Modern Stochastics: Theory and Applications 11 (4) (2024) 439–458
https://doi.org/10.15559/24-VMSTA254

Generalized fractional calculus and some models of generalized counting processes

Khrystyna Buchak[a], **Lyudmyla Sakhno**[b,*]

[a] *Faculty of Mathematics and Digital Technologies, Uzhgorod National University, Narodna Square 3, 88000 Uzhgorod, Ukraine*
[b] *Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska St. 64, 01601 Kyiv, Ukraine*

khrystyna.buchak@uzhnu.edu.ua (K. Buchak), lyudmyla.sakhno@knu.ua (L. Sakhno)

Received: 3 December 2023, Revised: 14 March 2024, Accepted: 14 April 2024,
Published online: 30 May 2024

Abstract Models of generalized counting processes time-changed by a general inverse subordinator are considered, their distributions are characterized, and governing equations for them are presented. The equations are given in terms of the generalized fractional derivatives, namely, convolution-type derivatives with respect to Bernštein functions. Some particular examples are presented.

Keywords Time-change, Poisson process, generalized counting process, subordinator, inverse subordinator, generalized fractional derivatives
2010 MSC 60G50, 60G51, 60G55

1 Introduction

Stochastic processes with random time-change has become a well-established and highly ramified branch of the modern theory of stochastic processes which attracts more and more attention due to various applications in financial, biological, ecological, physical, technical and other fields of research.

*Corresponding author.

Rich class of models is provided by time-changed Poisson processes, of which the most intensively studied are two fractional extensions of the Poisson process, namely, the space-fractional and time-fractional Poisson processes obtained by choosing a stable subordinator or its inverse in the role of time correspondingly. We refer, for example, to papers [8, 19, 21, 1, 2, 16, 20], among many others (see also references therein). In particular, in paper [21] a general class of time-changed Poisson processes $N(H^{\psi}(t))$, $t > 0$, was introduced and studied, where $N(t)$ is a Poisson process and $H^{\psi}(t)$ is an arbitrary subordinator with the Laplace exponent ψ, independent of $N(t)$, and their distributional properties, hitting times and governing equations were presented (see, also [8]). In paper [5] Poisson processes time-changed by general inverse subordinators were studied, the governing equations for their marginal distributions were presented and some other properties were described. The Poisson process itself, being in a sense a core object concerning applicability to count data and simple tractability, however, as a reverse side of its simplicity, is a rather restrictive model. Therefore, it has been quite natural to search for its extensions and generalizations with some new useful features and properties needed for applications. For example, time-changed processes $N(H^{\psi}(t))$ allow for jumps of arbitrary size and other interesting properties ([21, 8]). Another extension is provided by generalized counting processes introduced in [7]: such processes perform k kinds of jumps of amplitude $1, 2, \ldots, k$ with rates $\lambda_1, \ldots, \lambda_k$, for a fixed $k \geq 1$. Note that these processes and their time-changed versions are of interest in various applications, in particular, in risk theory (see, e.g., [13]).

Recently, generalized counting processes, their time-changed versions and fractional extensions have been intensively investigated, including Poisson and Skellam processes of order k, Pólya–Aeppli process of order k, Bell–Touchard process, Poisson-logarithmic process, etc. and their fractional extensions. We refer to papers [7, 15, 6, 10, 12, 11, 13], to mention only few, see also references therein.

In the present paper we consider several particular models of generalized counting processes time-changed by a general inverse subordinator, characterize their distributions and present governing equations for them, which are obtained following the technique presented in [5]. The equations are given in terms of the generalized Caputo–Djrbashian derivatives, which are called also convolution-type derivatives with respect to Bernštein functions. These generalized derivatives were introduced in [14] and [22] and have been widely used to describe various advanced stochastic models.

The convolution-type derivatives allow to study the properties of subordinators and their inverses in the unifying manner ([22, 17]), in particular, the governing equations for their densities can be given in terms of convolution-type derivatives. These properties will be the key tools for our study: by considering several models of time-changed processes, we elucidate the technique leading from the equations for the densities of inverse subordinators and their Laplace transforms to the equations for probabilities of processes time-changed by inverse subordinators and equations for some related functions.

Stable subordinators and their inverses are the most widely used tools for time-change and lead to so-called fractional in time/space or space-time fractional versions of original processes and corresponding fractional governing equations for probabilities of processes. In this paper we consider a general framework of double time-change by means of an arbitrary subordinator and an inverse to another subordinator, with a natural connection to generalized fractional calculus. To the best of our knowledge, the generalized counting processes with such general double time-change have not been considered in the literature. We also present a particular model of time-changed generalized counting process obtained by using the compound Poisson-Gamma subordinator. The state probabilities for this model can be given by the closed form expressions involving some special functions. In this way we complement the existing studies of time-changed processes.

The paper is organized as follows. Section 2 collects the basic definitions and facts on the convolution-type derivatives needed in our study. We also consider, as a warmup example, the time-changed Poisson process $N(Y^f(t))$, $t > 0$, where Y^f is the inverse to the subordinator with Laplace exponent f, independent of $N(t)$. We partly extend the results from [5] for this process. This simplest case allows to demonstrate very transparently the technique underpinning the derivations of the corresponding results for more general models considered in the next sections. Namely, in Sections 3 and 4 we study the processes $N^{\psi,f}(t) = N\left(H^{\psi}(Y^f(t))\right)$, $t > 0$, and $M^{\psi,f}(t) = M\left(H^{\psi}(Y^f(t))\right)$, $t > 0$, where M is the generalized counting process. Time-change is done by the independent subordinator H^{ψ} and inverse subordinator Y^f related to Bernštein functions ψ and f correspondingly. We present the differential-difference equations for probabilities of the processes $N^{\psi,f}$ and $M^{\psi,f}$, which are given in terms of the convolution-type derivative with respect to function f and the difference operator related to function ψ. Expressions for the probabilities of these processes are also presented in terms of the Laplace transform of the process Y^f. Some other properties of these processes are described and a comparison with related results in the literature is given. As examples, we consider the processes $N^{\psi,f}$ and $M^{\psi,f}$, where H^{ψ} is a compound Poisson-Gamma process.

2 Preliminaries

Generalized fractional derivatives

We review the main definitions and some facts on the generalized fractional derivatives, which will be important for our study. (For more details, see, e.g., [17, 22].)

Let $f(x)$ be a Bernštein function,

$$f(x) = a + bx + \int_0^{\infty} \left(1 - e^{-xs}\right) \overline{\nu}_f(ds), \quad x > 0, \ a, b \geq 0, \tag{1}$$

$\overline{\nu}_f(ds)$ is a nonnegative measure on $(0, \infty)$ (the Lévy measure for $f(x)$) such that

$$\int_0^{\infty} (s \wedge 1)\, \overline{\nu}_f(ds) < \infty.$$

The generalized Caputo–Djrbashian (C-D) derivative, or convolution-type derivative, with respect to the Bernštein function f is defined on the space of absolutely continuous functions as follows ([22], Definition 2.4):

$$\mathcal{D}_t^f u(t) = b\frac{d}{dt}u(t) + \int_0^t \frac{\partial}{\partial t}u(t-s)\nu_f(s)ds, \tag{2}$$

where $\nu_f(s) = a + \overline{\nu}_f(s,\infty)$ is the tail of the Lévy measure $\overline{\nu}_f(s)$ of the function f.

