it satisfies the assumptions of Lemma 5.2. Hence, of course, X satisfies the assumptions of Lemma 4.1. Substituting these, we have

$$
\begin{aligned}
&D_\Theta X_t \\
&= e^{\int_0^t b'(X_s)\mathrm{d}s} \left(\sigma_1^2 \int_0^t e^{-2\int_0^s b'(X_u)\mathrm{d}u} \mathrm{d}s + \sigma_2^2 \int_0^t \int_{|z|>0} e^{-2\int_0^s b'(X_u)\mathrm{d}u} \eta(z) N(\mathrm{d}s, \mathrm{d}z) \right) \\
&\ge e^{\int_0^t (b'(X_s) - 2\|b\|_{\mathrm{Lip}})\mathrm{d}s} \left(\sigma_1^2 t + \sigma_2^2 \int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z) \right).
\end{aligned}
$$

Noticing the condition $\sigma_1^2 + \sigma_2^2 \ne 0$ and the fact

$$
\mathbb{P}\left(\int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z) > 0, \ \forall t > 0 \right) = 1, \tag{6.2}
$$

we have

$$
\mathbb{P}(D_\Theta X_t > 0, \ \forall t \in (0, T]) = 1.
$$

See Section A.1 for a proof of equation (6.2). So we have

$$
1 = \mathbb{P}\big(D_\Theta X_t > 0, \ \forall t \in (0,T]\big) \le \mathbb{P}\big(D_\Theta X_t \ne 0 \text{ on } \{t \in (0,T] : X_t = X_T^*\}\big) = 1.
$$

Therefore, we conclude by Lemma 4.1 that the law of X_T^* is absolutely continuous with respect to the Lebesgue measure. □

A Appendices

We give some lemmas to show Theorem 2.1. In Subsection A.1, we prove equation (6.2). In Subsection A.3, we confirm equation (6.2) and in Subsection A.2 we provide some results useful for Subsection A.3.

A.1 A proof of equation (6.2)

In this section, we prove equation (6.2). We set $t > 0$, η as (6.1) and $\varepsilon_k = \frac{1}{2^k}$ for any $k \in \mathbb{N}$. Since

$$\lim_{k\to\infty} \int_0^t \int_{|z|>\varepsilon_k} \eta(z) N(\mathrm{d}s, \mathrm{d}z) = \int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z) \;\text{ in }\; L^2(\Omega),$$

$$\lim_{k\to\infty} \int_0^t \int_{|z|>\varepsilon_k} \eta(z) N(\mathrm{d}s, \mathrm{d}z) = \int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z) \;\text{ in distribution.}$$

Noting that the support of η, for any $t > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left(\int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z) = 0\right) \\
&= \lim_{k\to\infty} \mathbb{P}\left(\int_0^t \int_{|z|>\varepsilon_k} \eta(z) N(\mathrm{d}s, \mathrm{d}z) = 0\right) \\
&= \lim_{k\to\infty} \mathbb{P}\left(\int_{|z|>\varepsilon_k} \eta(z) N(t, \mathrm{d}z) = 0\right) \\
&\leq \lim_{k\to\infty} \mathbb{P}\left(N\left(t, \left(\varepsilon_k, \frac{1}{2}\right] \cup \left(-\frac{1}{2}, -\varepsilon_k\right]\right) = 0\right).
\end{aligned}$$

Here, since for any $A \in \mathcal{B}(\mathbb{R}) \setminus \{0\}$, $\{N(t, A)\}_{t\geq 0}$ is a Poisson process with intensity $\nu(A)$ (e.g., see [1], Th. 2.3.5), we have

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{P}\left(N\left(t, \left(\varepsilon_k, \frac{1}{2}\right] \cup \left(-\frac{1}{2}, -\varepsilon_k\right]\right) = 0\right) \\
&= \lim_{k\to\infty} \exp\left(-t\nu\left(\left(\varepsilon_k, \frac{1}{2}\right] \cup \left(-\frac{1}{2}, -\varepsilon_k\right]\right)\right) \\
&= 0.
\end{aligned}$$

The last equation follows from assumption (2.2). We set for each $t > 0$,

$$I_t = \int_0^t \int_{|z|>0} \eta(z) N(\mathrm{d}s, \mathrm{d}z);$$

from countable additivity we have

$$\mathbb{P}\left(\bigcup_{t\in(0,\infty)\cap\mathbb{Q}} \{I_t = 0\}\right) \leq \sum_{t\in(0,\infty)\cap\mathbb{Q}} \mathbb{P}\big(\{I_t = 0\}\big) = 0 \quad \big(\text{see, e.g., [16], 1.9(b)}\big).$$

