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Sample path properties of multidimensional integral with respect to stochastic measure

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Abstract The integral with respect to a multidimensional stochastic measure, assuming only its σ-additivity in probability, is studied. The continuity and differentiability of realizations of the integral are established.

Keywords Stochastic measure, trajectories of random functions, multivariable Haar functions, Besov space

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1 Introduction

The main purpose of this article is to study the regularity property of the integral of a deterministic function with respect to stochastic measure μ, where μ is defined on Borel subsets of $[0, 1]^d$. In basic statements of the paper, for μ we assume only σ-additivity in probability. We study the regularity with respect to a parameter and with respect to the upper limit of the integral. The case $d = 1$ was considered in [20, Section 2.3], but methods of [20] do not work for $d \geq 2$. In our paper, we study the integral using the Fourier–Haar expansion of the integrand.

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Sample path behavior of various stochastic processes was studied in many works; we mention only some of them. The conditions for sample boundedness and continuity of α-stable processes were established in [24, Chapter 10]. Hölder continuity of harmonizable operator scaling stable random field is given by Corollary 5.5 [5]. In Theorem 4.1 from [18] the uniform modulus of continuity for some self-similar $S\alpha S$ random field was obtained, while the upper bound of the modulus of continuity of stable random fields is given by Proposition 5.1 and Corollary 5.3 [6]. These results, for example, imply Hölder continuity of mentioned fields.

Conditions of continuity of Gaussian random fields may be found in Theorems 3.3.3, 3.4.1 and 3.4.3 [1]. Hölder continuity of centered Gaussian random fields under certain conditions is proved in [30, Theorem 4.2]. Theorem 2.1 [31] gives an estimate of the modulus of continuity of general random fields on metric space, these results are applied to harmonizable stable random fields.

In our article, we consider stochastic processes which are integrals with respect to a general stochastic measure. The definition of these measures, their properties and examples as well as information about the Fourier–Haar series may be found in Section 2. The continuity and differentiability of sample paths of the parameter-dependent integral are established in Section 3. The continuity of realizations of the integral as a function of the upper limit is studied in Section 4.

2 Preliminaries

2.1 Stochastic measures

In this subsection, we give basic information concerning stochastic measures in a general setting. In statements of Sections 3 and 4, this set function is defined on Borel subsets of $[0,1]^d$.

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and $\mathcal{B}$ a σ-algebra of subsets of X.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

In other words,

$$\mu(A \cup B) = \mu(A) + \mu(B) \text{ for } A \cap B = \emptyset,$$
$$\mu(A_n) \xrightarrow{\mathsf{P}} 0 \text{ for } A_n \downarrow \emptyset. \tag{1}$$

We do not assume the moment existence or martingale properties for SM. We can say that μ is an L_0-valued measure.

We note the following examples of SMs in multidimensional spaces. All measures, if nothing else is mentioned, are considered on Borel σ-algebra of sets in $[0,1]^d$.

Example 1. Let μ be a random orthogonal measure – it is defined, for example, in [9, Section 5.3] and [14, Section 2.3] – with a finite structural function. Then μ is an SM in the sense of Definition 1; condition (1) follows from [9, Section 5.3, Theorem 1(d)]. For example, the set function $\mu(A) = \int_{[0,1]^d} \mathbf{1}_A(t)\, dW(t)$, where we take multiple Wiener–Itô integral (see its properties, for example, in [17, Section 1.1]), is a random orthogonal measure with the structural function $\mathsf{m}(A) = d!\lambda_d(A)$, where λ_d is the d-dimensional Lebesgue measure.

Example 2. Let μ be an independently scattered α-stable random measure for $\alpha \in (0, 2]$ with a finite control measure. According to the definition – see [24, Definition 3.3.1] – for disjoint Borel sets $\{A_n : n \geq 1\}$ the following holds:

$$\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \quad \text{a.s.}$$

Therefore, μ is an SM.

Example 3. Consider the set function $\mu : \mathcal{B}([0, 1]^d) \to \mathsf{L}_0$ such that

$$\mu(A) = \int_{\mathbb{R}^d} \mathbf{1}_A(t)\, dZ_H^q(t), \tag{2}$$

where Z_H^q is the Hermite sheet (see, for example, [7] or [29, Section 4]), $H = (H_1, \ldots, H_d)$, $1/2 < H_i < 1$. Under these conditions, integral in (2) is well defined, and μ is an SM in the sense of Definition 1; the statement (1) can be proved using the representation of μ via Wiener integral (see [29, (4.12)]). Proposition 3 of [7] implies the Hölder continuity of μ.

If $q = 1$ then the Hermite sheet coincides with the fractional Brownian sheet.

Theorem 10.1.1 of [15] states the sufficient conditions under which the product of one-dimensional SMs generates an SM.

SMs may be used for study of stochastic dynamical systems (see [3]).

For deterministic measurable functions $f : \mathsf{X} \to \mathbb{R}$, an integral of the form $\int_{\mathsf{X}} f\, d\mu$ is studied in [15, Chapter 7], [20, Chapter 1]. In particular, every bounded measurable f is integrable w. r. t. any μ. An analogue of the Lebesgue dominated convergence theorem holds for this integral (see [15, Proposition 7.1.1] or [20, Theorem 1.5]).

Below we will use the following statement for SMs defined on arbitrary σ-algebra $\mathcal{B}$.

Lemma 1 (Lemma 3.1 [19])**.** *Let μ be an SM on $(\mathsf{X}, \mathcal{B})$, $f_k : \mathsf{X} \to \mathbb{R}$, $k \geq 1$, be measurable functions such that $\hat{f}(x) = \sum_{k=1}^{\infty} |f_k(x)|$ is integrable w. r. t. μ. Then*

$$\sum_{k=1}^{\infty} \Big(\int_{\mathsf{X}} f_k\, d\mu \Big)^2 < \infty \quad \textit{a.s.}$$

2.2 Besov spaces

We recall the definition of Besov spaces $B^{\alpha}_{p,p}([0, 1]^d)$ following [8] and [12]. Other approaches to the definition and properties of these spaces are considered in [4], [23, Sections 2.1 and 2.4], [27], [28, Sections 2 and 5]. The equivalence of different definitions is discussed in [8], [16, Section 17.2], [23, Section 2.3].

