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Heat equation with a general stochastic measure in a bounded domain

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Abstract A stochastic heat equation on $[0, T] \times B$, where B is a bounded domain, is considered. The equation is driven by a general stochastic measure, for which only σ-additivity in probability is assumed. The existence, uniqueness and Hölder regularity of the solution are proved.

Keywords Heat equation, mild solution, stochastic measure, Hölder regularity
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1 Introduction

In this paper we consider the following boundary value problem:

$$\begin{cases} du(t,x) = a^2\Delta_x u(t,x)dt + f(t,x,u(t,x))dx + \sigma(t,x)d\mu(t)\,,\,(t,x)\in\bar{D}, \\ u(t,x) = 0,\ (t,x)\in S, \quad u(0,x) = u_0(x),\ x\in B\,. \end{cases} \tag{1}$$

Here B is a bounded domain in $\mathbb{R}^d$, $D = (0,T)\times B$, $\bar{D}$ is a closure of D, $S = (0,T]\times\partial B$, Δ_x is the Laplace operator

$$\Delta_x g(x) = \sum_{i=1}^{d} \frac{\partial^2 g(x)}{\partial x_i^2}.$$

Stochastic measure μ is defined on sets of time variable. The conditions on f, u_0, σ and μ, as well as the definition of the solution of (1), are formulated in the following sections.

Various properties of the solutions of different stochastic partial differential equations, where stochastic noise is generated by a general stochastic measure, were previously investigated in many articles. For example, averaging principle for a fractional heat equation driven by a general stochastic measure was established in [21], the behavior of the solution of parabolic equation as time variable goes to infinity was studied in [14], the existence and uniqueness of the solution of the parabolic equation driven by a σ-finite stochastic measure were proved in [22]. In the mentioned articles the spatial variable took values in $\mathbb{R}$, while in [2] the stochastic cable equation on $[0, T] \times [0, 1]$ was considered. On the other hand, stochastic parabolic equation with random coefficients, where stochastic noise is generated by a two-parameter Wiener process, was studied in [1], stochastic parabolic equation driven by a Lévy process was considered in [10], various properties of the solution of stochastic heat equation on bounded polygonal domains in $\mathbb{R}^2$ were established in [13] and [4], the regularity of solutions of nonhomogeneous Dirichlet boundary value problems for stochastic parabolic equations on bounded domains in $\mathbb{R}^2$ was investigated in [5]. Note that the results and methods of [3] are widely used in this article; the difference between them is mentioned in the conclusion.

The rest of the paper is organized in the following way. In Section 2 some properties of stochastic measures and particular functional spaces are mentioned. The main result of the paper is formulated in Section 3 and proved in Section 4, along with related auxiliary statements.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space and $\mathcal{B}$ be an arbitrary σ-algebra on the sets of X. Denote by $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ the set of all real-valued random variables defined on $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0. In this paper we assume everywhere that $X = [0, T]$, $\mathcal{B}$ is a Borel σ-algebra on $[0, T]$.

Consider some examples of SMs. If M_t is a square integrable martingal then $\mu(A) = \int_0^T \mathbf{1}_A(t)\, dM_t$ is an SM. α-stable random measure on $\mathcal{B}$ for $\alpha \in (0, 1) \cup (1, 2]$, as it is defined in [20, Sections 3.2-3.3], is an SM by means of Definition 1. Let W_t^H be a fractional Brownian motion with the Hurst index $H > 1/2$ and $f : [0, T] \to \mathbb{R}$ be a bounded measurable function, then function of sets $\mu(A) = \int_0^T f(t)\mathbf{1}_A(t)\, dW_t^H$ is an SM, as follows from [15, Theorem 1.1]. More stochastic measures can be found in [19].

The definition of the integral $\int_A g\, d\mu$, where $g : \mathbb{R} \to \mathbb{R}$ is a deterministic measurable function, $A \in \mathcal{B}$ and μ is an SM, and its basic properties are given in [11, Chapter 7]. Note that every bounded measurable g is integrable with respect to (w. r. t.) any μ.

In the sequel, μ denotes an SM, C and $C(\omega)$ denote positive constant and positive random constant, respectively, whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a. s.).

Recall the following important lemma.

Lemma 1. *(Lemma 3.1 in [16]) Let $\phi_l : \mathbb{R} \to \mathbb{R}$, $l \geq 1$, be measurable functions such that $\tilde{\phi}(x) = \sum_{l=1}^{\infty} |\phi_l(x)|$ is integrable w.r.t. μ on $\mathbb{R}$. Then*

$$\sum_{l=1}^{\infty}\Big(\int_{\mathbb{R}} \phi_l\, d\mu\Big)^2 < \infty \quad a.\ s.$$

We consider the *Besov spaces* $B_{22}^{\alpha}([c,d])$, $0 < \alpha < 1$, with the standard norm

$$\|g\|_{B_{22}^{\alpha}([c,d])} = \|g\|_{\mathsf{L}_2([c,d])} + \Big(\int_0^{d-c} (\omega_{2,[c,d]}(g,r))^2 r^{-2\alpha-1}\, dr\Big)^{1/2}, \tag{2}$$

where

$$\omega_{2,[c,d]}(g,r) = \sup_{0\leq h\leq r}\Big(\int_c^{d-h} |g(s+h)-g(s)|^2\, ds\Big)^{1/2}.$$

For any $T > 0$ and all $n \geq 0$, put

$$d_{kn}^{(T)} = k2^{-n}T, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(T)} = (d_{(k-1)n}^{(T)}, d_{kn}^{(T)}], \quad 1 \leq k \leq 2^n .$$

For the estimates of stochastic integral we use the following result.

Lemma 2. *(Lemma 3 in [17] or Lemma 3.3 in [18]) Let Z be an arbitrary set, and function $q(z,s) : Z \times [0,T] \to \mathbb{R}$ be such that all paths $q(z,\cdot)$ are continuous on $[0,T]$. Denote*

$$q_n(z,s) = \sum_{1\leq k\leq 2^n} q\big(z, d_{(k-1)n}^{(T)}\big)\mathbf{1}_{\Delta_{kn}^{(T)}}(s).$$

Then the random function

$$\eta(z) = \int_A q(z,s)\, d\mu(s), \ z \in Z, \ A \subset [0,T],$$

has a version

$$\begin{aligned}\widetilde{\eta}(z) &= \int_A q_0(z,s)\, d\mu(s)\\ &+ \sum_{n\geq 1}\Big(\int_A q_n(z,s)\, d\mu(s) - \int_A q_{n-1}(z,s)\, d\mu(s)\Big)\end{aligned} \tag{3}$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned}|\widetilde{\eta}(z)| &\leq |q(z,0)\mu(A)| + \sum_{n\geq 1}\sum_{1\leq k\leq 2^n} |q(z,d_{(k-1)n}^{(T)}) - q(z,d_{(k'-1)(n-1)}^{(T)})||\mu(\Delta_{kn}^{(T)} \cap A)|\\ &\leq |q(z,0)\mu(A)| + \Big\{\sum_{n\geq 1} 2^{n\beta} \sum_{1\leq k\leq 2^n} |q(z,d_{kn}^{(T)}) - q(z,d_{(k-1)n}^{(T)})|^2\Big\}^{1/2}\end{aligned}$$

$$\times\Big\{\sum_{n\ge 1}2^{-n\beta}\sum_{1\le k\le 2^n}|\mu(\Delta_{kn}^{(T)}\cap A)|^2\Big\}^{1/2}, \tag{4}$$

where $\Delta_{kn}^{(T)}\subset\Delta_{k'(n-1)}^{(T)}$.