In the case where $f(x) = x^\alpha$, $x > 0$, $\alpha \in (0,1)$, the derivative (2) reduces to the fractional C-D derivative

$$\mathcal{D}_t^f u(t) = \frac{d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds. \tag{3}$$

For the Laplace transform of the derivative (2) the following relation holds ([22], Lemma 2.5):

$$\mathcal{L}\left[\mathcal{D}_t^f u\right](s) = f(s)\mathcal{L}[u](s) - \frac{f(s)}{s} u(0), \quad s > s_0,$$

for u such that $|u(t)| \le \mathsf{M} e^{s_0 t}$, M and s_0 being some constants. Similarly to the C-D fractional derivative, the convolution-type derivative can be alternatively defined via its Laplace transform.

The generalization of the classical Riemann–Liouville (R-L) fractional derivative is introduced in [22] by means of another convolution-type derivative with respect to f given by the formula

$$\mathbb{D}_t^f u(t) = b\frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(t-s)\nu_f(s) ds. \tag{4}$$

The derivatives $\mathcal{D}_t^f$ and $\mathbb{D}_t^f$ are related as follows (see, [22], Proposition 2.7):

$$\mathbb{D}_t^f u(t) = \mathcal{D}_t^f u(t) + \nu_f(t) u(0). \tag{5}$$

Bernštein functions are associated in a natural way with subordinators.

Let $H^f(t)$, $t \ge 0$, be a subordinator, that is, nondecreasing Lévy process. Its Laplace transform is of the form:

$$\mathcal{L}[H^f(t)](s) = \mathsf{E} e^{-sH^f(t)} = e^{-tf(s)},$$

where the function f, called the Laplace exponent, is a Bernštein function. Consider a subordinator H^f with the Laplace exponent f given by (1), and let Y^f be its inverse process defined as

$$Y^f(t) = \inf\left\{s \ge 0 : H^f(s) > t\right\}. \tag{6}$$

It was shown in [22] that the distribution of the inverse process Y^f has a density $\ell_f(t,x) = \mathrm{P}\{Y^f(t) \in dx\}/dx$ provided that the following condition holds.

Condition I. $\overline{\nu}_f(0,\infty) = \infty$ and the tail $\nu_f(s) = a + \overline{\nu}_f(s,\infty)$ is absolutely continuous.

The Laplace transform of the density with respect to t has the form ([22])

$$\mathcal{L}_t\left(\ell_f(t,x)\right)(r) = \frac{f(r)}{r} e^{-xf(r)}.$$

The density $\ell_f(t,u)$ of the inverse process Y^f satisfies the equation ([22], Theorem 4.1)

$$\begin{aligned} &\mathbb{D}_t^f \ell_f(t,u) = -\frac{\partial}{\partial u}\ell_f(t,u),\ t>0,\\ &0<u<\infty,\ \text{if } b=0,\ \text{and } 0<u<t/b,\ \text{if } b>0, \end{aligned} \tag{7}$$

subject to

$$\ell_f(t,u/b)=0,\ \ \ell_f(t,0)=\nu_f(t),\ \ \ell_f(0,u)=\delta(u). \tag{8}$$

The space Laplace transform of the density $\ell_f(t,x)$,

$$\tilde{\ell_f}(t,\lambda)=\int_0^\infty e^{-\lambda x}\ell_f(t,x)dx = \mathsf{E}e^{-\lambda Y^f(t)},\quad t\ge 0,\ \lambda>0, \tag{9}$$

is an eigenfunction of the operator $\mathcal{D}_t^f$, that is, satisfies the equation

$$\mathcal{D}_t^f \tilde{\ell_f}(t,\lambda) = -\lambda\tilde{\ell_f}(t,\lambda),\quad t>0, \tag{10}$$

with $\tilde{\ell_f}(0,\lambda)=1$ (see [5, 14, 17]).

In the case where $f(x)=x^\alpha$, $x>0$, $\alpha\in(0,1)$, we have $\tilde{\ell_f}(t,\lambda)=\mathcal{E}_\alpha(-\lambda t^\alpha)$, where $\mathcal{E}_\alpha(\cdot)$ is the Mittag-Leffler function

$$\mathcal{E}_\alpha(z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)},\quad z\in\mathcal{C},\ \alpha\in(0,\infty). \tag{11}$$

Equations (7)–(10) are important for the study of the processes time-changed by inverse subordinators as can be seen in what follows. We first consider the Poisson process with an inverse subordinator as a simplest example which demonstrates the technique to be applied further to more general models.

Poisson process time-changed by an inverse subordinator

Let $N(t)$ be the Poisson process with intensity λ, and $Y^f(t)$ be the inverse subordinator. Consider the process $N^f(t)=N(Y^f(t))$, $t>0$.

Theorem 1. *Let Condition I hold. Then the marginal distributions* $p_k^f(t) = P\left\{N(Y^f(t))=k\right\}$, $k=0,1,\dots$, *satisfy the differential equation*

$$\mathcal{D}_t^f p_k^f(t)=\lambda\left(p_k^f(t)-p_{k-1}^f(t)\right),\quad k\ge 0,\ t>0 \tag{12}$$

with $p_k^f(0)=1$ *for* $k=0$, $p_k^f(0)=0$ *for* $k\ge 1$, *and*

$$p_k^f(t)=\frac{(-\lambda\partial_\lambda)^k}{k!}\tilde{\ell}_f(t,\lambda), \tag{13}$$

where $\tilde{\ell}_f(t,\lambda)=\int_0^\infty e^{-\lambda u}\ell_f(t,u)du$ *is the Laplace transform of the density of the inverse subordinator* $Y^f(t)$.

The probability generating function of the process N^f has the form

$$G^f(u,t) = \tilde{\ell}(t, \lambda(1-u)), \quad |u| < 1, \tag{14}$$

and satisfies the equation

$$\mathcal{D}_t^f G^f(u,t) = -\lambda(1-u) G^f(u,t) \tag{15}$$

with $G^f(u,0) = 1$.

Proof. Proof of equation (12). For the probabilities $p_x^f(t)$ we have

$$\begin{aligned} p_x^f(t) &= \mathsf{P}\left\{N\left(Y^f(t)\right) = x\right\} = \int_0^\infty p_x(u)\ell_f(t,u)du \\ &= \int_0^\infty \frac{e^{-\lambda u}(\lambda u)^x}{x!}\ell_f(t,u)du, \quad x = 0,1,2,\ldots. \end{aligned} \tag{16}$$

We take the generalized R-L convolution-type derivative $\mathbb{D}_t^f$ given by (4) and use the equations (7)–(8) for the density $\ell^f(t,u)$ of the inverse subordinator. We obtain

$$\begin{aligned} \mathbb{D}_t^f p_x^f(t) &= \int_0^\infty p_x(u)\mathbb{D}_t^f \ell_f(t,u)du = -\int_0^\infty p_x(u)\frac{\partial}{\partial u}\ell_f(t,u)du \\ &= \int_0^\infty \ell_f(t,u)\frac{\partial}{\partial u}p_x(u)du - p_x(u)\ell_f(t,u)\Big|_{u=0}^\infty \\ &= \int_0^\infty \ell_f(t,u)(-\lambda[p_x(u) - p_{x-1}(u)])du + p_x(0)\ell_f(t,0) \\ &= -\lambda\left[p_x^f(t) - p_{x-1}^f(t)\right] + p_x(0)\nu_f(t). \end{aligned} \tag{17}$$

