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# Perturbation of an $\alpha$-stable type stochastic process by a pseudo-gradient

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**Abstract** A Markov process defined by some pseudo-differential operator of an order $1 < \alpha < 2$ as the process generator is considered. Using a pseudo-gradient operator, that is, the operator defined by the symbol $i\lambda|\lambda|^{\beta-1}$ with some $0 < \beta < 1$, the perturbation of the Markov process under consideration by the pseudo-gradient with a multiplier, which is integrable at some large enough power, is constructed. Such perturbation defines a family of evolution operators, properties of which are investigated. A corresponding Cauchy problem is considered.

**Keywords** $\alpha$-stable process, perturbation, pseudo-gradient, pseudo-process
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## 1 Introduction

Let $d$ be some fixed positive integer number. By $\mathbb{R}^d$ we denote the real $d$-dimensional Euclidean space. As usual, we denote by $(\cdot,\cdot)$ the inner product and by $|\cdot|$ the norm in $\mathbb{R}^d$ (we use the last notation for denoting the absolute value of a real number or a multi-index and the module of a complex number).

Let us consider a family of pseudo-differential operators $(A(t,x))_{t\geq 0, x\in\mathbb{R}^d}$ defined by the symbols $(a(t,x,\lambda))_{\lambda\in\mathbb{R}^d}$ for every $t\geq 0$, $x\in\mathbb{R}^d$. That is,

$$A(t,x)f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}a(t,x,\lambda)F(\lambda)e^{i(\lambda,x)}\,\mathrm{d}\lambda,\quad t\geq 0,\ x\in\mathbb{R}^d,$$

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where $F$ is the Fourier transform of the function $f$: $F(\lambda) = \int_{\mathbb{R}^d} f(x)e^{-i(\lambda,x)}\,\mathrm{d}x$, $\lambda \in \mathbb{R}^d$. Note that the function $a$ may be complex-valued.

We assume that the following conditions from [6, Ch. 4] about the function $a$ and the operator $A$ are satisfied (we denote by $Lf(r,\cdot)(x)$ the result of the operator $L$ used on $f(r,x)$ as the function of $x$).

$A_1$) The function $a$ is homogeneous of a degree $\alpha \in (1,2)$ with respect to the variable $\lambda$ and $\operatorname{Re} a(t,x,\lambda) \ge a_0 > 0$ for all $t \ge 0$, $x, \lambda \in \mathbb{R}^d$, $|\lambda| = 1$.

$A_2$) The function $a$ has $N \ge 2d+3$ (which is some natural number) continuous derivatives in $\lambda$ for all $\lambda \ne 0$, and

$$|\partial^{\varkappa} a(t,x,\cdot)(\lambda)| \le C_N|\lambda|^{\alpha-|\varkappa|},$$

$$|\partial^{\varkappa}[a(t,x,\cdot) - a(s,y,\cdot)](\lambda)| \le C_N\left(|x-y|^{\gamma} + |t-s|^{\gamma/\alpha}\right)|\lambda|^{\alpha-|\varkappa|}$$

for all multi-indexes $\varkappa$ with $|\varkappa| \le N$, $x, y, \lambda \in \mathbb{R}^d$, $\lambda \ne 0$, $s, t \in [0,+\infty)$. Here $\gamma \in (0,1)$ is some constant.

$A_3$) In the representation

$$A(t,x)f(x) = \int_{\mathbb{R}^d} \Omega\left(t, x, \frac{h}{|h|}\right) \frac{f(x) - 2f(x-h) + f(x-2h)}{|h|^{d+\alpha}}\,\mathrm{d}h,$$

the function $\Omega(t,x,\cdot)$ is even and nonnegative.

*Remark* 1. Assumptions $A_1$, $A_2$, $A_3$ coincide with assumptions ($A_{41}$), ($A_{42}$), ($A_{44}$) from [6], respectively (see pages 266 and 294 there).

Theorem 4.3 from [6] (see also Theorem 4.1 there) states that there exists a bounded nonterminating strict Markov process $(x(t))_{t\ge 0}$ without second kind discontinuities, and the fundamental solution $(g(s,x,t,y))_{0\le s<t, x,y\in\mathbb{R}^d}$ to the equation

$$\frac{\partial}{\partial t}u(t,x) + A(t,x)u(t,\cdot)(x) = 0, \quad t > 0,\ x \in \mathbb{R}^d,$$

is its transition probability density. The function $g$ can be constructed by the parametrix method (see [6, Sec. 4.1.4]).

If the function $a$ is defined by the equality

$$a(t,x,\lambda) = c|\lambda|^{\alpha}, \quad t \ge 0,\ x \in \mathbb{R}^d,\ \lambda \in \mathbb{R}^d,$$

where $c > 0$ is some constant, the corresponding Markov process is an isotropic $\alpha$-stable process. The function $g$ can be presented by the equality

$$g(s,x,t,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\{i(x-y,\lambda) - c(t-s)|\lambda|^{\alpha}\}\mathrm{d}\lambda$$

in this case. The operator $A$ (which does not depend on $t$ and $x$) is the generator of this process. This is the simplest example of the processes considered in this paper. Therefore we name the process $(x(t))_{t\ge 0}$ *the $\alpha$-stable type process*.

Choosing some $0 < \beta \le 1$, let us denote the $\beta$-order "pseudo-gradient" by $\nabla_\beta$, i.e. it is the operator which is defined by the symbol $(i\lambda|\lambda|^{\beta-1})_{\lambda\in\mathbb{R}^d}$. Note that $\nabla_\beta$ is a vector-valued operator. In the case of $\beta = 1$, it is the ordinary gradient. We restrict ourselves to the case $\beta \neq 1$ in this paper.

Our goal is to construct a perturbation of the considered process using the operator $\big(b, \nabla_\beta\big)$, where $b$ is some measurable $\mathbb{R}^d$-valued function. Under the perturbation of the Markov process $(x(t))_{t\ge 0}$ we understand the construction of a two-parameter family of operators $(\mathbb{T}_{st})_{0\le s<t}$, defined on the set $C_b(\mathbb{R}^d)$ of real-valued continuous bounded functions defined on $\mathbb{R}^d$, such that for every $\varphi \in C_b(\mathbb{R}^d)$ and $t > 0$, the function $u(s, x, t) = \mathbb{T}_{st}\varphi(x)$ satisfies in some sense the following Cauchy problem:

$$\frac{\partial}{\partial s}u(s, x, t) + L(s, x)u(s, \cdot, t)(x) = 0, \quad 0 \le s < t,\ x \in \mathbb{R}^d,$$

$$\lim_{s\uparrow t} u(s, x, t) = \varphi(x), \quad x \in \mathbb{R}^d,$$

where $L(s, x) = A(s, x) + (b(s, x), \nabla_\beta)$.

The problem of perturbation of Markov processes was and remains in the focus of attention of many researchers. The diffusion case, that is, if $\alpha = 2$, was considered by M. I. Portenko (see [14, 15] and the references there). The perturbation operators there are of the form $(b, \nabla)$, where the function $b$ belongs to some $L_p$-space of functions or is some generalized function of the delta function type. In [19], the parabolic equation of the type $\nabla(A\nabla u) + B\nabla u - u_t = 0$ was considered. The case $\alpha < 2$ was considered in the works of K. Bogdan and T. Jakubowski [2], who studied such a perturbation with the function $b$ from a Kato class. In [7], T. Jakubowski considered a fundamental solution of the fractional diffusion equation (like written above) with a singular drift, i.e. the case of $\beta = 1$ and some singular function $b$. In [8], an $\alpha$-stable process was perturbed by the gradient operator with a multiplier $b$ satisfying a certain integral space-time condition. S. I. Podolynny, M. I. Portenko [13] and M. I. Portenko [16–18] investigated such perturbations with the function $b$ from $L_p$. Y. Maekawa and H. Miura were concerned with nonlocal parabolic equations in the presence of a divergence free drift term in [10]. J.-U. Loebus and M. I. Portenko [9] perturbed the infinitesimal operator of a one-dimensional symmetric $\alpha$-stable process using the operator $(q\delta_0, \partial_{\alpha-1})$, where $\partial_{\alpha-1}$ is some pseudo-differential operator of the order $\alpha - 1$. Perturbation of an $\alpha$-stable process by a fractional Laplacian operator was considered in [5]. The results for $\alpha \in (1, 2)$ and perturbation operators of the type $(b, \nabla_{\alpha-1})$ can be found in [11, 12]. We studied the case of an $\alpha$-stable process and perturbation operators $(b, \nabla_\beta)$ with a $\mathbb{R}^d$-valued time independent function $b$ from $L_p(\mathbb{R}^d)$ and $0 < \beta < \alpha$ in [3]. In [4], the case of a generalized coefficient $b$ is considered.

This article is structured as follows. The next section is devoted to some auxiliary facts. The perturbation equation is solved in Section 3. In Section 4, we study some properties of the corresponding two-parameter evolutionary family of operators. The last section is devoted to the construction of a solution to a corresponding Cauchy problem.

We will use the notation $C$ for different constants, if their values are not important. If we need to emphasize the dependence of the constant $C$ on parameter $\pi$, we will write $C_\pi$. We will use the notation $(a \vee b)$ for $\max(a, b)$ and $(a \wedge b)$ for $\min(a, b)$.