Note that for $\alpha=(\beta+1)/2$

$$\Big\{\sum_{n\ge 1}2^{n\beta}\sum_{1\le k\le 2^n}|q\big(z,d_{kn}^{(T)}\big)-q\big(z,d_{(k-1)n}^{(T)}\big)|^2\Big\}^{1/2}\le C\|q(z,\cdot)\|_{B_{22}^{\alpha}([0,T])}, \tag{5}$$

as follows from Theorem 1.1 [9]. Moreover, Lemma 1 implies that for each $\beta>0$, $T>0$, $A\in\mathcal{B}([0,T])$

$$\sum_{n\ge 1}2^{-n\beta}\sum_{1\le k\le 2^n}|\mu(\Delta_{kn}^{(T)}\cap A)|^2<+\infty\quad\text{a. s.}$$

We also use the following notations, that were introduced, for example, in [7].

$$d(P,Q)=\big(|x_1-x_2|^2+|t_1-t_2|\big)^{1/2},\qquad P=(t_1,x_1),\ Q=(t_2,x_2);$$
$$\|u\|_{\alpha}^{D}=\sup_{D}|u|+\sup_{P,Q\in D}\frac{|u(P)-u(Q)|}{d(P,Q)^{\alpha}};$$
$$\|u\|_{1+\alpha}^{D}=\|u\|_{\alpha}^{D}+\Big\|\frac{\partial u}{\partial x}\Big\|_{\alpha}^{D}.$$

Let $R\subset S\cup\{0\}\times\bar{B}$, $S_\tau=(0,\tau]\times\partial B$. Denote

$$\bar{d}_P=d((S_\tau\cup\{0\}\times\bar{B})\setminus R,P);$$
$$\bar{d}_{PQ}=\min(\bar{d}_P,\bar{d}_Q);$$
$$M_{p,j}^{R,D}[g]=\sup_{P\in D}\bar{d}_P^{\,p+j}|D_x^j g(P)|;$$
$$M_{p,j+\alpha}^{R,D}[g]=\sup_{P,Q\in D}\bar{d}_{PQ}^{\,p+j+\alpha}\frac{|D_x^j g(P)-D_x^j g(Q)|}{d(P,Q)^{\alpha}};$$
$$\|g\|_{p,m}^{R,D}=\sum_{j=0}^{m}(M_{p,j+\alpha}^{R,D}[g]+M_{p,j}^{R,D}[g]).$$

It can be easily seen that functions $\|\cdot\|_{1+\alpha}^{D}$ and $\|\cdot\|_{p,m}^{R,D}$ are norms. The spaces of functions with finite norms $\|\cdot\|_{1+\alpha}^{D}$, $\|\cdot\|_{p,m}^{R,D}$ are Banach spaces.

3 Formulation of the problem and the main result

Denote $\mathcal{L}u=a^2\Delta_x u-\frac{\partial u}{\partial t}$. We consider the solution of (1) in the mild sense, i.e. the measurable random function $u(t,x)=u(t,x,\omega):[0,T]\times B\times\Omega\to\mathbb{R}$ that satisfies

$$u(t,x)=\int_B G(t,x;0,y)u_0(y)dy+\int_0^t ds\int_B G(t,x;s,y)f(s,y,u(s,y))dy$$

$$+\int_{(0,t]} d\mu(s)\int_B G(t,x;s,y)\sigma(s,y)dy, \quad (6)$$

where $G(t,x;s,y)$ is a Green's function of the equation $\mathcal{L}u = 0$ in D. According to [12, Chapter IV, §16, Theorem 16.3], the following inequalities hold for some constants λ, $M > 0$:

$$|G(t,x;s,y)| \le M(t-s)^{-d/2}e^{-\frac{\eta|x-y|^2}{t-s}}, \quad (7)$$

$$\left|\frac{\partial G(t,x;s,y)}{\partial x_i}\right| \le M(t-s)^{-(d+1)/2}e^{-\frac{\eta|x-y|^2}{t-s}}, \quad (8)$$

$$\left|\frac{\partial G(t,x;s,y)}{\partial t}\right| \le M(t-s)^{-d/2-1}e^{-\frac{\eta|x-y|^2}{t-s}}. \quad (9)$$

In our assertions we often refer to the following definition, which can be found in [8, p. 437].

Definition 2. The domain S belongs to a class $A^{m+\beta}$ (A^m) in $\mathbb{R}^d$ ($\mathbb{R}^{d+1}$) if for every point P of $\bar{S}$ there exists a sphere with center P and a function χ, which belongs to a class $A^{m+\beta}$ (A^m), such that for certain $i \le d$

$$x_i = \chi(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d) \qquad (x_i = \chi(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d,t))$$

inside the sphere.

We consider domain B, functions u_0, f, σ that satisfy the following assumptions.

Assumption 1. There exists $\beta \in (0,1)$ such that $\bar{S}$ belongs to a class $A^{1+\beta}$ in $\mathbb{R}^{d+1}$.

Assumption 2. Function $u_0 : \bar{B}\times\Omega \to \mathbb{R}$ is measurable and bounded for each fixed $\omega \in \Omega$.

Assumption 3. Function $f(s,y,z) : [0,T]\times\bar{B}\times\mathbb{R} \to \mathbb{R}$ is measurable, bounded and

$$|f(s,y_1,z_1) - f(s,y_2,z_2)| \le L_f\big(|y_1-y_2|^{\beta(f)} + |z_1-z_2|\big)$$

for some constants $L_f > 0$, $\beta(f) > 0$ and all $s \in [0,T]$, y_1 , $y_2 \in \bar{B}$, z_1 , $z_2 \in \mathbb{R}$.