Using the relation (5) between convolution derivatives of C-D and R-L types, we have

$$\mathcal{D}_t^f p_x^f(t) = \mathbb{D}_t^f p_x^f(t) - \nu_f(t)p_x^f(0), \tag{18}$$

and note also that

$$p_x^f(0) = \int_0^\infty p_x(u)\ell_f(0,u)du = \int_0^\infty p_x(u)\delta(u)du = p_x(0) = 1. \tag{19}$$

From (17), taking into account (18)–(19), we finally obtain

$$\mathcal{D}_t^f p_x^f(t) = -\lambda\left[p_x^f(t) - p_{x-1}^f(t)\right].$$

Proof of equation (13):

$$\begin{aligned} \mathsf{P}\left\{N(Y^f(t)) = k\right\} &= \int_0^\infty \mathsf{P}(N(s) = k)\mathsf{P}(Y^f(t) \in ds) \\ &= \int_0^\infty \frac{(-\lambda\partial_\lambda)^k}{k!}e^{-\lambda s}\mathsf{P}\left\{Y^f(t) \in ds\right\} \end{aligned}$$

$$= \mathsf{E}\left[\frac{(-\lambda\partial_\lambda)^k}{k!}\exp\left\{-\lambda Y^f(t)\right\}\right]$$
$$= \frac{(-\lambda\partial_\lambda)^k}{k!}\mathsf{E}\exp\left(-\lambda Y^f(t)\right) = \frac{(-\lambda\partial_\lambda)^k}{k!}\tilde{\ell}_f(t,\lambda).$$

We used above that the distribution of the Poisson process N with the rate λ can be written as

$$p_k(t) = \frac{(-\lambda\partial_\lambda)^k}{k!}e^{-\lambda t}.$$

For the probability generating function we have

$$G^f(u,t) = \mathsf{E}u^{N(Y^f(t))} = \int_0^\infty e^{-s\lambda(1-u)}\ell_f(t,s)ds$$
$$= \mathsf{E}e^{-\lambda(1-u)Y^f(t)} = \tilde{\ell}_f(t,\lambda(1-u)),$$

therefore, we obtain formula (14). Now we use the fact, that $\tilde{\ell}_f(t,\lambda)$ is an eigenfunction of the operator $\mathcal{D}_t^f$ (see (10)), from which we conclude that $G^f(u,t)$, being given by $\tilde{\ell}_f(t,\lambda(1-u))$, satisfies equation (15). □

Remark 1. Note that (10) can be deduced directly from equation (12) by taking $k=1$, since $p_0^f(t) = \mathsf{P}\left\{N\left(Y^f(t)\right)=0\right\} = \int_0^\infty e^{-\lambda u}\ell_f(t,u)du = \tilde{\ell}_f(t,\lambda)$.

3 Models of time-changed Poisson processes

Consider the time-changed Poisson processes $N^\psi(t) = N\left(H^\psi(t)\right)$, $t>0$, where $N(t)$ is the Poisson process with intensity $\lambda > 0$ and $H^\psi(t)$ is the subordinator with Bernštein function $\psi(u)$ (with $a=b=0$, see formula (1)), independent of $N(t)$. This class of processes was introduced and studied in [21] and called by the authors Poisson processes with Bernštein intertimes. It was shown in [21] that the distributions of $N^\psi(t)$, $t>0$, can be presented as

$$p_k^\psi(t) = \mathsf{P}\left\{N^\psi(t)=k\right\} = \frac{(-1)^k}{k!}\frac{d^k}{dt^k}e^{-t\psi(\lambda u)}|_{u=1}, \tag{20}$$

and satisfy the difference-differential equations

$$\frac{d}{dt}p_k^\psi(t) = -\psi(\lambda)p_k^\psi(t) + \sum_{m=1}^k \frac{\lambda^m}{m!}p_{k-m}^\psi(t)\int_0^\infty e^{-s\lambda}s^m\overline{\nu}_\psi(ds), \quad k\ge 0,\ t>0. \tag{21}$$

Note that since ψ is a Bernštein function, it is in C^∞, and the integrals in (21) with respect to its Lévy measure $\overline{\nu}_\psi$ are well defined and represent (up to the sign) the derivatives of ψ, so that (21) can be also written as

$$\frac{d}{dt}p_k^\psi(t) = -\psi(\lambda)p_k^\psi(t) - \sum_{m=1}^k \psi^{(m)}(\lambda)\frac{(-\lambda)^m}{m!}p_{k-m}^\psi(t), \quad k\ge 0,\ t>0, \tag{22}$$

with the usual initial conditions $p_0^{\psi}(0) = 1$, $p_k^{\psi}(0) = 0$ for $k \geq 1$. The last equation can be represented in the form

$$\frac{d}{dt} p_k^{\psi}(t) = -\psi(\lambda(I - B))p_k^{\psi}(t), \quad k \geq 0, \ t > 0, \tag{23}$$

where B is the backshift operator, $Bp_k^f(t) = p_{k-1}^f(t)$, and it is supposed that $p_{-1}(t) = 0$.

The probability generating function of the process N^{ψ} is of the form ([21])

$$G^{\psi}(u, t) = \mathsf{E} u^{N^{\psi}(t)} = e^{-t\psi(\lambda(1-u))}, \quad |u| < 1. \tag{24}$$

We refer for more detail on these processes to [21].

Consider the process N^{ψ} time-changed by an independent inverse subordinator Y^f. We can state the following result.

Theorem 2. *The process* $N^{\psi,f}(t) = N\left(H^{\psi}(Y^f(t))\right)$, $t > 0$, *has probability distribution function*

$$p_k^{\psi,f}(t) = \mathsf{P}\left\{N^{\psi,f}(t) = k\right\} = \frac{(-\lambda\partial_\lambda)^k}{k!} \tilde{\ell}_f(t, \psi(\lambda)), \tag{25}$$

and the probabilities $p_k^{\psi,f}$ *satisfy the equation*

$$\mathcal{D}_t^f p_k^{\psi,f}(t) = -\psi\left(\lambda\left(I - B\right)\right) p_k^{\psi,f}(t), \quad k \geq 0, \ t > 0 \tag{26}$$

with the usual initial conditions.

The probability generating function of the process $N^{\psi,f}$ *has the form*

$$G^{\psi,f}(u, t) = \tilde{\ell}(t, \psi(\lambda(1-u))), \quad |u| < 1, \tag{27}$$

and satisfies the equation

$$\mathcal{D}_t^f G^{\psi,f}(u, t) = -\psi(\lambda(1-u))G^{\psi,f}(u, t) \tag{28}$$

with $G^{\psi,f}(u, 0) = 1$.

