Modern Stochastics: Theory and Applications 11 (2) (2024) 195–216
https://doi.org/10.15559/24-VMSTA247

Asymptotic normality of the LSE for chirp signal parameters

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Received: 17 October 2023, Revised: 13 January 2024, Accepted: 13 January 2024,
Published online: 23 January 2024

Abstract A time continuous statistical model of chirp signal observed against the background of stationary Gaussian noise is considered in the paper. Asymptotic normality of the LSE for parameters of such a sinusoidal regression model is obtained.

Keywords Chirp signal, stationary Gaussian stochastic process, least squares estimate, strong consistency, asymptotic normality, Fresnel integrals

1 Introduction

In signal processing theory and nonlinear regression analysis the problem of detecting hidden periodicities, in particular, the problem of estimation of amplitudes and angular frequencies of the harmonic oscillations in the presence of random noise is intensively studied due to its wide application in various fields of knowledge. The solution to the problem of detecting hidden periodicities has a long history dating back to the works by Lagrange and, since the end of 19th century, there are numerous publications. Not being able to talk here about this subject in more detail, we will refer only to the works [2, 9, 26, 14, 15], where a lot of links to publications on the topic can be found.

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Over the past decades, in the literature on signal and image processing sinusoidal statistical models on the plane have been intensively studied due to their multiple applications in the analysis of the symmetrical textured surfaces. See, for example, [27, 16, 28, 17, 25, 12, 11]. Some of scalar and 2D results are generalized to trigonometric regression models on $\mathbb{R}^M$, $M \geq 3$, in the papers [3, 13, 10].

One more important extension of classical trigonometric models are models of frequency-modulated sinusoidal signals observed in the presence of additive random noises. Different problems of parametric estimation of such signals have been studied for a long time [4, 5]. Now the number of publications in this area has increased significantly. Some links to recent articles can be found in [8].

The most studied is the case of chirp signals, that is, linearly frequency-modulated signals. For continuous time t it can be written as the sum of functions of the form

$$A \cos\big(\phi t + \psi t^2\big) + B \sin\big(\phi t + \psi t^2\big), \quad t \geq 0, \tag{1}$$

where A and B are amplitudes, ϕ is the starting frequency, ψ is the chirp rate. For discrete time t and random noise, that is, for a linear time series, some results on consistency and asymptotic normality of LSE of chirp signal parameters have been obtained in many works. We will only point to publications [23, 18, 20, 21, 24, 19, 7] and references there in.

In the present paper we consider time continuous multiple chirp-signal observed with additive strongly or weakly dependent random noise and obtain a result on the LSE asymptotic normality.

Introduce the notation $\sin_j^0(t) = \sin(\phi_j^0 t + \psi_j^0 t^2)$, $\cos_j^0(t) = \cos(\phi_j^0 t + \psi_j^0 t^2)$, $(C_j^0)^2 = (A_j^0)^2 + (B_j^0)^2$. For vectors v and matrices M, we will denote by v^* and M^* the transposed ones.

Assume we observe a stochastic process

$$X(t) = g\big(t, \theta^0\big) + \varepsilon(t), \quad t \in \mathbb{R}_+, \tag{2}$$

where

$$g\big(t, \theta^0\big) = \sum_{j=1}^{N} \big(A_j^0 \cos_j^0(t) + B_j^0 \sin_j^0(t)\big), \tag{3}$$

$$\theta^0 = \big(A_1^0, B_1^0, \phi_1^0, \psi_1^0, \dots, A_N^0, B_N^0, \phi_N^0, \psi_N^0\big)^*, \tag{4}$$

$\varepsilon = \{\varepsilon(t), t \in \mathbb{R}\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and satisfying the following condition.

A1. ε is a sample-continuous stationary Gaussian process with zero mean and covariance function (c.f.) $B(t) = E\varepsilon(t)\varepsilon(0)$, $t \in \mathbb{R}$, having one of the properties:

(i) $B(t) = L(|t|)|t|^{-\alpha}$, $\alpha \in (0, 1)$, with nondecreasing slowly varying at infinity function L;

(ii) $B(\cdot) \in L_1(\mathbb{R})$.

In the paper [8] to estimate the parameters (4) we introduced special parametric sets that depend on the observation time T, which allows us to distinguish asymptotically the parameters of our statistical model. Assuming that true values of amplitudes $A_j^0, B_j^0 \neq 0$, $j = \overline{1, N}$, are different numbers and the true values of frequencies ϕ_j^0, $j = \overline{1, N}$, and chirp rates ψ_j^0, $j = \overline{1, N}$, are different positive numbers, we arrange the chirp rates $\psi^0 = (\psi_1^0, \dots, \psi_N^0)$ in increasing order and suppose that $\psi^0 \in \Psi(\underline{\psi}, \overline{\psi})$, where

$$\Psi(\underline{\psi}, \overline{\psi}) = \left\{\psi = (\psi_1, \dots, \psi_N) \in \mathbb{R}^N : 0 \leq \underline{\psi} < \psi_1 < \dots < \psi_N < \overline{\psi} < +\infty\right\}.$$

In turn, we also introduce the parametric set

$$\Phi(\underline{\phi}, \overline{\phi}) = \left\{\phi = (\phi_1, \dots, \phi_N) : 0 \leq \underline{\phi} < \phi_j < \overline{\phi} < +\infty, \, j = \overline{1, N}\right\}$$

such that $\phi^0 = (\phi_1^0, \dots, \phi_N^0) \in \Phi(\underline{\phi}, \overline{\phi})$.

Consider monotonically nondecreasing family of open sets $\Psi_T \subset \Psi(\underline{\psi}, \overline{\psi})$, $T > T_0 > 0$, containing vector ψ^0, such that $\bigcup_{T>T_0} \Psi_T = \widetilde{\Psi}$, $\widetilde{\Psi}^c = \Psi^c(\underline{\psi}, \overline{\overline{\psi}})$, with the following properties.

B. 1) $\lim_{T\to\infty} \inf_{\substack{1\leq j\leq N-1 \\ \psi\in\Psi_T}} T^2(\psi_{j+1} - \psi_j) = +\infty$;

2) $\lim_{T\to\infty} \inf_{\psi\in\Psi_T} T^2\psi_1 = +\infty$.

Definition 1. Any random vector

$$\theta_T = (A_{1T}, B_{1T}, \phi_{1T}, \psi_{1T}, \dots, A_{NT}, B_{NT}, \phi_{NT}, \psi_{NT})^* \tag{5}$$

such that it is an absolute minimum point of the functional

$$Q_T(\theta) = \int_0^T \left[X(t) - g(t, \theta)\right]^2 dt \tag{6}$$

on the parametric set $\Theta_T^c \subset \mathbb{R}^{4N}$, where amplitudes A_j, B_j, $j = \overline{1, N}$, can take any values and parameters (ϕ, ψ) take values in the set $\Phi^c(\underline{\phi}, \overline{\phi}) \times \Psi_T^c$, $T > T_0 > 0$, is called LSE of the parameter θ^0.

Remark 1. In the paper [8] integral (6) is preceded by the factor T^{-1}. In the present paper we keep the same notation $Q_T(\theta)$.

In [8] we obtained the following result.

Theorem 1. *Let the conditions **A1** and **B** be satisfied. Then LSE θ_T is a strongly consistent estimate of parameter θ^0 in the sense that $A_{jT} \to A_j^0$, $B_{jT} \to B_j^0$, $T(\phi_{jT} - \phi_j^0) \to 0$, $T^2(\psi_{jT} - \psi_j^0) \to 0$ a.s., as $T \to \infty$, $j = \overline{1, N}$.*

To formulate the main result of the paper we introduce additional assumptions regarding the stochastic process ε.

A2(*i*) The process ε that satisfies the condition **A1(*i*)** has a spectral density $f(\lambda) = \tilde{L}(\frac{1}{|\lambda|})|\lambda|^{\alpha-1}$, where $\tilde{L}$ is a slowly varying at infinity function, and f has the 4th spectral moment.

(*ii*) The spectral density of the process ε that satisfies the condition **A1(*ii*)** has the 4th spectral moment.

We give an example of the fulfillment of the condition **A2(*i*)**.

Example 1 (Bessel c.f. [22, 14, 15]). Let's consider the process ε with c.f. $B(t) = (1+t^2)^{-\frac{\alpha}{2}}$, $\alpha \in (0,1)$, $t \in \mathbb{R}$. This function satisfies condition **A1(*i*)**. Its spectral density is of the form

$$f(\lambda) = \frac{2^{\frac{1-\alpha}{2}}}{\sqrt{\pi}\Gamma(\frac{\alpha}{2})} K_{\frac{\alpha-1}{2}}(|\lambda|)|\lambda|^{\frac{\alpha-1}{2}}, \quad \lambda \in \mathbb{R},$$

where $K_\nu(z)$, $z > 0$, is the modified Bessel function of the 2nd kind of order ν. If $\lambda \to 0$, then $f(\lambda) \sim [2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}]^{-1}|\lambda|^{\alpha-1}(1-h(\lambda))$, $h(\lambda) \to 0$, as $\lambda \to 0$. For large argument ($\lambda \to +\infty$) $K_{\frac{\alpha-1}{2}} \sim \sqrt{\frac{\pi}{2\lambda}}e^{-\lambda}$ (see Hankel expansion [1, p. 378, 9.7.2]), that is, $f(\lambda)$ has all the spectral moments.