Assumption 4. Function $\sigma(s,y) : [0,T]\times\bar{B} \to \mathbb{R}$ is measurable, bounded and

$$|\sigma(s_1,y_1) - \sigma(s_2,y_2)| \le L_\sigma(|y_1-y_2|^{\beta(\sigma)} + |s_1-s_2|^{\beta(\sigma)})$$

for some constants $L_\sigma > 0$, $1 > \beta(\sigma) > 1/2$ and all $s_1, s_2 \in [0,T]$, y_1 , $y_2 \in \bar{B}$.

In some statements we refer to the following assumptions on μ.

Assumption 5. Stochastic measure μ has bounded paths:

$$|\mu((0,t])| \le C_\mu(\omega), \quad (10)$$

for random constant $C_\mu(\omega)$ and all $t \in [0,T]$.

Assumption 6. Stochastic measure μ has Hölder continuous paths:

$$|\mu((s_1,s_2])| \le C(\omega)|s_1-s_2|^{\beta(\mu)},$$

for random constant $C(\omega)$, deterministic constant $\beta(\mu)$ and all $s_1,\ s_2 \in [0,T]$.

For example, stochastic measure $\mu(A) = \int_0^T \mathbf{1}_A(t)\, dW_t^H$ satisfies Assumption 6 with $\beta(\mu) = H$. Also, note that Assumption 6 implies Assumption 5.

We can formulate the main result of the paper.

Theorem 1. *Let Assumptions 1–4 hold.*

1. *Then solution of* (6) *exists and is unique in the following sense: if $u_1(t,x)$ and $u_2(t,x)$ are two solutions of* (6)*, then, for each $(t,x) \in [0,T] \times \bar{B}$, $u_1(t,x) = u_2(t,x)$ a. s.*
2. *In addition, assume that Assumption 5 holds. Then, for each fixed $\delta > 0$, $\gamma_1 < \beta(\sigma)$ and set B', $d(\partial B, \bar{B}') > 0$, a random function $u(t,x)$, which is the solution of* (6)*, has a version $\tilde{u}^{(x)}(t,x)$, which satisfies*

$$|\tilde{u}^{(x)}(t,x_1) - \tilde{u}^{(x)}(t,x_2)| \le L_{\tilde{u}^{(x)}} |x_1 - x_2|^{\gamma_1}, \quad \forall t \in [\delta, T],\ x_1,\ x_2 \in \bar{B}', \tag{11}$$

for a random constant $L_{\tilde{u}^{(x)}} = L_{\tilde{u}^{(x)}}(\omega) > 0$.

3. *In addition, assume that Assumption 6 holds. Then, for each fixed $\delta > 0$, B', $d(\partial B, \bar{B}') > 0$, $\gamma_1 < \beta(\sigma)$, $\gamma_2 < \beta(\mu) \wedge \big(\beta(\sigma)/(4 - 2\beta(\sigma))\big)$, a random function $u(t,x)$, which is the solution of* (6)*, has a version $\tilde{u}(t,x)$, which satisfies*

$$|\tilde{u}(t_1,x_1) - \tilde{u}(t_2,x_2)| \le L_{\tilde{u}}\big(|x_1 - x_2|^{\gamma_1} + |t_1 - t_2|^{\gamma_2}\big),$$
$$\forall t_1,\ t_2 \in [0,T],\ x_1,\ x_2 \in \bar{B}',$$

for a random constant $L_{\tilde{u}} = L_{\tilde{u}}(\omega) > 0$.

4 Auxiliary lemmas and proof of the main result

To prove Theorem 1, we need the following results about stochastic integral.

Lemma 3. *Let Assumptions 1, 2, 4, 5 hold. Then, for arbitrary set B', $d(\partial B, \bar{B}') > 0$, the random process*

$$\zeta(x) = \int_{(0,t]} d\mu(s) \int_B G(t,x;s,y)\sigma(s,y)dy \tag{12}$$

has a version of a kind (3)*, which is Hölder continuous with the exponent γ_1 on B' for all $t \in [0,T]$, $\gamma_1 < \beta(\sigma)$.*

Proof. Let

$$q(z,s) = \begin{cases} \int_B (G(t,x_1;s,y) - G(t,x_2;s,y))\sigma(s,y)dy, \text{ if } 0 \le s < t, \\ \sigma(t,x_1) - \sigma(t,x_2), \text{ if } t \le s \le T. \end{cases} \tag{13}$$

Here $z = (t, x_1, x_2)$. The function (13) is continuous in $[0,T]$ as a function of s, as follows from

$$\int_B G(t,x;s,y)\sigma(s,y)dy \to \sigma(t,x), \quad s \to t-. \tag{14}$$

We give the brief proof of (14). Fix $\varepsilon > 0$. Then for all $0 \le r < t$

$$\Big|\int_B G(t,x;s,y)\sigma(s,y)dy - \sigma(t,x)\Big| \le \Big|\int_B G(t,x;s,y)\big(\sigma(s,y)-\sigma(r,y)\big)dy\Big|$$
$$+\Big|\int_B G(t,x;s,y)\sigma(r,y)dy - \sigma(r,x)\Big| + |\sigma(r,x)-\sigma(t,x)|$$
$$\le C|t-r|^{\beta(\sigma)} + \Big|\int_B G(t,x;s,y)\sigma(r,y)dy - \sigma(r,x)\Big|.$$

We can choose r such that $C|t-r|^{\beta(\sigma)} \le \varepsilon/2$. On the other hand,

$$\int_B G(t,x;s,y)\sigma(r,y)dy \to \sigma(r,x), \quad s \to t-,$$

as follows from [7, Chapter 3, Sec. 7, Definition]. Therefore, there exists $\delta > 0$ which may depend on t and x such that for all $s > t-\delta$,

$$\Big|\int_B G(t,x;s,y)\sigma(r,y)dy - \sigma(r,x)\Big| < \varepsilon/2,$$

, and the convergence (14) holds. Therefore, we can apply Lemma 2 for q, which is defined by (13). At first, we estimate $\omega_{2,[0,t]}(q,r)$. Consider the difference

$$q(z,s+h)-q(z,s) = \int_B \big(G(t,x_1;s,y)-G(t,x_2;s,y)\big)\big(\sigma(s+h,y)-\sigma(s,y)\big)dy$$
$$+\int_B \big(G(t,x_1;s+h,y)-G(t,x_2;s+h,y)$$
$$-G(t,x_1;s,y)+G(t,x_2;s,y)\big)\sigma(s+h,y)dy = I_1 + I_2.$$

