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Unsolved problem about stability of stochastic difference equations with continuous time and distributed delay

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Abstract Despite the fact that the theory of stability of continuous-time difference equations has a long history, is well developed and very popular in research, there is a simple and clearly formulated problem about the stability of stochastic difference equations with continuous time and distributed delay, which has not been solved for more than 13 years. This paper offers to readers some generalization on this unsolved problem in the hope that it will help move closer to its solution.

Keywords Difference equations, continuous time, distributed delay, stochastic perturbations, asymptotic mean square quasistability
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1 Introduction

The theory of stability for difference equations with continuous time has a long history in both deterministic and stochastic cases, it is well developed and very popular in research. See, for instance, [1–4, 6–18, 20, 21, 24, 25] and references therein.

However, there is a problem of stability of stochastic difference equations with continuous time and distributed delay, simply and clearly formulated more than 13 years ago [19], the solution of which has not yet been found. This problem was included also in the group of other unsolved problems that require solutions [22, 23]. Some generalization of this unsolved problem is offered here in the hope that it will help solve it.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space with the space of events Ω, the σ-algebra $\mathfrak{F}$ and the probability $\mathbf{P}$, $\{\mathfrak{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e. $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$ for $t_1 < t_2$, $\mathbf{E}$ be the expectation with respect to the probability $\mathbf{P}$ [5].

Consider the scalar stochastic difference equation with continuous time [18]

$$\begin{gathered} x(t+h) = a_0 x(t) + a_1 x(t-h) + b \int_{t-h}^{t} x(s)ds + \sigma x(t)\xi(t+h), \\ t > -h, \qquad x(\theta) = \phi(\theta), \qquad \theta \in \Theta = [-2h, 0], \end{gathered} \tag{1.1}$$

where a_0, a_1, b, σ, $h > 0$ are known constants, $\xi(t)$ is $\mathfrak{F}_t$-measurable stationary stochastic process [5] such that

$$\mathbf{E}\xi(t) = 0, \qquad \mathbf{E}\xi^2(t) = 1. \tag{1.2}$$

Below the following definitions of stability for Equation (1.1) are considered.

Definition 1.1. The zero solution of Equation (1.1) is called:

- mean square stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t;\phi)|^2 < \varepsilon$ for all $t \geq 0$ if
$$\|\phi\|^2 = \sup_{\theta \in \Theta} \mathbf{E}|\phi(\theta)|^2 < \delta;$$
- asymptotically mean square stable if it is mean square stable and for each initial function ϕ
$$\lim_{t \to \infty} \mathbf{E}|x(t;\phi)|^2 = 0;$$
- asymptotically mean square quasistable if it is mean square stable and for each $t \in [0, h)$, each initial function ϕ and a positive integer j
$$\lim_{j \to \infty} \mathbf{E}|x(t + jh;\phi)|^2 = 0.$$

Remark 1.1. Note that the asymptotic mean square quasistability follows from the asymptotic mean square stability but the converse statement is not true [18].

It is known [18] that in the case $b = 0$, $h = 1$ the necessary and sufficient conditions for asymptotic mean square quasistability of the zero solution of Equation (1.1) are

$$|a_1| < 1, \qquad |a_0| < 1 - a_1, \qquad \sigma^2 < \frac{1 + a_1}{1 - a_1}\left[(1 - a_1)^2 - a_0^2\right]. \tag{1.3}$$

The problem to get the necessary and sufficient stability conditions for Equation (1.1) in the case $a_1 = 0$ was first formulated more than 13 years ago (see [19, 22]), but it is unsolved until now. Although it is quite close to the known results, but, nevertheless, it seams that it cannot be solved by known methods. It is possible that to solve this problem it is necessary to use some principally new ideas.

Below some results on the stability of Equation (1.1) and some possibilities for their improvement are considered.