Thus we obtain

$$\mathbb{P}\left(\bigcap_{t\in(0,\infty)\cap\mathbb{Q}} \{I_t > 0\}\right) = 1.$$

Here, since $\eta \geq 0$, we obtain

$$\mathbb{P}\left(\bigcap_{0\leq s\leq u} \{I_s \leq I_u\}\right) = 1.$$

By denseness of rational numbers, we obtain the following:

$$\begin{aligned}\mathbb{P}(\forall t>0,\ I_t>0) &\geq \mathbb{P}\Big(\bigcap_{t\in(0,\infty)\cap\mathbb{Q}}\{I_t>0\}\cap\bigcap_{0\leq s\leq u}\{I_s\leq I_u\}\Big)\\ &=1.\end{aligned}$$

A.2 Preparation for proof of convergence of $X^{(n)}$

To prove Theorem 2.1, we apply a variation of the method introduced by Komatsu ([8], proof of Theorem 1) in order to prove the convergence of $X^{(n)}$. This technique has been used in [11].

Lemma A.1. *For $\varepsilon>0$, $\delta>1$ and $r\in(0,1]$, we can choose a smooth function $\psi_{\delta,\varepsilon}$ which satisfies the conditions*

$$\psi_{\delta,\varepsilon}(x)=\begin{cases}\text{between } 0 \text{ and } 2(x\log\delta)^{-1}, & \varepsilon\delta^{-1}<x<\varepsilon,\\ 0, & \text{otherwise},\end{cases}$$

and $\int_{\varepsilon\delta^{-1}}^{\varepsilon}\psi_{\delta,\varepsilon}(y)dy=1$. We define $u_r(x)=|x|^r$ and $u_{r,\delta,\varepsilon}=u_r*\psi_{\delta,\varepsilon}$. Then, $u_{r,\delta,\varepsilon}\in C^2$ and for any $x\in\mathbb{R}$,

$$|x|^r\leq\varepsilon^r+u_{r,\delta,\varepsilon}(x), \tag{A.1}$$

$$u_{r,\delta,\varepsilon}(x)\leq|x|^r+\varepsilon^r. \tag{A.2}$$

We introduce a quasimartingale and its properties. Let $T\in[0,\infty]$ and Z be a càdlàg adapted process defined on $[0,T]$. A finite subdivision of $[0,T]$ is defined by $\Delta t=(t_0,t_1,\ldots,t_{n+1})$ such that $0=t_0<t_1<\cdots<t_{n+1}=T$.

Definition A.2. The mean variation of X is defined by

$$V_T(X):=\sup_{\Delta t}\mathbb{E}\Big[\sum_{i=0}^{n}\big|\mathbb{E}[X_{t_i}-X_{t_{i+1}}|\mathscr{F}_{t_i}]\big|\Big].$$

Definition A.3. A càdlàg adapted process Z is a quasimartingale on $[0,T]$ if for each $t\in[0,T]$, $\mathbb{E}[|Z_t|]<\infty$ and $V_T(Z)<\infty$.

Kurtz [9] proved the following lemma by using Rao's theorem ([12], Section III, Theorem 17).

Lemma A.4 ([9], Lemma 5.3)**.** *Let Z be a càdlàg adapted process defined on $[0,T]$. Suppose that for each $t\in[0,T]$, $\mathbb{E}[|Z_t|]<\infty$ and $V_t(Z)<\infty$. Then, for each $h>0$,*

$$h\mathbb{P}\Big(\sup_{t\in[0,T]}|Z_t|>h\Big)\leq V_T(Z)+\mathbb{E}\big[|Z_T|\big].$$

A.3 *Proof of L^p-convergence of $X^{(n)}$*

In this section, we prove several lemmas in order to give complete proof of Lemma 5.2. To that purpose, we show the following two statements.

Lemma A.5. *We assume the same assumptions as in Theorem 2.1. Then*

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big|^p\Big] = 0 \text{ for any } p \in (1, \beta).$$

This is because the following two conditions are valid.

(i) $\sup_{t\in[0,T]} |X_t^{(n)} - X_t|^p \to 0$ *in probability as* $n \to \infty$.

(ii) The class of random variables

$$\Big\{\sup_{s\in[0,t]} \big|X_s^{(n)} - X_s\big|^p\Big\}_{t\in[0,T]}$$

is uniformly integrable.