Proof. To prove equation (25), we perform the calculations as follows:

$$\begin{aligned} p_k^{\psi,f}(t) &= \mathsf{P}\left\{N\left(H^{\psi}(Y^f(t))\right) = k\right\} \\ &= \int_0^{\infty} \mathsf{P}\left\{N(s) = k\right\} \mathsf{P}\left\{H^{\psi}(Y^f(t)) \in ds\right\} \\ &= \int_0^{\infty} \frac{(-\lambda\partial_\lambda)^k}{k!} e^{-\lambda s} \mathsf{P}\left\{H^{\psi}(Y^f(t)) \in ds\right\} \\ &= \mathsf{E}\left[\frac{(-\lambda\partial_\lambda)^k}{k!} \exp\left\{-\lambda H^{\psi}(Y^f(t))\right\}\right] \\ &= \frac{(-\lambda\partial_\lambda)^k}{k!} \mathsf{E} \exp\left\{-\lambda H^{\psi}(Y^f(t))\right\}. \end{aligned}$$

Now it is left to note that

$$\mathsf{E}\exp\left\{-\lambda H^{\psi}(Y^f(t))\right\} = \mathsf{E}\exp\left(-\psi(\lambda)Y^f(t)\right) = \tilde{\ell}_f(t,\psi(\lambda)).$$

Let us state equation (26). We have

$$p_k^{\psi,f}(t) = \mathsf{P}\left\{N\left(H^{\psi}(Y^f(t))\right) = k\right\} = \int_0^{\infty} p_k^{\psi}(u)\ell_f(t,u)du, \quad k = 0,1,2,\dots.$$

We repeat reasoning in the same lines as in the proof of Theorem 1. We take the generalized R-L derivative $\mathbb{D}_t^f$ and use equations (22)–(23):

$$\begin{aligned}
\mathbb{D}_t^f p_k^{\psi,f}(t) &= \int_0^{\infty} p_k^{\psi}(u)\mathbb{D}_t^f \ell_f(t,u)du = -\int_0^{\infty} p_k^{\psi}(u)\frac{d}{du}\ell_f(t,u)du \\
&= \int_0^{\infty} \ell_f(t,u)\frac{d}{du}p_k^{\psi}(u)du - p_k^{\psi}(u)\ell_f(t,u)\big|_0^{\infty} \\
&= \int_0^{\infty} \ell_f(t,u)\left(-\psi(\lambda(I-B))\right)p_k^{\psi}(u)du + p_k^{\psi}(0)\ell_f(t,0) \\
&= \int_0^{\infty} \ell_f(t,u)\Big(-\Big(\psi(\lambda)p_k^{\psi}(u) \\
&\quad + \sum_{m=1}^{k}\psi^{(m)}(\lambda)\frac{(-\lambda)^m}{m!}p_{k-m}^{\psi}(u)\Big)\Big)du + p_k^{\psi}(0)\nu_f(t) \\
&= -\psi(\lambda)p_k^{\psi,f}(t) - \sum_{m=1}^{k}\psi^{(m)}(\lambda)\frac{(-\lambda)^m}{m!}p_{k-m}^{\psi,f}(t) + p_k^{\psi}(0)\nu_f(t) \\
&= -\psi(\lambda(I-B))p_k^{\psi,f}(t) + p_k^{\psi}(0)\nu_f(t).
\end{aligned} \tag{29}$$

Using the relation between generalized derivatives of C-D and R-L types, we can write

$$\mathcal{D}_t^f p_k^{\psi,f}(t) = \mathbb{D}_t^f p_k^{\psi,f}(t) - \nu_f(t)p_k^{\psi,f}(0);$$

and we have (see (8))

$$p_k^{\psi,f}(0) = \int_0^{\infty} p_k^{\psi}(u)\ell_f(0,u)du = \int_0^{\infty} p_k^{\psi}(u)\delta(u)du = p_k^{\psi}(0) = 1.$$

Therefore, we obtain

$$\mathcal{D}_t^f p_k^{\psi,f}(t) = -\psi(\lambda(I-B))p_k^{\psi,f}(t).$$

Using the expression for $G^{\psi}(u,t)$ given by (24), we calculate $G^{\psi,f}$:

$$\begin{aligned}
G^{\psi,f}(u,t) = \mathsf{E}u^{N\left(H^{\psi}(Y^f(t))\right)} &= \int_0^{\infty} G^{\psi}(u,s)\ell_f(t,s)ds = \\
&= \int_0^{\infty} e^{-s\psi(\lambda(1-u))}\ell_f(t,s)ds = \mathsf{E}e^{-\psi(\lambda(1-u))Y^f(t)} = \\
&= \tilde{\ell}_f(t,\psi(\lambda(1-u)));
\end{aligned}$$

and since $\tilde{\ell}_f(t, \psi(\lambda(1-u)))$ is an eigenfunction of $\mathcal{D}_t^f$ with the eigenvalue $\psi(\lambda(1-u))$ (see (10)), it follows that $G^{\psi,f}(u,t)$ satisfies equation (28). □

Remark 2. We notice at once that equations (26) for the probabilities of the process $N^{\psi,f}(t) = N(H^{\psi}(Y^f(t)))$ mimic the corresponding equations for the process $N^{\psi}(t) = N(H^{\psi}(t))$, that is, the process before the time-change by an inverse subordinator, only the ordinary derivative in time is replaced with the generalized fractional derivative. This can be anticipated quite straightforwardly, in view of the technique used for the proof, and holds also for other models, like those below and in the next section. On the other side, the equation (26) tells us, that for the probabilities of the processes with double time-change like $N\left(H^{\psi}(Y^f(t))\right)$, the action (in time) of the operator $\mathcal{D}_t^f$ (which is related to Y^f) is equal to the action (in space) of the operator $-\psi\left(\lambda\left(I-B\right)\right)$ (which is related to H^{ψ}, and depends also on the outer process N). Some more insight on these operators can be found within the approach applied in the paper [1]. Let $Y^f = Y^{\beta}$ be the inverse stable subordinator with a parameter $\beta \in (0,1]$, that is, $f(\lambda) = \lambda^{\beta}$, then equation (26) becomes

$$\mathcal{D}_t^{\beta} p_k^{\psi,\beta}(t) = -\psi\left(\lambda\left(I-B\right)\right) p_k^{\psi,\beta}(t), \tag{30}$$

where $\mathcal{D}_t^{\beta}$ is the Caputo–Djrbashian fractional derivative (3). If we suppose furthermore that $H^{\psi} = H^{\alpha}$ is the stable subordinator with a parameter $\alpha \in (0,1]$, that is, $\psi(\lambda) = \lambda^{\alpha}$, then we come to the equation

$$\mathcal{D}_t^{\beta} p_k^{\alpha,\beta}(t) = -\lambda^{\alpha}\left(I-B\right)^{\alpha} p_k^{\alpha,\beta}(t). \tag{31}$$

Equations (30) and (31) were stated in paper [1], where the particular representation of the operator $\psi\left(\lambda\left(I-B\right)\right)$ was used, and the equations for the probabilities of time-changed processes were stated using the interplay between the operator $\psi\left(\lambda\left(I-B\right)\right)$ and the fractional operator $\mathcal{D}_t^{\beta}$ in the Fourier domain. The approach in [1] can be extended for the case where we deal with the generalized fractional operator $\mathcal{D}_t^f$, as soon as we know that its eigenfunction is given by $\tilde{\ell}_f(t,\lambda)$ (see (10)). We refer for more detail on this approach to [1].

Remark 3. One of the important properties of the Poisson process is that it is a renewal process having i.i.d. waiting times (or inter-arrival times) J_n with exponential distribution. Fractional in time Poisson process $N(Y^{\beta}(t))$, $t > 0$, where Y^{β} is the inverse stable subordinator with a parameter $\beta \in (0,1]$, keeps the renewal property with distribution of waiting times represented by means of the Mittag-Leffler function,

$$\mathsf{P}\{J_n > t\} = \mathcal{E}_{\beta}(-\lambda t^{\beta}),$$

where λ is a rate of the Poisson process N, the Mittag-Leffler function is given by (11). Moreover, this property is retained by the process $N(Y^f)$, where Y^f is a general inverse subordinator: $N(Y^f)$ is a renewal process with distribution of waiting times represented as

$$\mathsf{P}\{J_n > t\} = \mathsf{E}e^{-\lambda Y^f(t)} = \tilde{\ell}_f(t,\lambda)$$

(see, e.g., [18, 14]).

