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Fast L_2-approximation of integral-type functionals of Markov processes

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Abstract In this paper, we provide strong L_2-rates of approximation of the integral-type functionals of Markov processes by integral sums. We improve the method developed in [2]. Under assumptions on the process formulated only in terms of its transition probability density, we get the accuracy that coincides with that obtained in [3] for a one-dimensional diffusion process.

Keywords Markov processes, integral functional, rates of convergence, strong approximation

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1 Introduction

Let X_t, $t \geq 0$, be a Markov process with values in $\mathbb{R}^d$. Consider the following objects:

1) the integral functional

$$I_T(h) = \int_0^T h(X_t)\, dt$$

of this process;

2) the sequence of integral sums

$$I_{T,n}(h) = \frac{T}{n} \sum_{k=0}^{n-1} h(X_{(kT)/n}), \quad n \geq 1.$$

In this paper, we establish *strong L_2-approximation rates*, that is, the bounds for

$$E\left|I_T(h) - I_{T,n}(h)\right|^2.$$

The current research is mainly motivated by the recent papers [2] and [3].

In [3], strong L_p-approximation rates are considered for an important particular case where X is a one-dimensional diffusion. The approach developed in this paper contains both the Malliavin calculus tools and the Gaussian bounds for the transition probability density of the process X, and relies substantially on the structure of the process.

Another approach to that problem has been developed in [2]. This approach is, in a sense, a modification of Dynkin's theory of continuous additive functionals (see [1], Chap. 6) and also involves the technique similar to that used in the proof of the classical Khasminskii lemma (see, e.g., [4, Lemma 2.1]). This approach allows us to obtain strong L_p-approximation rates under assumptions on the process X formulated only in terms of its transition probability density.

For a bounded function h, the strong L_p-rates of approximation of the integral functional $I_T(h)$ obtained in [2] essentially coincide with those established in [3]. However, under additional regularity assumptions on the function h (e.g., when h is Hölder continuous), the rates obtained in [3] are sharper (see [2, Thm. 2.2] and [3, Thm. 2.3]).

In this note, we improve the method developed in [2], so that under the assumption of the Hölder continuity of h, the strong L_2-approximation rates coincide with those obtained in [3], preserving at the same time the advantage of the method that the assumptions on the process X are quite general and do not essentially rely on the structure of the process.

2 Main result

In what follows, P_x denotes the law of the Markov process X conditioned by $X_0 = x$, and $\mathbb{E}_x$ denotes the expectation with respect to this law. Both the absolute value of a real number and the Euclidean norm in $\mathbb{R}^d$ are denoted by $|\cdot|$.

We make the following assumption on the process X.

A. The process X possesses a transition probability density $p_t(x,y)$ that is differentiable with respect to t and satisfies the following estimates:

$$p_t(x,y) \le C_T t^{-d/\alpha} Q\big(t^{-1/\alpha}(x-y)\big), \quad t \le T, \tag{1}$$

$$\big|\partial_t p_t(x,y)\big| \le C_T t^{-1-d/\alpha} Q\big(t^{-1/\alpha}(x-y)\big), \quad t \le T, \tag{2}$$

$$\big|\partial^2_{tt} p_t(x,y)\big| \le C_T t^{-2-d/\alpha} Q\big(t^{-1/\alpha}(x-y)\big), \quad t \le T, \tag{3}$$

for some fixed $\alpha \in (0,2]$ and some distribution density Q such that $\int_{\mathbb{R}^d} |z|^{2\gamma} Q(z)\,dz < \infty$. Without loss of generality, we assume that in (1)–(3) $C_T \ge 1$.

We assume that the function h satisfies the Hölder condition with exponent $\gamma \in (0, \alpha/2]$, that is,

$$\|h\|_\gamma := \sup_{x\ne y} \frac{|h(x)-h(y)|}{|x-y|^\gamma} < \infty.$$

Now we formulate the main result of the paper.