Theorem 2. *Let the conditions **A1**, **A2** and **B** be fulfilled. Then the normed LSE*

$$\begin{aligned} &T^{\frac{1}{2}} D_T\big(\theta_T - \theta^0\big) = \\ &= \big(T\big(A_{1T} - A_1^0\big), T\big(B_{1T} - B_1^0\big), T^2\big(\phi_{1T} - \phi_1^0\big), T^3\big(\psi_{1T} - \psi_1^0\big), \ldots \\ &\ldots, T\big(A_{NT} - A_N^0\big), T\big(B_{NT} - B_N^0\big), T^2\big(\phi_{NT} - \phi_N^0\big), T^3\big(\psi_{NT} - \psi_N^0\big)\big)^* \end{aligned} \tag{7}$$

is asymptotically, as $T \to \infty$, normal $N(0, W)$, where W is a block matrix consisting of square blocks (60) of order 4 with elements given by the formulas (59), (61) and (62).

In the 2nd section some auxiliary statements necessary to prove the asymptotic normality of the LSE θ_T are obtained.

2 Two lemmas

Set

$$\frac{\partial}{\partial\theta_i} g(t,\theta) = g_i(t,\theta), \qquad \frac{\partial^2}{\partial\theta_i\partial\theta_j} g(t,\theta) = g_{ij}(t,\theta), \quad i,j = \overline{1,4N}, \tag{8}$$

and write down the system of normal equations for θ_T:

$$0 = Q_T'(\theta_T) = \left(-2\int_0^T \big[X(t) - g(t,\theta_T)\big] g_i(t,\theta_T)dt\right)_{i=1}^{4N}. \tag{9}$$

Consider the Hesse matrix

$$\begin{aligned} Q_T''(\theta) &= \left(-2\int_0^T \big[X(t) - g(t,\theta)\big] g_{ij}(t,\theta)dt + 2\int_0^T g_i(t,\theta)g_j(t,\theta)dt\right)_{i,j=1}^{4N} \\ &= Q_{1T}''(\theta) + Q_{2T}''(\theta). \end{aligned} \tag{10}$$

and the Taylor expansion

$$-\frac{1}{2}Q_T'\big(\theta^0\big)=\frac{1}{2}Q_T'(\theta_T)-\frac{1}{2}Q_T'\big(\theta^0\big)=\frac{1}{2}Q_T''(\overline{\theta})\big(\theta_T-\theta^0\big). \tag{11}$$

Since the Taylor formula for vectors does not exist, equality (11) needs clarification. We write $4N$ Taylor formulas for coordinates of the vector $Q_T'(\theta_T)$ and obtain for each such expansion its own intermediate value $\overline{\theta_i}$ depending on T, which has the property $\|\overline{\theta_i}-\theta^0\|\le\|\theta_T-\theta^0\|$, $i=\overline{1,4N}$. Then, for convenience, we use the notation $\overline{\theta_i}=\overline{\theta}$, $i=\overline{1,4N}$. It goes without saying, after that $Q_T''(\overline{\theta})$ is no longer a Hesse matrix.

Let's introduce a block-diagonal matrix D_T that contains N blocks, and each block, in turn, is the diagonal matrix

$$d_T=\operatorname{diag}\big(T^{1/2},T^{1/2},T^{3/2},T^{5/2}\big). \tag{12}$$

Then the relation (11) can be rewritten in the form

$$D_T\big(\theta_T-\theta^0\big)=\left(D_T^{-1}\left(\frac{1}{2}Q_T''(\overline{\theta})\right)D_T^{-1}\right)^{-1}D_T^{-1}\left(-\frac{1}{2}Q_T'\big(\theta^0\big)\right). \tag{13}$$

Lemma 1. *Under conditions **A1** and **B***

$$D_T^{-1}\left(\frac{1}{2}Q_T''(\overline{\theta})\right)D_T^{-1}\to H \ \textit{a.s.},\quad \textit{as } T\to\infty, \tag{14}$$

where H is a block-diagonal matrix with blocks

$$H_m=\frac{1}{2}\begin{bmatrix}1 & 0 & \frac{B_m^0}{2} & \frac{B_m^0}{3}\\ 0 & 1 & -\frac{A_m^0}{2} & -\frac{A_m^0}{3}\\ \frac{B_m^0}{2} & -\frac{A_m^0}{2} & \frac{(C_m^0)^2}{3} & \frac{(C_m^0)^2}{4}\\ \frac{B_m^0}{3} & -\frac{A_m^0}{3} & \frac{(C_m^0)^2}{4} & \frac{(C_m^0)^2}{5}\end{bmatrix},\quad m=\overline{1,N}. \tag{15}$$

Proof. Note that

$$D_T^{-1}\left(\frac{1}{2}Q_{1T}''(\overline{\theta})\right)D_T^{-1}$$

is a block-diagonal matrix and consists of N blocks

$$d_T^{-1}\left(\frac{1}{2}Q_{1T,k}''(\overline{\theta})\right)d_T^{-1}=d_T^{-1}\left(-\int_0^T\big[X(t)-g(t,\overline{\theta})\big]g_{ij}(t,\overline{\theta})dt\right)_{i,j=4k+1}^{4(k+1)}d_T^{-1},$$
$$k=\overline{0,N-1}. \tag{16}$$

Let us show that each matrix (16) converges to zero matrix. Since the proof of this fact is the same for all k, we will carry it out for $k=0$.

Denote the matrix $d_T^{-1}(\frac{1}{2}Q_{1T,0}''(\overline{\theta}))d_T^{-1}$ by $J^1(\overline{\theta})$. Then

$$J^1(\overline{\theta})=d_T^{-1}\left(\int_0^T\big[g(t,\overline{\theta})-g\big(t,\theta^0\big)\big]g_{ij}(t,\overline{\theta})dt\right)_{i,j=1}^{4}d_T^{-1}$$

$$- d_T^{-1}\left(\int_0^T \varepsilon(t) g_{ij}(t,\overline{\theta})dt\right)_{i,j=1}^4 d_T^{-1} = J_1^1 + J_2^1. \tag{17}$$

For further evaluation of matrices J_1^1 and J_2^1 consider the matrix

$$d_T^{-1}\big(g_{ij}(t,\overline{\theta})\big)_{i,j=1}^4 d_T^{-1} =$$
$$\begin{bmatrix} 0 & 0 & \frac{-t\sin_1^-(t)}{T^2} & \frac{-t^2\sin_1^-(t)}{T^3} \\ 0 & 0 & \frac{t\cos_1^-(t)}{T^2} & \frac{t^2\cos_1^-(t)}{T^3} \\ \frac{-t\sin_1^-(t)}{T^2} & \frac{t\cos_1^-(t)}{T^2} & \frac{-t^2(\overline{A}_1\cos_1^-(t)+\overline{B}_1\sin_1^-(t))}{T^3} & \frac{-t^3(\overline{A}_1\cos_1^-(t)+\overline{B}_1\sin_1^-(t))}{T^4} \\ \frac{-t^2\sin_1^-(t)}{T^3} & \frac{t^2\cos_1^-(t)}{T^3} & \frac{-t^3(\overline{A}_1\cos_1^-(t)+\overline{B}_1\sin_1^-(t))}{T^4} & \frac{-t^4(\overline{A}_1\cos_1^-(t)+\overline{B}_1\sin_1^-(t))}{T^5} \end{bmatrix}, \tag{18}$$

where $\sin_j^-(t) = \sin(\overline{\phi}_j t + \overline{\psi}_j t^2)$, $\cos_j^-(t) = \cos(\overline{\phi}_j t + \overline{\psi}_j t^2)$.

By Theorem 1, for $t \in [0, T]$,

$$\begin{aligned} \left|g(t,\overline{\theta}) - g\big(t,\theta^0\big)\right| &\le \sum_{k=1}^N \big(|\overline{A}_k - A_k^0| + |\overline{B}_k - B_k^0| \\ &+ \big(|A_k^0| + |B_k^0|\big)\big(t|\overline{\phi}_k - \phi_k^0| + t^2|\overline{\psi}_k - \psi_k^0|\big)\big) \le \sum_{k=1}^N \big(|A_{kT} - A_k^0| + |B_{kT} - B_k^0| \\ &+ \big(|A_k^0| + |B_k^0|\big)\big(T|\phi_{kT} - \phi_k^0| + T^2|\psi_{kT} - \psi_k^0|\big)\big) = \zeta_T \to 0 \text{ a.s.,} \quad \text{as } T \to \infty. \end{aligned} \tag{19}$$

On the other hand, due to (18), $J_{1,ij}^1 = 0$, $i, j = 1, 2$,

$$\int_0^T |J_{1,13}^1|dt \le \frac{\zeta_T}{2};\ \int_0^T |J_{1,14}^1|dt \le \frac{\zeta_T}{3};\ \int_0^T |J_{1,23}^1|dt \le \frac{\zeta_T}{2};\ \int_0^T |J_{1,24}^1|dt \le \frac{\zeta_T}{3};$$
$$\begin{aligned} &\int_0^T \big(|J_{1,33}^1| + |J_{1,34}^1| + |J_{1,44}^1|\big)dt \\ &\qquad \le \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)\big(|A_{1T} - A_1^0| + |B_{1T} - B_1^0| + |A_1^0| + |B_1^0|\big)\zeta_T. \end{aligned} \tag{20}$$

So,

$$J_1^1 \to 0 \text{ a.s.,} \quad \text{as } T \to \infty. \tag{21}$$

Consider the matrix J_2^1 in (17). The elements $J_{2,ij}^1 = 0$, $i, j = 1, 2$. Successively we obtain

$$\begin{aligned} \left|J_{2,13}^1\right| = \left|T^{-2}\int_0^T \varepsilon(t)t\sin_1^-(t)dt\right| &\le \left|T^{-2}\int_0^T \varepsilon(t)t\big(\sin_1^-(t) - \sin_1^0(t)\big)dt\right| \\ &+ \left|T^{-2}\int_0^T \varepsilon(t)t\sin_1^0(t)dt\right| = I_1(T) + I_2(T); \end{aligned} \tag{22}$$

$$I_1(T) \le T^{-1} \int_0^T |\varepsilon(t)| dt \big(T|\phi_{1T} - \phi_1^0| + T^2 |\psi_{1T} - \psi_1^0| \big),$$

where the 2nd factor vanishes a.s., as $T \to \infty$, and under condition **A1**

$$T^{-1} \int_0^T |\varepsilon(t)| dt \le \frac{1}{2}\left(1 + T^{-1} \int_0^T \varepsilon^2(t) dt\right) \to \frac{1}{2}\big(1 + B(0)\big) \text{ a.s.,} \quad \text{as } T \to \infty,$$

that is, $I_1(T) \to 0$ a.s., as $T \to \infty$.