Proof. (i) For clarity, we write $r = \frac{p}{\beta}$. By using the triangle inequality and Jensen's inequality, we have

$$\big|X_t^{(n)} - X_t\big| \le \int_0^t \big|b\big(X_s^{(n)}\big) - b(X_s)\big| \mathrm{d}s + \sigma_2 \big|L_t^{(n)} - L_t\big|.$$

By the definition of supremum, we have

$$\sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big| \le T \sup_{t\in[0,T]} \big|b\big(X_t^{(n)}\big) - b(X_t)\big| + \sigma_2 \sup_{t\in[0,T]} \big|L_t^{(n)} - L_t\big|.$$

Since b is Lipschitz continuous, we have

$$\sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big| \le CT \sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big| \mathrm{d}s + \sigma_2 \sup_{t\in[0,T]} \big|L_t^{(n)} - L_t\big|.$$

By using Gronwall's inequality and Jensen's inequality, we have

$$\sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big| \le C\sigma_2 \exp(CT) \sup_{t\in[0,T]} \big|L_t^{(n)} - L_t\big|. \tag{A.3}$$

Here, by the above inequality and Lemma A.4, for any $h > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \big|X_t^{(n)} - X_t\big|^p > h\Big) \\
&\le \mathbb{P}\left(\sup_{t\in[0,T]} \big|L_t^{(n)} - L_t\big|^r > \left(\frac{h}{C\sigma_2 \exp(CT)}\right)^{\frac{1}{\beta}}\right) \\
&\le \mathbb{P}\left(\sup_{t\in[0,T]} \big(\varepsilon^p + u_{r,\delta,\varepsilon}\big(L_t^{(n)} - L_t\big)\big) > \left(\frac{h}{C\sigma_2 \exp(CT)}\right)^{\frac{1}{\beta}}\right)
\end{aligned}$$

$$\leq \left(\frac{C\sigma_2 \exp(CT)}{h}\right)^{\frac{1}{\beta}} \left(V_T\left(\varepsilon^p + u_{r,\delta,\varepsilon}\left(L^{(n)} - L\right)\right)\right.$$
$$\left. + \mathbb{E}\left[\left|\varepsilon^p + u_{r,\delta,\varepsilon}\left(L_T^{(n)} - L_T\right)\right|\right]\right).$$

Here, by the definition of the mean variation and (A.2), we have

$$V_T\left(\varepsilon^p + u_{r,\delta,\varepsilon}\left(L^{(n)} - L\right)\right) = V_T\left(u_{r,\delta,\varepsilon}\left(L^{(n)} - L\right)\right),$$
$$\mathbb{E}\left[\left|\varepsilon^p + u_{r,\delta,\varepsilon}\left(L_t^{(n)} - L_t\right)\right|\right] = \mathbb{E}\left[\varepsilon^p + u_{r,\delta,\varepsilon}\left(L_T^{(n)} - L_T\right)\right].$$

By using the Lévy–Itô decomposition ([1], Theorem 2.4.16), we have

$$\begin{aligned}
L_t^{(n)} - L_t &= \int_0^t \int_{|z|>1} z \mathbb{1}_{\{0<|z|\leq n\}} N(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{0<|z|\leq 1} z \mathbb{1}_{\{0<|z|\leq n\}} \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\quad - \int_0^t \int_{|z|>1} z N(\mathrm{d}s, \mathrm{d}z) - \int_0^t \int_{0<|z|\leq 1} z \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \\
&= - \int_0^t \int_{|z|>1} z \mathbb{1}_{\{|z|>n\}} N(\mathrm{d}s, \mathrm{d}z) - \int_0^t \int_{0<|z|\leq 1} z \mathbb{1}_{\{|z|>n\}} \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \\
&= - \int_0^t \int_{|z|>1} z \mathbb{1}_{\{|z|>n\}} N(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

The last equality follows by $n \geq 1$. Using the Itô formula ([1], Theorem 4.4.7), $N(\mathrm{d}t, \mathrm{d}z) = \widetilde{N}(\mathrm{d}t, \mathrm{d}z) + \nu(\mathrm{d}z)\mathrm{d}t$ and the function $u_{r,\delta,\varepsilon}$ defined in Lemma A.1, we have

$$\begin{aligned}
&u_{r,\delta,\varepsilon}\left(L_t^{(n)} - L_t\right) \\
&= \int_0^t \int_{|z|\geq 1} \left\{u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right)\right\} N(\mathrm{d}s, \mathrm{d}z) \\
&= \int_0^t \int_{|z|\geq 1} \left\{u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right)\right\} \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t \int_{|z|\geq 1} \left\{u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right)\right\} \mathrm{d}s\,\nu(\mathrm{d}z) \\
&=: M_t^{\delta,\varepsilon} + I_t^{\delta,\varepsilon}.
\end{aligned}$$