For functions $f \in \mathsf{L}_p([0, 1]^d)$, we consider the value

$$\| f \|_{B^{\alpha}_{p,p}([0,1]^d)} = \| f \|_{\mathsf{L}_p([0,1]^d)} + \Big(\int_0^1 (\omega_p(f, r))^p r^{-\alpha p - 1}\, dr \Big)^{1/p},$$

where ω_p denotes the L_p-modulus of continuity,

$$\omega_p(f, r) = \sup_{|h|\le r} \Big(\int_{I_h} |f(x+h) - f(x)|^p \, dx \Big)^{1/p},$$
$$\text{where} \quad I_h = \{x \in [0,1]^d : x + h \in [0,1]^d\},$$

and $|h|$ denotes the Euclidean norm in $\mathbb{R}^d$. Then

$$B^{\alpha}_{p,p}([0,1]^d) = \{f \in \mathsf{L}_p([0,1]^d) : \|f\|_{B^{\alpha}_{p,p}([0,1]^d)} < +\infty\},$$

and $\|\cdot\|_{B^{\alpha}_{p,p}([0,1]^d)}$ is a norm in this space.

Let μ be an SM defined on the Borel σ-algebra of $[0,1]^d$. For $x = (x_1, x_2, \dots, x_d) \in [0,1]^d$ set

$$\mu(x) = \mu\Big(\prod_{i=1}^{d} [0, x_i]\Big).$$

In the sequel, we will refer to the following assumption on stochastic measure μ.

*Assumption A*1. There exists a real-valued finite measure m on $(\mathsf{X}, \mathcal{B})$ with the following property: if a measurable function $g : \mathsf{X} \to \mathbb{R}$ satisfies $\int_{\mathsf{X}} g^2 \, d\mathsf{m} < +\infty$ then g is integrable w. r. t. μ on X.

For orthogonal measure, we can take its structural measure as m. For an α-stable random measure, Assumption A1 holds for the control measure m (see (3.4.1) [24]). For an SM from Example 3 we can take measure m such that

$$\mathsf{m}(A) = \int_A dv \int_{[0,1]^d} \prod_{i=1}^{d} |u_i - v_i|^{2H_i - 2} du,$$

where u and v are d-dimensional vectors with the components u_i, v_i.

We have the following statement concerning the Besov regularity of SMs defined on Borel subsets of $[0,1]^d$.

Theorem 1 (Theorem 5.1 [19]). *Let the random function*

$$\mu(x) = \mu\Big(\prod_{i=1}^{d} [0, x_i]\Big), \quad x \in [0,1]^d,$$

have continuous paths and Assumption A1 hold.

Then for any $1 \le p < +\infty$, $0 < \alpha < \min\{1/p, 1/2\}$ *the realization* $\mu(x)$, $x \in [0,1]^d$, *with probability 1 belongs to the Besov space* $B^{\alpha}_{p,p}([0,1]^d)$.

2.3 Haar functions

In order to construct a version of the stochastic integral, we use the approximation of the integrand with Fourier–Haar series. In [21] Fourier–Haar series were already used in approximation of the solution to stochastic wave equation when $d = 1$.

We follow the definition of one-dimensional Haar functions $\chi_n(x)$, $x \in [0,1]$, from [13, Section 3]. For $n = 2^j + i$, $1 \le i \le 2^j$, $j \ge 0$, we write

$$
\begin{aligned}
&\Delta_n = \Delta_j^i = \big((i-1)2^{-j}, i2^{-j}\big), \quad \overline{\Delta}_n = \big[(i-1)2^{-j}, i2^{-j}\big],\\
&\Delta_1 = \Delta_0^0 = (0,1), \quad \overline{\Delta}_1 = [0,1],\\
&\Delta_n^+ = \big(\Delta_j^i\big)^+ = \big((i-1)2^{-j}, (2i-1)2^{-j-1}\big) = \Delta_{j+1}^{2i-1},\\
&\Delta_n^- = \big(\Delta_j^i\big)^- = \big((2i-1)2^{-j-1}, i2^{-j}\big) = \Delta_{j+1}^{2i}.
\end{aligned}
\tag{3}
$$

Now let $\chi_1 = 1$, and

$$
\chi_n(x) = \begin{cases} 0, & x \notin \overline{\Delta}_n,\\ 2^{j/2}, & x \in \Delta_n^+,\\ -2^{j/2}, & x \in \Delta_n^-, \end{cases}
$$

for $2^j + 1 \le n \le 2^{j+1}$. In discontinuity points and at the endpoints of $[0,1]$ we define

$$
\begin{aligned}
&\chi_n(x) = \big(\chi_n(x-) + \chi_n(x+)\big)/2, \quad x \in (0,1),\\
&\chi_n(0) = \chi_n(0+), \quad \chi_n(1) = \chi_n(1-).
\end{aligned}
\tag{4}
$$

For $g \in \mathsf{L}_1([0,1])$, we define the Fourier–Haar coefficients and sums in a standard way:

$$
\begin{aligned}
&c_n(g) = \int_{[0,1]} g(x)\chi_n(x)\,dx = 2^{j/2}\Big(\int_{\Delta_n^+} g(x)\,dx - \int_{\Delta_n^-} g(x)\,dx\Big),\\
&S_N(g,x) = \sum_{n=1}^{N} c_n(g)\chi_n(x).
\end{aligned}
$$

We will approximate g taking the Fourier–Haar sums for values $N = 2^k$, $k \in \mathbb{Z}_+$.

By (III.8), (III.8′) of [13], for $x \ne i2^{-k}$, $0 \le i \le 2^k$, it holds

$$
S_{2^k}(g,x) = \sum_{i=1}^{2^k} 2^k \mathbf{1}_{\Delta_k^i}(x) \int_{\Delta_k^i} g(t)\,dt,
$$

and

$$
\begin{aligned}
&S_{2^k}\Big(g, \frac{i}{2^k}\Big) = 2^{k-1}\Big(\int_{\Delta_k^i} g(t)\,dt + \int_{\Delta_k^{i+1}} g(t)\,dt\Big), \quad i = 1, 2, \dots 2^k - 1,\\
&S_{2^k}(g,0) = 2^k \int_{\Delta_k^1} g(t)\,dt, \quad S_{2^k}(g,1) = 2^k \int_{\Delta_k^{2^k}} g(t)\,dt.
\end{aligned}
$$

For integrable functions $f : [0,1]^d \to \mathbb{R}$ we will use approximation by multivariate Fourier–Haar sums.