On the other hand, under condition **A1(*i*)**,

$$\begin{aligned} EI_2^2(T) &= T^{-4} \int_0^T \int_0^T tsB(t-s) \sin_1^0(t) \sin_1^0(s) dt ds \le T^{-2} \int_0^T \int_0^T B(t-s) dt ds \\ &= \int_0^1 \int_0^1 B\big(T(t-s)\big) dt ds = \int_{-1}^1 (1 - |t|) B(Tt) dt \le 2 \int_0^1 B(Tt) dt \\ &\le \frac{2}{1-\alpha} \frac{L(T)}{T^\alpha}. \end{aligned}$$

Let's take $T_n = n^\beta$ with $\beta\alpha > 1$. Then $I_2(T_n) \to 0$ a.s., as $n \to \infty$. Consider

$$\begin{aligned} \sup_{T_n \le T \le T_{n+1}} \big| I_2(T) - I_2(T_n) \big| &\le \left(\left(\frac{T_{n+1}}{T_n} \right)^2 - 1 \right) I_2(T_n) + I_3(T_n), \\ I_3(T_n) &= T_n^{-1} \int_{T_n}^{T_{n+1}} |\varepsilon(t)| dt. \end{aligned}$$

As far as

$$EI_3^2(T_n) \le T_n^{-2} \int_{T_n}^{T_{n+1}} \int_{T_n}^{T_{n+1}} E|\varepsilon(t)\varepsilon(s)| dt ds \le B(0) \left(\frac{T_{n+1} - T_n}{T_n} \right)^2 = O\big(n^{-2}\big),$$

then $I_3(T_n) \to 0$ a.s., and $J_{2,13}^1 \to 0$ a.s., as $T \to \infty$.

Similarly, $J_{2,14}^1$, $J_{2,23}^1$, $J_{2,24}^1 \to 0$ a.s., as $T \to \infty$.

Under condition **A1(*ii*)** the proofs of these facts are almost the same taking into account the relation

$$T^{-2} \int_0^T \int_0^T |B(t-s)| dt ds = O\big(T^{-1}\big).$$

The proof of a.s. convergence of $J_{2,33}^1$, $J_{2,34}^1$ and $J_{2,44}^1$ to zero is the same, and therefore we will prove it for $J_{2,44}^1$ only:

$$\begin{aligned} \big| J_{2,44}^1 \big| &= \bigg| T^{-5} \int_0^T t^4 \varepsilon(t) \big[\big(\overline{A}_1 - A_1^0\big) \cos_1^-(t) + A_1^0 \big(\cos_1^-(t) - \cos_1^0(t)\big) \\ &\quad + A_1^0 \cos_1^0(t) + \big(\overline{B}_1 - B_1^0\big) \sin_1^-(t) + B_1^0 \big(\sin_1^-(t) - \sin_1^0(t)\big) + B_1^0 \sin_1^0(t) \big] dt \bigg| \\ &\le \big[|A_{1T} - A_1^0| + |B_{1T} - B_1^0| + \big(|A_1^0| + |B_1^0|\big)\big(T|\phi_{1T} - \phi_1^0| + T^2 |\psi_{1T} - \psi_1^0|\big) \big] \end{aligned}$$

$$\times T^{-1}\int_0^T |\varepsilon(t)|dt + \left|T^{-5}\int_0^T t^4\varepsilon(t)\big(A_1^0\cos_1^0(t) + B_1^0\sin_1^0(t)\big)dt\right|. \quad (23)$$

The a.s. convergence to zero of the last integral on the right-hand side of the inequality (23) can be proved similarly to the convergence of $I_2(T)$. So,

$$D_T^{-1}\left(\frac{1}{2}Q_{1T}''(\overline{\theta})\right)D_T^{-1} \to 0 \text{ a.s.,} \quad \text{as } T\to\infty. \quad (24)$$

Consider next the behavior of the normed matrix (see (10))

$$D_T^{-1}\left(\frac{1}{2}Q_{2T}''(\overline{\theta})\right)D_T^{-1} = D_T^{-1}\left(\int_0^T g_i(t,\overline{\theta})g_j(t,\overline{\theta})dt\right)_{i,j=1}^{4N} D_T^{-1}. \quad (25)$$

Assume first that for fixed $k_1, k_2 \in \overline{0, N-1}$, $k_1 \neq k_2$, the indices $i = \overline{4k_1+1, 4(k_1+1)}$, $j = \overline{4k_2+1, 4(k_2+1)}$. This means that for some $m_1 \neq m_2$ we extract in the matrix (25) the 4th order submatrix $J^2 = (J_{lr}^2)_{l,r=1}^4$. Then

$$J_{11}^2 = T^{-1}\int_0^T \cos_{m_1}^-(t)\cos_{m_2}^-(t)dt; \quad J_{12}^2 = T^{-1}\int_0^T \cos_{m_1}^-(t)\sin_{m_2}^-(t)dt;$$

$$J_{13}^2 = T^{-2}\int_0^T \cos_{m_1}^-(t)\big(-\overline{A}_{m_2}t\sin_{m_2}^-(t) + \overline{B}_{m_2}t\cos_{m_2}^-(t)\big)dt;$$

$$J_{14}^2 = T^{-3}\int_0^T \cos_{m_1}^-(t)\big(-\overline{A}_{m_2}t^2\sin_{m_2}^-(t) + \overline{B}_{m_2}t^2\cos_{m_2}^-(t)\big)dt;$$

$$J_{21}^2 = T^{-1}\int_0^T \sin_{m_1}^-(t)\cos_{m_2}^-(t)dt; \quad J_{22}^2 = T^{-1}\int_0^T \sin_{m_1}^-(t)\sin_{m_2}^-(t)dt;$$

$$J_{23}^2 = T^{-2}\int_0^T \sin_{m_1}^-(t)\big(-\overline{A}_{m_2}t\sin_{m_2}^-(t) + \overline{B}_{m_2}t\cos_{m_2}^-(t)\big)dt;$$

$$J_{24}^2 = T^{-3}\int_0^T \sin_{m_1}^-(t)\big(-\overline{A}_{m_2}t^2\sin_{m_2}^-(t) + \overline{B}_{m_2}t^2\cos_{m_2}^-(t)\big)dt;$$

$$J_{31}^2 = T^{-2}\int_0^T \big(-\overline{A}_{m_1}t\sin_{m_1}^-(t) + \overline{B}_{m_1}t\cos_{m_1}^-(t)\big)\cos_{m_2}^-(t)dt;$$

$$J_{32}^2 = T^{-2}\int_0^T \big(-\overline{A}_{m_1}t\sin_{m_1}^-(t) + \overline{B}_{m_1}t\cos_{m_1}^-(t)\big)\sin_{m_2}^-(t)dt;$$

$$J_{33}^2 = T^{-3}\int_0^T \big(-\overline{A}_{m_1}t\sin_{m_1}^-(t) + \overline{B}_{m_1}t\cos_{m_1}^-(t)\big) \times \big(-\overline{A}_{m_2}t\sin_{m_2}^-(t) + \overline{B}_{m_2}t\cos_{m_2}^-(t)\big)dt;$$

$$J_{34}^2 = T^{-4}\int_0^T \big(-\overline{A}_{m_1}t\sin_{m_1}^-(t) + \overline{B}_{m_1}t\cos_{m_1}^-(t)\big) \times \big(-\overline{A}_{m_2}t^2\sin_{m_2}^-(t) + \overline{B}_{m_2}t^2\cos_{m_2}^-(t)\big)dt;$$

$$J_{41}^2 = T^{-3}\int_0^T \big(-\overline{A}_{m_1}t^2\sin_{m_1}^-(t) + \overline{B}_{m_1}t^2\cos_{m_1}^-(t)\big)\cos_{m_2}^-(t)dt;$$

$$J^2_{42} = T^{-3}\int_0^T \big(-\overline{A}_{m_1} t^2 \sin^{-}_{m_1}(t) + \overline{B}_{m_1} t^2 \cos^{-}_{m_1}(t)\big)\sin^{-}_{m_2}(t)dt;$$

$$J^2_{43} = T^{-4}\int_0^T \big(-\overline{A}_{m_1} t^2 \sin^{-}_{m_1}(t) + \overline{B}_{m_1} t^2 \cos^{-}_{m_1}(t)\big)$$
$$\times \big(-\overline{A}_{m_2} t \sin^{-}_{m_2}(t) + \overline{B}_{m_2} t \cos^{-}_{m_2}(t)\big)dt;$$

$$J^2_{44} = T^{-5}\int_0^T \big(-\overline{A}_{m_1} t^2 \sin^{-}_{m_1}(t) + \overline{B}_{m_1} t^2 \cos^{-}_{m_1}(t)\big)$$
$$\times \big(-\overline{A}_{m_2} t^2 \sin^{-}_{m_2}(t) + \overline{B}_{m_2} t^2 \cos^{-}_{m_2}(t)\big)dt. \qquad (26)$$