Theorem 1 (equation (12)) shows that the distributions of the process $N(Y^f)$ solve an analogue of the Kolmogorov forward equation for the Poisson process, with the ordinary derivative in time replaced with the convolution-type derivative (or the C-D fractional derivative for the case of process $N(Y^\beta)$).

Equation (26) for distributions of processes $N^{\psi,f}$ appears in the form which is substantially different from the case of the processes N and $N(Y^f)$: this equation contain additionally the probabilities $p^{\psi,f}_{k-j}(t)$ for $2 \le j \le k$.

For the processes with double time-change $N^{\psi,f}(t) = N(H^\psi(Y^f(t))), t > 0$, the renewal property does not hold in general. In particular, in [8] it was shown that for the space-time fractional Poisson processes $N(H^\alpha(Y^\beta(t)))$ (with H^α and Y^β being stable and inverse stable subordinators) the renewal property does not hold, opposite to the case of time-fractional Possion process $N(Y^\beta(t))$.

The waiting times of k-th event of the processes $N(H^\alpha(Y^\beta(t)))$ were investigated in [1]. Namely, for

$$T_k^{\alpha,\beta} = \inf\{s \ge 0 \,:\, N(H^\alpha(Y^\beta(t))) > k\}$$

the density of distribution was derived in the form ([1], Theorem 2)

$$\mathsf{P}\{T_k^{\alpha,\beta} \in dt\}/dt = \frac{\beta k}{\alpha t}\frac{(-\lambda\partial_\lambda)^k}{k!}\mathcal{E}_\beta(-t^\beta\lambda^\alpha) = \frac{\beta k}{\alpha t}p_k^{\alpha\beta}(t). \tag{32}$$

Following [1], for the general case of process $N^{\psi,f}$, we can consider the waiting times

$$T_k^{\psi,f} = \inf\{s \ge 0 \,:\, N(H^\psi(Y^f(t))) > k\}$$

and write their distribution in the form

$$\mathsf{P}\{T_k^{\psi,f} \le t\} = \mathsf{P}\{N(H^\psi(Y^f(t))) \ge k\} = \sum_{m=k}^{\infty} p_m^{\psi,f}(t)$$
$$= \sum_{m=k}^{\infty}\frac{(-\lambda)^m}{m!}\partial_\lambda^m\tilde{\ell}_f(t,\psi(\lambda)),$$

where for the last equality formula (25) was used. We note that the part of calculations in the proof of Theorem 2 in [1] still holds for the general case of the process $N^{\psi,f}$ and allows to simplify the sum in the formula above to the expression

$$\mathsf{P}\{T_k^{\psi,f} \le t\} = \frac{(-\lambda)^k}{(k-1)!}\partial_\lambda^{k-1}\Big(\frac{\tilde{\ell}_f(t,\psi(\lambda))}{\lambda}\Big). \tag{33}$$

For the case of space-time fractional Poisson process $N(H^\alpha(Y^\beta(t)))$ we have in (33) $\tilde{\ell}_f(t,\psi(\lambda)) = \mathcal{E}_\beta(-t^\beta\lambda^\alpha)$ and the derivation of formula (32) relies on particular properties of the Mittag-Leffler function and the simple form of the Bernštein function of a stable subordinator (see the proof of Theorem 2 in [1]). It would be interesting to obtain expressions for other particular cases of the process $N^{\psi,f}$. We address this problem to further research.

Consider a generalization of counting processes with Bernštein intertimes, which was introduced in the paper [8]:

$$N^{\psi_1,\psi_2,\dots,\psi_n}(t) = N\left(\sum_{j=1}^{n} H^{\psi_j}(t)\right), \quad t \geq 0,$$

where $H^{\psi_i}, i = 1, \dots n$, are n independent subordinators with Bernštein functions ψ_i, independent of the Poisson process N. In [8] the following result was stated.

Proposition 1 ([8]). *The distribution of the subordinated process $N^{\psi_1,\psi_2,\dots,\psi_n}(t)$, $t \geq 0$, is the solution to the Cauchy problem*

$$\frac{d}{dt} p_k^{\psi_1,\psi_2,\dots,\psi_n}(t) = -\sum_{j=1}^{n} \psi_j(\lambda(I - B)) p_k^{\psi_1,\psi_2,\dots,\psi_n}(t), \quad k \geq 0, \; t > 0,$$

with the usual initial conditions

$$p_k^{\psi_1,\psi_2,\dots,\psi_n}(0) = \begin{cases} 1, & k > 0, \\ 0, & k = 0. \end{cases}$$

Consider the process $N^{\psi_1,\psi_2,\dots,\psi_n,f}(t) = N^{\psi_1,\psi_2,\dots,\psi_n}(Y^f(t)), t \geq 0$.

Theorem 3. *The process $N^{\psi_1,\psi_2,\dots,\psi_n,f}$ has probability distribution function*

$$p_k^{\psi_1,\psi_2,\dots,\psi_n,f}(t) = \frac{(-\lambda\partial_\lambda)^k}{k!} \tilde{\ell}_f(t, \sum_{j=1}^{n} \psi_j(\lambda)),$$

and the probabilities $p_k^{\psi_1,\psi_2,\dots,\psi_n,f}(t)$ satisfy the equation

$$\mathcal{D}_t^f \, p_k^{\psi_1,\psi_2,\dots,\psi_n,f}(t) = -\sum_{j=1}^{n} \psi_j \left(\lambda \left(I - B\right)\right) p_k^{\psi_1,\psi_2,\dots,\psi_n,f}(t), \quad k \geq 0, \; t > 0,$$

with the usual initial conditions.

Proof of Theorem 3 is obtained by the same reasoning as that of Theorem 2, using Proposition 1.

Example 1. Consider the time-changed process

$$N^{GN}(t) = N_1(G_N(t)) = N_1(G(N(t))), \quad t > 0,$$

where $N_1(t)$ is the Poisson process with intensity λ_1, and

$$G_N(t) = G(N(t)), \quad t > 0,$$

is the compound Poisson-Gamma subordinator with parameters λ, α, β, that is, with the Laplace exponent

$$\psi_{GN}(u) = \lambda\beta^\alpha \left(\beta^{-\alpha} - (\beta + u)^{-\alpha}\right).$$

In the case when $\alpha = 1$ we have the compound Poisson-exponential subordinator, which we will denote as $E_N(t)$, and the corresponding time-changed process as $N^E(t)$. The detailed study of the processes $N^{GN}(t)$ and $N^E(t)$ was presented in [3, 4].

We now consider the processes

$$N^{GN,f}(t) = N^{GN}(Y^f(t)) \quad \text{and} \quad N^{E,f}(t) = N^E(Y^f(t)), \quad t \geq 0.$$

The next theorem is obtained as a corollary of Theorem 2.