The multivariate Haar functions are defined with equalities

$$
\chi^{(d)}_{n_1,\dots,n_d}(x) = \chi_{n_1}(x_1)\chi_{n_2}(x_2)\dots\chi_{n_d}(x_d),
$$

where χ_{n_s} are one-dimensional Haar functions, $x = (x_1, \ldots, x_d) \in [0, 1]^d$. Consider the Fourier–Haar coefficients

$$c^{(d)}_{n_1,\ldots,n_d}(f) = \int_{[0,1]^d} f(x)\chi^{(d)}_{n_1,\ldots,n_d}(x)\,dx,$$

and sums

$$S^{(d)}_{2^k}(f, x) = \sum_{n_1,\ldots,n_d \in \{1,2,3,\ldots 2^k\}} c^{(d)}_{n_1,\ldots,n_d}(f)\chi^{(d)}_{n_1,\ldots,n_d}(x).$$

In the sequel, we denote by $\mathbb{N}_k$ the set of elements $(n_1, \ldots, n_d)$ such that $1 \le n_s \le 2^k$ for all s, $1 \le s \le d$. In other words, $\mathbb{N}_k = \{1, 2, 3, \ldots, 2^k - 1, 2^k\}^d$.

The following statement on the uniform convergence of multivariate Fourier–Haar sums for continuous functions follows from formula (14) in [22, Lemma 2] for $p = \infty$. For $d = 1$ this fact may be found, for example, in [13, Theorem III.2]. A similar statement for multivariate periodic functions was proved in [2].

Lemma 2. *If $f \in \mathbb{C}([0, 1]^d)$ then*

$$\sup_{x\in[0,1]^d} |S^{(d)}_{2^k}(f, x) - f(x)| \to 0, \quad k \to \infty.$$

3 Parameter dependent integral

In this section, we study the properties of random function

$$\eta(z) = \int_{[0,1]^d} f(z, x)\,d\mu(x), \quad z \in Z, \tag{5}$$

where Z is a metric space. Note that a parameter dependent stochastic integral w.r.t. Brownian sheet was considered in [25, 26].

If f is continuous in x then Lemma 2 and analogue of the dominated convergence theorem ([20, Theorem 1.5]) imply that for each fixed z

$$\int_{[0,1]^d} f(z, x)\,d\mu(x) = \operatorname*{p\text{-}lim}_{k\to\infty} \int_{[0,1]^d} S^{(d)}_{2^k}(f, x)\,d\mu(x),$$

where p-lim is considered as the limit in probability. Therefore,

$$\tilde{\eta}(z) = \int_{[0,1]^d} S^{(d)}_1(f, x)\,d\mu(x) + \sum_{k=1}^{\infty} \int_{[0,1]^d} (S^{(d)}_{2^k}(f, x) - S^{(d)}_{2^{k-1}}(f, x))\,d\mu(x) \tag{6}$$

is the version of $\eta(z)$.

In (6) we have integrals of simple functions, and each integral is equal to a respective linear combination of values of μ. We fix the same version of μ for all these values.

We will use notation $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 stays on i-th position, $e_i \in \mathbb{R}^d$.

Lemma 3. *Assume that there exists $l \geq 0$ such that function $f(z,x): Z \times [0,1]^d \to \mathbb{R}$ is continuously differentiable l times on $[0,1]^d$ for each z, and*

$$\Big|\frac{\partial^r f(z,x)}{\partial x_{s_1} \dots \partial x_{s_r}}\Big| \leq C_f, \quad x \in [0,1]^d, \tag{7}$$

for some constant $C_f > 0$, which is independent of z, and all $0 \leq r \leq l$, $s_1, \dots, s_r \subset \{1, \dots, d\}$. Moreover, let l-th derivatives be Hölder continuous with an exponent $\alpha > 0$, i. e.

$$\Big|\frac{\partial^l f(z,x^{(1)})}{\partial x_{s_1} \dots \partial x_{s_l}} - \frac{\partial^l f(z,x^{(2)})}{\partial x_{s_1} \dots \partial x_{s_l}}\Big| \leq C_f |x^{(1)} - x^{(2)}|^\alpha. \tag{8}$$

If $l + \alpha > d/2$, then the version (6) satisfies the inequality

$$|\tilde{\eta}(z)| \leq C_f C_\mu^{(d)}(\omega),$$

where $C_\mu^{(d)}(\omega)$ denotes a random constant that depends only of dimension d and SM μ, $C_\mu^{(d)}(\omega) < \infty$ a.s.

Moreover, series (6) converges absolutely and uniformly, i. e.

$$\sum_{k=1}^{\infty} \sup_{z \in Z} \Big| \int_{[0,1]^d} (S_{2^k}^{(d)}(f,x) - S_{2^{k-1}}^{(d)}(f,x))\, d\mu(x) \Big| \leq C_f C_\mu^{(d)}(\omega). \tag{9}$$

Proof. We have that

$$\int_{[0,1]^d} S_{2^k}^{(d)}(f,x)\, d\mu(x) = \sum_{(n_1,\dots,n_d) \in \mathbb{N}_k} c_{n_1,\dots,n_d}^{(d)}(f) \int_{[0,1]^d} \chi_{n_1,\dots,n_d}^{(d)}(x)\, d\mu(x). \tag{10}$$