Let us show that each element (26) of the matrix J^2 converges to zero a.s. Since the proofs for all elements are similar, we will prove this fact for the awkward element J^2_{44}, rewriting it in the form

$$J^2_{44} = T^{-5}\int_0^T t^4\Big(\big[-\big(\overline{A}_{m_1} - A^0_{m_1}\big)\sin^{-}_{m_1}(t) + \big(\overline{B}_{m_1} - B^0_{m_1}\big)\cos^{-}_{m_1}(t)\big]$$
$$+ \big[-A^0_{m_1}\sin^{-}_{m_1}(t) + B^0_{m_1}\cos^{-}_{m_1}(t)\big]\Big)$$
$$\times \Big(\big[-\big(\overline{A}_{m_2} - A^0_{m_2}\big)\sin^{-}_{m_2}(t) + \big(\overline{B}_{m_2} - B^0_{m_2}\big)\cos^{-}_{m_2}(t)\big]$$
$$+ \big[-A^0_{m_2}\sin^{-}_{m_2}(t) + B^0_{m_2}\cos^{-}_{m_2}(t)\big]\Big)dt = T^{-5}\int_0^T t^4(z_1+z_2)(z_3+z_4)dt.$$

Then

$$T^{-5}\int_0^T t^4|z_1z_3|dt \le \frac{1}{5}\big(|A_{m_1T} - A^0_{m_1}| + |B_{m_1T}B^0_{m_1}|\big)$$
$$\times \big(|A_{m_2T} - A^0_{m_2}| + |B_{m_2T} - B^0_{m_2}|\big) \to 0 \text{ a.s.,} \quad \text{as } T \to \infty;$$

$$T^{-5}\int_0^T t^4|z_1z_4|dt \le \frac{1}{5}\big(|A_{m_1T} - A^0_{m_1}| + |B_{m_1T} - B^0_{m_1}|\big)$$
$$\times \big(|A^0_{m_2}| + |B^0_{m_2}|\big) \to 0 \text{ a.s.,} \quad \text{as } T \to \infty;$$

$$T^{-5}\int_0^T t^4|z_2z_3|dt \le \frac{1}{5}\big(|A_{m_2T} - A^0_{m_2}| + |B_{m_2T} - B^0_{m_2}|\big)$$
$$\times \big(|A^0_{m_1}| + |B^0_{m_1}|\big) \to 0 \text{ a.s.,} \quad \text{as } T \to \infty;$$

Write further

$$T^{-5}\int_0^T t^4 z_2z_4dt =$$
$$T^{-5}\int_0^T t^4\Big(\big[-A^0_{m_1}\big(\sin^{-}_{m_1}(t) - \sin^{0}_{m_1}(t)\big) + B^0_{m_1}\big(\cos^{-}_{m_1}(t) - \cos^{0}_{m_1}(t)\big)\big]$$
$$+ \big[-A^0_{m_1}\sin^{0}_{m_1}(t) + B^0_{m_1}\cos^{0}_{m_1}(t)\big]\Big)$$
$$\times \Big(\big[-A^0_{m_2}\big(\sin^{-}_{m_2}(t) - \sin^{0}_{m_2}(t)\big) + B^0_{m_2}\big(\cos^{-}_{m_2}(t) - \cos^{0}_{m_2}(t)\big)\big]$$
$$+ \big[-A^0_{m_2}\sin^{0}_{m_2}(t) + B^0_{m_2}\cos^{0}_{m_2}(t)\big]\Big)dt = T^{-5}\int_0^T t^4(z_5+z_6)(z_7+z_8)dt;$$

$$T^{-5}\int_0^T t^4|z_5z_7|dt \le \frac{1}{5}\big(T|\phi_{m_1T}-\phi^0_{m_1}|+T^2|\psi_{m_1T}-\psi^0_{m_1}|\big)$$
$$\times\big(T|\phi_{m_2T}-\phi^0_{m_2}|+T^2|\psi_{m_2T}-\psi^0_{m_2}|\big)\big(|A^0_{m_1}|+|B^0_{m_1}|\big)$$
$$\times\big(|A^0_{m_2}|+|B^0_{m_2}|\big)\to 0 \text{ a.s.,}\quad \text{as } T\to\infty;$$
$$T^{-5}\int_0^T t^4|z_5z_8|dt \le \frac{1}{5}\big(|A^0_{m_1}|+|B^0_{m_1}|\big)\big(|A^0_{m_2}|+|B^0_{m_2}|\big)$$
$$\times\big(T|\phi_{m_1T}-\phi^0_{m_1}|+T^2|\psi_{m_1T}-\psi^0_{m_1}|\big)\to 0 \text{ a.s.,}\quad \text{as } T\to\infty;$$
$$T^{-5}\int_0^T t^4|z_6z_7|dt \le \frac{1}{5}\big(|A^0_{m_1}|+|B^0_{m_1}|\big)\big(|A^0_{m_2}|+|B^0_{m_2}|\big)$$
$$\times\big(T|\phi_{m_2T}-\phi^0_{m_2}|+T^2|\psi_{m_2T}-\psi^0_{m_2}|\big)\to 0 \text{ a.s.,}\quad \text{as } T\to\infty.$$

The integral remains

$$T^{-5}\int_0^T t^4|z_6z_8|dt = T^{-5}\int_0^T t^4\big(-A^0_{m_1}\sin^0_{m_1}(t)+B^0_{m_1}\cos^0_{m_1}(t)\big)$$
$$\times\big(-A^0_{m_2}\sin^0_{m_2}(t)+B^0_{m_2}\cos^0_{m_2}(t)\big)dt. \tag{27}$$

Similarly, all the integrals (26) can be represented as a random part that converges to zero a.s. plus the same integral with the true values of the parameters.

As it follows from the formulas 2.655 in the integral tables [6, p. 226], for any $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}\backslash\{0\}$, $n\in\mathbb{N}$,

$$\int_0^T t^n \begin{matrix}\cos\\ \sin\end{matrix}\ \big(\alpha t+\beta t^2\big)dt = O\big(T^{n-1}\big),\quad \text{as } T\to\infty. \tag{28}$$

Taking into account that all the parameters ϕ^0_m, ψ^0_m, $m=\overline{1,N}$, are different, the integral (27) reduces to integrals of the form (28) and is the quantity of order $O(T^{-2})$, $T\to\infty$. The convergence to zero, as $T\to\infty$, is established similarly for all integrals (26), except for J^2_{ij}, $i,j=1,2$. However the last ones are of order $O(T^{-1})$, if we substitute in them the true values of the parameters.

It remains to consider in matrix (25) a block-diagonal submatrix with blocks $(J^2_{ij})^{4(k+1)}_{i,j=4k+1}$, $k=\overline{0,N-1}$. Let's look at one such block $J^2(m)$ that corresponds to the m-th term of the regression function g. The elements of this block can be obtained from formulae (26) taking $m_1=m_2=m$. As above, using the consistency property of the LSE parameters from Theorem 1, we can reduce the problem of calculating the limit of $J^2(m)$ by considering the elements of the matrix $J^2(m)$, where the true values of the parameters are substituted. After that, obtaining the limit matrix H_m from the formulation of Lemma 1 becomes trivial if (28) is applied.

Let's explain this with an example:

$$J^2_{44}(m)=T^{-5}\int_0^T t^4\big(-A^0_m\sin^0_m(t)+B^0_m\cos^0_m(t)\big)^2dt\to\frac{(C^0_m)^2}{10},\quad \text{as } T\to\infty,$$

and so on. On the other hand,

$$\begin{bmatrix} J^2_{11}(m) & J^2_{12}(m)\\ J^2_{21}(m) & J^2_{22}(m)\end{bmatrix}\to\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix},\quad \text{as } T\to\infty,$$

due to the convergence of the Fresnel integrals. □

Now we introduce one more block-diagonal matrix S_T consisting of N similar diagonal blocks,

$$s_T = \operatorname{diag}\big(T^{1/2}, T^{1/2}, T^{3/2}, T^{3/2}\big). \tag{29}$$

Consider the Gaussian random vector (see equation (13)) $S_T D_T^{-1}(-\frac{1}{2}Q'_T(\theta^0))$, consisting of N Gaussian vectors of dimension 4. We write one of them down omitting in the true values of parameters the index $m \in \{1, \dots, N\}$:

$$\xi_T = \bigg(\int_0^T \varepsilon(t)\cos^0(t)dt, \int_0^T \varepsilon(t)\sin^0(t)dt,$$
$$\int_0^T \varepsilon(t)t\big(-A^0\sin^0(t) + B^0\cos^0(t)\big)dt,$$
$$T^{-1}\int_0^T \varepsilon(t)t^2\big(-A^0\sin^0(t) + B^0\cos^0(t)\big)dt\bigg)^* = \big(\xi_T^1, \xi_T^2, \xi_T^3, \xi_T^4\big)^*. \tag{30}$$

Denote by $G_T = (G_{ij,T})_{i,j=1}^4 = E\xi_T\xi_T^*$ the covariance matrix of the vector ξ_T and study its asymptotic behavior as $T \to \infty$. It is convenient to write further $\sin(\alpha^0 + \phi^0 t + \psi^0 t^2) = \sin^0(t, \alpha^0)$, $\cos(\alpha^0 + \phi^0 t + \psi^0 t^2) = \cos^0(t, \alpha^0)$, $-A^0\sin^0(t) + B^0\cos^0(t) = C^0\cos^0(t, \alpha^0)$, $\tan\alpha^0 = \frac{A^0}{B^0}$.