Theorem 1. *Suppose that Assumption* **A** *holds. Then*

$$\mathbb{E}_x\big|I_T(h)-I_{T,n}(h)\big|^2 \le \begin{cases} D_{T,\gamma,\alpha,Q}C_{\gamma,\alpha}\|h\|_\gamma^2 n^{-(1+2\gamma/\alpha)}, & \gamma\ne\alpha/2,\\ D_{T,\gamma,\alpha,Q}\|h\|_\gamma^2 n^{-2}\ln n, & \gamma=\alpha/2,\end{cases}$$

where

$$D_{T,\gamma,\alpha,Q}=8C_T^2T^{2+2\gamma/\alpha}\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz,$$

$$C_{\gamma,\alpha}=\max\biggl\{(1-2\gamma/\alpha)^{-1}(2\gamma/\alpha)^{-1},\ \max_{n\ge1}\biggl(\frac{(\ln n)^2}{n^{1-2\gamma/\alpha}}\biggr)\biggr\}.$$

We provide the proof of Theorem 1 in Section 3.

Remark 1. Any diffusion process satisfies conditions (1)–(3) with $\alpha=2$, $Q(x)=c_1e^{-c_2|x|^2}$, and properly chosen c_1,c_2 (see [2]). In the case where X is a one-dimensional diffusion, Theorem 1 provides the same rates of convergence as those obtained in [3] (see Theorem 2.3 in [3]).

Remark 2. Similarly to [2], we formulate the assumption on the process X only in terms of its transition probability density. Condition **A**, compared with condition **X** (cf. [2]), contains the additional assumption (3).

3 Proof of Theorem 1

Proof. For $t\in[kT/n,(k+1)T/n)$, denote

$$\eta_n(t)=\frac{kT}{n},\qquad \zeta_n(t)=\frac{(k+1)T}{n},$$

and put $\Delta_n(s):=h(X_s)-h(X_{\eta_n(s)})$, $s\in[0,T]$.

By the Markov property of X, for any $r<s$, we have

$$\begin{aligned}\mathbb{E}_x|X_s-X_r|^{2\gamma}&=\mathbb{E}_x\int_{\mathbb{R}^d}p_{s-r}(X_r,z)|X_r-z|^{2\gamma}\,dz\\ &\le C_T\mathbb{E}_x\int_{\mathbb{R}^d}(s-r)^{-d/\alpha}Q\big((s-r)^{-1/\alpha}(X_r-z)\big)|X_r-z|^{2\gamma}\,dz\\ &=C_T(s-r)^{2\gamma/\alpha}\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz.\end{aligned}$$

Therefore, using the inequality $s-\eta_n(s)\le T/n$, $s\in[0,T]$ and the Hölder continuity of the function h, we obtain:

$$\mathbb{E}_x\big|\Delta_n(s)\big|^2\le C_TT^{2\gamma/\alpha}\biggl(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\biggr)\|h\|_\gamma^2n^{-2\gamma/\alpha}. \tag{4}$$

Split

$$\mathbb{E}_x\big|I_T(h)-I_{T,n}(h)\big|^2=2\mathbb{E}_x\int_0^T\int_s^T\Delta_n(s)\Delta_n(t)\,dt\,ds=J_1+J_2+J_3, \tag{5}$$

where

$$J_1 = 2\mathbb{E}_x \int_0^T \int_s^{\zeta_n(s)+T/n} \Delta_n(s)\Delta_n(t)\,dt\,ds,$$

$$J_2 = 2\mathbb{E}_x \int_0^{T/n} \int_{\zeta_n(s)+T/n}^T \Delta_n(s)\Delta_n(t)\,dt\,ds,$$

$$J_3 = 2\mathbb{E}_x \int_{T/n}^T \int_{\zeta_n(s)+T/n}^T \Delta_n(s)\Delta_n(t)\,dt\,ds.$$

For $|J_1|$ and $|J_2|$, the estimates can be obtained in the same way. Indeed, using the Cauchy inequality and (4), we get

$$\begin{aligned}|J_1| &\le 2\int_0^T \int_s^{\zeta_n(s)+T/n} \big(\mathbb{E}_x\big|\Delta_n(s)\big|^2\big)^{1/2}\big(\mathbb{E}_x\big|\Delta_n(t)\big|^2\big)^{1/2}\,dt\,ds \\ &\le 2C_T T^{2\gamma/\alpha}\|h\|_\gamma^2\bigg(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\bigg)n^{-2\gamma/\alpha}\int_0^T \big(T/n+\zeta_n(s)-s\big)\,ds \\ &\le 4C_T T^{2+2\gamma/\alpha}\|h\|_\gamma^2\bigg(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\bigg)n^{-(1+2\gamma/\alpha)}.\end{aligned}$$