2 Characteristic equation and regions of stability

Substituting the solution $x(t)$ of Equation (1.1) in the form $x(t) = Ce^{\lambda t}$, where C and λ are constants, into Equation (1.1) with $\sigma = 0$, i.e. in the deterministic case, we get the so-called characteristic equation of Equation (1.1):

$$e^{\lambda h} = a_0 + a_1 e^{-\lambda h} + \frac{b}{\lambda}\left(1 - e^{-\lambda h}\right). \tag{2.1}$$

The characteristic Equation (2.1) can be presented (see Appendix A.1) in the form of the system of two equations ($\omega \in \mathbf{R}$):

$$\begin{aligned}(1-a_1)\cos\omega h &= a_0 + \frac{b}{\omega}\sin\omega h,\\ (1+a_1)\sin\omega h &= -\frac{b}{\omega}(1-\cos\omega h).\end{aligned} \tag{2.2}$$

The system (2.2) defines (see Appendix A.2) three parts of the bound of the exact region of asymptotic stability of the zero solution of Equation (1.1) in the deterministic case ($\sigma = 0$):

$$\begin{gathered} a_0 + a_1 + bh = 1,\\ a_0 + a_1 = 1, \qquad b = -a_0\omega\tan\frac{\omega h}{2},\\ a_0 = 1 + a_1 + 2\cos\omega h, \qquad b = -(1+a_1)\omega\cot\frac{\omega h}{2}.\end{gathered} \tag{2.3}$$

Immediately from (1.1) it follows (see Appendix A.3) that

$$\mathbf{E}x^2(t+h) \le \left[(|a_0| + |a_1| + |b|h)^2 + \sigma^2\right] \sup_{s\in[t-h,t]} \mathbf{E}x^2(s). \tag{2.4}$$

From (2.4) the following sufficient condition for asymptotic mean square stability of the zero solution of Equation (1.1) follows:

$$(|a_0| + |a_1| + |b|h)^2 + \sigma^2 < 1$$

or

$$|a_0| + |a_1| + |b|h < \sqrt{1-\sigma^2}. \tag{2.5}$$

Remark 2.1. Note that in the case $b = 0$ the sufficient stability condition (2.5) for arbitrary $\sigma^2 < 1$ coincides with the necessary and sufficient stability condition (1.3) if $a_1 = 0$, i.e. $a_0^2 + \sigma^2 < 1$, or if $a_0 = 0$, i.e. $a_1^2 + \sigma^2 < 1$.

In Figure 1 the stability regions defined by the condition (1.3) (the regions 1 and 3) and by the condition (2.5) (the regions 2 and 4) are shown in the space (a_0, a_1) for $b = 0$, $h = 1$ and different values of σ^2:

1) $\sigma^2 = 0$ (the regions 1 and 2); 2) $\sigma^2 = 0.7$ (the regions 3 and 4).

One can see that in correspondence with Remark 2.1 on the coordinate axes the bound of stability region, defined by the condition (2.5), coincides with the bound of stability region, defined by the condition (1.3).

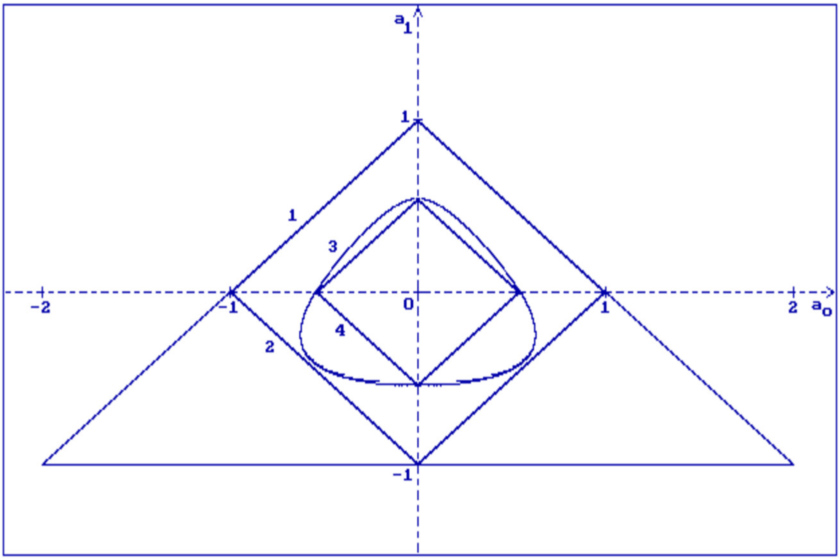

Fig. 1. Stability regions in the space (a_0, a_1) for $b = 0$, $h = 1$, defined by the condition (1.3) (the regions 1 ($\sigma = 0$) and 3 ($\sigma = 0.7$)) and by the condition (2.5) (the regions 2 ($\sigma = 0$) and 4 ($\sigma = 0.7$))