Let's do some preliminary calculations. We will use the notation

$$\mu_T\big(\lambda, \alpha^0\big) = \begin{bmatrix} \mu_{11,T}(\lambda, \alpha^0) & \mu_{12,T}(\lambda, \alpha^0) \\ \mu_{21,T}(\lambda, \alpha^0) & \mu_{22,T}(\lambda, \alpha^0) \end{bmatrix}$$
$$= \begin{bmatrix} \int_0^T \cos(\lambda t)\cos^0(t, \alpha^0)dt & \int_0^T \cos(\lambda t)\sin^0(t, \alpha^0)dt \\ \int_0^T \sin(\lambda t)\cos^0(t, \alpha^0)dt & \int_0^T \sin(\lambda t)\sin^0(t, \alpha^0)dt \end{bmatrix}. \tag{31}$$

Extracting the full square under the signs of sine and cosine [8] we get

$$\begin{matrix}\mu_{11,T}(\lambda, \alpha^0) \\ \mu_{22,T}(\lambda, \alpha^0)\end{matrix} =$$
$$= \frac{1}{2}\int_0^T \cos\big(\alpha^0 + \big(\phi^0 - \lambda\big)t + \psi^0 t^2\big)dt \pm \frac{1}{2}\int_0^T \cos\big(\alpha^0 + \big(\phi^0 + \lambda\big)t + \psi^0 t^2\big)dt$$
$$= \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\bigg(\frac{(\phi^0 - \lambda)^2}{4\psi^0} - \alpha^0\bigg)\int_{\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}} \cos\big(t^2\big)dt$$
$$+ \sin\bigg(\frac{(\phi^0 - \lambda)^2}{4\psi^0} - \alpha^0\bigg)\int_{\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}} \sin\big(t^2\big)dt\Bigg]$$
$$\pm \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\bigg(\frac{(\phi^0 + \lambda)^2}{4\psi^0} - \alpha^0\bigg)\int_{\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}} \cos\big(t^2\big)dt$$

$$+\sin\left(\frac{(\phi^0+\lambda)^2}{4\psi^0}-\alpha^0\right)\int_{\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}\sin\left(t^2\right)dt\Bigg]; \tag{32}$$

$$\begin{aligned}
&\begin{matrix}\mu_{12,T}(\lambda,\alpha^0)\\ \mu_{21,T}(\lambda,\alpha^0)\end{matrix} = \\
&= \frac{1}{2}\int_0^T \sin\left(\alpha^0+\left(\phi^0+\lambda\right)t+\psi^0t^2\right)dt \pm \frac{1}{2}\int_0^T \sin\left(\alpha^0+\left(\phi^0-\lambda\right)t+\psi^0t^2\right)dt \\
&= \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\left(\frac{(\phi^0+\lambda)^2}{4\psi^0}-\alpha^0\right)\int_{\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}\sin\left(t^2\right)dt \\
&\qquad -\sin\left(\frac{(\phi^0+\lambda)^2}{4\psi^0}-\alpha^0\right)\int_{\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0+\lambda}{2\sqrt{\psi^0}}}\cos\left(t^2\right)dt\Bigg] \\
&\pm \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\left(\frac{(\phi^0-\lambda)^2}{4\psi^0}-\alpha^0\right)\int_{\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}\sin\left(t^2\right)dt \\
&\qquad -\sin\left(\frac{(\phi^0-\lambda)^2}{4\psi^0}-\alpha^0\right)\int_{\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}^{T\sqrt{\psi^0}+\frac{\phi^0-\lambda}{2\sqrt{\psi^0}}}\cos\left(t^2\right)dt\Bigg].
\end{aligned} \tag{33}$$

Denote by $S(x)=\int_0^x\sin(t^2)dt$, $C(x)=\int_0^x\cos(t^2)dt$, $x\in\mathbb{R}$, the Fresnel integrals and take $\gamma_\pm(\lambda)=\frac{\phi^0\pm\lambda}{2\sqrt{\psi^0}}$. Then

$$\lim_{T\to\infty}\mu_T\left(\lambda,\alpha^0\right)=\begin{bmatrix}\mu_{11}(\lambda,\alpha^0) & \mu_{12}(\lambda,\alpha^0)\\ \mu_{21}(\lambda,\alpha^0) & \mu_{22}(\lambda,\alpha^0)\end{bmatrix},$$

$$\begin{aligned}
&\begin{matrix}\mu_{11}(\lambda,\alpha^0)\\ \mu_{22}(\lambda,\alpha^0)\end{matrix} = \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\left(\gamma_-^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\gamma_-(\lambda)\right)\right) \\
&+\sin\left(\gamma_-^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\gamma_-(\lambda)\right)\right)\pm\cos\left(\gamma_+^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\gamma_+(\lambda)\right)\right) \\
&\qquad \pm\sin\left(\gamma_+^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\gamma_+(\lambda)\right)\right)\Bigg];
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\begin{matrix}\mu_{12}(\lambda,\alpha^0)\\ \mu_{21}(\lambda,\alpha^0)\end{matrix} = \frac{1}{2\sqrt{\psi^0}}\Bigg[\cos\left(\gamma_+^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\gamma_+(\lambda)\right)\right) \\
&-\sin\left(\gamma_+^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\gamma_+(\lambda)\right)\right)\pm\cos\left(\gamma_-^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\gamma_-(\lambda)\right)\right) \\
&\qquad \mp\sin\left(\gamma_-^2(\lambda)-\alpha^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\gamma_-(\lambda)\right)\right)\Bigg].
\end{aligned} \tag{35}$$

From (32), (33) and properties of Fresnel integrals it follows that uniformly in $T, \lambda \geq 0$

$$\left|\mu_{ij,T}\left(\lambda, \alpha^0\right)\right| \leq \frac{4}{\sqrt{\psi^0}}, \quad i, j = 1, 2. \tag{36}$$

Besides, for any $\Lambda > 0$ and $i, j = 1, 2$,

$$2\sqrt{\psi^0} \sup_{\lambda\in[0,\Lambda]} \left|\mu_{ij,T}\left(\lambda, \alpha^0\right) - \mu_{ij}\left(\lambda, \alpha^0\right)\right| \leq \sup_{\lambda\in[0,\Lambda]} \left|\int_0^{T\sqrt{\psi^0}+\gamma_+(\lambda)} \cos\left(t^2\right)dt - \sqrt{\frac{\pi}{8}}\right|$$
$$+ \sup_{\lambda\in[0,\Lambda]} \left|\int_0^{T\sqrt{\psi^0}+\gamma_-(\lambda)} \cos\left(t^2\right)dt - \sqrt{\frac{\pi}{8}}\right| + \sup_{\lambda\in[0,\Lambda]} \left|\int_0^{T\sqrt{\psi^0}+\gamma_+(\lambda)} \sin\left(t^2\right)dt - \sqrt{\frac{\pi}{8}}\right|$$
$$+ \sup_{\lambda\in[0,\Lambda]} \left|\int_0^{T\sqrt{\psi^0}+\gamma_-(\lambda)} \sin\left(t^2\right)dt - \sqrt{\frac{\pi}{8}}\right| \to 0, \quad \text{as } T \to \infty. \tag{37}$$

To continue the study of the asymptotic behavior, as $T \to \infty$, of the matrix G_T under condition **A2** we will use the standard formula $B(t) = 2\int_0^\infty \cos(\lambda t) f(\lambda)d\lambda$. Then from (30) by the Lebesque majorized convergence theorem

$$\begin{aligned} G_{11,T} &= 2\int_0^\infty f(\lambda) \int_0^T \int_0^T \cos\left(\lambda(t-s)\right) \cos^0(t) \cos^0(s) dt ds d\lambda \\ &= 2\int_0^\infty f(\lambda)\left[\left(\mu_{11,T}(\lambda, 0)\right)^2 + \left(\mu_{21,T}(\lambda, 0)\right)^2\right]d\lambda \\ &\xrightarrow[T\to\infty]{} 2\int_0^\infty f(\lambda)\left[\left(\mu_{11}(\lambda, 0)\right)^2 + \left(\mu_{21}(\lambda, 0)\right)^2\right]d\lambda = G_{11}. \end{aligned} \tag{38}$$

Similarly,

$$G_{22} = 2\int_0^\infty f(\lambda)\left[\left(\mu_{12}(\lambda, 0)\right)^2 + \left(\mu_{22}(\lambda, 0)\right)^2\right]d\lambda; \tag{39}$$

$$G_{12} = 2\int_0^\infty f(\lambda)\left[\mu_{11}(\lambda, 0)\mu_{12}(\lambda, 0) + \mu_{22}(\lambda, 0)\mu_{21}(\lambda, 0)\right]d\lambda. \tag{40}$$

Bellow we will use the notation $\gamma^0 = \frac{(C^0)^2}{(\psi^0)^2}$ and formulae (64), (65) from Appendix.