Theorem 4. *The process $N^{GN,f}(t)$ has the probability distribution function*

$$p_k^{GN,f}(t) = \frac{\left(-\lambda_1 \partial_{\lambda_1}\right)^k}{k!} \tilde{\ell}_f(t, \psi_{GN}(\lambda_1)),$$

and $p_k^{GN,f}(t)$ satisfy the equation

$$\mathcal{D}_t^f p_k^{GN,f}(t) = \left(\frac{\lambda \beta^\alpha}{(\lambda_1 + \beta)^\alpha} - \lambda \right) p_k^{GN,f}(t) + \frac{\lambda \beta^\alpha}{(\lambda_1 + \beta)^\alpha} \sum_{m=1}^{k} \frac{\lambda_1^m}{(\lambda_1 + \beta)^m} \frac{\Gamma(m+\alpha)}{m!\Gamma(\alpha)} p_{k-m}^{GN,f}(t).$$

The process $N^{E,f}(t) = N_1(E_N(Y^f(t)))$ has the probability distribution function

$$p_k^{E,f}(t) = \frac{\left(-\lambda_1 \partial_{\lambda_1}\right)^k}{k!} \tilde{\ell}_f(t, \psi_E(\lambda_1)),$$

where $\psi_E(u) = \frac{\lambda u}{\beta + u}$, and $p_k^{E,f}(t)$ satisfy the equation

$$\mathcal{D}_t^f p_k^{E,f}(t) = -\lambda \frac{\lambda_1}{\lambda_1 + \beta} p_k^{E,f}(t) + \frac{\lambda \beta}{\lambda_1 + \beta} \sum_{m=1}^{k} \left(\frac{\lambda_1}{\lambda_1 + \beta} \right)^m p_{k-m}^{E,f}(t).$$

4 Models of time-changed generalized counting processes

Consider the generalized counting process (GCP) $M(t), t \geq 0$, which was introduced in [7]. The probabilities $\tilde{p}_n(t) = \mathsf{P}\{M(t) = n\}$ depend on k parameters $\lambda_1, \ldots, \lambda_k$ and are given by the formula

$$\tilde{p}_n(t) = \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\left(\lambda_j t\right)^{x_j}}{x_j!} e^{-\Lambda t}, \quad n \geq 0, \tag{34}$$

where $\Omega(k,n) = \left\{ (x_1, \ldots, x_k) \,:\, \sum_{j=1}^k j x_j = n, x_j \in N_0 \right\}$, $\Lambda = \sum_{j=1}^k \lambda_j$.

GCP performs k kinds of jumps of amplitude $1, 2, \ldots, k$ with rates $\lambda_1, \ldots, \lambda_k$. Note that GCP comprises as particular cases such important for applications models as the Poisson process of order k and Pólya–Aeppli process of order k (see, e.g., [11, 10]).

The probability generating function of GCP is given by ([12])

$$\tilde{G}(u,t) = \mathsf{E}u^{M(t)} = \exp\Big\{ -\sum_{j=1}^{k} \lambda_j (1-u^j)t \Big\}, \quad |u| < 1. \tag{35}$$

The probabilities (34) satisfy

$$\frac{d\tilde{p}_n(t)}{dt} = -\Lambda \tilde{p}_n(t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j \tilde{p}_{n-k}(t), \quad n \geq 0,$$

with the usual initial condition. The probabilities $\tilde{p}_n(t)$ can be also written as

$$\tilde{p}_n(t) = \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} (-\partial_\Lambda)^{z_k} e^{-\Lambda t}, \quad n \geq 0, \tag{36}$$

where $z_k = \sum_{j=1}^{k} x_j$ (see [13]).

Recently various models of time-changed GCP and fractional GCP were studied, in particular, with time-change given by a Lévy subordinators, inverse subordinators, including the cases of some specific subordinators, and also the fractional extensions of processes of the form $M(H^\psi(t))$ (see, for example, [7, 10, 12, 11, 13], and references therein).

Following the lines of the previos section, we consider the time-changed process $\mathsf{M}^{\psi,f}(t) = M(H^\psi(Y^f(t)))$, that is, with double time-change by an independent subordinator H^ψ and an inverse subordinator Y^f, which are independent of M. To the best of our knowledge, such general case has not been presented in the literature. In the next theorem we characterize its probabilities $\tilde{p}_n^{\psi,f}(t) = \mathsf{P}\left\{\mathsf{M}^{\psi,f}(t) = n\right\}$ and probability generating function.

Theorem 5. *The process $\mathsf{M}^{\psi,f}$ has the probability distribution function*

$$\tilde{p}_n^{\psi,f}(t) = \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} (-\partial_\Lambda)^{z_k} \tilde{\ell}_f(t, \psi(\Lambda)), \quad n \geq 0, \tag{37}$$

and $\tilde{p}_n^{\psi,f}(t)$ satisfy the equation

$$\mathcal{D}_t^f \tilde{p}_n^{\psi,f}(t) = -\psi(\Lambda)\tilde{p}_n^{\psi,f}(t) - \sum_{m=1}^{n} \sum_{\Omega(k,m)} \psi^{(z_k)}(\Lambda) \prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} \tilde{p}_{n-m}^{\psi,f}(t),$$
$$n \geq 0,\ t > 0, \tag{38}$$

with the initial conditions

$$\tilde{p}_n^{\psi,f}(0) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

The probability generating function of the process $\mathsf{M}^{\psi,f}$ *is of the form*

$$\tilde{G}^{\psi,f}(u,t) = \tilde{\ell}_f\Big(t, \psi\Big(\sum_{j=1}^{k}\lambda_j(1-u^j)\Big)\Big), \quad |u| < 1, \tag{39}$$

and satisfies the equation

$$\mathcal{D}_t^f \tilde{G}^{\psi,f}(u,t) = -\psi\Big(\sum_{j=1}^{k}\lambda_j(1-u^j)\Big)\tilde{G}^{\psi,f}(u,t) \tag{40}$$

with $\tilde{G}^{\psi,f}(u,0) = 1$.

Proof. For calculating $\tilde{p}_n^{\psi,f}(t)$ we use the formula (36):

$$\begin{aligned}\tilde{p}_n^{\psi,f}(t) &= \int_0^\infty \tilde{p}_n(s)P\left\{H^\psi(Y^f(t)) \in ds\right\} = \\ &= \int_0^\infty \sum_{\Omega(k,n)}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}(-\partial_\Lambda)^{z_k} e^{-\Lambda t}P\left\{H^\psi(Y^f(t)) \in ds\right\} = \\ &= \sum_{\Omega(k,n)}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\mathsf{E}\,(-\partial_\Lambda)^{z_k}\exp\left\{-\Lambda H^\psi(Y^f(t))\right\} = \\ &= \sum_{\Omega(k,n)}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}(-\partial_\Lambda)^{z_k}\,\mathsf{E}\exp\left\{-\Lambda H^\psi(Y^f(t))\right\}.\end{aligned}$$

This implies formula (37), since

$$\mathsf{E}\exp\left\{-\Lambda H^\psi(Y^f(t))\right\} = \mathsf{E}\exp\left\{-\psi(\Lambda)Y^f(t)\right\} = \tilde{\ell}_f(t,\psi(\Lambda)).$$

To derive equation (38), we follow the same lines as in the proof of Theorem 2, but now, instead of equation (22) (or (23)) for probabilities of the process $N^\psi = N(H^\psi)$, we use the following equation for the probabilities $\tilde{p}_n^\psi(t)$ of the process $M^\psi = M(H^\psi)$ (see [13]):

$$\frac{d}{dt}\tilde{p}_n^\psi(t) = -\psi(\Lambda)\tilde{p}_n^\psi(t) - \sum_{m=1}^{n}\sum_{\Omega(k,m)}\psi^{z_k}(\Lambda)\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\tilde{p}_{n-m}^\psi(t).$$

Thus, we obtain

$$\begin{aligned}\mathbb{D}_t^f\tilde{p}_n^{\psi,f}(t) = \psi(\Lambda)\tilde{p}_n^{\psi,f}(t) &- \sum_{m=1}^{n}\sum_{\Omega(k,m)}\psi^{z_k}(\Lambda)\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\tilde{p}_{n-m}^{\psi,f}(t)\\ &+ \tilde{p}_n^{\psi,f}(0)\nu_f(t),\end{aligned}$$

and then apply the same reasonings as those after formula (29) to come to (38).