## 2 Auxiliaries

First of all let us note that if $0 < \beta < 1$ the operator $\nabla_\beta$ can be represented in the integral form:

$$\nabla_\beta f(x) = n_\beta \int_{\mathbb{R}^d} (f(x+y) - f(x)) \frac{y}{|y|^{d+\beta+1}}\, \mathrm{d}y, \tag{1}$$

where $n_\beta = -2^{-1}\pi^{-(d-1)/2}\Gamma(-\beta/2)\,\Gamma((d+\beta+1)/2)\,/\,\Gamma(-\beta)$ (here one have to use the equality $\Gamma(1+x) = x\Gamma(x)$ to expand the Euler gamma function $\Gamma$ to negative noninteger arguments). Representation (1) is true if a function $f$ is at least Lipschitz continuous and bounded. To obtain the normalizing factor $n_\beta$, it is sufficient to apply the operator $\nabla_\beta$ to the function $f_\lambda(x) = e^{i(\lambda,x)}$, $x \in \mathbb{R}^d$, for any $\lambda \in \mathbb{R}^d$. Moreover, this is the way to prove (1).

Next, we will need some auxiliary statements. The following lemma is proved in [6] (see Lemma 1.11 there). It will be used frequently in this article.

**Lemma 1.** *The inequality ($0 \le s < t$, $x, y \in \mathbb{R}^d$, remind that $1 < \alpha < 2$)*

$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{\lambda/\alpha}}{((\tau-s)^{1/\alpha} + |z-x|)^{d+l}} \frac{(t-\tau)^{\varkappa/\alpha}}{((t-\tau)^{1/\alpha} + |y-z|)^{d+k}}\, \mathrm{d}z \le$$
$$C\left[ B\left(1 + \frac{\varkappa - k}{\alpha}, 1 + \frac{\lambda}{\alpha}\right) \frac{(t-s)^{1+(\varkappa+\lambda-k)/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+l}} + \right.$$
$$\left. B\left(1 + \frac{\varkappa}{\alpha}, 1 + \frac{\lambda - l}{\alpha}\right) \frac{(t-s)^{1+(\varkappa+\lambda-l)/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+k}} \right] \tag{2}$$

*holds with some constant $C > 0$ that depends only on $d$, $\alpha$, $k$ and $l$ for all $\varkappa$, $\lambda$, $k$, $l$, satisfying the inequalities $0 < k < \alpha + \varkappa$, $0 < l < \alpha + \lambda$. Here $B(\cdot,\cdot)$ is the Euler beta function.*

*Remark* 2. Analyzing the proof of Lemma 1, one can see that the following inequality holds for all $0 \le s < \tau < t$, $x, y \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \frac{(\tau-s)^{\lambda/\alpha}}{((\tau-s)^{1/\alpha} + |z-x|)^{d+l}} \frac{(t-\tau)^{\varkappa/\alpha}}{((t-\tau)^{1/\alpha} + |y-z|)^{d+k}}\, \mathrm{d}z \le$$
$$C\left[ \frac{(\tau-s)^{\lambda/\alpha}(t-\tau)^{(\varkappa-k)/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+l}} + \frac{(\tau-s)^{(\lambda-l)/\alpha}(t-\tau)^{\varkappa/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+k}} \right]. \tag{3}$$

The following lemma can be obtained from the results of [6, Ch. 4]).

**Lemma 2.** *If the assumptions $A_1$–$A_3$ hold, then the transition probability density of the process $(x(t))_{t\ge0}$ has the following properties ($0 \le s < t \le T$, $x, y \in \mathbb{R}^d$): the function $g$ is continuously differentiable with respect to $x \in \mathbb{R}^d$ and*

$$|\partial^k g(s,\cdot,t,y)(x)| \le N_{k,T} \frac{(t-s)^{1-(\gamma+k)/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+\alpha-\gamma}}, \quad k = 0, 1, \tag{4}$$

*where $\partial^k$ means some derivative of the order k;*

$$|\nabla_\beta g(s,\cdot,t,y)(x)| \le N_{\beta,T}\left(\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}} + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right). \tag{5}$$

*Here, positive constants $N_{k,T}$ and $N_{\beta,T}$ can be depended on $T$.*

**Proof.** Let us note that the function $g$ can be constructed by the parametrix method (see [6, Th. 4.1]). That is, it can be presented by the equality

$$g(s,x,t,y) = g_0(s,x,t,y) + h(s,x,t,y),$$

where

$$g_0(s,x,t,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\{i(x-y,\lambda) - a(t,y,\lambda)(t-s)\}\mathrm{d}\lambda,$$

$$h(s,x,t,y) = \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g_0(s,x,\tau,z)\Phi(\tau,z,t,y)\mathrm{d}z, \tag{6}$$

and the function $\Phi$ is determined by some integral equation.

Theorem 4.1 cited above states that the following inequalities hold:

$$|h(s,x,t,y)| \le C\left(\frac{(t-s)^{1+\gamma/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha}} + \frac{t-s}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right), \tag{7}$$

$$|\Phi(s,x,t,y)| \le C\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}. \tag{8}$$

The function $g_0$ satisfies the inequality (see [6, eq. (4.1.25)])

$$|\partial^k g_0(s,\cdot,t,y)(x)| \le N_k\frac{t-s}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha+k}} \tag{9}$$

in which $\partial^k$ means some derivative of the integer order $0 \le k \le N-2d-1$ (the constant $N$ is defined in assumption $A_2$), $N_k > 0$ are some constants. Note that $k$ can be at least 0, 1, or 2.

Inequality (4) is the simple consequence of (7) and (9) in the case of $k=0$.

To prove inequality (4) for the case of $k=1$, we have to show the equality

$$\partial\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} g_0(s,\cdot,\tau,z)\Phi(\tau,z,t,y)\,\mathrm{d}z\,(x) = \int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} \partial g_0(s,\cdot,\tau,z)(x)\Phi(\tau,z,t,y)\,\mathrm{d}z. \tag{10}$$

Due to (8) and (9),

$$|\partial g_0(s,\cdot,\tau,z)(x)\Phi(\tau,z,t,y)| \le$$

$$\frac{N_1(\tau - s)}{((\tau - s)^{1/\alpha} + |z - x|)^{d+\alpha+1}} \frac{C}{((t - \tau)^{1/\alpha} + |y - z|)^{d+\alpha-\gamma}} \leq$$
$$N_1(\tau - s)^{-1-(d+1)/\alpha} \frac{C}{((t - \tau)^{1/\alpha} + |y - z|)^{d+\alpha-\gamma}}.$$

The right hand side of the last inequality is integrable with respect to $z$ on $\mathbb{R}^d$ for every fixed $0 \leq s < \tau < t$, $y \in \mathbb{R}^d$ and we can write down the equality

$$\partial \int_{\mathbb{R}^d} g_0(s, \cdot, \tau, z)\Phi(\tau, z, t, y)\, \mathrm{d}z\, (x) = \int_{\mathbb{R}^d} \partial g_0(s, \cdot, \tau, z)(x)\Phi(\tau, z, t, y)\, \mathrm{d}z.$$

Inequality (3) leads us to the following ($0 \leq s < \tau < t \leq T$, $x, y \in \mathbb{R}^d$) inequalities:

$$\left| \partial \int_{\mathbb{R}^d} g_0(s, \cdot, \tau, z)\Phi(\tau, z, t, y)\, \mathrm{d}z\, (x) \right| \leq$$
$$C \left[ \frac{(\tau - s)(t - \tau)^{-1+\gamma/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha+1}} + \frac{(\tau - s)^{-1/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha-\gamma}} \right] \leq$$
$$C_T \left( (\tau - s)(t - \tau)^{-1+\gamma/\alpha} + (\tau - s)^{-1/\alpha} \right) (t - s)^{-1-(d+1)/\alpha}.$$

The last expression is integrable with respect to $\tau$ on $(s, t)$, so we have (10).

Using (2) we can obtain the following inequalities valid for $k = 0, 1$:

$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \left| \partial^k g_0(s, \cdot, \tau, z)(x)\Phi(\tau, z, t, y) \right| \mathrm{d}z \leq$$
$$C_k \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{\tau - s}{((\tau - s)^{1/\alpha} + |z - x|)^{d+\alpha+k}} \frac{1}{((t - \tau)^{1/\alpha} + |y - z|)^{d+\alpha-\gamma}}\, \mathrm{d}z \leq$$
$$C_k \left( \frac{(t - s)^{1-k/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha+k}} + \frac{(t - s)^{1+\gamma/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha-\gamma}} \right) \leq$$
$$C_k \frac{(t - s)^{1-k/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha-\gamma}},$$

where $C_k > 0$ are some constants. Therefore (use (9)),

$$|\partial^k g(s, \cdot, t, y)(x)| \leq (N_k \vee C_k) \frac{(t - s)^{1-k/\alpha}}{((t - s)^{1-(k+\gamma)/\alpha} + |y - x|)^{d+\alpha+-\gamma}} \times$$
$$\left[ \frac{(t - s)^{(k+\gamma)/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{k+\gamma}} + (t - s)^{\gamma/\alpha} \right] \leq$$
$$C_{k,T} \frac{(t - s)^{1-(\gamma+k)/\alpha}}{((t - s)^{1/\alpha} + |y - x|)^{d+\alpha-\gamma}},$$

for all $0 \leq s < t \leq T$, $x, y \in \mathbb{R}^d$ and every $T > 0$. Here the constants $C_{k,T} > 0$ depend on $T$.