For each $n_s \geq 2$, take j_s such that $2^{j_s} + 1 \leq n_s \leq 2^{j_s+1}$, $j_s \geq 0$. Denote by $\{j^{(k)}, 1 \leq k \leq d\}$ the permutation of $\{j_k, 1 \leq k \leq d\}$ such that

$$j^{(1)} \leq j^{(2)} \leq \dots \leq j^{(d)}.$$

If $n_{s_1} = \dots = n_{s_i} = 1$, we set $j^{(1)} = \dots = j^{(i)} = 0$. We rewrite the expression for $c_{n_1,\dots,n_d}^{(d)}(f)$ in the following way:

$$\begin{aligned}
c_{n_1,\dots,n_d}^{(d)}(f) &= \int_{[0,1]^d} f(z,t) \chi_{n_1,\dots,n_d}^{(d)}(t)\, dt = \int_{[0,1]^d} f(z,t) \prod_{1 \leq s \leq d} \chi_{n_s}(t_s)\, dt \\
&= 2^{\sum_{1 \leq s \leq d, n_s \geq 2} j_s/2} \int_{[0,1]^d} f(z,t) \prod_{1 \leq s \leq d, n_s \geq 2} \Big(\mathbf{1}_{\Delta_{n_s}^+}(t_s) - \mathbf{1}_{\Delta_{n_s}^-}(t_s)\Big)\, dt \\
&= 2^{\sum_{1 \leq s \leq d, n_s \geq 2} j_s/2} \int_{[0,1]^d} f(z,t) \prod_{1 \leq s \leq d, n_s \geq 2} \Big(\mathbf{1}_{\Delta_{n_s}^+}(t_s) - \mathbf{1}_{\Delta_{n_s}^+}(t_s - 2^{-j_s - 1})\Big)\, dt \\
&\overset{(*)}{=} 2^{\sum_{1 \leq s \leq d, n_s \geq 2} j_s/2} \\
&\quad \times \int_{[0,1]^d} \sum_{\varepsilon_s \in \{0,1\}, n_s \geq 2} (-1)^{\varepsilon_1 + \dots + \varepsilon_d} f(z, t_1 + \varepsilon_1 2^{-j_1 - 1}, \dots, t_d + \varepsilon_d 2^{-j_d - 1})
\end{aligned}$$

$$\times \prod_{1\le s\le d, n_s\ge 2} \mathbf{1}_{\Delta_{n_s}^+}(t_s)\, dt = 2^{\sum_{1\le s\le d, n_s\ge 2} j_s/2}$$
$$\times \int_{\prod_{1\le s\le d, n_s\ge 2}\Delta_{n_s}^+} \sum_{\varepsilon_s\in\{0,1\}, n_s\ge 2} (-1)^{\varepsilon_1+\cdots+\varepsilon_d}$$
$$\times f(z, t_1+\varepsilon_1 2^{-j_1-1}, \ldots, t_d+\varepsilon_d 2^{-j_d-1})\, dt. \tag{11}$$

In equality (*) we opened the brackets in the product

$$\prod_{1\le s\le d, n_s\ge 2}\Big(\mathbf{1}_{\Delta_{n_s}^+}(t_s) - \mathbf{1}_{\Delta_{n_s}^+}(t_s - 2^{-j_s-1})\Big)$$

and used the change of the variables $t_s - 2^{-j_s-1} \to t_s$ in $\mathbf{1}_{\Delta_{n_s}^+}(t_s - 2^{-j_s-1})$.

The last sum in (11) can be represented as the sum of at most 2^{d-1} summands of the form

$$f(z,t^*) - f(z, t^* + 2^{-j^{(d)}-1} e_{s_d}) = \int_0^{2^{-j^{(d)}-1}} \frac{\partial f(z, t^*+h_{s_d})}{\partial t_{s_d}} dh_{s_d}, \quad j^{(d)} = j_{s_d}.$$

Now we repeat the same thoughts $l-1$ times and use the properties of a function f, which leads to the inequality

$$|c_{n_1,\ldots,n_d}^{(d)}(f)| \le C_f 2^{d-l-1} 2^{\sum_{1\le s\le d, n_s\ge 2} -j_s/2} 2^{-(j^{(d)}+\cdots+j^{(d-l+1)}+\alpha j^{(d-l)})}. \tag{12}$$

Further, we have the estimate

$$\sum_{k=1}^{\infty} \sup_{z\in Z}\Big|\int_{[0,1]^d} (S_{2^k}^{(d)}(f,x) - S_{2^{k-1}}^{(d)}(f,x))\, d\mu(x)\Big|$$
$$\le \sum_{k=1}^{\infty} \sup_{z\in Z}\Big| \sum_{(n_1,\ldots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} c_{n_1,\ldots,n_d}^{(d)}(f) \int_{[0,1]^d} \chi_{n_1,\ldots,n_d}^{(d)}(x)\, d\mu(x)\Big|$$
$$\overset{(12)}{\le} 2^d C_f \sum_{k=1}^{\infty} \sum_{(n_1,\ldots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} 2^{\sum_{1\le s\le d, n_s\ge 2} -j_s/2} 2^{-(j^{(d)}+\cdots+j^{(d-l+1)}+\alpha j^{(d-l)})}$$
$$\times \Big|\int_{[0,1]^d} \chi_{n_1,\ldots,n_d}^{(d)}(x)\, d\mu(x)\Big|$$
$$\le 2^d C_f \Big(\sum_{k=1}^{\infty} \sum_{(n_1,\ldots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} 2^{\beta\sum_{1\le s\le d, n_s\ge 2} j_s} 2^{-2(j^{(d)}+\cdots+j^{(d-l+1)}+\alpha j^{(d-l)})}\Big)^{1/2}$$
$$\times \Big(\sum_{k=1}^{\infty} \sum_{(n_1,\ldots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} 2^{(-\beta-1)\sum_{1\le s\le d, n_s\ge 2} j_s} \Big(\int_{[0,1]^d} \chi_{n_1,\ldots,n_d}^{(d)}(x)\, d\mu(x)\Big)^2\Big)^{1/2}$$
$$:= 2^d C_f P_1^{1/2} P_2^{1/2}.$$

For a sufficiently small $\beta > 0$ we have the following estimate for P_1:

$$P_1 = \sup_{k\ge 1} \sum_{(n_1,\ldots,n_d)\in\mathbb{N}_k} 2^{\beta\sum_{1\le s\le d, n_s\ge 2} j_s} 2^{-2(j^{(d)}+\cdots+j^{(d-l+1)}+\alpha j^{(d-l)})}$$

$$
\begin{aligned}
&= \sup_{k\geq 1} \sum_{A\subset\{1,\dots,d\}} \sum_{\substack{0\leq j_i\leq k,\\ i\in A}} 2^{(\beta+1)\sum_{i\in A} j_i - 2(j^{(d)}+\dots+j^{(d-l+1)}+\alpha j^{(d-l)})} \\
&\leq 2^d \sup_{k\geq 1} \sum_{0\leq j_i\leq k} 2^{(\beta+1-2(l+\alpha)/d)\sum_{i=1}^d j_i} = 2^d C. \qquad (13)
\end{aligned}
$$