I_1 is estimated in the same way as $A_2(s,h)$ in [3], where we estimate the derivatives using (8). More precisely, we get

$$|I_1| \le Ch^{\beta(\sigma)}\int_B |G(t,x_1;s,y)-G(t,x_2;s,y)|dy$$
$$\le Ch^{\beta(\sigma)}|x_1-x_2|\int_{\mathbb{R}^d} dy\int_0^1 \big|\mathrm{grad}_x G(t,\theta x_1+(1-\theta)x_2,s,y)\big|d\theta$$
$$\le Ch^{\beta(\sigma)}|x_1-x_2|\int_{\mathbb{R}^d} dy\int_0^1 (t-s)^{-\frac{d+1}{2}} e^{-\frac{\eta(\theta x_1+(1-\theta)x_2-y)}{t-s}}d\theta$$
$$\le C\frac{h^{\beta(\sigma)}|x_1-x_2|}{(t-s)^{1/2}}\int_0^1 d\theta\int_{\mathbb{R}^d} e^{-\frac{\eta(\theta x_1+(1-\theta)x_2-y)}{t-s}}\frac{dy}{(t-s)^{d/2}} = C\frac{h^{\beta(\sigma)}|x_1-x_2|}{(t-s)^{1/2}}.$$

Therefore, we obtain that

$$\int_0^{t-h} I_1^2 ds \le Ch^{2\beta(\sigma)}|x_1-x_2|^2(C+|\ln h|) \le Ch^{2\gamma}|x_1-x_2|^2,\ \gamma > 1/2. \quad (15)$$

Denote

$$v(t,x,s) = \int_B G(t+\tau,x;\tau,y)\sigma(s,y)dy.$$

Now we apply the definition in [7, Chapter 3, Sec. 7] to obtain the properties of v:

$$
\begin{aligned}
\mathcal{L}v = \int_B \mathcal{L}G(t+\tau, x; \tau, y)\sigma(s, y)dy = 0,\\
v(t, x, s)|_{(t,x)\in S} = \Big(\int_B G(t+\tau, x; \tau, y)\sigma(s, y)dy\Big)\Big|_{(t,x)\in S}\\
= \Big(\int_B G(t+\tau, x; \tau, y)\sigma(s, y)dy\Big)|_{(t+\tau,x)\in S} \overset{[7],(7.4)}{=} 0,\ t \le T-\tau,\\
v(0, x, s) = \lim_{t\to 0}\int_B G(t+\tau, x; \tau, y)\sigma(s, y)dy \overset{[7],(7.3)}{=} \sigma(s, x).
\end{aligned}
\tag{16}
$$

Now consider (16) as a boundary value problem for each fixed s. Theorem 11 in [8, Sec. 1] implies that it has unique solution; consequently, v does not depend on τ. Therefore,

$$
I_2 = v(t-s-h, x_1, s+h) - v(t-s-h, x_2, s+h) - v(t-s, x_1, s+h) + v(t-s, x_2, s+h).
$$

We can construct the extension of a function $\sigma(s, y)$, which is bounded and Hölder continuous in $[0, T]\times\mathbb{R}^d$ with the same exponent. This follows, for example, from [7, Chapter 3, Theorem 2, p. 60]. Now we note that $v(t, x, s) = v^{(1)}(t, x, s) - v^{(2)}(t, x, s)$, where $v^{(1)}$ is a solution of the Cauchy problem

$$
\begin{cases}
\mathcal{L}v^{(1)}(t, x, s) = 0,\\
v^{(1)}(t, x, s)|_{t=0} = \sigma(s, x),
\end{cases}
$$

in $[0, T]\times\mathbb{R}^d$, and $v^{(2)}$ is a solution of a boundary value problem

$$
\begin{cases}
\mathcal{L}v^{(2)} = 0,\\
v^{(2)}|_{t=0} = 0, \quad v^{(2)}|_S = v^{(1)},
\end{cases}
$$

in $[0, T]\times B$. We represent I_2 in a form $I_{21} - I_{22}$, where

$$
\begin{aligned}
I_{2i} = v^{(i)}(t-s-h, x_1, s+h) - v^{(i)}(t-s-h, x_2, s+h)\\
- v^{(i)}(t-s, x_1, s+h) + v^{(i)}(t-s, x_2, s+h), \quad i = 1, 2.
\end{aligned}
$$

According to [8, Sec. 4, Theorem 2], $v^{(1)}$ can be represented in the form

$$
v^{(1)}(t, x, s) = \int_{\mathbb{R}^d} p(t, x-y)\sigma(s, y)dy,
$$

where

$$
p(t, x) = \frac{1}{(4a^2\pi t)^{d/2}} e^{-\frac{|x|^2}{4a^2 t}}.
$$

Therefore,

$$
I_{21} = \int_{\mathbb{R}^d}\big(p(t-s-h, x_1-y) - p(t-s-h, x_2-y) - p(t-s, x_1-y) + p(t-s, x_2-y)\big)
$$

$$\times \sigma(s+h,y)dy = A_1(s,h)$$

in the notations of [3]. Recall the estimates for $A_1(s,h)$ from the mentioned article:

$$\begin{aligned}|I_{21}| &\le \int_{\mathbb{R}^d} |p(t-s,x_1-y)-p(t-s,x_2-y)||\sigma(s+h,y)-\sigma(s,y)|dy\\ &\le C|x_1-x_2|^{\beta(\sigma)} \int_{\mathbb{R}^d} dy \int_{t-s-h}^{t-s} \tau^{-\frac{d}{2}-1} e^{-\frac{C|y|^2}{\tau}} d\tau\\ &= C|x_1-x_2|^{\beta(\sigma)} \int_{t-s-h}^{t-s} \tau^{-1} d\tau \int_{\mathbb{R}^d} \tau^{-\frac{d}{2}} e^{-\frac{C|y|^2}{\tau}} dy\\ &= C|x_1-x_2|^{\beta(\sigma)} \ln\frac{t-s}{t-s-h}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^{t-h} I_{21}^2 ds \le C|x_1-x_2|^{2\beta(\sigma)} \int_0^{t-h} \ln^2\frac{t-s}{t-s-h} ds\\ &\le Ch|x_1-x_2|^{2\beta(\sigma)} \int_0^{+\infty} \ln^2(1+1/u)du = Ch|x_1-x_2|^{\beta(\sigma)}.\end{aligned} \tag{17}$$