For proving inequality (5), we use representation (1) of the operator $\nabla_\beta$ (remind that $0 < \beta < 1$). So, since $a(t, x, \cdot)$ is a homogeneous function and

$$\nabla_\beta g_0(s, \cdot, t, y)(x) =$$

$$\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\lambda|\lambda|^{\beta-1}\exp\{i(x-y,\lambda)-a(t,y,(t-s)^{1/\alpha}\lambda)\}\mathrm{d}\lambda =$$
$$\frac{i}{(2\pi)^d(t-s)^{(d+\beta)/\alpha}}\int_{\mathbb{R}^d}\lambda|\lambda|^{\beta-1}\exp\left\{i\left(\frac{x-y}{(t-s)^{1/\alpha}},\lambda\right)-a(t,y,\lambda)\right\}\mathrm{d}\lambda,$$

the result of [6, Lemma 4.2] leads us to the inequality

$$|\nabla_\beta g_0(s,\cdot,t,y)(x)| \le C_\beta\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}} \tag{11}$$

valid for all $0\le s<t$, $x,y\in\mathbb{R}^d$ with some positive constant $C_\beta$.

Let us consider the function

$$f(x)=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} g_0(s,x,\tau,z)\Phi(\tau,z,t,y)\,\mathrm{d}z,\quad x\in\mathbb{R}^d,$$

for every fixed $0\le s<t$, $y\in\mathbb{R}^d$. Using representation (1) we can write down the equality

$$\nabla_\beta f(x)=k_\beta\int_{\mathbb{R}^d}\frac{u}{|u|^{d+\beta+1}}(f(x+u)-f(x))\,\mathrm{d}u = I_1+I_2,$$

where $I_1$ and $I_2$ are the integrals of the same integrand as on the left hand side but over $B_\varepsilon(0)=\{u\in\mathbb{R}^d : |u|\le\varepsilon\}$ and $B_\varepsilon(0)^C$, respectively.

Since, for $u\in B_\varepsilon(0)$ and fixed $x\in\mathbb{R}^d$, $f(x+u)-f(x)=(\nabla f(x+\theta u),u)$, where $\theta=\theta(u)\in(0,1)$,

$$I_1 =$$
$$k_\beta\int_{B_\varepsilon(0)}\frac{u}{|u|^{d+\beta+1}}\left(u,\nabla\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d} g_0(s,\cdot,\tau,z)\Phi(\tau,z,t,y)\,\mathrm{d}z\,(x+\theta u)\right)\mathrm{d}u =$$
$$k_\beta\int_{B_\varepsilon(0)}\frac{u}{|u|^{d+\beta+1}}\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}(u,\nabla g_0(s,\cdot,\tau,z)(x+\theta u))\,\Phi(\tau,z,t,y)\,\mathrm{d}z\,\mathrm{d}u.$$

Due to (8) and (9), the integrand here can be estimated as follows:

$$\left|\frac{u}{|u|^{d+\beta+1}}(u,\nabla g_0(s,\cdot,\tau,z)(x+\theta u))\,\Phi(\tau,z,t,y)\right| \le$$
$$\frac{C}{|u|^{d+\beta-1}}\frac{\tau-s}{((\tau-s)^{1/\alpha}+|z-x-\theta u|)^{d+\alpha+1}}\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}.$$

For fixed $u$ the last function is integrable with respect to $(\tau,z)$ on $(s,t)\times\mathbb{R}^d$ and the integral is not greater then

$$\frac{C}{|u|^{d+\beta-1}}\frac{(t-s)^{1-1/\alpha}}{((t-s)^{1/\alpha}+|y-x-\theta u|)^{d+\alpha-\gamma}}.$$

Let us take $\varepsilon<\frac{1}{2}(t-s)^{1/2}$. The obtained estimator is not greater then the following function (for fixed $0\le s<t$, $x,y\in\mathbb{R}^d$) integrable with respect to $u$ on $B_\varepsilon(0)$:

$$\frac{C}{|u|^{d+\beta-1}}\frac{(t-s)^{1-1/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}.$$

Here we have used the obvious inequalities

$$(t-s)^{1/\alpha} + |y-x-\theta u| \geq (t-s)^{1/\alpha} + |y-x| - \theta|u| \geq \\ \frac{1}{2}\left((t-s)^{1/\alpha} + 2|y-x|\right) \geq \frac{1}{2}\left((t-s)^{1/\alpha} + |y-x|\right).$$

The integrand in $I_2$ can be estimated as follows ($|u| \geq \varepsilon$):

$$\left|\frac{u}{|u|^{d+\beta+1}}(g_0(s,x+u,\tau,z) - g_0(s,x,\tau,z))\Phi(\tau,z,t,y)\right| \leq \\ \frac{C}{|u|^{d+\beta}}\left(\frac{\tau-s}{((\tau-s)^{1/\alpha}+|z-x-u|)^{d+\alpha}} + \frac{\tau-s}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha}}\right) \times \\ \frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}.$$

Integrating this function with respect to $(\tau, z)$ on $(s,t) \times \mathbb{R}^d$ for every fixed $u \in B_\varepsilon(0)^C$ one can obtain the estimator

$$\frac{C}{|u|^{d+\beta}}\left(\frac{t-s}{((t-s)^{1/\alpha}+|y-x-u|)^{d+\alpha-\gamma}} + \frac{t-s}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right) \leq \\ \frac{C}{|u|^{d+\beta}}(t-s)^{-(d-\gamma)/\alpha},$$

which is integrable with respect to $u$ on $B_\varepsilon(0)^C$ for all fixed $0 \leq s < t$.

Therefore, integrals $I_1$ and $I_2$ are absolutely convergent. So, using the Fubini theorem we can state that

$$\nabla_\beta \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g_0(s,\cdot,\tau,z)\Phi(\tau,z,t,y)\,\mathrm{d}z\,(x) = \\ \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \nabla_\beta g_0(s,\cdot,\tau,z)(x)\Phi(\tau,z,t,y)\,\mathrm{d}z.$$

Now, using (8), (11) and Lemma 1, one can obtain the estimate

$$|\nabla_\beta h(s,\cdot,t,y)(x)| \leq C\left(\frac{(t-s)^{\gamma/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}} \right. \\ \left. + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right),$$

and inequality (5). The lemma is proved. □

## 3 Solving of perturbation equation

Let us consider the perturbation equation ($0 \leq s < t$, $x, y \in \mathbb{R}^d$)

$$G(s,x,t,y) = g(s,x,t,y) +$$

$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), \nabla_\beta G(\tau,\cdot,t,y)(z))\,\mathrm{d}z. \tag{12}$$

Formally applying the operator $\nabla_\beta$ to both sides of equation (12) with respect to the variable $x$ we can search for its solution in the form

$$G(s,x,t,y) = g(s,x,t,y) + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), v(\tau,z,t,y))\,\mathrm{d}z, \tag{13}$$

where the function $(v(s,x,t,y))_{0\le s<t, x,y\in\mathbb{R}^d}$ is a solution to the equation

$$v(s,x,t,y) = v_0(s,x,t,y) + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z), v(\tau,z,t,y))\,\mathrm{d}z, \tag{14}$$

in which

$$v_0(s,x,t,y) = \nabla_\beta g(s,\cdot,t,y)(x). \tag{15}$$

**Theorem 1.** *Let assumptions $A_1$–$A_3$ hold and the $\mathbb{R}^d$-valued function $(b(t,x))_{t>0,x\in\mathbb{R}^d}$ belong to $L_p([0,T]\times\mathbb{R}^d)$ for every $T>0$ with some $p>\frac{d+\alpha}{\alpha-1}$, including $p=+\infty$. Then the following statements are true: there exists a unique solution to equation (14) in the class of functions satisfying the estimate*

$$|v(s,x,t,y)| \le C_T\left(\frac{1}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}} + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right); \tag{16}$$

*the function (13) is a solution to equation (12) and satisfies the estimate*

$$|G(s,x,t,y)| \le C_T\left(\frac{(t-s)^{\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}} + \frac{(t-s)^{1-\gamma/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right) \tag{17}$$

*for all $0\le s<t\le T$, $x,y\in\mathbb{R}^d$ and every $T>0$, where a positive constant $C_T$ can depend on $T$.*

Before proving this theorem, let us prove the following auxiliary statement.