Here we denoted by A the set of indexes i, for which $n_s = 1$. It is left to prove that for each fixed $\beta > 0$, $P_2 \leq C_\mu^{(d)}(\omega)$. As follows from Lemma 1, it is sufficient to show that

$$
\tilde{P}_2 = \sum_{k=1}^{\infty} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} 2^{(-\beta/2-1/2)\sum_{1\leq s\leq d, n_s\geq 2} j_s} |\chi^{(d)}_{n_1,\dots,n_d}(x)| \leq C^{(d)}, \qquad (14)
$$

where by $C^{(d)}$ we denote positive constant that depends only on d. The inequality (14) holds true, as is proved below.

$$
\begin{aligned}
\tilde{P}_2 &= \sup_{k\geq 1} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k} 2^{(-\beta/2-1/2)\sum_{1\leq s\leq d, n_s\geq 2} j_s} |\chi_{n_1}(x_1)\dots\chi_{n_d}(x_d)| \\
&\leq \sup_{k\geq 1} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k} 2^{(-\beta/2)\sum_{1\leq s\leq d, n_s\geq 2} j_s} \\
&\quad \times \prod_{s=1}^{d} \Big(\mathbf{1}_{\Delta_{n_s}}(x_s) + \frac{1}{2}\mathbf{1}_{\{(i-1)2^{-j_s}\}}(x_s) + \frac{1}{2}\mathbf{1}_{\{i2^{-j_s}\}}(x_s) \Big) \\
&\overset{(*)}{\leq} \sup_{k\geq 1} \sum_{A\subset\{1,\dots,d\}} \sum_{\substack{0\leq j_s\leq k,\\ s\in A}} 2^{(-\beta/2)\sum_{s\in A} j_s} \leq 2^d \sup_{k\geq 1} \sum_{0\leq j_s\leq k} 2^{(-\beta/2)\sum_{s=1}^d j_s} = C^{(d)}.
\end{aligned}
$$

Here in inequality (*) we used that for each set $j_1, \dots, j_d$, for each x, $x_s \neq i2^{-j_s}$, exists exactly one set $(n_1, \dots, n_d)$ such that $\chi^{(d)}_{n_1,\dots,n^d}(x) \neq 0$ and $2^{j_s} + 1 \leq n_s \leq 2^{j_s+1}$. If we have coordinates $x_s = i2^{-j_s}$, we take into account that $\mathbf{1}_{\{i2^{-j_s}\}}(x_s)$ has the coefficient $1/2$, as follows from (4). Now the statement of the lemma is a consequence of (13) and (14). $\square$

Remark 1. Assume that constants in inequalities (7) and (8) depend not only on f, but also on z. Then the series (6) converges a.s. for each fixed $z \in Z$ and the following analogue of (9):

$$
\sum_{k=1}^{\infty} \Big| \int_{[0,1]^d} (S^{(d)}_{2^k}(f,x) - S^{(d)}_{2^{k-1}}(f,x))\, d\mu(x) \Big| \leq C_{f,z} C^{(d)}_\mu(\omega).
$$

This statement is proved similarly to Lemma 3. We just refer to $C_{f,z}$ everywhere instead of C_f.

The following statement gives the conditions of the continuity with respect to parameter $z \in Z$.

Theorem 2. *Let Z be a metric space, and the function $f(z, x) : Z \times [0, 1]^d \to \mathbb{R}$ be such that all paths $f(\cdot, x)$ are continuous on Z for each x, $f(z, \cdot)$ is continuously differentiable l times on $[0, 1]^d$ for fixed $l \geq 0$ and all $z \in Z$, while*

$$\Big|\frac{\partial^r f(z, x)}{\partial x_{s_1} \dots \partial x_{s_r}}\Big| \leq C_f, \quad x \in [0, 1]^d,$$

for some constant $C_f > 0$ (independent of z) and all $0 \leq r \leq l$, $s_1, \dots, s_r \subset \{1, \dots, d\}$. Let l-th derivatives be Hölder continuous with the exponent α and

$$\Big|\frac{\partial^l f(z, x^{(1)})}{\partial x_{s_1} \dots \partial x_{s_l}} - \frac{\partial^l f(z, x^{(2)})}{\partial x_{s_1} \dots \partial x_{s_l}}\Big| \leq C_f |x^{(1)} - x^{(2)}|^\alpha.$$

If, in addition, $l + \alpha > d/2$, then the random function η defined by (5) has a version (6) with continuous paths on Z a.s.

Proof. From (11) it follows that

$$c^{(d)}_{n_1,\dots,n_d}(f) = 2^{\sum_{1\leq s\leq d, n_s\geq 2} j_s/2} \int_{\prod_{1\leq s\leq d, n_s\geq 2} \Delta^+_{n_s}} \sum_{\varepsilon_s\in\{0,1\}, n_s\geq 2} (-1)^{\varepsilon_1+\dots+\varepsilon_d}$$
$$\times f(z, t_1 + \varepsilon_1 2^{-j_1-1}, \dots, t_d + \varepsilon_d 2^{-j_d-1})\, dt,$$

and coefficients $c^{(d)}_{n_1,\dots,n_d}$ are continuous functions of variable z. Now we can see from representation (10) that $\int_{[0,1]^d} S^{(d)}_{2^k}(f, x)\, d\mu(x)$ is continuous in variable z for all $k \geq 1$, $\omega \in \Omega$.