$$G_{33,T} = 2\left(C^0\right)^2 \int_0^\infty f(\lambda) \int_0^T \int_0^T \cos\left(\lambda(t-s)\right) ts \cos^0\left(t, \alpha^0\right) \sin^0\left(s, \alpha^0\right) ds dt d\lambda$$
$$= 2\left(C^0\right)^2 \int_0^\infty f(\lambda) \left[\left(\int_0^T t\cos(\lambda t) \cos^0\left(t, \alpha^0\right)dt\right)^2\right.$$
$$\left. + \left(\int_0^T t\sin(\lambda t) \cos^0\left(t, \alpha^0\right)dt\right)^2\right] d\lambda = \frac{\gamma^0}{2}\int_0^\infty f(\lambda)\left[\left(\sin^0\left(T, \alpha^0\right)\cos(\lambda T)\right.\right.$$
$$\left. - \sin\left(\alpha^0\right) - \phi^0\mu_{11,T}\left(\lambda, \alpha^0\right) + \lambda\mu_{22,T}\left(\lambda, \alpha^0\right)\right)^2$$
$$\left. + \left(\sin^0\left(T, \alpha^0\right)\sin(\lambda T) - \phi^0\mu_{21,T}\left(\lambda, \alpha^0\right) - \lambda\mu_{12,T}\left(\lambda, \alpha^0\right)\right)^2\right]d\lambda$$

$$= \frac{\gamma^0}{2}\left(\sin^0\left(T,\alpha^0\right)\right)^2 \int_0^\infty f(\lambda)\cos^2(\lambda T)d\lambda$$
$$- \gamma^0 \sin^0\left(T,\alpha^0\right)\int_0^\infty f(\lambda)\left[\sin\left(\alpha^0\right) + \phi^0\mu_{11,T}\left(\lambda,\alpha^0\right) - \lambda\mu_{22,T}\left(\lambda,\alpha^0\right)\right]\cos(\lambda T)d\lambda$$
$$+ \frac{\gamma^0}{2}\int_0^\infty f(\lambda)\left[-\sin\left(\alpha^0\right) - \phi^0\mu_{11,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{22,T}\left(\lambda,\alpha^0\right)\right]^2 d\lambda$$
$$+ \frac{\gamma^0}{2}\left(\sin^0\left(T,\alpha^0\right)\right)^2 \int_0^\infty f(\lambda)\sin^2(\lambda T)d\lambda$$
$$- \gamma^0 \sin^0\left(T,\alpha^0\right)\int_0^\infty f(\lambda)\left[\phi^0\mu_{21,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{12,T}\left(\lambda,\alpha^0\right)\right]\sin(\lambda T)d\lambda$$
$$+ \frac{\gamma^0}{2}\int_0^\infty f(\lambda)\left[\phi^0\mu_{21,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{12,T}\left(\lambda,\alpha^0\right)\right]^2 d\lambda.$$

The 2nd and 5th terms of the last sum tend to zero, as $T \to \infty$, due to the uniform convergence (37) and the well-known property of the Fourier transform of functions from L_1. Besides, the adding of the 1st and 4th terms give the following result:

$$\lim_{T\to\infty}\left(G_{33,T} - \frac{1}{4}\gamma^0 B(0)\left(\sin^0\left(T,\alpha^0\right)\right)^2\right)$$
$$= \frac{\gamma^0}{2}\int_0^\infty f(\lambda)\Big[\left(-\sin\left(\alpha^0\right) - \phi^0\mu_{11,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{22,T}\left(\lambda,\alpha^0\right)\right)^2$$
$$+ \left(\phi^0\mu_{21,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{12,T}\left(\lambda,\alpha^0\right)\right)^2\Big]d\lambda. \tag{41}$$

The next elements are

$$G_{23,T} = 2C^0\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos\left(\lambda(t-s)\right)t\cos^0\left(t,\alpha^0\right)\sin^0(s)dsdtd\lambda$$
$$= 2C^0\int_0^\infty f(\lambda)\left[\int_0^T t\cos(\lambda t)\cos^0\left(t,\alpha^0\right)dt\int_0^T \cos(\lambda t)\sin^0(t)dt\right.$$
$$\left. + \int_0^T t\sin(\lambda t)\cos^0\left(t,\alpha^0\right)dt\int_0^T \sin(\lambda t)\sin^0(t)dt\right]d\lambda$$
$$= \sqrt{\gamma^0}\int_0^\infty f(\lambda)\Big[\left(\sin^0\left(T,\alpha^0\right)\cos(\lambda T) - \sin\left(\alpha^0\right) - \phi^0\mu_{11,T}\left(\lambda,\alpha^0\right)\right.$$
$$\left. + \lambda\mu_{22,T}\left(\lambda,\alpha^0\right)\right)\mu_{12,T}(\lambda,0)$$
$$+ \left(\sin^0\left(T,\alpha^0\right)\sin(\lambda T) - \phi^0\mu_{21,T}\left(\lambda,\alpha^0\right) - \lambda\mu_{12,T}\left(\lambda,\alpha^0\right)\right)\mu_{22,T}(\lambda,0)\Big]d\lambda$$
$$= \sqrt{\gamma^0}\sin^0\left(T,\alpha^0\right)\int_0^\infty f(\lambda)\mu_{12,T}(\lambda,0)\cos(\lambda T)d\lambda$$
$$+ \sqrt{\gamma^0}\int_0^\infty f(\lambda)\left[-\sin\left(\alpha^0\right) - \phi^0\mu_{11,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{22,T}\left(\lambda,\alpha^0\right)\right]\mu_{12,T}(\lambda,0)d\lambda$$
$$+ \sqrt{\gamma^0}\sin^0\left(T,\alpha^0\right)\int_0^\infty f(\lambda)\mu_{22,T}(\lambda,0)\sin(\lambda T)d\lambda$$
$$- \sqrt{\gamma^0}\int_0^\infty f(\lambda)\left[\phi^0\mu_{21,T}\left(\lambda,\alpha^0\right) + \lambda\mu_{12,T}\left(\lambda,\alpha^0\right)\right]\mu_{22,T}(\lambda,0)d\lambda$$

$$\xrightarrow[T\to\infty]{} \sqrt{\gamma^0}\int_0^\infty f(\lambda)\big[\big(-\sin\big(\alpha^0\big)-\phi^0\mu_{11}\big(\lambda,\alpha^0\big)+\lambda\mu_{22}\big(\lambda,\alpha^0\big)\big)\mu_{12}(\lambda,0) \\ -\big(\phi^0\mu_{21}\big(\lambda,\alpha^0\big)+\lambda\mu_{12}\big(\lambda,\alpha^0\big)\big)\mu_{22}(\lambda,0)\big]d\lambda. \quad (42)$$

Similarly to (42)

$$\lim_{T\to\infty} G_{13,T} = \sqrt{\gamma^0}\int_0^\infty f(\lambda)\big[\big(-\sin\big(\alpha^0\big)-\phi^0\mu_{11}\big(\lambda,\alpha^0\big)+\lambda\mu_{22}\big(\lambda,\alpha^0\big)\big)\mu_{11}(\lambda,0) \\ -\big(\phi^0\mu_{11}\big(\lambda,\alpha^0\big)+\lambda\mu_{12}\big(\lambda,\alpha^0\big)\big)\mu_{21}(\lambda,0)\big]d\lambda. \quad (43)$$

To find the last diagonal element, we will use formulae (66), (67) and the results of previous calculation:

$$G_{44,T} = \frac{2(C^0)^2}{T^2}\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos\big(\lambda(t-s)\big)t^2s^2\cos^0\big(t,\alpha^0\big)\sin^0\big(s,\alpha^0\big)ds\,dt\,d\lambda \\ = \frac{2(C^0)^2}{T^2}\int_0^\infty f(\lambda)\left[\left(\int_0^T t^2\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt\right)^2 \right. \\ \left. + \left(\int_0^T t^2\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt\right)^2\right]d\lambda = \frac{\gamma^0}{2T^2}\int_0^\infty f(\lambda)\Big[\Big(T\sin^0\big(T,\alpha^0\big)\cos(\lambda T) \\ -\mu_{12,T}\big(\lambda,\alpha^0\big)-\phi^0\int_0^T t\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt+\lambda\int_0^T t\sin(\lambda t)\sin^0\big(t,\alpha^0\big)dt\Big)^2 \\ +\Big(T\sin^0\big(T,\alpha^0\big)\sin(\lambda T)-\mu_{22,T}\big(\lambda,\alpha^0\big)-\phi^0\int_0^T t\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt \\ -\lambda\int_0^T t\cos(\lambda t)\sin^0\big(t,\alpha^0\big)dt\Big)^2\Big]d\lambda = \frac{\gamma^0}{4}B(0)\big(\sin^0\big(T,\alpha^0\big)\big)^2+O\big(T^{-1}\big), \quad (44)$$

as $T\to\infty$.

On the other hand,

$$G_{34,T} = \frac{2(C^0)^2}{T}\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos\big(\lambda(t-s)\big)t^2s\cos^0(t,\alpha)\cos^0(s,\alpha)dt\,ds\,d\lambda \\ = \frac{2(C^0)^2}{T}\int_0^\infty f(\lambda)\left[\int_0^T t^2\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt\int_0^T t\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt \right. \\ \left. +\int_0^T t^2\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt\int_0^T t\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt\right]d\lambda \\ = \frac{(C^0)^2}{\psi^0 T}\int_0^\infty f(\lambda)\Big[\int_0^T t\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt\Big(T\sin^0\big(T,\alpha^0\big)\cos(\lambda T) \\ -\mu_{12,T}\big(\lambda,\alpha^0\big)-\phi^0\int_0^T t\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt+\lambda\int_0^T t\sin(\lambda t)\sin^0\big(t,\alpha^0\big)dt\Big) \\ +\int_0^T t\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt\Big(T\sin^0\big(T,\alpha^0\big)\sin(\lambda T)-\mu_{22,T}\big(\lambda,\alpha^0\big)$$

$$-\phi^0 \int_0^T t\sin(\lambda t)\cos^0(t,\alpha^0)dt - \lambda \int_0^T t\cos(\lambda t)\sin^0(t,\alpha^0)dt\Big)\Big]d\lambda$$
$$= \frac{(C^0)^2}{\psi^0}\sin^0(T,\alpha^0)\int_0^\infty f(\lambda)\Big[\int_0^T t\cos(\lambda t)\cos^0(t,\alpha^0)dt\Big]\cos(\lambda T)d\lambda$$
$$+ \frac{(C^0)^2}{\psi^0}\sin^0(T,\alpha^0)\int_0^\infty f(\lambda)\Big[\int_0^T t\sin(\lambda t)\cos^0(t,\alpha^0)dt\Big]\sin(\lambda T)d\lambda + O(T^{-1})$$
$$= \frac{\gamma^0}{4}B(0)\big(\sin^0(T,\alpha^0)\big)^2 + O(T^{-1})$$
$$- \frac{\gamma^0}{2}\sin^0(T,\alpha^0)\int_0^\infty f(\lambda)\big[\sin(\alpha^0) + \phi^0\mu_{11,T}(\lambda,\alpha^0) - \lambda\mu_{22,T}(\lambda,\alpha^0)\big]\cos(\lambda T)d\lambda$$
$$- \frac{\gamma^0}{2}\sin^0(T,\alpha^0)\int_0^\infty f(\lambda)\big[\phi^0\mu_{21,T}(\lambda,\alpha^0) + \lambda\mu_{12,T}(\lambda,\alpha^0)\big]\sin(\lambda T)d\lambda. \tag{45}$$

From (45) it follows that

$$\lim_{T\to\infty}\left(G_{34,T} - \frac{\gamma^0}{4}B(0)\big(\sin^0(T,\alpha^0)\big)^2\right) = 0. \tag{46}$$

Using the same approach, we get

$$G_{14,T},\, G_{24,T} \xrightarrow[T\to\infty]{} 0. \tag{47}$$

Collecting formulas (34), (35), (38)–(47), we arrive at the following statement.