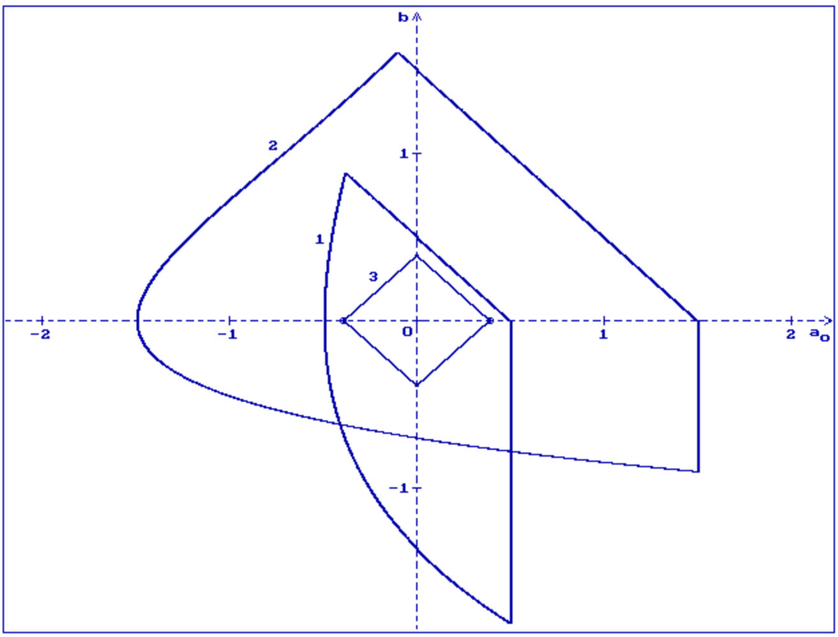

Fig. 2. Stability regions in the space (a_0, b) for $h = 1$, defined by the condition (2.3) (the regions 1 ($a_1 = 0.5$) and 2 ($a_1 = -0.5$)) and by the condition (2.5) (the region 3, $|a_1| = 0.5$, $\sigma^2 = 0.2$)

Consider now stability regions given by the conditions (2.3) and (2.5) in the space (a_0, b) by the fixed a_1.

In Figure 2 the stability regions defined by the condition (2.3) (the regions 1 and 2) and the condition (2.5) (the line 3, $\sigma^2 = 0.2$) are shown in the space (a_0, b) for $h = 1$ and $a_1 = 0.5$ (the line 1), $a_1 = -0.5$ (the line 2). One can see that even for small level of stochastic perturbations ($\sigma^2 = 0.2$) the stability region obtained by the condition (2.5) is essentially less than in the deterministic case (the regions 1 and 2), in particular, for negative a_1 (the region 2).

The unsolved problem: to get the necessary and sufficient stability conditions for Equation (1.1) or at least to improve the existing sufficient stability condition (2.5) by increasing the region of stability defined by it.

3 Conclusions

Trying to solve a problem unsolved for many years, one does not know whether it can be solved by known methods or whether it is necessary to find a new, unknown until now ideas and methods to solve it. But in any case any success in this direction generates both new problems and new methods and leads to the development of the theory. So, all potential readers are invited to participate in the discussion and in the solution of the offered here unsolved problem.