Finally we refer to Lemma 3 and obtain that $\int_{[0,1]^d} S^{(d)}_{2^k}(f, x)\, d\mu(x)$ converges to $\tilde{\eta}(z)$ uniformly a.s. as $k \to \infty$, that implies the statement of our theorem. □

Theorem 3. *Let the function $f(z, x) : Z \times [0, 1]^d \to \mathbb{R}$, $Z = [a, b]$, be continuous differentiable $l + 1$ times on $Z \times [0, 1]^d$ while*

$$\Big|\frac{\partial^{r+1} f(z, x)}{\partial z \partial x_{s_1} \dots \partial x_{s_r}}\Big| \leq C_f, \quad x \in [0, 1]^d,$$

for all $0 \leq r \leq l$, $s_1, \dots, s_r \subset \{1, \dots, d\}$. Moreover, let $l + 1$-th derivatives be Hölder continuous with the exponent $\alpha > 0$, i. e.

$$\Big|\frac{\partial^{l+1} f(z, x^{(1)})}{\partial z \partial x_{s_1} \dots \partial x_{s_l}} - \frac{\partial^{l+1} f(z, x^{(2)})}{\partial z \partial x_{s_1} \dots \partial x_{s_l}}\Big| \leq C_f |x^{(1)} - x^{(2)}|^\alpha.$$

If, in addition, $l + \alpha > d/2$, then paths of a random function $\tilde{\eta}$, which is defined by (6), have bounded derivatives on Z,

$$\frac{d\tilde{\eta}(z)}{dz} = \int_{[0,1]^d} \frac{\partial f(z, x)}{\partial z}\, d\mu(x).$$

Proof. Lemma 3 implies that

$$\int_{[0,1]^d} \frac{\partial f(z, x)}{\partial z}\, d\mu(x) = \int_{[0,1]^d} S^{(d)}_1\Big(\frac{\partial f}{\partial z}, x\Big)\, d\mu(x)$$

$$+\sum_{k=1}^{\infty}\int_{[0,1]^d}\Big(S_{2^k}^{(d)}\Big(\frac{\partial f}{\partial z},x\Big)-S_{2^{k-1}}^{(d)}\Big(\frac{\partial f}{\partial z},x\Big)\Big)\,d\mu(x),$$

where the series on the right-hand side converges uniformly. According to equalities

$$\begin{aligned}
&\int_{[0,1]^d} S_{2^k}^{(d)}\Big(\frac{\partial f}{\partial z},x\Big)\,d\mu(x)\\
&\quad=\sum_{(n_1,\dots,n_d)\in\mathbb{N}_k} c_{n_1,\dots,n_d}^{(d)}\Big(\frac{\partial f}{\partial z}\Big)\int_{[0,1]^d}\chi_{n_1,\dots,n_d}^{(d)}(x)\,d\mu(x)\\
&\quad\overset{(11)}{=}\sum_{(n_1,\dots,n_d)\in\mathbb{N}_k}\frac{\partial c_{n_1,\dots,n_d}^{(d)}(f)}{\partial z}\int_{[0,1]^d}\chi_{n_1,\dots,n_d}^{(d)}(x)\,d\mu(x)\\
&\quad=\frac{d}{dz}\int_{[0,1]^d} S_{2^k}^{(d)}(f,x)\,d\mu(x)
\end{aligned}$$

and Remark 1, we can differentiate the series in (6) and obtain that

$$\begin{aligned}
\frac{d\tilde{\eta}(z)}{dz}&=\frac{d}{dz}\int_{[0,1]^d} S_1^{(d)}(f,x)\,d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\frac{d}{dz}\int_{[0,1]^d}\big(S_{2^k}^{(d)}(f,x)-S_{2^{k-1}}^{(d)}(f,x)\big)\,d\mu(x)\\
&=\int_{[0,1]^d}\frac{\partial f(z,x)}{\partial z}\,d\mu(x),
\end{aligned}$$

which finishes the proof. □

4 Integral as a function of upper limit

For a continuous function $f(x):[0,1]^d\to\mathbb{R}$ and $y=(y_1,\dots,y_d)\in[0,1]^d$ we consider the random function

$$\xi(y)=\int_{\prod_{s=1}^d[0,y_s]} f(x)\,d\mu(x). \tag{15}$$

Lemma 2 implies that

$$\begin{aligned}
\tilde{\xi}(y)&=\int_{\prod_{s=1}^d[0,y_s]} S_1^{(d)}(f,x)\,d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\int_{\prod_{s=1}^d[0,y_s]}(S_{2^k}^{(d)}(f,x)-S_{2^{k-1}}^{(d)}(f,x))\,d\mu(x)
\end{aligned} \tag{16}$$

is the version of $\xi(y)$.

We refer to the following assumptions on SM μ.

Assumption A2. Random function $\mu(x)=\mu\Big(\prod_{i=1}^d[0,x_i]\Big)$, $x\in[0,1]^d$, has continuous paths.

In the following condition on the uniform modulus of continuity we take the continuous version of μ.

*Assumption A*3. If $d \geq 2$ then $\sum_{k=1}^{\infty} k^{d-2}\omega(\mu, 2^{-k}) < \infty$ a.s.

It is easy to see that Asumption A3 holds for $\mu(x)$ with Hölder continuous paths. It also holds if, for example, $\omega(\mu, \tau) \leq C|\ln\tau|^{-\epsilon}$, $\epsilon > d-1$.

Example 4. Denote

$$q(\tau) = |\ln\tau|^{-\gamma}, \quad \gamma > d - 1/2, \qquad \mathcal{K}(z) = \sqrt{\frac{dq^2(z)}{dz}}.$$

We introduce the stochastic process

$$B^q(x) = \int_{\prod_{s=1}^{d}[0,x_s]} \prod_{s=1}^{d} \mathcal{K}(x_s - y_s)dW(y), \quad x = (x_1, \dots, x_d),$$

where W is a d-dimensional Wiener process. Theorem 3.1 in [11] implies the existence of rectangle $[t, T] = \prod_{i=1}^{d}[t_i, T_i] \subset [0, 1]^d$ such that

$$\lim_{\varepsilon\to 0+} \sup_{\substack{x,\bar{x}\in[t,T] \\ \delta_{x,\bar{x}}\leq\varepsilon}} \frac{|B^q(x) - B^q(\bar{x})|}{\delta_{x,\bar{x}}\sqrt{\ln\left(\frac{D}{q^{-1}(\delta_{x,\bar{x}})}\right)}} = C \quad \text{a.s.} \tag{17}$$

Here $\delta_{x,\bar{x}} = \|B^q(x) - B^q(\bar{x})\|_{L^2(\Omega)} \leq Cq(|x-\bar{x}|)$, D is a diameter of $[t, T]$. From (17) it follows that for all $x, \bar{x} \in [t, T]$,

$$\begin{aligned} |B^q(x) - B^q(\bar{x})| &\leq C\delta_{x,\bar{x}}\sqrt{\ln D + \delta_{x,\bar{x}}^{-1/\gamma}} \\ &\overset{\gamma>1/2}{\leq} C|\ln|x-\bar{x}||^{-\gamma}\sqrt{\ln D + |\ln|x-\bar{x}||} \\ &\leq C|\ln|x-\bar{x}||^{1/2-\gamma}. \end{aligned}$$

Therefore, $\omega_{[t,T]}(B^q, \tau) \leq C|\ln\tau|^{1/2-\gamma}$.