Using the expression for the probability generating function $\tilde{G}^{\psi}(u,t)$ of the time-changed process $M^{\psi}(t) = M(H^{\psi}(t))$ (see [12]),

$$\tilde{G}^{\psi}(u,t) = \exp\Big\{ - t\psi\Big(\sum_{j=1}^{k} \lambda_j (1-u^j)\Big)\Big\},$$

we calculate $\tilde{G}^{\psi,f}$ as

$$\begin{aligned}\tilde{G}^{\psi,f}(u,t) &= \int_0^\infty \tilde{G}^{\psi}(u,s)\ell_f(t,s)ds = \\ &= \int_0^\infty \exp\Big\{ - t\psi\Big(\sum_{j=1}^{k} \lambda_j (1-u^j)\Big)\Big\}\ell_f(t,s)ds = \\ &= \mathsf{E}e^{-\psi\left(\sum_{j=1}^{k} \lambda_j (1-u^j)\right)Y^f(t)} = \tilde{\ell}_f\Big(t, \psi\Big(\sum_{j=1}^{k} \lambda_j (1-u^j)\Big)\Big),\end{aligned}$$

that is, (39) holds. Equation (40) follows in view of (39) and (10). □

Remark 4. Following the same calculation as in [13, formula (5.3)], equation (38) can also be written in the form

$$\mathcal{D}_t^f \tilde{p}_n^{\psi,f}(t) = -\psi\Big(\Lambda\Big(I - \frac{1}{\Lambda}\sum_{j=1}^{n\wedge k} \lambda_j B^j\Big)\Big)\tilde{p}_{n-m}^{\psi,f}(t).$$

Remark 5. Taking $k = 1$ in Theorem 5 we obtain the results of Theorem 2.

Example 2. As a continuation of Example 1, consider GCP with time-change given by a compound Poisson-Gamma subordinator: $M^{GN}(t) = M(G_N(t))$.

Probabilities of this process can be calculated and expressed in terms of special functions.

We will use the fact that the transition probability measure of the process $G_N(t)$ can be written in the closed form ([3])

$$\begin{aligned}P\{G_N(t) \in ds\} &= e^{-\lambda t}\delta_{\{0\}}(ds) + \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t \beta^\alpha)^n}{n!\Gamma(\alpha n)} s^{\alpha n - 1} e^{-\beta s} ds \\ &= e^{-\lambda t}\delta_{\{0\}}(ds) + e^{-\lambda t - \beta s}\frac{1}{s}\Phi\left(\alpha, 0, \lambda t(\beta s)^\alpha\right) ds, \qquad (41)\end{aligned}$$

therefore, the probability law of $G_N(t)$ has atom $e^{-\lambda t}$ at zero, that is, has a discrete part $P\{G_N(t) = 0\} = e^{-\lambda t}$, and the density of the absolutely continuous part is expressed in terms of the Wright function, more precisely, by means of its particular case defined as

$$\Phi(\rho, 0, z) = \sum_{k=1}^{\infty} \frac{z^k}{k!\Gamma(\rho k)}, \quad z \in C,\ \rho \in (-1,0) \cup (0,\infty).$$

Using formula (41), we obtain:

$$\tilde{p}_n^{GN}(t) = \mathsf{P}\{M(G_N(t)) = n\} = \int_0^\infty \mathsf{P}(M(s) = n)\mathsf{P}(G_N(t) \in ds)$$

$$= \int_0^\infty \sum_{\Omega(k,n)} \prod_{j=1}^k \frac{(\lambda_j s)^{x_j}}{x_j!} e^{-\Lambda s}$$

$$\times \left\{ e^{-\lambda t}\delta_{\{0\}}(ds) + e^{-\lambda t - \beta s}\frac{1}{s}\Phi\left(\alpha, 0, \lambda t(\beta s)^\alpha\right)\right\} ds$$

$$= e^{-\lambda t}\int_0^\infty \sum_{\Omega(k,n)} \prod_{j=1}^k \frac{(\lambda_j s)^{x_j}}{x_j!} e^{-\Lambda s} e^{-\beta s}\frac{1}{s}\Phi\left(\alpha, 0, \lambda t(\beta s)^\alpha\right) ds$$

$$= e^{-\lambda t}\sum_{\Omega(k,n)} \int_0^\infty \prod_{j=1}^k \frac{(\lambda_j s)^{x_j}}{x_j!} e^{-(\Lambda+\beta)s}\frac{1}{s}\sum_{l=1}^\infty \frac{(\lambda t\beta^\alpha)^l}{l!\Gamma(\alpha l)} s^{\alpha l - 1} ds$$

$$= e^{-\lambda t}\sum_{\Omega(k,n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\sum_{l=1}^\infty \frac{(\lambda t\beta^\alpha)^l}{l!\Gamma(\alpha l)}\int_0^\infty e^{-(\Lambda+\beta)s} s^{\sum_{j=1}^k x_j + \alpha l - 1} ds$$

$$= e^{-\lambda t}\sum_{\Omega(k,n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\sum_{l=1}^\infty \frac{(\lambda t\beta^\alpha)^l}{l!\Gamma(\alpha l)}\frac{\Gamma(z_k + \alpha l)}{(\Lambda+\beta)^{z_k + \alpha l}}$$

$$= e^{-\lambda t}\sum_{\Omega(k,n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\frac{1}{(\Lambda+\beta)^{z_k}}\sum_{l=1}^\infty \frac{(\lambda t\beta^\alpha)^l}{l!(\Lambda+\beta)^{\alpha l}}\frac{\Gamma(z_k + \alpha l)}{\Gamma(\alpha l)},$$

which can be written in the form

$$\tilde{p}_n^{GN}(t) = e^{-\lambda t}\sum_{\Omega(k,n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\frac{1}{(\Lambda+\beta)^{z_k}}\, {}_1\Psi_1\left((z_k, \alpha), (0, \alpha), \frac{\lambda t\beta^\alpha}{(\Lambda+\beta)^\alpha}\right),$$

where ${}_p\Psi_q$ is the generalized Wright function

$${}_p\Psi_q((a_i, \alpha_i), (b_j, \beta_j), z) = \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)}\frac{z^k}{k!}$$

defined for $z \in C$, $a_i, b_i \in C$, $\alpha_i, \beta_i \in R$, $\alpha_i, \beta_i \neq 0$ and $\sum \alpha_i - \sum \beta_i > -1$ (see, e.g., [9]).

If $\alpha = 1$, that is, $G_N(t) = E_N(t)$, then probabilities take the form

$$\tilde{p}_n^E(t) = e^{-\lambda t}\sum_{\Omega(k,n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\frac{1}{(\Lambda+\beta)^{z_k}}\mathcal{E}_{1,2}^{z_k+1}\left(\frac{\lambda t\beta}{\Lambda+\beta}\right),$$

where $\mathcal{E}_{\rho,\delta}^\gamma$ is the three-parameter generalized Mittag-Leffler function, which is defined as

$$\mathcal{E}^{\gamma}_{\rho,\delta}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}\frac{z^k}{k!\Gamma(\rho k+\delta)},\quad z\in C,\ \rho,\delta,\gamma\in C,$$

with Re$(\rho)>0$, Re$(\delta)>0$, Re$(\gamma)>0$ (see, e.g., [9]). For $n=0$ we have

$$\tilde{p}_0^{GN}(t)=e^{-\lambda t}+e^{-\lambda t}\sum_{l=1}^{\infty}\frac{(\lambda t\beta^{\alpha})^l}{(\Lambda+\beta)^{\alpha l}}\frac{\Gamma(\alpha l)}{l!\Gamma(\alpha l)}=\exp\Big\{-\lambda t\Big(1-\frac{\beta^{\alpha}}{(\Lambda+\beta)^{\alpha}}\Big)\Big\},$$

and, if $\alpha=1$,

$$\tilde{p}_0^{E}(t)=\exp\Big\{-\frac{\lambda\Lambda t}{\Lambda+\beta}\Big\}.$$

Consider now the process $M^{GN,f}(t)=M(G_N(Y^f(t)))$, $t\ge 0$. The next theorem follows from Theorem 5.