On the other hand, estimating $A_1(s,h)$ in a similar way to estimation [3.54] in [18], we get

$$\begin{aligned}|I_{21}| &\le \Big|\int_{\mathbb{R}^d} \big(p(t-s-h,x_1-y)-p(t-s,x_1-y)\big)\sigma(s+h,y)dy\Big|\\ &+\Big|\int_{\mathbb{R}^d} \big(p(t-s-h,x_2-y)-p(t-s,x_2-y)\big)\sigma(s+h,y)dy\Big|\\ &= C\Big|\int_{\mathbb{R}^d} e^{-|v|^2}\big(\sigma(s+h,x_1+2av\sqrt{t-s-h})-\sigma(s+h,x_1+2av\sqrt{t-s})\big)dv\Big|\\ &+C\Big|\int_{\mathbb{R}^d} e^{-|v|^2}\big(\sigma(s+h,x_2+2av\sqrt{t-s-h})-\sigma(s+h,x_2+2av\sqrt{t-s})\big)dv\Big|\\ &\le C\int_{\mathbb{R}^d} e^{-|v|^2}|v(\sqrt{t-s-h}-\sqrt{t-s})|^{\beta(\sigma)}dv \le Ch^{\beta(\sigma)}(t-s)^{-\beta(\sigma)/2},\end{aligned} \tag{18}$$

where for the i-th summand we used the substitutions

$$v = \frac{y-x_i}{2a\sqrt{t-s-h}}, \quad v = \frac{y-x_i}{2a\sqrt{t-s}}.$$

From (4) and (18) it follows that

$$\int_0^{t-h} I_{21}^2 ds \le Ch^{2\beta(\sigma)+\lambda(1-2\beta(\sigma))}|x_1-x_2|^{2\lambda\beta(\sigma)},\ 0<\lambda<1. \tag{19}$$

Now we estimate I_{22}. Fix $\alpha \in (0,1)$. As the functions $v(t,x,s)$ and $v^{(1)}(t,x,s)$ are bounded in $\bar{Q}$ uniformly on $t,\ x,\ s$, the same holds for $v^{(2)}(t,x,s)$. Let us prove it, for example, for v:

$$|v(t,x,s)| \le \int_B |G(t,x;0,y)||\sigma(s,y)|dy \overset{(7)}{\le} C\int_{\mathbb{R}^d} t^{-d/2} e^{-\frac{\eta(x-y)^2}{t}} dy \le C.$$

It is possible to take the domains B'' and B''' such that $\bar{B}'' \subset B$, $\bar{B}''' \subset B''$, $\bar{B}' \subset B'''$ and $\partial B''$, $\partial B''' \in A^3$ (see Definition 2).

Remark 1. *The sets B'' and B''' can be easily constructed; let us do it, for example, for B''. Introduce the notations*

$$\begin{aligned}
\eta_1(x) &= Ce^{\frac{1}{|x|^2-1}}\mathbf{1}_{\{|x|<1\}},\ x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \eta_1(x)\,dx = 1,\\
\eta_\varepsilon(x) &= \varepsilon^{-d}\eta_1(x\varepsilon^{-1}),\\
B'_\varepsilon &= B' \cup \{x \in \mathbb{R}^d : d(x, \partial B') < \varepsilon\},
\end{aligned}$$

and take $\kappa_\varepsilon(x) = \int_{B'_\varepsilon} \eta_\varepsilon(x-y)dy$. For a sufficiently small $\varepsilon > 0$, $\kappa_\varepsilon(x) = 1$, $x \in B'$, $\kappa_\varepsilon(x) = 0$, $x \notin B$. Let $B'' = \kappa_\varepsilon^{-1}((1/2, 1])$ and consider arbitrary $x^* \in \partial B''$. Obviously, there exists an index j such that $\frac{\partial \kappa_\varepsilon(x^*)}{\partial x_j} \neq 0$, and, consequently, a function $h \in C^\infty(\mathbb{R}^{d-1})$ such that $\partial B''$ can be locally represented in a form $x_j = h(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$.

Denote $S'' = [0, T] \times \partial B''$, $B''_0 = \{0\} \times B''$. It is obvious that $v^{(2)} \in C([0, T] \times \bar{B}'')$. As $[0, T] \times \bar{B}''$ is a compact, there exist polynomials Ψ_m such that $\Psi_m \to v^{(2)}$ in $C([0, T] \times \bar{B}'')$. Therefore, there exists a sequence $\psi_m(t, x) = \Psi_m(t, x) - \Psi_m(0, x)$ such that $\psi_m \in C^3(S'' \cup B''_0)$, $\psi_m = 0$ on B''_0 and $\psi_m \to v^{(2)}$ on $C(S'' \cup B''_0)$. Let $v_m^{(2)}$ be a solution of the boundary value problem

$$\begin{cases} \mathcal{L}v_m^{(2)} = 0,\\ v_m^{(2)}|_{S''\cup B''_0} = \psi_m, \end{cases}$$

on $[0, T] \times B''$. Theorem 7 in [7, Chap. III, Sec. 3] implies that $v_m^{(2)} \in C_{2+\alpha}([0, T] \times \bar{B}''_0)$. Therefore, we can apply Theorem 4 in [7, Chap. IV, Sec. 7] for the functions $v_m^{(2)} - v_n^{(2)}$, where B'', B''' and $(0, T) \times B''$ are the sets R, R_0 and D in the formulation of the theorem, respectively. Using, in addition, the maximum principle, we obtain

$$|v_m^{(2)} - v_n^{(2)}|_{0,2+\alpha}^{R_0,D} \leq K|v_m^{(2)} - v_n^{(2)}|_0 \leq K|\psi_m - \psi_n|_0^{S''\cup B''_0} \to 0, m, n \to \infty,$$

and sequence $\{v_m^{(2)} : m \geq 1\}$ converges in $\|\cdot\|_{0,2+\alpha}^{R_0,D}$ to a limit function $\tilde{v}^{(2)}$; for example, $M_{0,0}^{R_0,D}[v_m^{(2)} - \tilde{v}^{(2)}] \to 0$, $m \to 0$. On the other hand, according to [7, Chap. III, Sec. 6, Corollary of Theorem 15], sequence $\{v_m^{(2)} : m \geq 1\}$ converges uniformly to $v^{(2)}$ on $[0, T] \times \bar{B}''$. Therefore, $\tilde{v}^{(2)} = v^{(2)}$ and

$$|v^{(2)}|_{0,2+\alpha}^{R_0,D} \leq K|v^{(2)}|_0 =: K_1,$$

where constants K and K_1 depend only on a, α, B''' and B''. This implies the inequality

$$|I_{22}| = \int_{t-s-h}^{t-s}\left|\frac{\partial v^{(2)}(w, x_1, s+h)}{\partial w} - \frac{\partial v^{(2)}(w, x_2, s+h)}{\partial w}\right|dw$$

$$\leq \int_{t-s-h}^{t-s} K_1 \frac{|x_1 - x_2|^{\alpha}}{\bar{d}_{x_1 x_2}^{2+\alpha}} dw \leq C \int_{t-s-h}^{t-s} \frac{|x_1 - x_2|^{\alpha}}{d(\bar{B}', \partial B''')^{2+\alpha}} dw = Ch|x_1 - x_2|^{\alpha}. \quad (20)$$

We can choose γ in (15), λ in (19), α in (20) such that

$$\omega_{2,[0,t]}(q, r) \leq C r^{\theta_1} |x_1 - x_2|^{\gamma_1}, \quad \theta_1 > 1/2.$$