**Lemma 3.** *If the assumptions of Theorem 1 hold and a $\mathbb{R}^d$-valued function $(v(s,x,t,y))_{0\le s<t,x,y\in\mathbb{R}^d}$ satisfies inequality (16), then the equality*

$$\nabla_\beta\left(\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,\cdot,\tau,z)(b(\tau,z), v(\tau,z,t,y))\,\mathrm{d}z\right)(x) = \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \nabla_\beta g(s,\cdot,\tau,z)(x)(b(\tau,z), v(\tau,z,t,y))\,\mathrm{d}z \tag{18}$$

*is valid.*

**Proof.** To prove this lemma we use calculations similar to those used in the proof of Lemma 2. First of all, let us prove the differentiability of the function

$$f(x)=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z,\qquad x\in\mathbb{R}^d,$$

for every fixed $0\le s<t\le T$, $y\in\mathbb{R}^d$. Since

$$|\partial g(s,\cdot,\tau,z)(x)(b(\tau,z),v(\tau,z,t,y))|\le C_T\frac{(\tau-s)^{1-(\gamma+1)/\alpha}}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\times \\ \left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right)|b(\tau,z)|$$

for all fixed $0\le s<\tau<t\le T$, $x,y,z\in\mathbb{R}^d$ we obtain that the last expression is not greater then

$$C_T(\tau-s)^{-(d+1)/\alpha}|b(\tau,z)|\times \\ \left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right).$$

Moreover[1] (we take $q=p/(p-1)$),

$$\int_{\mathbb{R}^d}|b(\tau,z)|\left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right)\mathrm{d}z\le \\ C\left(\int_{\mathbb{R}^d}|b(\tau,z)|^p\,\mathrm{d}z\right)^{1/p}\left(\int_{\mathbb{R}^d}\left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{q(d+\beta)}}+\right.\right. \\ \left.\left.\frac{(t-\tau)^{q(1-\beta/\alpha)}}{((t-\tau)^{1/\alpha}+|y-z|)^{q(d+\alpha-\gamma)}}\right)\mathrm{d}z\right)^{1/q}\le \\ C\left(\int_{\mathbb{R}^d}|b(\tau,z)|^p\,\mathrm{d}z\right)^{1/p}(t-\tau)^{(d-1)/q\alpha-(d+\beta)/\alpha}<+\infty$$

almost everywhere on $(s,t)$. Therefore, we have

$$\partial\int_{\mathbb{R}^d} g(s,\cdot,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z\,(x)= \\ \int_{\mathbb{R}^d}\partial g(s,\cdot,\tau,z)(x)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z$$

and (see Remark 2)

$$\left|\partial\int_{\mathbb{R}^d} g(s,\cdot,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z\,(x)\right|\le$$

[1] Here and further, we use the obvious inequality $(f+g)^r\le 2^r(f^r+g^r)$, valid for all $r\ge 0$ and nonnegative functions $f$ and $g$, without any references.

$$C\int_{\mathbb{R}^d}\frac{(\tau-s)^{1-(\gamma+1)/\alpha}|b(\tau,z)|}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\times$$

$$\left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right)\mathrm{d}z\le$$
$$C_T\left(\int_{\mathbb{R}^d}|b(\tau,z)|^p\,\mathrm{d}z\right)^{1/p}(\tau-s)^{-(d+p)/\alpha p}(t-\tau)^{-(d+\beta p)/\alpha p}\times$$
$$\left(\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\vee\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}\right)\le$$
$$C_T\left(\int_{\mathbb{R}^d}|b(\tau,z)|^p\,\mathrm{d}z\right)^{1/p}\times$$
$$(\tau-s)^{-(d+p)/\alpha p}(t-\tau)^{-(d+\beta p)/\alpha p}\left((t-s)^{-(d+\alpha-\gamma)/\alpha}\vee(t-s)^{-(d+\beta)/\alpha}\right).$$

The last function is integrable with respect to $\tau$ on $(s,t)$ for every fixed $0\le s<t\le T$ in the assumptions of the lemma.

Therefore,

$$\partial\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}g(s,\cdot,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z\,(x)=$$
$$\int_s^t\mathrm{d}\tau\,\partial\int_{\mathbb{R}^d}g(s,\cdot,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z\,(x)=$$
$$\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}\partial g(s,\cdot,\tau,z)(x)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z.$$

Consequently, we can use representation (1) of the operator $\nabla_\beta$:

$$\nabla_\beta f(x)=k_\beta\int_{\mathbb{R}^d}\frac{u}{|u|^{d+\beta+1}}(f(x+u)-f(x))\,\mathrm{d}u=I_1+I_2,$$

where $I_1$ and $I_2$ are the integrals of the same integrand as on the left hand side but over $B_\varepsilon(0)$ and $B_\varepsilon(0)^C$, respectively. Namely,

$$I_1=k_\beta\int_{B_\varepsilon(0)}\frac{u\,\mathrm{d}u}{|u|^{d+\beta+1}}\times$$
$$\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}(g(s,x+u,\tau,z)-g(s,x,\tau,z))(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z=$$
$$k_\beta\int_{B_\varepsilon(0)}\frac{u\,\mathrm{d}u}{|u|^{d+\beta+1}}\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}(\nabla g(s,\cdot,\tau,z)(x+\theta u),u)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z,$$

where $\theta\in(0,1)$ is some constant dependent of $s,t,x,y$ and $u$ only;

$$I_2=k_\beta\int_{B_\varepsilon(0)^C}\frac{u}{|u|^{d+\beta+1}}\times$$
$$\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}(g(s,x+u,\tau,z)-g(s,x,\tau,z))(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z\,\mathrm{d}u.$$

The integrand of $I_1$ satisfies the inequality

$$\left|\frac{u}{|u|^{d+\beta+1}}(\nabla g(s,\cdot,\tau,z)(x+\theta u),u)(b(\tau,z),v(\tau,z,t,y))\right| \leq \\ \frac{C_T}{|u|^{d+\beta-1}}\frac{(\tau-s)^{1-(\gamma+1)/\alpha}|b(\tau,z)|}{((\tau-s)^{1/\alpha}+|z-x-\theta u|)^{d+\alpha-\gamma}} \times \\ \left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right). \tag{19}$$

For the integral $I_2$ we have

$$\left|\frac{u}{|u|^{d+\beta+1}}(g(s,x+u,\tau,z)-g(s,x,\tau,z))(b(\tau,z),v(\tau,z,t,y))\right| \leq \\ \frac{C_T|b(\tau,z)|}{|u|^{d+\beta}} \times \\ \left(\frac{(\tau-s)^{1-(\gamma+1)/\alpha}}{((\tau-s)^{1/\alpha}+|z-x-u|)^{d+\alpha-\gamma}}+\frac{(\tau-s)^{1-(\gamma+1)/\alpha}}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\right) \times \\ \left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right). \tag{20}$$

The right hand sides of (19) and (20) are integrable on the domains of integrals $I_1$ and $I_2$, respectively. Thus, using the Fubini theorem, we obtain equality (18) and the lemma is proved. □

**Proof of Theorem 1.** Let us solve equation (14) for all fixed $0 \leq s < t \leq T$, $x, y \in \mathbb{R}^d$, using the method of successive approximations. Namely, consider the sequence of functions $(v_k(s,x,t,y))_{0\leq s<t\leq T, x,y\in\mathbb{R}^d}$, $k = 0, 1, 2, \ldots$, given by the recurrence relation

$$v_{k+1}(s,x,t,y) = \int_s^t d\tau \int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z),v_k(\tau,z,t,y))\, dz,$$

where $v_0$ is defined by (15).

Using inequality (5) one can obtain the relation ($0 \leq s < t \leq T$, $x, y \in \mathbb{R}^d$)

$$|v_{k+1}(s,x,t,y)| \leq \\ 2N_{\beta,T}\|b\|_p^T \left(\int_s^t d\tau \int_{\mathbb{R}^d} \frac{|v_k(\tau,z,t,y)|^q}{((\tau-s)^{1/\alpha}+|x-z|)^{(d+\beta)q}}\, dz + \\ \int_s^t d\tau \int_{\mathbb{R}^d} \frac{|v_k(\tau,z,t,y)|^q(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha}+|x-z|)^{(d+\alpha-\gamma)q}}\, dz\right)^{1/q}. \tag{21}$$

Iterating relation (21) by using inequality (2), we come to the estimate

$$|v_k(s,x,t,y)| \leq R_{k,T} \times \\ \left(\frac{(t-s)^{k\theta}}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}}+\frac{(t-s)^{k\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right), \tag{22}$$

where $\theta = 1 - ((d+\alpha)/p + \beta)/\alpha$ (note that $\theta > (1-\beta)/\alpha$) and the sequence $\{R_{k,T} : k = 0, 1, 2, \dots\}$ satisfies the relations $R_0 = N_{\beta,T}$ and

$$R_{k+1,T} = R_{k,T} 8^{1+1/q} N_{\beta,T} \|b\|_p^T C^{1/q} (1 + T^{\gamma/\alpha}) \times \\ \Big( B\left((k+1)\theta q, 1\right)^{1/q} \vee B\left(1 + k\theta q, \theta q\right)^{1/q} \Big). \tag{23}$$

for all $k = 1, 2, \dots$. Here $C$ is the maximum of a finite number of constants taken from inequality (2). Indeed, when $k = 0$, inequality (22) coincides with inequality (5). If (22) is correct for some $k \in \mathbb{N}$, we obtain

$$|v_{k+1}(s,x,t,y)| \le 8N_{\beta,T} \|b\|_p^T R_{k,T} \times$$
$$\Bigg[ \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha} + |x-z|)^{(d+\alpha-\gamma)q}} \frac{(t-\tau)^{k\theta q}}{((t-\tau)^{1/\alpha} + |z-y|)^{(d+\beta)q}} \,\mathrm{d}z +$$
$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha} + |x-z|)^{(d+\alpha-\gamma)q}} \frac{(t-\tau)^{k\theta q + (1-\beta/\alpha)q}}{((t-\tau)^{1/\alpha} + |z-y|)^{(d+\alpha-\gamma)q}} \,\mathrm{d}z +$$
$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{1}{((\tau-s)^{1/\alpha} + |x-z|)^{(d+\beta)q}} \frac{(t-\tau)^{k\theta q + (1-\beta/\alpha)q}}{((t-\tau)^{1/\alpha} + |z-y|)^{(d+\alpha-\gamma)q}} \,\mathrm{d}z +$$
$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{1}{((\tau-s)^{1/\alpha} + |x-z|)^{(d+\beta)q}} \frac{(t-\tau)^{k\theta q}}{((t-\tau)^{1/\alpha} + |z-y|)^{(d+\beta)q}} \,\mathrm{d}z \Bigg]^{1/q}.$$