Here, by (A.1), for any $x, y \in \mathbb{R}$,

$$\begin{aligned}
-u_{r,\delta,\varepsilon}(y) &\leq \varepsilon^r - |y|^r, \\
u_{r,\delta,\varepsilon}(x) - u_{r,\delta,\varepsilon}(y) &\leq 2\varepsilon^r + |x|^r - |y|^r \\
&\leq 2\varepsilon^r + \left||x|^r - |y|^r\right| \\
&\leq 2\varepsilon^r + |x - y|^r.
\end{aligned}$$

So we have

$$u_{r,\delta,\varepsilon}\left(L_T^{(n)} - L_T\right)$$

$$\leq \int_0^T \int_{|z|\geq 1} \left\{ u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right) \right\} \widetilde{N}(\mathrm{d}s, \mathrm{d}z)$$
$$+ \int_0^T \int_{|z|\geq 1} \left\{ 2\varepsilon^r + |z|^r \mathbb{1}_{\{|z|>n\}} \right\} \mathrm{d}s\, \nu(\mathrm{d}z).$$

Also,

$$\int_0^T \int_{|z|\geq 1} \left\{ 2\varepsilon^r + |z|^r \mathbb{1}_{\{|z|>n\}} \right\} \mathrm{d}s\, \nu(\mathrm{d}z) \leq 2CT\varepsilon^r + T \int_{\mathbb{R}\setminus\{0\}} |z|^r \mathbb{1}_{\{|z|>n\}} \nu(\mathrm{d}z).$$

We can evaluate

$$\begin{aligned}
&V_T\left(u_{r,\delta,\varepsilon}\left(L^{(n)} - L\right)\right) \\
&= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^n \left| \mathbb{E}\left[u_{r,\delta,\varepsilon}\left(L_{t_i}^{(n)} - L_{t_i}\right) - u_{r,\delta,\varepsilon}\left(L_{t_{i+1}}^{(n)} - L_{t_{i+1}}\right) \mid \mathscr{F}_{t_i} \right] \right| \right] \\
&= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^n \left| \mathbb{E}\left[M_{t_i}^{\delta,\varepsilon} + I_{t_i}^{\delta,\varepsilon} - M_{t_{i+1}}^{\delta,\varepsilon} - I_{t_{i+1}}^{\delta,\varepsilon} \mid \mathscr{F}_{t_i} \right] \right| \right] \\
&= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^n \left| \mathbb{E}\left[I_{t_i}^{\delta,\varepsilon} - I_{t_{i+1}}^{\delta,\varepsilon} \mid \mathscr{F}_{t_i} \right] \right| \right].
\end{aligned}$$

The last equality is valid, since $(M_t^{\delta,\varepsilon})_{t\in[0,T]}$ is a martingale. By using Jensen's inequality, we have

$$\begin{aligned}
&V_T\left(u_{r,\delta,\varepsilon}\left(L^{(n)} - L\right)\right) \\
&\leq \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^n \mathbb{E}\left[\left| I_{t_i}^{\delta,\varepsilon} - I_{t_{i+1}}^{\delta,\varepsilon} \right| \mid \mathscr{F}_{t_i} \right] \right] \\
&= \sup_{\Delta t} \sum_{i=0}^n \mathbb{E}\left[\left| \int_{t_i}^{t_{i+1}} \int_{|z|\geq 1} \left\{ u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) \right.\right.\right. \\
&\qquad \left.\left.\left. - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right) \right\} \mathrm{d}s\, \nu(\mathrm{d}z) \right| \right] \\
&\leq \sup_{\Delta t} \sum_{i=0}^n \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{|z|\geq 1} \left| u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-} - z\mathbb{1}_{\{|z|>n\}}\right) \right.\right. \\
&\qquad \left.\left. - u_{r,\delta,\varepsilon}\left(L_{s-}^{(n)} - L_{s-}\right) \right| \mathrm{d}s\, \nu(\mathrm{d}z) \right] \\
&\leq \sup_{\Delta t} \sum_{i=0}^n \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{|z|\geq 1} \left(2\varepsilon^r + |z|^r \mathbb{1}_{\{|z|>n\}} \right) \mathrm{d}s\, \nu(\mathrm{d}z) \right] \\
&\leq \int_0^T \int_{|z|\geq 1} \left(2\varepsilon^r + |z|^r \mathbb{1}_{\{|z|>n\}} \right) \mathrm{d}s\, \nu(\mathrm{d}z).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]}\big|X_t^{(n)}-X_t\big|^p>h\Big)\\
&\le\left(\frac{C\sigma_2\exp(CT)}{h}\right)^{\frac{1}{\beta}}\left\{(4CT+1)\varepsilon^r+2T\int_{\mathbb{R}\setminus\{0\}}|z|^r\mathbb{1}_{\{|z|>n\}}\nu(\mathrm{d}z)\right\}.
\end{aligned}$$