Estimating $q(z, s)$ in the same way as I_2 for $s < t$ and using Hölder continuity of σ for $t \leq s \leq T$, we obtain that for each $\tilde{\gamma}_1 < \beta(\sigma)$,

$$|q(z, s)| \leq C|x_1 - x_2|^{\tilde{\gamma}_1}. \quad (21)$$

Now we proceed to the estimating of $\omega_{2,[0,T]}(q, r)$. We obtain that

$$\omega_{2,[0,T]}(q, r) = \sup_{0 \leq h \leq r} \|q(\cdot + h) - q(\cdot)\|_{L_2([0,T-h])}$$
$$\leq \sup_{0 \leq h \leq r} \big(\|q(\cdot + h) - q(\cdot)\|_{L_2([0,t-h])} + \|q(\cdot + h) - q(\cdot)\|_{L_2([t-h,t])}$$
$$+ \|q(\cdot + h) - q(\cdot)\|_{L_2([t,T-h])} \big) \leq \omega_{2,[0,t]}(q, r) + \tilde{I}(r),$$

where

$$\tilde{I}(r) = \Big(\int_{t-r}^{t} |q(z, t) - q(z, s)|^2 \, ds \Big)^{1/2}.$$

Triangle inequality for the norm $\|\cdot\|_{L_2}$ together with (21) implies that

$$\tilde{I}(r) \leq \Big(\int_{t-r}^{t} |q(z, t)|^2 \, ds \Big)^{1/2} + \Big(\int_{t-r}^{t} |q(z, s)|^2 \, ds \Big)^{1/2} \leq C r^{1/2} |x_1 - x_2|^{\tilde{\gamma}_1}, \quad (22)$$

where we take $\tilde{\gamma}_1 \in (\gamma_1, \beta(\sigma))$. On the other hand, the difference $q(z, t) - q(z, s)$ can be rewritten in the following way:

$$q(z, t) - q(z, s) = \sigma(t, x_1) - \sigma(t, x_2) - \int_B G(t, x_1; s, y)\sigma(s, y) dy$$
$$+ \int_B G(t, x_2; s, y)\sigma(s, y) dy$$
$$= v(0, x_1, t) - v(0, x_2, t) - v(t - s, x_1, s) + v(t - s, x_2, s)$$
$$= \sum_{i=1}^{2} v^{(i)}(0, x_1, t) - v^{(i)}(0, x_2, t) - v^{(i)}(t - s, x_1, s) + v^{(i)}(t - s, x_2, s). \quad (23)$$

Remark that

$$|v^{(1)}(0, x_1, t) - v^{(1)}(t - s, x_1, s)| = \Big| \sigma(t, x_1) - \int_{\mathbb{R}^d} p(t - s, x_1 - y)\sigma(s, y) dy \Big|$$
$$= \Big| \sigma(t, x_1) - \frac{1}{\big(4a^2\pi(t - s)\big)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{(x_1 - y)^2}{4a^2(t-s)}} \sigma(s, y) dy \Big|$$
$$= \frac{1}{\pi^{d/2}} \Big| \int_{\mathbb{R}^d} e^{-|v|^2} \sigma(t, x_1) dv - \int_{\mathbb{R}^d} e^{-|v|^2} \sigma(s, 2av\sqrt{t - s} + x_1) dv \Big|$$

$$\leq C\int_{\mathbb{R}^d} e^{-|v|^2}\big((t-s)^{\beta(\sigma)}+|v|(t-s)^{\beta(\sigma)/2}\big)dv \leq C(t-s)^{\beta(\sigma)/2}.$$

The same estimates can be applied for $|v^{(1)}(0,x_2,t)-v^{(1)}(t-s,x_2,s)|$. That leads to the inequality

$$|v^{(1)}(0,x_1,t)-v^{(1)}(0,x_2,t)-v^{(1)}(t-s,x_1,s)+v^{(1)}(t-s,x_2,s)| \leq C(t-s)^{\beta(\sigma)/2}. \tag{24}$$

For the second summand in (23) we can use the same estimates as in (20) and obtain that

$$\begin{aligned}&|v^{(2)}(0,x_1,t)-v^{(2)}(0,x_2,t)-v^{(2)}(t-s,x_1,s)+v^{(2)}(t-s,x_2,s)|\\ &=|v^{(2)}(0,x_1,s)-v^{(2)}(0,x_2,s)-v^{(2)}(t-s,x_1,s)+v^{(2)}(t-s,x_2,s)| \leq C(t-s).\end{aligned} \tag{25}$$

Here we also applied the fact $v^{(2)}(0,x_i,t)=v^{(2)}(0,x_i,s)=0$. Eqs. (24) and (25) imply that

$$\tilde{I}^2(r) \leq C\int_{t-r}^{t}(t-s)^{\beta(\sigma)}ds = Cr^{\beta(\sigma)+1}. \tag{26}$$

Together with (22), (26) leads to the estimate

$$\tilde{I}(r) \leq Cr^{\theta_2}|x_1-x_2|^{\gamma_1},$$

where $\theta_2 > 1/2$. In conclusion,

$$\omega_{2,[0,T]}(q,r) \leq Cr^{\theta}|x_1-x_2|^{\gamma_1}, \quad \theta=\min\{\theta_1,\theta_2\} > 1/2.$$

As a result,

$$\|q(z,\cdot)\|_{B^{\varepsilon}_{22}([0,t])} \leq C|x_1-x_2|^{\gamma_1}+C|x_1-x_2|^{\gamma_1}\left(\int_0^t r^{-2\varepsilon-1+2\theta}\,dr\right)^{1/2} \leq C|x_1-x_2|^{\gamma_1}$$

for a sufficiently small ε. The only fact left to prove is that

$$\sum_{n\geq 1}2^{-n\beta}\sum_{1\leq k\leq 2^n}|\mu(\Delta^{(T)}_{kn}\cap(0,t])|^2 < C(\omega) \quad \text{a. s.},$$

where $C(\omega)$ does not depend on t. Assume that for each n, $t\in\Delta^{(T)}_{k_n n}$; then by Assumption 5

$$\begin{aligned}&\sum_{n\geq 1}2^{-n\beta}\sum_{1\leq k\leq 2^n}|\mu(\Delta^{(T)}_{kn}\cap(0,t])|^2\\ &\quad\leq \sum_{n\geq 1}2^{-n\beta}\sum_{1\leq k\leq 2^n}|\mu(\Delta^{(T)}_{kn})|^2+\sum_{n\geq 1}2^{-n\beta}|\mu(\Delta^{(T)}_{k_n n}\cap(0,t])|^2 \leq C(\omega).\end{aligned}$$

□

Lemma 4. *Let Assumptions 1, 2, 4, 6 hold. Then the random process*

$$\hat{\xi}(t) = \int_{(0,t]} d\mu(s) \int_B G(t, x; s, y)\sigma(s, y)dy \tag{27}$$

has a version of a kind (3), which is Hölder continuous on $[\delta, T]$ with the exponent γ_2 for all $x \in B$, $T > \delta > 0$, $\gamma_2 < \beta(\mu)$, $\gamma_2 < \beta(\sigma)/(4 - 2\beta(\sigma))$. If $x \in B'$, where $\bar{B}' \subset B$, we can choose Hölder constant that depends only on σ, μ, γ_2, δ and B'.