Theorem 6. *The process $M^{GN,f}$ has the probability distribution function given by formula (37) with $\psi=\psi_{GN}$ and the probabilities $\tilde{p}_n^{GN,f}(t)$ satisfy the equation*

$$\mathcal{D}_t^f\,\tilde{p}_n^{GN,f}(t)=\left(\frac{\lambda\beta^{\alpha}}{(\Lambda+\beta)^{\alpha}}-\lambda\right)\tilde{p}_n^{GN,f}(t)+$$
$$+\frac{(\lambda\beta)^{\alpha}}{(\Lambda+\beta)^{\alpha}\,\Gamma(\alpha)}\sum_{m=1}^{n}\sum_{\Omega(k,m)}\frac{\Gamma(z_k+\alpha)}{(\Lambda+\beta)^{z_k}}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\tilde{p}_{n-m}^{GN,f}(t).$$

The process $M^{E,f}$ has the probability distribution function given by (37) with $\psi=\psi_{EN}$ and the probabilities $\tilde{p}_n^{E,f}(t)$ satisfy the equation

$$\mathcal{D}_t^f\,\tilde{p}_n^{E,f}(t)=-\lambda\frac{\Lambda}{\Lambda+\beta}\tilde{p}_n^{E,f}(t)+$$
$$+\frac{\lambda\beta}{\Lambda+\beta}\sum_{m=1}^{n}\sum_{\Omega(k,m)}\frac{\Gamma(z_k+1)}{(\Lambda+\beta)^{z_k}}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\tilde{p}_{n-m}^{E,f}(t).$$

Acknowledgement

The authors are grateful to the reviewers for their valuable remarks and suggestions, which helped to improve the paper.

References

[1] Beghin, L., D'Ovidio, M.: Fractional Poisson process with random drift. Electron. J. Probab. **19**, 1–26 (2014) MR3304182. https://doi.org/10.1214/EJP.v19-3258

[2] Beghin, L., Orsingher, E.: Fractional Poisson processes and related planar random motions. Electron. J. Probab. **14**, 1790–1826 (2009) MR2535014. https://doi.org/10.1214/EJP.v14-675

[3] Buchak, K., Sakhno, L.: Compositions of Poisson and Gamma processes. Mod. Stoch. Theory Appl. **4**(2), 161–188 (2017) MR3668780. https://doi.org/10.15559/17-VMSTA79

[4] Buchak, K., Sakhno, L.: Properties of Poisson processes directed by compound Poisson-Gamma subordinators. Mod. Stoch. Theory Appl. **5**(2), 167–189 (2018) MR3813090. https://doi.org/10.15559/18-vmsta101

[5] Buchak, K., Sakhno, L.: On the governing equations for Poisson and Skellam processes time-changed by inverse subordinators. Theory Probab. Math. Stat. **98**, 91–104 (2019) MR3824680. https://doi.org/10.1090/tpms/1064

[6] Chukova, S., Minkova, L.: Pólya-Aeppli of order k risk model. Commun. Stat., Simul. Comput. **44**(3), 551–564 (2015) MR3257726. https://doi.org/10.1080/03610918.2013.784987

[7] Di Crescenzo, A., Martinucci, B., Meoli, A.: A fractional counting process and its connection with the Poisson process. ALEA Lat. Am. J. Probab. Math. Stat. **13**, 291–307 (2016) MR3479478

[8] Garra, R., Orsingher, E., Scavino, M.: Some probabilistic properties of fractional point processes. Stoch. Anal. Appl. **35**(4), 701–718 (2017) MR3651139. https://doi.org/10.1080/07362994.2017.1308831

[9] Haubold, H.J., Mathai, A.M., Saxena, R.K.: Mittag-Leffler functions and their applications. J. Appl. Math. **51** (2011) MR2800586. https://doi.org/10.1155/2011/298628

[10] Kadankova, T., Leonenko, N., Scalas, E.: Fractional non-homogeneous Poisson and Pólya-Aeppli processes of order k and beyond. Commun. Stat., Theory Methods **52**(8), 2682–2701 (2023) MR4559257. https://doi.org/10.1080/03610926.2021.1958228

[11] Kataria, K.K., Khandakar, M.: Generalized fractional counting process. J. Theor. Probab. **35**(4), 2784–2805 (2022) MR4509086. https://doi.org/10.1007/s10959-022-01160-6

[12] Kataria, K.K., Khandakar, M.: Skellam and time-changed variants of the generalized fractional counting process. Fract. Calc. Appl. Anal. **25**, 1873–1907 (2022) MR4493600. https://doi.org/10.1007/s13540-022-00091-7

[13] Kataria, K.K., Khandakar, M., Vellaisamy, P.: Generalized counting process: its non-homogeneous and time-changed versions. Preprint arXiv:2210.03981 (2022)

[14] Kochubei, A.N.: General fractional calculus, evolution equations, and renewal processes. Integral Equ. Oper. Theory **71**(4), 583–600 (2011) MR2854867. https://doi.org/10.1007/s00020-011-1918-8

[15] Kostadinova, K., Minkova, L.: On the Poisson process of order k. Pliska Stud. Math. Bulgar. **22**, 117–128 (2019) MR3203700

[16] Leonenko, N., Scalas, E., Trinh, M.: The fractional non-homogeneous Poisson process. Stat. Probab. Lett. **120**, 147–156 (2017) MR3567934. https://doi.org/10.1016/j.spl.2016.09.024

[17] Meerschaert, M.M., Toaldo, B.: Relaxation patterns and semi-Markov dynamics. Stoch. Process. Appl. **129**(8), 2850–2879 (2019) MR3980146. https://doi.org/10.1016/j.spa.2018.08.004

[18] Meerschaert, M.M., Nane, E., Vellaisamy, P.: The fractional Poisson process and the inverse stable subordinator. Electron. J. Probab. **16**(59), 1600–1620 (2011) MR2835248. https://doi.org/10.1214/EJP.v16-920

[19] Orsingher, E., Polito, F.: The space-fractional Poisson process. Stat. Probab. Lett. **82**, 852–858 (2012) MR2899530. https://doi.org/10.1016/j.spl.2011.12.018

[20] Orsingher, E., Polito, F.: On the integral of fractional Poisson processes. Stat. Probab. Lett. **83**(4), 1006–1017 (2013) MR3041370. https://doi.org/10.1016/j.spl.2012.12.016

[21] Orsingher, E., Toaldo, B.: Counting processes with Bernštein intertimes and random jumps. J. Appl. Probab. **52**, 1028–1044 (2015) MR3439170. https://doi.org/10.1239/jap/1450802751

[22] Toaldo, B.: Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups. Potential Anal. **42**, 115–140 (2015) MR3297989. https://doi.org/10.1007/s11118-014-9426-5