In the last inequality, we have used the inequality $\zeta_n(s) - s \le T/n$, $s \in [0, T]$. Similarly,

$$|J_2| \le 2C_T T^{2+2\gamma/\alpha}\|h\|_\gamma^2\bigg(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\bigg)n^{-(1+2\gamma/\alpha)}.$$

Now we proceed to the estimation of $|J_3|$, which is the main part of the proof. Observe that the following identities hold:

$$\begin{aligned}\int_{\mathbb{R}^d}\partial^2_{uv}p_u(x,y)p_{v-u}(y,z)\,dz &= \partial^2_{uv}p_u(x,y)\int_{\mathbb{R}^d}p_{v-u}(y,z)\,dz \\ &= \partial^2_{uv}p_u(x,y) = 0, \quad y\in\mathbb{R}^d, \end{aligned}\tag{6}$$

$$\begin{aligned}\int_{\mathbb{R}^d}\partial^2_{uv}p_u(x,y)p_{v-u}(y,z)\,dy &= \partial^2_{uv}\int_{\mathbb{R}^d}p_u(x,y)p_{v-u}(y,z)\,dy \\ &= \partial^2_{uv}p_v(x,z) = 0, \quad z\in\mathbb{R}^d, \end{aligned}\tag{7}$$

where in (6) we used that $\int_{\mathbb{R}^d} p_r(y,z)\,dz = 1$, $r > 0$, $y \in \mathbb{R}^d$, and in (7) we used the Chapman–Kolmogorov equation.

We have:

$$\begin{aligned}J_3 = 2\int_{T/n}^T \int_{\zeta_n(s)+T/n}^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(y)h(z)\big[&p_s(x,y)p_{t-s}(y,z) \\ &- p_{\eta_n(s)}(x,y)p_{t-\eta_n(s)}(y,z) - p_s(x,y)p_{\eta_n(t)-s}(y,z) \\ &+ p_{\eta_n(s)}(x,y)p_{\eta_n(t)-\eta_n(s)}(y,z)\big]\,dz\,dy\,dt\,ds\end{aligned}$$

$$= 2\int_{T/n}^{T}\int_{\zeta_n(s)+T/n}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\eta_n(s)}^{s}\int_{\eta_n(t)}^{t} h(y)h(z)\partial^2_{uv}\big(p_u(x,y) \times p_{v-u}(y,z)\big)\,dv\,du\,dz\,dy\,dt\,ds$$

$$= -\int_{T/n}^{T}\int_{\zeta_n(s)+T/n}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\eta_n(s)}^{s}\int_{\eta_n(t)}^{t} \big(h(y)-h(z)\big)^2\partial^2_{uv}\big(p_u(x,y) \times p_{v-u}(y,z)\big)\,dv\,du\,dz\,dy\,dt\,ds, \tag{8}$$

where in the last identity we have used (6) and (7).

Further, we have

$$\partial^2_{uv}p_u(x,y)p_{v-u}(y,z) = p_u(x,y)\partial^2_{rr}p_r(y,z)\big|_{r=v-u} + \partial_u p_u(x,y)\partial_r p_r(y,z)\big|_{r=v-u}.$$

Then, using condition **A** and the Hölder continuity of the function h, we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(h(y)-h(z)\big)^2\big|\partial^2_{uv}\big(p_u(x,y)p_{v-u}(y,z)\big)\big|\,dz\,dy \le C_T^2\|h\|_\gamma^2\bigg(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\bigg)\big((v-u)^{2\gamma/\alpha-2}+(v-u)^{2\gamma/\alpha-1}u^{-1}\big). \tag{9}$$

Therefore, according to (8) and (9),

$$|J_3| \le C_T^2\|h\|_\gamma^2\bigg(\int_{\mathbb{R}^d}|z|^{2\gamma}Q(z)\,dz\bigg) \times \int_{T/n}^{T}\int_{\zeta_n(s)+T/n}^{T}\int_{\eta_n(s)}^{s}\int_{\eta_n(t)}^{t}\big((v-u)^{2\gamma/\alpha-2}+(v-u)^{2\gamma/\alpha-1}u^{-1}\big)\,dv\,du\,dt\,ds. \tag{10}$$