Using inequality (2), we can write

$$|v_{k+1}(s,x,t,y)| \le 8N_{\beta,T} \|b\|_p^T R_{k,T} C^{1/q} \times$$
$$\Bigg[ B\left((k+1)\theta q, 1 + \left(1 - \frac{\beta}{\alpha}\right) q\right) \frac{(t-s)^{(k+1)\theta q + (1-\beta/\alpha)q}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\alpha-\gamma)q}} +$$

$$B\left(1 + k\theta q, \theta q + \frac{\gamma}{\alpha} q\right) \frac{(t-s)^{(k+1)\theta q + q\gamma/\alpha}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\beta)q}} +$$
$$B\left((k+1)\theta q + \frac{\gamma}{\alpha} q, 1 + \left(1 - \frac{\beta}{\alpha}\right) q\right) \frac{(t-s)^{(k+1)\theta q + (1-\beta/\alpha+\gamma/\alpha)q}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\alpha-\gamma)q}} +$$

$$B\left(1 + k\theta q + \left(1 - \frac{\beta}{\alpha}\right), \theta q + \frac{\gamma}{\alpha} q\right) \frac{(t-s)^{(k+1)\theta q + (1-\beta/\alpha+\gamma/\alpha)q}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\alpha-\gamma)q}} +$$
$$B\left((k+1)\theta q + \frac{\gamma}{\alpha} q, 1\right) \frac{(t-s)^{(k+1)\theta q + q\gamma/\alpha}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\beta)q}} +$$

$$B\left(1 + k\theta q + \left(1 - \frac{\beta}{\alpha}\right) q, \theta q\right) \frac{(t-s)^{(k+1)\theta q + (1-\beta/\alpha)q}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\alpha-\gamma)q}} +$$
$$B\left((k+1)\theta q, 1\right) \frac{(t-s)^{(k+1)\theta q}}{((t-s)^{1/\alpha} + |x-y|)^{(d+\beta)q}} +$$

$$B\left(1+k\theta q,\theta q\right)\frac{(t-s)^{(k+1)\theta q}}{((t-s)^{1/\alpha}+|x-y|)^{(d+\beta)q}}\Bigg]^{1/q}.$$

Since the function $(B(x,y))_{x>0,y>0}$ decreases with respect to each of its arguments, we have the inequality

$$|v_{k+1}(s,x,t,y)|\le 8N_{\beta,T}\|b\|_p^T R_{k,T}C^{1/q}\times$$
$$\left(B\left((k+1)\theta q,1\right)^{1/q}\vee B\left(1+k\theta q,\theta q\right)^{1/q}\right)8^{1/q}(1+T^{\gamma/\alpha})\times$$
$$\left(\frac{(t-s)^{(k+1)\theta}}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}}+\frac{(t-s)^{(k+1)\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right),$$

that is, (22) holds for $k+1$. Therefore, it is correct for all $k=0,1,2,\dots$, and relation (23) is true.

Since $\lim_{k\to\infty}B(1+k\theta q,\theta q)=\lim_{k\to\infty}B((k+1)\theta q,1)=0$, the series $\sum_{k=0}^{\infty}v_k(s,x,t,y)$ converges uniformly with respect to $x,y\in\mathbb{R}^d$ and locally uniformly with respect to $0\le s<t$. Let $(v(s,x,t,y))_{0\le s<t,x,y\in\mathbb{R}^d}$ be the sum of this series. The function $v$ is a solution to equation (14) and the estimate

$$|v(s,x,t,y)|\le C_T\left(\frac{1}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}}+\frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right)$$

holds for all $0\le s<t\le T$, $x,y\in\mathbb{R}^d$ and every $T>0$ with some constant $C_T>0$ depended on $T$.

The uniqueness of this solution follows from the fact that the difference $v^*$ of every two such solutions satisfies the equation

$$v^*(s,x,t,y)=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}v_0(s,x,\tau,z)(b(\tau,z),v^*(\tau,z,t,y))\,\mathrm{d}z.$$

Therefore, since $v^*$ satisfies estimate (16), using inequality (2), one can obtain the inequality ($0\le s<t\le T$, $x,y\in\mathbb{R}^d$)

$$|v^*(s,x,t,y)|\le R_{k,T}\times$$
$$\left(\frac{(t-s)^{k\theta}}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}}+\frac{(t-s)^{k\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right),$$

for all $k\in\mathbb{N}$, $T>0$, where $R_{k,T}$ is defined by (23). Since $\lim_{k\to\infty}R_{k,T}T^{k\theta}=0$ for all $T>0$, this means that $v^*(s,x,t,y)\equiv 0$.

Let us define the function $G$ by equality (13) with the function $v$ just constructed. Using (2), (4) and Hölder's inequality we can write the following chain of inequalities:

$$|G(s,x,t,y)|\le N_{0,T}\frac{(t-s)^{1-\gamma/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}+2^{1+1/q}N_{0,T}C_T\|b\|_p^T\times$$
$$\Bigg[\Bigg(\int_s^t\mathrm{d}\tau\int_{\mathbb{R}^d}\frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha}+|x-z|)^{(d+\alpha-\gamma)q}}\times$$

$$\left.\frac{1}{((t-\tau)^{1/\alpha}+|z-y|)^{(d+\beta)q}}\,\mathrm{d}z\right)^{1/q}+$$

$$\left(\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha}+|x-z|)^{(d+\alpha-\gamma)q}}\times\right.$$

$$\left.\left.\frac{(t-\tau)^{(1-\beta/\alpha)q}}{((t-\tau)^{1/\alpha}+|z-y|)^{(d+\beta)q}}\,\mathrm{d}z\right)^{1/q}\right]\le$$

$$C_T\left[\frac{(t-s)^{\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}}+\frac{(t-s)^{1-\gamma/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}}\right],$$

where $C_T$ (in the last expression) is some positive constant, which might be depended on $T$.

Let us prove that $\nabla_\beta G(s,\cdot,t,y)(x)\equiv v(s,x,t,y)$. Using the statement of Lemma 3, we obtain the equality

$$\nabla_\beta G(s,\cdot,t,y)(x)=v_0(s,x,t,y)+$$
$$\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z),v(\tau,z,t,y))\,\mathrm{d}z=v(s,x,t,y),$$

which is true for all $0\le s<t$, $x,y\in\mathbb{R}^d$.

Consequently, the function $G$ is a solution to equation (12) and it satisfies estimate (17). The theorem is proved. □

## 4 Family of evolution operators

Let us define the two-parameter family of operators $\{\mathbb{T}_{st}:0\le s<t\}$ in the space of continuous bounded functions $C_b(\mathbb{R}^d)$ defined by the equality

$$\mathbb{T}_{st}\varphi(x)=\int_{\mathbb{R}^d} G(s,x,t,y)\varphi(y)\,\mathrm{d}y,\quad \varphi\in C_b(\mathbb{R}^d),\ x\in\mathbb{R}^d. \tag{24}$$

Similarly to the proof of Theorem 1 one can prove the following statement.

**Lemma 4.** *The function $w(s,x,t,\varphi)=\int_{\mathbb{R}^d} v(s,x,t,y)\varphi(y)\,\mathrm{d}y$, $0\le s<t$, $x\in\mathbb{R}^d$ and $\varphi\in C_b(\mathbb{R}^d)$ ($v$ is defined in Theorem 1), is a unique (in the class of functions, which satisfy the inequality $|w(s,x,t,\varphi)|\le C_T(t-s)^{-\beta/\alpha}$) solution to the equation*

$$w(s,x,t,\varphi)=w_0(s,x,t,\varphi)+$$
$$\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z),w(\tau,z,t,\varphi))\,\mathrm{d}z, \tag{25}$$

*for all $0\le s<t\le T$, $x\in\mathbb{R}^d$ and every $T>0$. Here $v_0$ is defined by (15) and $w_0(s,x,t,\varphi)=\int_{\mathbb{R}^d} v_0(s,x,t,y)\varphi(y)\,\mathrm{d}y$.*