Now we are ready to formulate the main result of the section.

Theorem 4. *Let Assumptions A2 and A3 hold, and the function $f(x) : [0, 1]^d \to \mathbb{R}$ be continuously differentiable d times on* $[0, 1]^d$.

Then, for the random function ξ defined by (15)*, version* (16) *has continuous paths on* $[0, 1]^d$ *a.s.*

Proof. For version (16), we have that

$$\tilde{\xi}(y) = \lim_{k\to\infty} \int_{\prod_{s=1}^{d}[0,y_s]} S_{2^k}^{(d)}(f, x)\, d\mu(x). \tag{18}$$

Here $S_{2^k}^{(d)}(f, x)$ is a simple function. By our assumption, SM μ has continuous paths. Therefore, for each k, the random function of variable y

$$\int_{\prod_{s=1}^{d}[0,y_s]} S_{2^k}^{(d)}(f, x)\, d\mu(x)$$

has continuous paths. We will prove that the convergence in (18) is uniform in $y \in [0,1]^d$ a.s., and this will imply the continuity of $\tilde{\xi}$.

We aim to show that

$$\sum_{k=1}^{\infty} \sup_{y\in[0,1]^d} \left| \int_{\prod_{s=1}^d [0,y_s]} (S_{2^k}^{(d)}(f,x) - S_{2^{k-1}}^{(d)}(f,x))\, d\mu(x) \right| \le \tilde{C}_{f,\mu}^{(d)}(\omega)$$

for some random constant $\tilde{C}_{f,\mu}^{(d)}(\omega) < \infty$ a.s. that may depend on d, f, μ.

Recall that paths of $\mu(x_1,\dots,x_d)$ are continuous, therefore for any cut of the set $\prod_{s=1}^d [y_{s1}, y_{s2}] \subset [0,1]^d$ we have

$$\mu\Big(\prod_{s=1}^d [y_{s1}, y_{s2}] \cap \{x_s = a\}\Big) = 0. \tag{19}$$

Thus, we obtain

$$\begin{aligned}
&\int_{\prod_{s=1}^d [0,y_s]} (S_{2^k}^{(d)}(f,x) - S_{2^{k-1}}^{(d)}(f,x))\, d\mu(x)\\
&\quad = \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} c_{n_1,\dots,n_d}^{(d)}(f) \int_{\prod_{s=1}^d [0,y_s]} \chi_{n_1,\dots,n_d}^{(d)}(x)\, d\mu(x)\\
&\quad \overset{(19)}{=} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} c_{n_1,\dots,n_d}^{(d)}(f) 2^{(j_1+\dots+j_d)/2}\\
&\qquad \times \int_{\prod_{s=1}^d [0,y_s]} \prod_{1\le s\le d, n_s\ge 2} \Big(\mathbf{1}_{\Delta_{n_s}^+}(x_s) - \mathbf{1}_{\Delta_{n_s}^-}(x_s)\Big)\, d\mu(x)\\
&\quad = \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} c_{n_1,\dots,n_d}^{(d)}(f) 2^{(j_1+\dots+j_d)/2}\\
&\qquad \times \sum_{\varepsilon_s\in\{+,-\}} \mu\Big(\prod_{1\le s\le d} \Delta_{n_s}^{\varepsilon_s} \cap \prod_{1\le s\le d} [0,y_s]\Big)\\
&\quad := A_{k1}(y) + A_{k2}(y).
\end{aligned}$$

Here $A_{k1}(y)$ is the sum of terms with $(n_1,\dots,n_d) \in \mathbb{N}_k \setminus \mathbb{N}_{k-1}$, $\varepsilon_s \in \{+,-\}$ such that

$$\prod_{s=1}^d \Delta_{n_s}^{\varepsilon_s} \subset \prod_{s=1}^d [0,y_s]$$

(here for $n_s = 1$ we take only $\varepsilon_s = +$ and $\Delta_{n_s}^{\varepsilon_s} = (0,1)$). Taking into account the definition of $\Delta_{n_s}^+$ and $\Delta_{n_s}^-$ in (3), we get that in $A_{k1}(y)$

$$\prod_{s=1}^d \Delta_{n_s}^{\varepsilon_s} \cap \prod_{s=1}^d [0,y_s] = \prod_{s=1}^d \Delta_{n_s}^{\varepsilon_s} = \prod_{s=1}^d \Delta_{m_s}$$

for some $(m_1,\dots,m_d) \in \mathbb{N}_{k+1} \setminus \mathbb{N}_k$. Therefore,

$$\sum_{k=1}^{\infty} \sup_{y\in[0,1]^d} |A_{k1}(y)|$$

$$\le \sum_{k=1}^{\infty} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} |c^{(d)}_{n_1,\dots,n_d}(f)| 2^{(j_1+\dots+j_d)/2} \sum_{\varepsilon_s\in\{+,-\}} \Big|\mu\Big(\prod_{1\le s\le d} \Delta^{\varepsilon_s}_{n_s}\Big)\Big|$$

$$\overset{(12)}{\le} C\sum_{k=1}^{\infty} \sum_{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1}} 2^{-(j_1+\dots+j_d)} \sum_{\varepsilon_s\in\{+,-\}} \Big|\mu\Big(\prod_{1\le s\le d} \Delta^{\varepsilon_s}_{n_s}\Big)\Big|$$

$$\le C\sum_{k=1}^{\infty} \sum_{(m_1,\dots,m_d)\in\mathbb{N}_{k+1}\setminus\mathbb{N}_k} 2^d 2^{-(j'_1+\dots+j'_d)} \Big|\mu\Big(\prod_{1\le s\le d} \Delta_{m_s}\Big)\Big|.$$