Denote $a_{\alpha,\gamma}(u,v) := (v-u)^{2\gamma/\alpha-2}+(v-u)^{2\gamma/\alpha-1}u^{-1}$. Then

$$\begin{aligned}
&\int_{T/n}^{T}\int_{\zeta_n(s)+T/n}^{T}\int_{\eta_n(s)}^{s}\int_{\eta_n(t)}^{t} a_{\alpha,\gamma}(u,v)\,dv\,du\,dt\,ds\\
&\quad= \sum_{i=1}^{n-1}\sum_{j=i+2}^{n-1}\int_{iT/n}^{(i+1)T/n}\int_{jT/n}^{(j+1)T/n}\int_{iT/n}^{s}\int_{jT/n}^{t} a_{\alpha,\gamma}(u,v)\,dv\,du\,dt\,ds\\
&\quad= \sum_{i=1}^{n-1}\sum_{j=i+2}^{n-1}\int_{iT/n}^{(i+1)T/n}\int_{jT/n}^{(j+1)T/n}\int_{u}^{(i+1)T/n}\int_{v}^{(j+1)T/n} a_{\alpha,\gamma}(u,v)\,dt\,ds\,dv\,du\\
&\quad\le T^2n^{-2}\sum_{i=1}^{n-1}\sum_{j=i+2}^{n-1}\int_{iT/n}^{(i+1)T/n}\int_{jT/n}^{(j+1)T/n} a_{\alpha,\gamma}(u,v)\,dv\,du\\
&\quad= T^2n^{-2}\sum_{i=1}^{n-1}\int_{iT/n}^{(i+1)T/n}\int_{(i+2)T/n}^{T} a_{\alpha,\gamma}(u,v)\,dv\,du,
\end{aligned}$$

where in the fourth line we used that, for $u \in [iT/n, (i+1)T/n)$ and $v \in [jT/n, (j+1)T/n)$, we always have $(i+1)T/n - u \le T/n$ and $(j+1)T/n - v \le T/n$.

Thus, from (10) we obtain

$$|J_3| \le C_T^2 T^2 \|h\|_\gamma^2 \left(\int_{\mathbb{R}^d} |z|^{2\gamma} Q(z)\, dz \right) n^{-2} (S_1 + S_2), \tag{11}$$

where

$$S_1 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{(i+1)T/n}^{T} (v-u)^{2\gamma/\alpha - 2}\, dv\, du,$$
$$S_2 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{(i+2)T/n}^{T} (v-u)^{2\gamma/\alpha - 1} u^{-1}\, dv\, du.$$

We estimate each term separately. In what follows, we consider the case $\gamma < \alpha/2$; the case of $\gamma = \alpha/2$ is similar and therefore omitted. We have

$$\begin{aligned} S_1 &\le (1 - 2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \big((i+1)T/n - u\big)^{2\gamma/\alpha - 1}\, du \\ &= (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} \big((i+1)T/n - iT/n\big)^{2\gamma/\alpha} \\ &\le (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha} \le C_{\gamma,\alpha} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha}. \end{aligned} \tag{12}$$

Finally, since $v - u \le T$ for $0 \le u < v \le T$, we have

$$\begin{aligned} S_2 &\le T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{(i+2)T/n}^{T} (v-u)^{-1} u^{-1}\, dv\, du \\ &\le T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \left(\int_{iT/n}^{(i+1)T/n} u^{-1}\, du \right) \left(\int_{(i+2)T/n}^{T} \big(v - (i+1)T/n\big)^{-1}\, dv \right) \\ &\le T^{2\gamma/\alpha} \ln n \sum_{i=1}^{n-1} \left(\int_{iT/n}^{(i+1)T/n} u^{-1}\, du \right) = T^{2\gamma/\alpha} (\ln n)^2 \\ &\le C_{\gamma,\alpha} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha}. \end{aligned} \tag{13}$$

Combining inequality (11) with (12) and (13), we derive

$$|J_3| \le 2 C_{\gamma,\alpha} C_T^2 T^{2+2\gamma/\alpha} \|h\|_\gamma^2 \left(\int_{\mathbb{R}^d} |z|^{2\gamma} Q(z)\, dz \right) n^{-(1+2\gamma/\alpha)}. \qquad \square$$

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