Lemma 2. *Under conditions* ***A1*** *and* ***A2***

$$\lim_{T\to\infty}(G_T - \tilde{G}_T) = \begin{bmatrix} G_{11} & G_{12} & G_{13} & 0 \\ G_{21} & G_{22} & G_{23} & 0 \\ G_{31} & G_{32} & G_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{48}$$

where

$$\tilde{G}_T = \frac{\gamma^0}{4}B(0)\big(\sin^0(T,\alpha^0)\big)^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}; \tag{49}$$

$$G_{11} = 2\int_0^\infty f(\lambda)\big[\big(\mu_{11}(\lambda,0)\big)^2 + \big(\mu_{21}(\lambda,0)\big)^2\big]d\lambda;$$
$$G_{12} = 2\int_0^\infty f(\lambda)\big[\mu_{11}(\lambda,0)\mu_{12}(\lambda,0) + \mu_{22}(\lambda,0)\mu_{21}(\lambda,0)\big]d\lambda;$$
$$G_{13} = \sqrt{\gamma^0}\int_0^\infty f(\lambda)\big[\big(-\sin(\alpha^0) - \phi^0\mu_{21}(\lambda,\alpha^0) + \lambda\mu_{22}(\lambda,\alpha^0)\big)\mu_{11}(\lambda,0)$$
$$- \big(\phi^0\mu_{11}(\lambda,\alpha^0) + \lambda\mu_{12}(\lambda,\alpha^0)\big)\mu_{21}(\lambda,0)\big]d\lambda;$$
$$G_{22} = 2\int_0^\infty f(\lambda)\big[\big(\mu_{12}(\lambda,0)\big)^2 + \big(\mu_{22}(\lambda,0)\big)^2\big]d\lambda;$$

$$G_{23} = \sqrt{\gamma^0}\int_0^\infty f(\lambda)\big[\big(-\sin\big(\alpha^0\big) - \phi^0\mu_{11}\big(\lambda,\alpha^0\big) + \lambda\mu_{22}\big(\lambda,\alpha^0\big)\big)\mu_{12}(\lambda,0) \\ - \big(\phi^0\mu_{21}\big(\lambda,\alpha^0\big) + \lambda\mu_{12}\big(\lambda,\alpha^0\big)\big)\mu_{22}(\lambda,0)\big]d\lambda;$$

$$G_{33} = \frac{\gamma^0}{2}\int_0^\infty f(\lambda)\big[\big(\sin\big(\alpha^0\big) + \phi^0\mu_{11,T}\big(\lambda,\alpha^0\big) - \lambda\mu_{22,T}\big(\lambda,\alpha^0\big)\big)^2 \\ + \big(\phi^0\mu_{21,T}\big(\lambda,\alpha^0\big) + \lambda\mu_{12,T}\big(\lambda,\alpha^0\big)\big)^2\big]d\lambda. \qquad (50)$$

3 Proof of Theorem 2

Proof. Consider the matrices H_m from the formulation of Lemma 1. Standard calculation shows that $\det H_m = \frac{(C_m^0)^4}{34560}$, and

$$H_m^{-1} = \frac{2}{(C_m^0)^2} \\ \times \begin{bmatrix} (A_m^0)^2 + 9(B_m^0)^2 & -8A_m^0B_m^0 & -36B_m^0 & 30B_m^0 \\ -8A_m^0B_m^0 & 9(A_m^0)^2 + (B_m^0)^2 & 36A_m^0 & -30A_m^0 \\ -36B_m^0 & 36A_m^0 & 192 & -180 \\ 30B_m^0 & -30A_m^0 & -180 & 180 \end{bmatrix}, \quad m = \overline{1,N}. \qquad (51)$$

Thus, H^{-1} is a block-diagonal matrix with blocks (51), and

$$K_T = \left(D_T^{-1}\left(\frac{1}{2}Q_T''(\overline{\theta})\right)D_T^{-1}\right)^{-1} - H^{-1} \to 0 \text{ a.s.}, \quad \text{as } T \to \infty. \qquad (52)$$

For $\delta \in [0,1)$ rewrite equation (13) in the form

$$T^{\frac{1}{2}-\delta}D_T\big(\theta_T - \theta^0\big) = T^{\frac{1}{2}-\delta}K_TS_T^{-1}\left(S_TD_T^{-1}\left(-\frac{1}{2}Q_T'\big(\theta^0\big)\right)\right) \\ + T^{\frac{1}{2}-\delta}H^{-1}S_T^{-1}\left(S_TD_T^{-1}\left(-\frac{1}{2}Q_T'\big(\theta^0\big)\right)\right). \qquad (53)$$

Add to the notation (30) the subscripts m:

$$\xi_{mT} = \big(\xi_{mT}^1, \xi_{mT}^2, \xi_{mT}^3, \xi_{mT}^4\big)^* = \\ \left(\int_0^T \varepsilon(t)\cos_m^0(t)dt, \int_0^T \varepsilon(t)\sin_m^0(t)dt, \int_0^T \varepsilon(t)t\big(-A_m^0\sin_m^0(t) + B_m^0\cos_m^0(t)\big)dt,\right. \\ \left. T^{-1}\int_0^T \varepsilon(t)t^2\big(-A_m^0\sin_m^0(t) + B_m^0\cos_m^0(t)\big)dt\right)^*, \quad m = \overline{1,N};$$

and put

$$S_TD_T^{-1}\left(-\frac{1}{2}Q_T'\big(\theta^0\big)\right) = \big(\xi_{1T}^*, \ldots, \xi_{NT}^*\big)^* = \Xi_T. \qquad (54)$$

Then equation (53) takes the form

$$T^{\frac{1}{2}-\delta} D_T\left(\theta_T-\theta^0\right)=T^{\frac{1}{2}-\delta} K_T S_T^{-1} \Xi_T+T^{\frac{1}{2}-\delta} H^{-1} S_T^{-1} \Xi_T=V_{1T}(\delta)+V_{2T}(\delta). \tag{55}$$

Rewrite equation (55) by setting $\delta=0$:

$$T^{\frac{1}{2}} D_T\left(\theta_T-\theta^0\right)=T^{\frac{1}{2}} K_T S_T^{-1} \Xi_T+T^{\frac{1}{2}} H^{-1} S_T^{-1} \Xi_T=V_{1T}(0)+V_{2T}(0). \tag{56}$$

Obviously, each element of the matrix K_T is a stochastic process depending on the parameter T and converging to zero a.s., as $T \to \infty$. Thus each coordinate of the vector $V_{1T}(0)$ is a linear combination of elements of the matrix K_T and one of the stochastic processes ξ_{mT}^1, ξ_{mT}^2, $T^{-1}\xi_{mT}^3$, $T^{-1}\xi_{mT}^4$, $m=\overline{1,N}$. The processes ξ_{mT}^1, ξ_{mT}^2 by Lemma 2 converge weakly to Gaussian random variables with zero mean and variances G_{11}, G_{22} (see formulae (50)), which depend on the parameters $(A_m^0, B_m^0, \phi_m^0, \psi_m^0)$. Besides, $T^{-1}\xi_{mT}^3$, $T^{-1}\xi_{mT}^4$ converge in mean square to 0, that is, coordinates of the vector $V_{1T}(0)$ tend to 0, at least, in probability, as $T \to \infty$.

On the other hand, as follows from (51), the vector $V_{2T}(0)=((V_{2T}^1(0))^*, \ldots, (V_{2T}^N(0))^*)^*$, where for $m=\overline{1,N}$

$$V_{2T}^m(0)=H_m^{-1}\begin{bmatrix} \xi_{mT}^1 \\ \xi_{mT}^2 \\ T^{-1}\xi_{mT}^3 \\ T^{-1}\xi_{mT}^4 \end{bmatrix}. \tag{57}$$

The 3rd and the 4th terms in each row of (57) converge in mean square to zero, and therefore vector $V_{2T}(0)$ weakly converges to the Gaussian vector $V_2(0)=((V_2^1(0))^*, \ldots, (V_2^N(0))^*)^*$, as $T \to \infty$, with

$$V_2^m(0)=\frac{2}{(C_m^0)^2}\begin{bmatrix} ((A_m^0)^2+9(B_m^0)^2)\xi_m^1-8A_m^0B_m^0\xi_m^2 \\ -8A_m^0B_m^0\xi_m^1+(9(A_m^0)^2+(B_m^0)^2)\xi_m^2 \\ -36B_m^0\xi_m^1+36A_m^0\xi_m^2 \\ 30B_m^0\xi_m^1-30A_m^0\xi_m^2 \end{bmatrix}, \quad m=\overline{1,N}. \tag{58}$$

Introduce the matrices of order 4×2

$$R_m=\frac{2}{(C_m^0)^2}\begin{bmatrix} (A_m^0)^2+9(B_m^0)^2 & -8A_m^0B_m^0 \\ -8A_m^0B_m^0 & 9(A_m^0)^2+(B_m^0)^2 \\ -36B_m^0 & 36A_m^0 \\ 30B_m^0 & -30A_m^0 \end{bmatrix}, \quad m=\overline{1,N}, \tag{59}$$

and the random vectors $r_m=(\xi_m^1,\xi_m^2)^*$, $m=\overline{1,N}$, with the covariance matrices $R_{lm}=Er_lr_m^*$, $l,m=\overline{1,N}$.