A Appendix

A.1 Appendix 1. Proof of the system (2.2)

Rewrite the characteristic Equation (2.1) in the form

$$e^{\lambda h} = a_0 + \frac{b}{\lambda} + \left(a_1 - \frac{b}{\lambda}\right) e^{-\lambda h}.$$

Putting here $\lambda = i\omega$, $i^2 = -1$, and using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain

$$\cos\omega h + i\sin\omega h = a_0 + \frac{b}{i\omega} + \left(a_1 - \frac{b}{i\omega}\right)(\cos\omega h - i\sin\omega h)$$

or

$$\cos\omega h + i\sin\omega h = a_0 - \frac{bi}{\omega} + \left(a_1 + \frac{bi}{\omega}\right)(\cos\omega h - i\sin\omega h)$$

or

$$(1 - a_1)\cos\omega h - a_0 - \frac{b}{\omega}\sin\omega h + i\left((1 + a_1)\sin\omega h + \frac{b}{\omega}(1 - \cos\omega h)\right) = 0,$$

from where the system (2.2) follows.

A.2 Appendix 2. Proof of the Equations (2.3)
1) If $\omega = 0$ then from the first Equation (2.2) it follows that $a_0 + a_1 + bh = 1$.
2) If $1 - a_1 = a_0$ then from the first Equation (2.2) it follows

$$-a_0(1 - \cos\omega h) = \frac{b}{\omega}\sin\omega h,$$
$$-a_0 2\sin^2\frac{\omega h}{2} = \frac{b}{\omega}2\sin\frac{\omega h}{2}\cos\frac{\omega h}{2},$$
$$b = -a_0\omega\tan\frac{\omega h}{2}.$$

3) From the second Equation (2.2) we have

$$(1 + a_1)2\sin\frac{\omega h}{2}\cos\frac{\omega h}{2} = -\frac{b}{\omega}2\sin^2\frac{\omega h}{2},$$

from where it follows

$$b = -(1 + a_1)\omega\cot\frac{\omega h}{2}$$

and via the first Equation (2.2)

$$\begin{aligned} a_0 =&(1 - a_1)\cos\omega h + (1 + a_1)\cot\frac{\omega h}{2}\sin\omega h \\ =&(1 - a_1)\cos\omega h + (1 + a_1)2\cos^2\frac{\omega h}{2} \\ =&(1 - a_1)\cos\omega h + (1 + a_1)(1 + \cos\omega h) \\ =&1 + a_1 + 2\cos\omega h. \end{aligned}$$

A.3 Appendix 3. Proof of the inequality (2.4)
Using (1.1) and (1.2), we obtain

$$\begin{aligned} \mathbf{E}x^2(t+h) &= \mathbf{E}\left(a_0x(t) + a_1x(t-h) + b\int_{t-h}^{t} x(s)ds\right)^2 + \sigma^2\mathbf{E}x^2(t) \\ &\le a_0^2\mathbf{E}x^2(t) + a_1^2\mathbf{E}x^2(t-h) + b^2h\int_{t-h}^{t}\mathbf{E}x^2(s)ds \\ &\quad + 2|a_0a_1|\mathbf{E}|x(t)x(t-h)| + 2|a_0b|\int_{t-h}^{t}\mathbf{E}|x(t)x(s)|ds \\ &\quad + 2|a_1b|\int_{t-h}^{t}\mathbf{E}|x(t-h)x(s)|ds + \sigma^2\mathbf{E}x^2(t). \end{aligned} \tag{A.1}$$

Note that for $t_1 = t$ or $t_1 = t - h$ and $s \in [t-h, t]$ we have

$$\mathbf{E}|x(t_1)x(s)| \le \sqrt{\mathbf{E}x^2(t_1)\mathbf{E}x^2(s)} \le \sup_{s\in[t-h,t]}\mathbf{E}x^2(s). \tag{A.2}$$

Substituting (A.2) into (A.1), we obtain (2.4):

$$\begin{aligned} \mathbf{E}x^2(t+h) &\le [a_0^2 + a_1^2 + b^2h^2 + 2|a_0a_1| + 2|a_0b| + 2|a_1b| + \sigma^2]\sup_{s\in[t-h,t]}\mathbf{E}x^2(s) \\ &= \left[(|a_0| + |a_1| + |b|h)^2 + \sigma^2\right]\sup_{s\in[t-h,t]}\mathbf{E}x^2(s). \end{aligned}$$

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