**Proof.** Equation (25) is obtained from equation (14) multiplying it by the function $\varphi$ and using the Fubini theorem. The justification of the usage of the Fubini theorem is based on estimates (5), (16) (see also (15)). Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}y \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} |v_0(s,x,\tau,z)||b(\tau,z)||v(\tau,z,t,y)||\varphi(y)|\,\mathrm{d}z \le \\
&\quad 4\|\varphi\|\|b\|_p^T N_{\beta,T} C_T \times \\
&\int_{\mathbb{R}^d} \mathrm{d}y \Bigg( \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{1}{((\tau-s)^{1/\alpha}+|z-x|)^{(d+\beta)q}} \times \\
&\quad \frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{(d+\beta)q}}\,\mathrm{d}z + \\
&\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{1}{((\tau-s)^{1/\alpha}+|z-x|)^{(d+\beta)q}} \frac{(t-\tau)^{(1-\beta/\alpha)q}}{((t-\tau)^{1/\alpha}+|y-z|)^{(d+\alpha-\gamma)q}}\,\mathrm{d}z + \\
&\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha}+|z-x|)^{(d+\alpha-\gamma)q}} \frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{q(d+\beta)}}\,\mathrm{d}z + \\
&\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{(\tau-s)^{(1-\beta/\alpha)q}}{((\tau-s)^{1/\alpha}+|z-x|)^{(d+\alpha-\gamma)q}} \times \\
&\quad \frac{(t-\tau)^{(1-\beta/\alpha)q}}{((t-\tau)^{1/\alpha}+|y-z|)^{(d+\alpha-\gamma)q}}\,\mathrm{d}z \Bigg)^{1/q} \le \\
&C_T \int_{\mathbb{R}^d} \left( \frac{1}{((t-s)^{1/\alpha}+|x-y|)^{d+\beta}} + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|x-y|)^{d+\alpha-\gamma}} \right) \mathrm{d}y \le \\
&\quad C_T (t-s)^{-\beta/\alpha}
\end{aligned}
$$

for each $T > 0$ and every $0 \le s < t \le T$, $x \in \mathbb{R}^d$ with some constants $C_T > 0$. Here we used the well-known formula[2] $\int_{\mathbb{R}^d}(a+|x|)^{-d-\varkappa}\,\mathrm{d}x = a^{-\varkappa} B(d,\varkappa)\frac{2\pi^{d/2}}{\Gamma(d/2)}$, which is valid for all $a > 0$ and $\varkappa > 0$. □

The next theorem contains the properties of the family of operators (24).

**Theorem 2.** *Let the assumptions of Theorem 1 hold. Then the following statements are true:*

- *the operators* $\mathbb{T}_{st}$, $0 \le s < t$, *are linear and bounded on* $C_b(\mathbb{R}^d)$*;*
- *if* $\varphi(x) \equiv 1$ *then* $\mathbb{T}_{st}\varphi(x) \equiv 1$*;*
- *the family of operators* $\{\mathbb{T}_{st} : 0 \le s < t\}$ *has an evolution property, that is,* $\mathbb{T}_{s\tau}\mathbb{T}_{\tau t} = \mathbb{T}_{st}$ *for all* $0 \le s < \tau < t$*;*
- $w\text{-}\lim_{s\uparrow t} \mathbb{T}_{st} = I$, *where $I$ is the identical operator, i.e.* $\lim_{s\uparrow t} \mathbb{T}_{st}\varphi(x) = \varphi(x)$, $x \in \mathbb{R}^d$ *for all* $\varphi \in C_b(\mathbb{R}^d)$.

[2]Below we will use this formula without any references.

**Proof.** The linearity of operator $\mathbb{T}_{st}$ is evident. Let us prove its boundedness. If $\varphi \in C_b(\mathbb{R}^d)$, then using inequality (17) we can write ($\|\varphi\| = \max_{x\in\mathbb{R}^d} |\varphi(x)|$)

$$|\mathbb{T}_{st}\varphi(x)| \le \|\varphi\| \int_{\mathbb{R}^d} |G(s,x,t,y)|\,\mathrm{d}y \le$$
$$C_T\|\varphi\| \int_{\mathbb{R}^d} \left( \frac{(t-s)^{\beta/\alpha}}{((t-s)^{1/\alpha} + |x-y|)^{d+\beta}} + \frac{(t-s)^{1-\gamma/\alpha}}{((t-s)^{1/\alpha} + |x-y|)^{d+\alpha-\gamma}} \right) \mathrm{d}y \le$$
$$C_T\|\varphi\|$$

for all $0 \le s < t \le T$ and each $T > 0$. Therefore the operators $\mathbb{T}_{st}$ are bounded.

Next, if $\varphi(x) \equiv 1$, then

$$\mathbb{T}_{st}\varphi(x) = \int_{\mathbb{R}^d} g(s,x,t,y)\,\mathrm{d}y +$$
$$\int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), \int_{\mathbb{R}^d} v(\tau,z,t,y)\,\mathrm{d}y)\,\mathrm{d}z.$$

The function $w(s,x,t,1) = \int_{\mathbb{R}^d} v(\tau,z,t,y)\,\mathrm{d}y$, $0 \le s < t$, $x \in \mathbb{R}^d$, is a solution to equation (25) with $w_0(s,x,t,1) \equiv 0$. So, $w(s,x,t,1) \equiv 0$ and $\mathbb{T}_{st}\varphi(x) = \int_{\mathbb{R}^d} g(s,x,t,y)\,\mathrm{d}y \equiv 1$.

Although the evolution property can be proved in a standard way (see, for example, [11, 12, 14, 16, 17]), we will provide it here. For this, let us choose arbitrary $0 \le s < u < t$, $\varphi \in C_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and consider

$$\mathbb{T}_{st}\varphi(x) = \mathbb{T}^0_{st}\varphi(x) + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z =$$
$$\mathbb{T}^0_{su}(\mathbb{T}^0_{ut}\varphi)(x) + \int_s^u \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z +$$
$$\int_{\mathbb{R}^d} g(s,x,u,y)\,\mathrm{d}y \int_u^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(u,y,\tau,z)(b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z =$$
$$\mathbb{T}^0_{su}(\mathbb{T}_{ut}\varphi)(x) + \int_s^u \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z,$$

where

$$\mathbb{T}^0_{st}\varphi(x) = \int_{\mathbb{R}^d} g(s,x,t,y)\varphi(y)\,\mathrm{d}y. \tag{26}$$

Using (25), one can obtain (by changing the order of integration)

$$w(s,x,t,\varphi) = \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y \int_{\mathbb{R}^d} \nabla_\beta g(s,\cdot,u,z)(x) g(u,z,t,y)\,\mathrm{d}z +$$

$$\int_s^u \mathrm{d}\tau \int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z +$$
$$\int_u^t \mathrm{d}\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_0(s,x,u,y) g(u,y,\tau,z)\,\mathrm{d}y (b(\tau,z), w(\tau,z,t,\varphi))\,\mathrm{d}z =$$

$$w_0(s,x,u,\mathbb{T}_{ut}\varphi)+\int_s^u \mathrm{d}\tau\int_{\mathbb{R}^d} v_0(s,x,\tau,z)(b(\tau,z),w(\tau,z,t,\varphi))\,\mathrm{d}z.$$

Here we used the well-known relation (the Chapman–Kolmogorov equation)

$$g(s,x,t,y)=\int_{\mathbb{R}^d} g(s,x,u,z)g(u,z,t,y)\,\mathrm{d}z,$$

which is true for all $0\le s<u<t$, $x,y\in\mathbb{R}^d$.

Since equation (25) has a unique solution, we state that

$$w(s,x,t,\varphi)=w(s,x,u,\mathbb{T}_{ut}\varphi)$$

for all $\varphi\in C_b(\mathbb{R}^d)$, $x\in\mathbb{R}^d$, $0\le s<u<t$. Therefore $\mathbb{T}_{st}\varphi(x)=\mathbb{T}_{su}(\mathbb{T}_{ut}\varphi)(x)$ and the evolution property is proved.

The last statement of this theorem follows from these two facts: first, using (26) we have the equality $w\text{-}\lim_{s\uparrow t}\mathbb{T}^0_{st}=I$, and second (note that $1\le q<\frac{d+\alpha}{d+1}$),

$$\begin{aligned}\left|\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z),w(\tau,z,t,\varphi))\,\mathrm{d}z\right| &\le\\
\|b\|_p^T N_{0,T}C_T\left(\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}\frac{(\tau-s)^{(1-\gamma/\alpha)q}}{((\tau-s)^{1/\alpha}+|x-z|)^{(d+\alpha-\gamma)q}}(t-\tau)^{-q\beta/\alpha}\,\mathrm{d}z\right)^{1/q} &\le\\
C_T\left(\int_s^t(\tau-s)^{-(q-1)d/\alpha}(t-\tau)^{-q\beta/\alpha}\,\mathrm{d}\tau\right)^{1/q} &=\\
C_T(t-s)^{1-q\beta/\alpha-(q-1)d/\alpha}\to 0,\ \text{as } s\uparrow t.&\end{aligned}$$

This completes the proof of the theorem. □

*Remark* 3. We cannot state that the operators $\mathbb{T}_{st}$ preserve the cone of nonnegative functions. We have no proof that the function $\mathbb{T}_{st}\varphi(x)$ can have negative values if $\varphi(x)\ge 0$, $x\in\mathbb{R}^d$. But the example of an $\alpha$-stable process and $b(t,x)\equiv b\in\mathbb{R}^d$ confirms this fact. Exactly analogously to how it is done in [1] for the case $\beta=\alpha-1$, we can obtain the equality

$$G(s,x,t,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\{i(x-y-(t-s)b|\lambda|^{\beta-1},\lambda)-c(t-s)|\lambda|^\alpha\}\mathrm{d}\lambda$$

in our case. The function $G$ here has negative values as the Fourier transform of a nonpositive definite function.

Thus, the family of evolution operators $\{\mathbb{T}_{st}:0\le s<t\}$ does not define any of the Markov processes but only a pseudo-process possessing a "Markov property".