Proof. Let $t_1 \le t_2$. We represent the difference of the integrals (27) in the form

$$\begin{aligned}\hat{\xi}(t_2) - \hat{\xi}(t_1) &= \int_{(0,t_2]} d\mu(s) \int_B G(t_2, x; s, y)\sigma(s, y)dy \\ &\quad - \int_{(0,t_1]} d\mu(s) \int_B G(t_1, x; s, y)\sigma(s, y)dy \\ &= \int_{(t_1,t_2]} \bar{q}(z, s)d\mu(s) + \int_{(0,t_1]} \bar{Q}(z, s)d\mu(s) = J_1 + J_2,\end{aligned} \tag{28}$$

where

$$\bar{q}(z, s) = \int_B G(t_2, x; s, y)\sigma(s, y)dy, \quad z = (t_2, x),\ s \in [t_1, t_2],$$

$$\bar{Q}(z, s) = \int_B (G(t_2, x; s, y) - G(t_1, x; s, y))\sigma(s, y)dy, \quad z = (t_1, t_2, x),\ s \in [0, t_1].$$

We fix a domain B' such that $x \in B'$, $\bar{B}' \subset B$ and, in the notations of Lemma 3, obtain that

$$\begin{aligned}&|\bar{q}(z, s)| \le C, \\ &|\bar{q}(z, s + h) - \bar{q}(z, s)| \le \int_B |G(t_2, x; s + h, y)||\sigma(s + h, y) - \sigma(s, y)|dy \\ &+\left|\int_B (G(t_2, x; s + h, y) - G(t_2, x; s, y))\sigma(s + h, y)dy\right| \le Ch^{\beta(\sigma)} \\ &+|v^{(1)}(t_2 - s - h, x, s + h) - v^{(1)}(t_2 - s, x, s + h)| \\ &+|v^{(2)}(t_2 - s - h, x, s + h) - v^{(2)}(t_2 - s, x, s + h)| \\ &\le C(h^{\beta(\sigma)} + h^{\beta(\sigma)}(t_2 - s)^{-\beta(\sigma)/2} + h) \le Ch^{\beta(\sigma)}(t_2 - s)^{-\beta(\sigma)/2},\end{aligned} \tag{29}$$

where the constant C in the last inequality depends on B'. We take k_{n1} and k_{n2} such that $t_1 \in \Delta_{k_{n1}n}^{(T)}$ and $t_2 \in \Delta_{k_{n2}n}^{(T)}$ and choose n_0 that satisfies the inequality

$$2^{-n_0}T < t_2 - t_1 \le 2^{-n_0+1}T.$$

For such n_0, $k_{n_01} + 1 = k_{n_02}$ or $k_{n_01} + 2 = k_{n_02}$, while for smaller n, $k_{n1} + 1 = k_{n2}$ or $k_{n1} = k_{n2}$. We can easily obtain by induction that for each $n \ge n_0$

$$k_{n2} - k_{n1} \le 2^{n-n_0+1} - 1 + T^{-1}(t_2 - t_1)2^n \le T^{-1}(t_2 - t_1)2^{n+1}.$$

The function $\bar{q}(z,s)$ was already defined on $[t_1,t_2]$, let $\bar{q}(z,s)=\bar{q}(z,t_1)$ for $s<t_1$ and $\bar{q}(z,s)=\bar{q}(z,t_2)$ for $s>t_2$. Now we can use Lemma 2 to estimate integral J_1:

$$|J_1| \le |\bar{q}(z,0)\mu((t_1,t_2])| + \sum_{n\ge 1}\sum_{1\le k\le 2^n} |\bar{q}(z,d^{(T)}_{(k-1)n}) - \bar{q}(z,d^{(T)}_{(k-2)n})||\mu(\Delta^{(T)}_{kn}\cap(t_1,t_2])|.$$

For each n we can omit summands for $k\le k_{n1}$, as for such k, $\bar{q}(z,d^{(T)}_{(k-1)n}) = \bar{q}(z,t_1)=\bar{q}(z,d^{(T)}_{(k-2)n})$, and summands for $k>k_{n2}$, as for such k, $\Delta^{(T)}_{kn}\cap(t_1,t_2]=\emptyset$:

$$\begin{aligned}
&|J_1| \le C(t_2-t_1)^{\gamma_2}\\
&+\sum_{n\ge 1}\sum_{k=k_{n1+1}}^{k_{n2}} |\bar{q}(z,d^{(T)}_{(k-1)n}) - \bar{q}(z,d^{(T)}_{(k-2)n})||\mu(\Delta^{(T)}_{kn}\cap(t_1,t_2])| \le C(\omega)(t_2-t_1)^{\gamma_2}\\
&+\sum_{n\ge 1}|\bar{q}(z,d^{(T)}_{(k_{n2}-1)n}) - \bar{q}(z,d^{(T)}_{(k_{n2}-2)n})||\mu(d^{(T)}_{(k_{n2}-1)n},t_2])\\
&+\sum_{n\ge n_0}\sum_{k=k_{n1+1}}^{k_{n2}-1} |\bar{q}(z,d^{(T)}_{(k-1)n}) - \bar{q}(z,d^{(T)}_{(k-2)n})||\mu(\Delta^{(T)}_{kn})|\\
&= C(\omega)(t_2-t_1)^{\gamma_2} + S_1 + S_2.
\end{aligned}$$

Now we estimate the sums S_1 and S_2, using (29).