Here j'_s are taken such that $2^{j'_s}+1 \le m_s \le 2^{j'_s+1}$, and $j'_s = j_s + 1$ for respective m_s. From estimates in (13), (14) it follows that

$$\sum_{k=1}^{\infty} \sup_{y\in[0,1]^d} |A_{k1}(y)| \le \tilde{C}^{(d)}_{\mu}(\omega)$$

for some random constant $\tilde{C}^{(d)}_{\mu}(\omega)$ that depends only of d and μ. It is easy to see that all estimates in (13), (14) remain valid if we change $\chi^{(d)}_{n_1,\dots,n_d}(x)$ to

$$2^{(j_1+\dots+j_d)/2} \prod_{1\le s\le d} \mathbf{1}_{\Delta_{m_s}}(x_s).$$

Further, we estimate $A_{k2}(y)$, i. e. the sum of terms with $(n_1,\dots,n_d) \in \mathbb{N}_k \setminus \mathbb{N}_{k-1}$ such that $y_s \in \Delta_{n_s}$ for some s.

We get

$$|A_{k2}(y)| \le \sum_{\substack{(n_1,\dots,n_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1},\\ \exists y_s\in\Delta^{\varepsilon_s}_{n_s}}} 2^{-\sum_{1\le s\le d, n_s\ge 2} j_s} \Big|\mu\Big(\prod_{1\le s\le d} ((\Delta_{n_s})^{\varepsilon_s} \cap [0, y_s])\Big)\Big|$$

$$= 2^d \sum_{\substack{(m_1,\dots,m_d)\in\mathbb{N}_k\setminus\mathbb{N}_{k-1},\\ \exists y_s\in\Delta_{m_s}}} 2^{-\sum_{1\le s\le d, m_s\ge 3} j'_s} \Big|\mu\Big(\prod_{1\le s\le d} (\Delta_{m_s} \cap [0, y_s])\Big)\Big|$$

$$\le 2^d \omega(\mu, 2^{-k-1}) \sum_{\substack{(m_1,\dots,m_d)\in\mathbb{N}_{k+1}\setminus\mathbb{N}_k,\\ \exists y_s\in\Delta_{m_s}}} 2^{-\sum_{1\le s\le d, m_s\ge 3} j'_s} =: D_{k+1}.$$

Now we check the convergence of the series $\sum_{k=1}^{\infty} D_k$ with the help of the sum

$$\sum_{\substack{(m_1,\dots,m_d)\in\mathbb{N}_k,\\ \exists y_s\in\Delta_{m_s}}} 2^{-\sum_{1\le s\le d, m_s\ge 3} j'_s} := T_k.$$

Notice that for each fixed set $(j'_1,\dots,j'_d)$ there exist at most

$$2^{\sum_{1\le s\le d, m_s\ge 3} j'_s} - \prod_{1\le s\le d, m_s\ge 3} \big(2^{j'_s} - 1\big)$$

sets $(m_1 \ldots, m_d)$, for which $\exists y_s \in \Delta_{m_s}$. Denoting by A the set of indices s, which satisfy the equality $n_s = 1$, we obtain that

$$
\begin{aligned}
\sup_y T_k &\le \sum_{A\subset\{1,\ldots,d\}} \sum_{\substack{1\le j_i'\le k,\\ i\in A}} \left(1-\prod_{i\in A}(1-2^{-j_i'})\right)\\
&= \sum_{A\subset\{1,\ldots,d\}} \sum_{\substack{1\le j_i'\le k,\\ i\in A}} \sum_{\substack{B\subset A,\\ B\neq\emptyset}} (-1)^{|B|+1} 2^{-\sum_{i\in B} j_i'}\\
&= \sum_{A\subset\{1,\ldots,d\}} \sum_{\substack{B\subset A,\\ B\neq\emptyset}} (-1)^{|B|+1} \sum_{\substack{1\le j_i'\le k,\\ i\in A}} 2^{-\sum_{i\in B} j_i'}\\
&\overset{(*)}{=} \sum_{u=1}^{d}\sum_{v=1}^{u}(-1)^{v+1}\binom{d}{u}\binom{u}{v}\sum_{\substack{1\le j_i'\le k,\\ 1\le i\le u}} 2^{-\sum_{1\le i\le v} j_i'}\\
&= \sum_{u=1}^{d}\sum_{v=1}^{u}(-1)^{v+1}\binom{d}{u}\binom{u}{v}(1-2^{-k})^{v}k^{u-v}\\
&= \sum_{u=1}^{d}\binom{d}{u}\left(k^u-(k-1+2^{-k})^u\right)\\
&= (k+1)^d-(k+2^{-k})^d.
\end{aligned}
$$

Here in $(*)$ we used the fact that sum $(-1)^{|B|+1}\sum_{\substack{1\le j_i'\le k,\\ i\in A}} 2^{-\sum_{i\in B} j_i'}$ depends only on $|A|$ and $|B|$. Therefore,

$$
\begin{aligned}
D_k &\le 2^d\omega(\mu,2^{-k})\big((k+1)^d-(k+2^{-k})^d-k^d+(k-1+2^{-k+1})\big)\\
&= 2^d\omega(\mu,2^{-k})\big((k+1)^d-2k^d+(k-1)^d+O(2^{-\beta k})\big),
\end{aligned}
$$

where $0<\beta<1$. Applying Assumption A3 we get the statement of the theorem. □

If $\mu(x)$ have a Hölder continuous paths with exponent $\gamma>0$, then the statement of Theorem 4 follows from Theorem 16 [10]. Moreover, Theorem 16 [10] states that ξ has the version that is Hölder continuous with the same exponent $\gamma>0$. In [10], integral is considered in the Young sense, and its value coincides with the value of our integral with respect to Hölder continuous μ.

Theorem 5. *Let Assumptions A1, A2 and A3 hold, and the function $f(x):[0,1]^d\to\mathbb{R}$ be continuously differentiable d times on $[0,1]^d$.*

Then, for the random function ξ defined by (15), for any $1\le p<+\infty$, $0<\alpha<\min\{1/p,1/2\}$ version (16) with probability 1 belongs to the Besov space $B^{\alpha}_{p,p}([0,1]^d)$.

Proof. The statement follows from Theorems 1 and 4. □

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