## 5 Cauchy problem

In this section we fix some $T>0$ and prove that the function $u(s,x,t)=\mathbb{T}_{st}\varphi(x)$, $0\le s<t\le T$, $x\in\mathbb{R}^d$, is a solution (in some sense) to the following Cauchy problem:

$$\frac{\partial}{\partial s}u(s,x,t)+L(s,x)u(s,\cdot,t)(x)=0,\quad 0\le s<t,\ x\in\mathbb{R}^d, \tag{27}$$

$$\lim_{s\uparrow t} u(s,x,t) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{28}$$

for every $t \in (0,T]$, where $L(s,x) = A(s,x) + (b(s,x), \nabla_\beta)$ and $\varphi \in C_b(\mathbb{R}^d)$.

Let a sequence $\{b_n : n \in \mathbb{N}\} \subset L_p([0,T] \times \mathbb{R}^d)$ be such that $L_p$-$\lim_{n\to\infty} b_n = b$ and the functions $b_n$ satisfy the assumptions of Theorem 1. Let $u_n(s,x,t) = \mathbb{T}_{st}^{(n)}\varphi(x)$, where the operators $\mathbb{T}_{st}^{(n)}$ are constructed using Theorems 1, 2 and functions $b_n$ instead of $b$. If the functions $u_n$ satisfy the Cauchy problem (27), (28) with the function $b_n$ instead of $b$ and the pointwise limit $u = \lim_{n\to\infty} u_n$ exists, we will call the function $u$ by the generalized solution to the Cauchy problem (27), (28).

The following auxiliary statement will be useful in constructing this generalized solution.

**Lemma 5.** *Let functions $(\tilde{b}(s,x))_{s\ge 0, x\in\mathbb{R}^d}$ and $(\hat{b}(s,x))_{s\ge 0, x\in\mathbb{R}^d}$ satisfy the assumptions of Theorem 1 with the same $p$. Then the corresponding functions $\tilde{G}$ and $\hat{G}$ satisfy the inequality*

$$|\tilde{G}(s,x,t,y) - \hat{G}(s,x,t,y)| \le C_T \|\tilde{b} - \hat{b}\|_p^T (1 + \|\tilde{b}\|_p^T + \|\hat{b}\|_p^T) \times \frac{1}{((t-s)^{1/\alpha} + |x-y|)^{d+\beta-\gamma}}, \tag{29}$$

*for all $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$ and each $T > 0$ with some positive constant $C_T$.*

**Proof.** Using (13), we can obtain the equality

$$\hat{G}(s,x,t,y) - \tilde{G}(s,x,t,y) = \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} g(s,x,\tau,z) W(\tau,z,t,y))\, \mathrm{d}z \tag{30}$$

for all $0 \le s < t$, $x, y \in \mathbb{R}^d$, where $W(s,x,t,y) = (\hat{b}(s,x), \hat{v}(s,x,t,y)) - (\tilde{b}(s,x), \tilde{v}(s,x,t,y))$, in which $\hat{v}$ and $\tilde{v}$ are solutions to equations obtained from (14) replacing the function $b$ by the functions $\hat{b}$ and $\tilde{b}$, respectively. Relation (14) leads us to the equation

$$W(s,x,t,y) = W_0(s,x,t,y) + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} (v_0(s,x,\tau,z), \hat{b}(s,x)) W(\tau,z,t,y))\, \mathrm{d}z + \int_{\mathbb{R}^d} W_0(s,x,\tau,z)(\tilde{b}(\tau,z), \tilde{v}(\tau,z,t,y))\, \mathrm{d}z,$$

where $W_0(s,x,t,y) = (\hat{b}(s,x) - \tilde{b}(s,x), v_0(s,x,t,y))$. Considering (5), we obtain the inequality

$$|W_0(s,x,t,y) \le |\hat{b}(s,x) - \tilde{b}(s,x)| N_{\beta,T} \left( \frac{1}{((t-s)^{1/\alpha} + |y-x|)^{d+\beta}} + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha} + |y-x|)^{d+\alpha-\gamma}} \right),$$

valid for all $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$ and each $T > 0$. Moreover,

$$\left| \int_s^t \mathrm{d}\tau \int_{\mathbb{R}^d} W_0(s,x,\tau,z)(\tilde{b}(\tau,z), \tilde{v}(\tau,z,t,y))\, \mathrm{d}z \right| \le$$

$$N_{\beta,T}|\hat{b}(s,x)-\tilde{b}(s,x)|C_T\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}|\tilde{b}(\tau,z)|\times$$
$$\left(\frac{1}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\beta}}+\frac{(\tau-s)^{1-\beta/\alpha}}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\right)\times$$
$$\left(\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}+\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\right)\mathrm{d}z=$$
$$C_T|\hat{b}(s,x)-\tilde{b}(s,x)|\sum_{k=1}^{4}I_k,$$

where

$$I_1=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}|\tilde{b}(\tau,z)|\frac{1}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\beta}}\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}\,\mathrm{d}z,$$

$$I_2=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}|\tilde{b}(\tau,z)|\frac{1}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\beta}}\times$$
$$\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\,\mathrm{d}z,$$

$$I_3=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}|\tilde{b}(\tau,z)|\frac{(\tau-s)^{1-\beta/\alpha}}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\times$$
$$\frac{1}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\beta}}\,\mathrm{d}z,$$

$$I_4=\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}|\tilde{b}(\tau,z)|\frac{(\tau-s)^{1-\beta/\alpha}}{((\tau-s)^{1/\alpha}+|z-x|)^{d+\alpha-\gamma}}\times$$
$$\frac{(t-\tau)^{1-\beta/\alpha}}{((t-\tau)^{1/\alpha}+|y-z|)^{d+\alpha-\gamma}}\,\mathrm{d}z.$$

Since $\tilde{b}\in L_p([0,T]\times\mathbb{R}^d)$ for each $T>0$, using inequality (2) we obtain the estimates

$$I_1\le\|\tilde{b}\|_p^T(2CB(1,\theta q))^{1/q}\frac{(t-s)^{\theta}}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}},$$

$$I_2\le\|\tilde{b}\|_p^T\left(2C\left(B\left(1,\theta q+q\frac{\gamma}{\alpha}\right)\vee B\left(1+\left(1-\frac{\beta}{\alpha}\right)q,\theta q\right)\right)\right)^{1/q}\times$$
$$\left(\frac{(t-s)^{\theta+\gamma/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}+\frac{(t-s)^{\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right),$$

$$I_3\le\|\tilde{b}\|_p^T\left(2C\left(B\left(1,\theta q+q\frac{\gamma}{\alpha}\right)\vee B\left(1+\left(1-\frac{\beta}{\alpha}\right)q,\theta q\right)\right)\right)^{1/q}\times$$

$$\left(\frac{(t-s)^{\theta+\gamma/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}+\frac{(t-s)^{\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right),$$

$$I_4 \le \|\tilde{b}\|_p^T\left(2CB\left(1+\left(1-\frac{\beta}{\alpha}\right)q,\theta q+q\frac{\gamma}{\alpha}\right)\right)^{1/q}\frac{(t-s)^{\theta+1-\beta/\alpha+\gamma/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}},$$

where, as it was above, $\theta = 1-((d+\alpha)/p+\beta)/\alpha$, $q = p/(p-1)$ and $C$ is maximum of positive constants derived from inequality (2) in considered four cases. Therefore,

$$\left|\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} W_0(s,x,\tau,z)(\tilde{b}(\tau,z),\tilde{v}(\tau,z,t,y))\,\mathrm{d}z\right| \le C_T\|\tilde{b}\|_p^T|\hat{b}(s,x)-\tilde{b}(s,x)|\times$$
$$\left(\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}+\frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right).$$

Let us denote

$$W_0^*(s,x,t,y) = W_0(s,x,t,y)+\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d} W_0(s,x,\tau,z)(\tilde{b}(\tau,z),\tilde{v}(\tau,z,t,y))\,\mathrm{d}z.$$

Then we can write the equality

$$W(s,x,t,y) = W_0^*(s,x,t,y)+$$
$$\int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}(v_0(s,x,\tau,z),\hat{b}(s,x))W(\tau,z,t,y))\,\mathrm{d}z. \tag{31}$$

This equation can be solved by the method of successive approximations, i.e. a solution of it can be found in the form $W(s,x,t,y)=\sum_{k=0}^{\infty}W_k^*(s,x,t,y)$. The terms of this series satisfy the relation ($k=1,2,\dots$)

$$W_k^*(s,x,t,y) = \int_s^t \mathrm{d}\tau\int_{\mathbb{R}^d}(v_0(s,x,\tau,z),\hat{b}(s,x))W_{k-1}^*(\tau,z,t,y))\,\mathrm{d}z.$$

To justify this, note that

$$|W_0^*(s,x,t,y)| \le C_T(1+\|\tilde{b}\|_p^T)|\hat{b}(s,x)-\tilde{b}(s,x)|\times$$
$$\left(\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}+\frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right).$$

Moreover, by the method of mathematical induction one can prove the estimate

$$|W_k^*(s,x,t,y)| \le R_k|\hat{b}(s,x)|\|\hat{b}-\tilde{b}\|_p^T\times$$
$$\left(\frac{(t-s)^{k\theta}}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}}+\frac{(t-s)^{k\theta+1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right)$$

valid for all $k\ge 1$, where $R_1 = C_T(1+\|\tilde{b}\|_p^T)$ and $C_T$ is some positive constant depended on $T$, and, for $k\ge 2$,

$$R_k = R_{k-1}C_T(1+T^{\gamma/\alpha})(1+\|\tilde{b}\|_p^T)\|\hat{b}\|_p^T\times$$

$$(8C(B(1,k\theta q) \vee B(\theta q,(k-1)\theta q+1)))^{1/q}.$$

Therefore, the series $\sum_{k=0}^{\infty} W_k^*(s,x,t,y)$ converges uniformly with respect to $x, y \in \mathbb{R}^d$ and locally uniformly with respect to $0 \le s < t$. So, its sum $W$ is a solution to equation (31). In addition, we obtain the estimate

$$|W(s,x,t,y)| \le C_T \left((1+\|\tilde{b}\|_p^T)|\hat{b}(s,x)-\tilde{b}(s,x)| + |\hat{b}(s,x)|\|\hat{b}-\tilde{b}\|_p^T\right) \times \left(\frac{1}{((t-s)^{1/\alpha}+|y-x|)^{d+\beta}} + \frac{(t-s)^{1-\beta/\alpha}}{((t-s)^{1/\alpha}+|y-x|)^{d+\alpha-\gamma}}\right), \quad (32)$$

valid for all $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$ and each $T > 0$, where $C_T$ is some positive constant depended on $T$.