Then the covariance matrix $\Sigma=EV_2(0)(V_2(0))^*$ can be presented in the form of block matrix with square blocks of order 4:

$$\Sigma=\left(R_lR_{lm}R_m^*\right)_{l,m=1}^N. \tag{60}$$

The elements of the matrices R_{lm} are of the form

$$R_{lm} = \begin{bmatrix} E\xi_l^1\xi_m^1 & E\xi_l^1\xi_m^2 \\ E\xi_l^2\xi_m^1 & E\xi_l^2\xi_m^2 \end{bmatrix}, \tag{61}$$

where

$$\begin{aligned}
E\xi_l^1\xi_m^1 &= 2\int_0^\infty f(\lambda)\big[\mu_{11,l}(\lambda,0)\mu_{11,m}(\lambda,0) + \mu_{21,l}(\lambda,0)\mu_{21,m}(\lambda,0)\big]d\lambda;\\
E\xi_l^1\xi_m^2 &= 2\int_0^\infty f(\lambda)\big[\mu_{11,l}(\lambda,0)\mu_{12,m}(\lambda,0) + \mu_{21,l}(\lambda,0)\mu_{22,m}(\lambda,0)\big]d\lambda;\\
E\xi_l^2\xi_m^1 &= 2\int_0^\infty f(\lambda)\big[\mu_{11,m}(\lambda,0)\mu_{12,l}(\lambda,0) + \mu_{21,m}(\lambda,0)\mu_{22,l}(\lambda,0)\big]d\lambda;\\
E\xi_l^2\xi_m^2 &= 2\int_0^\infty f(\lambda)\big[\mu_{12,l}(\lambda,0)\mu_{12,m}(\lambda,0) + \mu_{22,l}(\lambda,0)\mu_{22,m}(\lambda,0)\big]d\lambda;
\end{aligned} \tag{62}$$

$\mu_{ij,m}(\lambda,0)$, $i, j = 1, 2$, $m = \overline{1, N}$, are given by formulae (34), (35) by adding the subscripts m corresponding to the parameters (ϕ_m^0, ψ_m^0).

Suppose $l = m$, then $E(\xi_m^1)^2 = G_{11,m}$, $E\xi_m^1\xi_m^2 = G_{12,m}$, $E(\xi_m^2)^2 = G_{22,m}$, where the last 3 values are given by formulae (38)–(40), and we substitute in them the true parameters values ϕ_m^0 and ψ_m^0. □

Corollary 1. *Under condition **A1**, **A2** and **B** for any* $\delta \in (0, 1)$

$$T^{1-\delta}\big(A_{jT} - A_j^0\big), T^{1-\delta}\big(B_{jT} - B_j^0\big), T^{2-\delta}\big(\phi_{jT} - \phi_j^0\big), T^{3-\delta}\big(\psi_{jT} - \psi_j^0\big) \xrightarrow{P} 0,$$

as $T \to \infty$, $j = \overline{1, N}$.

Corollary 1 follows from Theorem 2 and equation (56).

Note also that the limiting Gaussian distribution in Theorem 2 is singular.

Corollary 2. *If* $rank(\mathcal{R}) = 2N$ *(see bellow), then* $rank(\Sigma) = 2N$.

Proof. The matrix Σ from (60) can be written as the product of three block matrices as follows:

$$\Sigma = \begin{bmatrix} R_1 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & R_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & R_N \end{bmatrix} \times \begin{bmatrix} R_{11} & R_{12} & R_{13} & \dots & R_{1N} \\ R_{21} & R_{22} & R_{23} & \dots & R_{2N} \\ R_{31} & R_{32} & R_{33} & \dots & R_{3N} \\ \dots & \dots & \dots & \dots & \dots \\ R_{N1} & R_{N2} & R_{N3} & \dots & R_{NN} \end{bmatrix}$$
$$\times \begin{bmatrix} R_1^* & 0 & 0 & \dots & 0 \\ 0 & R_2^* & 0 & \dots & 0 \\ 0 & 0 & R_3^* & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & R_N^* \end{bmatrix} = R\mathcal{R}R^*, \tag{63}$$

where R is of order $4N \times 2N$, $\mathcal{R}$ is the square matrix of order $2N$, R^* is of order $2N \times 4N$. The matrix $\mathcal{R}$ is the covariance matrix of the random vector $\xi =$

$(\xi_1^1, \xi_1^2, \ldots, \xi_m^1, \xi_m^2, \ldots, \xi_N^1, \xi_N^2)^*$, and in Corollary 2 we assume that it is not singular. Let's find the *rank* of the matrix R.

Consider a square submatrix M of R consisting of all its rows that contain the first two rows of each matrix R_m, $m = \overline{1, N}$. Then

$$\det M = \prod_{m=1}^{N} \det\left(\frac{2}{(C_m^0)^2}\begin{bmatrix}(A_m^0)^2 + 9(B_m^0)^2 & -8A_m^0 B_m^0 \\ -8A_m^0 B_m^0 & 9(A_m^0)^2 + (B_m^0)^2\end{bmatrix}\right) = 36^N.$$

Thus, $rank(R) = rank(R^*) = 2N$, $rank(R\mathcal{R}) = rank(R) = 2N$, and $rank(\Sigma) = rank(R\mathcal{R}R^*) = rank(R\mathcal{R}) = 2N$. □

Appendix

To find the elements of the matrix G from Lemma 2, we need the following integrals.

$$1) \int_0^T t\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt$$
$$= \frac{1}{2\psi^0}\big[\sin^0\big(T,\alpha^0\big)\cos(\lambda T) - \sin\big(\alpha^0\big) - \phi^0\mu_{11,T}\big(\lambda,\alpha^0\big) + \lambda\mu_{22,T}\big(\lambda,\alpha^0\big)\big];$$
$$2) \int_0^T t\sin(\lambda t)\cos^0\big(t,\alpha^0\big)dt$$
$$= \frac{1}{2\psi^0}\big[\sin^0\big(T,\alpha^0\big)\sin(\lambda T) - \phi^0\mu_{21,T}\big(\lambda,\alpha^0\big) - \lambda\mu_{12,T}\big(\lambda,\alpha^0\big)\big];$$
$$3) \int_0^T t\cos(\lambda t)\sin^0\big(t,\alpha^0\big)dt$$
$$= \frac{1}{2\psi^0}\big[-\cos^0\big(T,\alpha^0\big)\cos(\lambda T) + \cos\big(\alpha^0\big) - \phi^0\mu_{12,T}\big(\lambda,\alpha^0\big) - \lambda\mu_{21,T}\big(\lambda,\alpha^0\big)\big];$$
$$4) \int_0^T t\sin(\lambda t)\sin^0\big(t,\alpha^0\big)dt$$
$$= \frac{1}{2\psi^0}\big[-\cos^0\big(T,\alpha^0\big)\sin(\lambda T) - \phi^0\mu_{22,T}\big(\lambda,\alpha^0\big) + \lambda\mu_{11,T}\big(\lambda,\alpha^0\big)\big]. \quad (64)$$

Obviously, integrals 1)–4) are bounded in T by the value

$$\frac{1}{2\psi^0}\left(2 + \phi^0\frac{4}{\sqrt{\psi^0}} + \lambda\frac{4}{\sqrt{\psi^0}}\right) = \frac{1}{\psi^0} + 2\frac{\phi^0 + \lambda}{(\psi^0)^{\frac{3}{2}}}. \quad (65)$$

Below we calculate integrals similar to the previous ones, but with factors t^2 instead of t. Consider the function

$$\rho(x) = \int_0^T t\sin(xt)\cos^0\big(t,\alpha^0\big)dt, \quad x \in \mathbb{R},$$

and the integral

$$5) \int_0^T t^2\cos(\lambda t)\cos^0\big(t,\alpha^0\big)dt = \left.\frac{d\rho(x)}{dx}\right|_{x=\lambda} = \frac{d}{dx}\left[\frac{1}{2\psi^0}\Big(\sin^0\big(T,\alpha^0\big)\sin(xT)\right.$$

$$- \phi^0 \int_0^T \sin(xt) \cos^0\big(t, \alpha^0\big) dt - x \int_0^T \cos(xt) \sin^0\big(t, \alpha^0\big) dt \Big) \Big]_{x=\lambda}$$

$$= \frac{1}{2\psi^0} \Big[T \sin^0\big(T, \alpha^0\big) \cos(\lambda T) - \mu_{12,T}\big(\lambda, \alpha^0\big) - \phi^0 \int_0^T t \cos(\lambda t) \cos^0\big(t, \alpha^0\big) dt$$

$$+ \lambda \int_0^T t \sin(\lambda t) \sin^0\big(t, \alpha^0\big) dt \Big]. \qquad (66)$$

Using the same approach, we get

$$6) \int_0^T t^2 \sin(\lambda t) \cos^0\big(t, \alpha^0\big) dt = \frac{1}{2\psi^0} \Big[T \sin^0\big(T, \alpha^0\big) \sin(\lambda T) - \mu_{22,T}\big(\lambda, \alpha^0\big)$$

$$- \phi^0 \int_0^T t \sin(\lambda t) \cos^0\big(t, \alpha^0\big) dt - \lambda \int_0^T t \cos(\lambda t) \sin^0\big(t, \alpha^0\big) dt \Big]. \qquad (67)$$

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