Since the above inequality holds for any $\varepsilon>0$, we obtain

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\big|X_t^{(n)}-X_t\big|^p>h\Big)\le 2T\left(\frac{C\sigma_2\exp(CT)}{h}\right)^{\frac{1}{\beta}}\int_{\mathbb{R}\setminus\{0\}}|z|^r\mathbb{1}_{\{|z|>n\}}\nu(\mathrm{d}z).$$

By the assumption, for any $h>0$, we have

$$\lim_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]}\big|X_t^{(n)}-X_t\big|^p>h\Big)=0.$$

(ii) Next, we show that the process

$$\Big\{\sup_{t\in[0,T]}\big|X_t^{(n)}-X_t\big|^p\Big\}_{T\ge 0}$$

is uniformly integrable. To show this, by using inequality (A.3) it suffices to show for some $q>1$,

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]}\big|L_t^{(n)}-L_t\big|^p\vee 1\Big)^q\Big]<\infty.$$

By assumption of (2.1) and the denseness of rational numbers, we can set $q>1$ such that $pq<\beta$ so that for each $n\in\mathbb{N}$,

$$\int_{|z|>n}|z|^{pq}\nu(\mathrm{d}z)<\infty. \tag{A.4}$$

Here, we set $g(x)=|x|^{pq}\vee 1$; then g is a nonnegative increasing submultiplicative function and $\lim_{x\to\infty}g(x)=\infty$. Using Theorem 25.18 in [13], we will show that

$$\mathbb{E}\big[g\big(\big|L_t^{(n)}-L_t\big|\big)\big]<\infty \qquad \text{for some } t>0.$$

For some $t>0$,

$$\begin{aligned}
\mathbb{E}\big[g\big(\big|L_t^{(n)}-L_t\big|\big)\big]&=\mathbb{E}[\mathbb{1}_{\{|L_t^{(n)}-L_t|\le 1\}}]+\mathbb{E}\big[\big|L_t^{(n)}-L_t\big|^{pq}\mathbb{1}_{\{|L_t^{(n)}-L_t|>1\}}\big]\\
&\le 1+\mathbb{E}\big[\big|L_t^{(n)}-L_t\big|^{pq}\big]\\
&<\infty.
\end{aligned}$$

The last inequality does not depend on $n\in\mathbb{N}$ because of (A.4) and Example 25.10 in [13]. Since we have shown (i) and (ii) from the above, the proof is completed. □

We recall

$$C_t = \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{|z|>0} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z),$$
$$C_t^{(n)} = \sigma_1 \int_0^t h(s)\mathrm{d}s + \sigma_2 \int_0^t \int_{0<|z|\le n} \mathbf{v}(s,z) N(\mathrm{d}s, \mathrm{d}z).$$

To complete the proof of Lemma 5.2, we show the following.

Lemma A.6. *In the setup of Lemma 5.2, the following convergence holds:*

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|C_t^{(n)} - C_t\big|^p\Big] = 0.$$

Proof. It can be proved in the same way as in Lemma A.5. In fact, by using the function $u_{r,\delta,\varepsilon}$ defined in Lemma A.1 with $r = \frac{p}{\beta}$, we have

$$u_{r,\delta,\varepsilon}\big(C_t^{(n)} - C_t\big)$$
$$= \int_0^t \int_{|z|\ge 1} \big\{u_{r,\delta,\varepsilon}\big(C_t^{(n)} - C_t + \mathbf{v}(s,z)\mathbb{1}_{\{|z|>n\}}\big) - u_{r,\delta,\varepsilon}\big(C_t^{(n)} - C_t\big)\big\} N(\mathrm{d}s, \mathrm{d}z).$$

Note that $\{C_t^{(n)} - C_t\}_{t\ge 0}$ is a compound Poisson process and assumptions of Lemma 5.2 are fulfilled. We can confirm the $L^p(\Omega \times [0,T])$ convergence by showing convergence in probability and uniform integrability in the same way as in Lemma A.5. □

Conflict of interest

The authors have no relevant financial or nonfinancial interests to disclose.

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