Using (30), (32) and (2) with the Hölder inequality, we obtain (29). ☐

**Lemma 6.** *Let the function $w$ be defined in Lemma 4. For every $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$, the inequality*

$$|w(s,x,t,\varphi) - w(s,y,t,\varphi)| \le C_T|x-y|(t-s)^{-(\beta+1)/\alpha}$$

*holds with some positive constant $C_T$ depended on $T$.*

**Proof.** Using (25) we can write down the relation ($0 \le s < t$, $x, y \in \mathbb{R}^d$)

$$w(s,x,t,\varphi) - w(s,y,t,\varphi) = w_0(s,x,t,\varphi) - w_0(s,y,t,\varphi) + \int_s^t d\tau \int_{\mathbb{R}^d} (v_0(s,x,\tau,z) - v_0(s,y,\tau,z))(b(\tau,z), w(\tau,z,t,\varphi))\, dz. \quad (33)$$

Let us remark that (see (6))

$$v_0(s,x,t,y) = \nabla_\beta g_0(s,\cdot,t,y)(x) + \int_s^t d\tau \int_{\mathbb{R}^d} \nabla_\beta g_0(s,\cdot,\tau,z)(x)\Phi(\tau,z,t,y)\, dz.$$

Moreover, for all $0 \le s < t$, $x, y, z \in \mathbb{R}^d$,

$$\begin{aligned}
&\nabla_\beta g_0(s,\cdot,t,z)(x) - \nabla_\beta g_0(s,\cdot,t,z)(y) = \\
&\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\left(e^{i(x-z,\lambda)} - e^{i(y-z,\lambda)}\right)\lambda|\lambda|^{\beta-1}\exp\{-a(t,z,\lambda)(t-s)\}\, d\lambda = \\
&-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(\eta-z,\lambda)}(\lambda, x-y)\lambda|\lambda|^{\beta-1}\exp\{-a(t,z,\lambda)(t-s)\}\, d\lambda = \\
&|x-y|D_{\beta+1}g_0(s,\cdot,t,z)(\eta),
\end{aligned}$$

where $\eta = \theta x + (1-\theta)y$ with some $\theta \in (0,1)$ and the operator $D_{\beta+1}$ is defined by the symbol $\left(\left(\lambda, \frac{x-y}{|x-y|}\right)\lambda|\lambda|^{\beta-1}\right)_{\lambda\in\mathbb{R}^d}$. This symbol satisfies the assumptions of [6, Lemma 4.2], which leads us to the estimate

$$|D_{\beta+1}g_0(s,\cdot,t,z)(\eta)| \le \frac{C}{((t-s)^{1/\alpha}+|z-\eta|)^{d+\beta+1}}.$$

Thus, using (2), we obtain the inequality

$$|v_0(s,x,t,z) - v_0(s,y,t,z)| \le \frac{C|x-y|}{((t-s)^{1/\alpha} + |z-\eta|)^{d+\beta+1}} + \int_s^t d\tau \int_{\mathbb{R}^d} \frac{C|x-y|}{((\tau-s)^{1/\alpha} + |u-\eta|)^{d+\beta+1}} \frac{C}{((t-\tau)^{1/\alpha} + |z-u|)^{d+\alpha-\gamma}}\, du \le |x-y| C_T \left( \frac{1}{((t-s)^{1/\alpha} + |z-\eta|)^{d+\beta+1}} + \frac{(t-s)^{1-(\beta+1)/\alpha}}{((t-s)^{1/\alpha} + |z-\eta|)^{d+\alpha-\gamma}} \right),$$

valid for all $0 \le s < t \le T$, $x, y, z \in \mathbb{R}^d$. And, as consequence, the following inequalities are true for all $0 \le s < t \le T$, $x, y \in \mathbb{R}^d$ and $\varphi \in C_b(\mathbb{R}^d)$:

$$|v_0(s,x,t,\varphi) - v_0(s,y,t,\varphi)| \le C_T |x-y| \|\varphi\| (t-s)^{-(\beta+1)/\alpha};$$

$$\left| \int_s^t d\tau \int_{\mathbb{R}^d} (v_0(s,x,\tau,z) - v_0(s,y,\tau,z))(b(\tau,z), w(\tau,z,t,\varphi))\, dz \right| \le C_T |x-y| \|b\|_p^T \left( \int_s^t (t-\tau)^{-\beta q/\alpha} d\tau \int_{\mathbb{R}^d} \left( \frac{1}{((t-s)^{1/\alpha} + |z-\eta|)^{d+\beta+1}} + \frac{(t-s)^{1-(\beta+1)/\alpha}}{((t-s)^{1/\alpha} + |z-\eta|)^{d+\alpha-\gamma}} \right)^q dz \right)^{1/q} \le C_T |x-y| (t-s)^{1-(1+d/\alpha)/p-(2\beta+1)/\alpha},$$

where $q = p/(p-1)$.

Therefore, using (33) and the fact that $1 - (1 + d/\alpha)/p - \beta/\alpha > (1-\beta)/\alpha$, we obtain the statement of the lemma. □

In the next theorem, we construct a generalized solution to the Cauchy problem formulated at the beginning of the section.

**Theorem 3.** *Let the assumptions of Theorem 1 hold and the function $G$ be constructed there. Then the function*

$$u(s,x,t) = \int_{\mathbb{R}^d} G(s,x,t,y)\varphi(y)\, dy, \quad 0 \le s < t \le T,\ x \in \mathbb{R}^d,$$

*is a generalized solution to the Cauchy problem (27), (28) for each $\varphi \in C_b(\mathbb{R}^d)$.*

**Proof.** Note that equality (28) is proved in Theorem 2 (see the last statement there).

Let us consider a sequence $\{b_n : n \in \mathbb{N}\}$ of $\mathbb{R}^d$-valued functions, which are infinitely continuously differentiable, have compact supports and belong to $L_p([0,T] \times \mathbb{R}^d)$, where $p$ is defined in Theorem 1. Assume that $L_p\text{-}\lim_{n\to\infty} b_n = b$, where $b$ is the function from Theorem 1.

We denote by $v_n$, $w_n$ and $G_n$ the objects that are defined as $v$, $w$ and $G$, respectively, using $b_n$ instead of $b$.

Lemma 5 allows us to state that the sequence of corresponding functions $G_n$ constructed in Theorem 1 using the functions $b_n$ instead of $b$ converges to the function $G$ uniformly with respect to $y \in \mathbb{R}^d$ for each fixed $0 \le s < t \le T$ and $x \in \mathbb{R}^d$.

Let us consider the function

$$f_n(s,x,t)=\int_{\mathbb{R}^d}(b_n(s,x),\nabla_\beta G_n(s,\cdot,t,y)(x))\varphi(y)\,\mathrm{d}y,\quad 0\le s<t\le T,\ x\in\mathbb{R}^d.$$

Remind that $\nabla_\beta G_n(s,\cdot,t,y)(x)=v_n(s,x,t,y)$. Moreover,

$$|f_n(s,x,t)-f_n(s,y,t)|\le|b_n(s,x)||w_n(s,x,t,\varphi)-w_n(s,y,t,\varphi)|+ \\ |b_n(s,x)-b_n(s,y)||w_n(s,y,t,\varphi)|.$$

Since the function $b_n$ is Lipschitz continuous with respect to $x$, uniformly to $s$ and has a compact support, then, as follows from Lemma 6, the function $f_n(s,x,t)$ is Lipschitz continuous with respect to $x$, uniformly to $s\in[0,t)$ for every fixed $t>0$.

Therefore, the function $u_n(s,x,t)=\int_{\mathbb{R}^d}G_n(s,x,t,y)\varphi(y)\,\mathrm{d}y$ is a solution to the Cauchy problem (27), (28) with the function $b_n$ instead of $b$ for every $t\in(0,T]$ (see [6, Th. 4.1.]).

So, since $u(s,x,t)=\lim_{n\to\infty}u_n(s,x,t)$, the function $u$ is a generalized solution to the Cauchy problem (27), (28) for each $\varphi\in C_b(\mathbb{R}^d)$. □

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