$$\begin{aligned}
S_1 &\le C(\omega)\sum_{n\ge 1} 2^{-n\beta(\sigma)}(t_2-d^{(T)}_{(k_{n2}-2)n})^{-\beta(\sigma)/2}(t_2-d^{(T)}_{(k_{n2}-1)n})^{\beta(\mu)}\\
&\le C(\omega)(t_2-t_1)^{\gamma_2}\sum_{n\ge 1}2^{-n(\beta(\mu)-\gamma_2)} = C(t_2-t_1)^{\gamma_2},
\end{aligned}$$

$$\begin{aligned}
S_2 &\le C\Big(\sum_{n\ge n_0}2^{-n\beta}\sum_{k=1}^{2^n}|\mu(\Delta^{(T)}_{kn})|^2\Big)^{1/2}\\
&\times\Big(\sum_{n\ge n_0}2^{n\beta}2^{-2n\beta(\sigma)}\sum_{k=k_{n1+1}}^{k_{n2}-1}(t_2-(k-2)2^{-n}T)^{-\beta(\sigma)}\Big)^{1/2}\\
&\le C(\omega)\Big(\sum_{n\ge n_0}2^{-n(2\beta(\sigma)-\beta)}\sum_{i=1}^{k_{n2}-k_{n1}}(i2^{-n}T)^{-\beta(\sigma)}\Big)^{1/2}\\
&\le C(\omega)\Big(\sum_{n\ge n_0}2^{-n(\beta(\sigma)-\beta)}(k_{n2}-k_{n1})^{1-\beta(\sigma)}\Big)^{1/2}\\
&\le C(\omega)(t_2-t_1)^{(1-\beta(\sigma))/2}2^{-n_0(2\beta(\sigma)-\beta-1)/2}\\
&\le C(\omega)(t_2-t_1)^{(\beta(\sigma)-\beta)/2} \le C(\omega)(t_2-t_1)^{\gamma_2},
\end{aligned}$$

where we choose $\beta>0$ such that

$$(\beta(\sigma)-\beta)/2 > \beta(\sigma)/(4-2\beta(\sigma)) > \gamma_2;$$

such β exists as $1 > \beta(\sigma)$. Therefore,

$$J_1 \le C(\omega)(t_2 - t_1)^{\gamma_2}. \tag{30}$$

In order to estimate J_2, we need to prove some properties of the function $\bar{Q}$. Firstly, notice that in the notations of Lemma 3 $\bar{Q}(z, s) = v(t_2 - s, x, s) - v(t_1 - s, x, s)$ and

$$\begin{aligned} |\bar{Q}(z, s)| &\le |v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)| \\ &\quad + |v^{(2)}(t_2 - s, x, s) - v^{(2)}(t_1 - s, x, s)| \\ &\le |v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)| + C(t_2 - t_1). \end{aligned}$$

The difference $|v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)|$ was already estimated in [3], see formulas (13)–(15):

$$\begin{aligned} |v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)| &\le C(t_2 - t_1)(t_1 - s)^{-1}, \\ |v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)| &\le C(t_2 - t_1)^{\beta(\sigma)}(t_1 - s)^{-\beta(\sigma)/2}, \\ |v^{(1)}(t_2 - s, x, s) - v^{(1)}(t_1 - s, x, s)| &\le C(t_2 - t_1)^{\beta(\sigma)/2}. \end{aligned}$$

This leads to the following estimates for $|\bar{Q}(z, s)|$:

$$|\bar{Q}(z, s)| \le C(t_2 - t_1)(t_1 - s)^{-1}, \tag{31}$$

$$|\bar{Q}(z, s)| \le C(t_2 - t_1)^{\beta(\sigma)}(t_1 - s)^{-\beta(\sigma)/2}, \tag{32}$$

$$|\bar{Q}(z, s)| \le C(t_2 - t_1)^{\beta(\sigma)/2}. \tag{33}$$

Eqs. (31) and (32) directly imply that

$$|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \le C(t_2 - t_1)(t_1 - s - h)^{-1}, \tag{34}$$

$$|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \le C(t_2 - t_1)^{\beta(\sigma)}(t_1 - s - h)^{-\beta(\sigma)/2}. \tag{35}$$

Rewrite the difference $\bar{Q}(z, s + h) - \bar{Q}(z, s)$ in a form

$$\begin{aligned} &\bar{Q}(z, s + h) - \bar{Q}(z, s) \\ &= \int_B \big(G(t_2, x; s, y) - G(t_1, x; s, y)\big)\big(\sigma(s + h, y) - \sigma(s, y)\big)dy \\ &\quad + \int_B \big(G(t_2, x; s + h, y) - G(t_2, x; s, y)\big)\sigma(s + h, y)dy \\ &\quad - \int_B \big(G(t_1, x; s + h, y) - G(t_1, x; s, y)\big)\sigma(s + h, y)dy = F_1 + F_2 - F_3. \end{aligned}$$

Using (9), we obtain that

$$\begin{aligned} |F_1| &\le Ch^{\beta(\sigma)} \int_B dy \int_{t_1}^{t_2} \frac{1}{(\tau - s)^{d/2+1}} e^{-\frac{\lambda(x-y)^2}{\tau - s}} d\tau \\ &\le Ch^{\beta(\sigma)} \int_{t_1}^{t_2} \frac{ds}{(\tau - s)^{d/2+1}} \int_{\mathbb{R}^d} e^{-\frac{\lambda(x-y)^2}{\tau - s}} dy \end{aligned}$$

$$\leq Ch^{\beta(\sigma)} \int_{t_1}^{t_2} \frac{ds}{(\tau - s)^{d/2+1}} \int_0^{+\infty} e^{-\frac{\lambda v^2}{\tau - s}} v^{d-1} dv$$

$$= Ch^{\beta(\sigma)} \int_{t_1}^{t_2} (\tau - s)^{-1} d\tau \leq Ch^{\beta(\sigma)} (t_2 - t_1)(t_1 - s - h)^{-1}.$$

F_2 can be estimated similarly to (29):

$$\begin{aligned}
|F_2| &= |v(t_2 - s - h, x, s + h) - v(t_2 - s, x, s + h)| \\
&\leq |v^{(1)}(t_2 - s - h, x, s + h) - v^{(1)}(t_2 - s, x, s + h)| \\
&+ |v^{(2)}(t_2 - s - h, x, s + h) - v^{(2)}(t_2 - s, x, s + h)| \\
&\leq C(h(t_1 - s - h)^{-1} + h) \leq Ch(t_1 - s - h)^{-1}.
\end{aligned}$$

The estimates hold for F_3, too. That leads to the following analogue of formula (19) in [3]:

$$|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \leq C\big(h^{\beta(\sigma)}(t_2 - t_1) + h\big)(t_1 - s - h)^{-1}. \tag{36}$$

The next inequality is proved with the help of (29):

$$\begin{aligned}
&|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \\
&\leq \left| \int_B \big(G(t_2, x; s + h, y)\sigma(s + h, y) - G(t_2, s; s, y)\sigma(s, y)\big) dy \right| \\
&+ \left| \int_B \big(G(t_1, x; s + h, y)\sigma(s + h, y) - G(t_1, s; s, y)\sigma(s, y)\big) dy \right| \\
&\leq Ch^{\beta(\sigma)}(t_1 - s)^{-\beta(\sigma)/2}.
\end{aligned} \tag{37}$$

Raising (35) to the power λ and (34) to the power $1 - \lambda$, where $\lambda \in (1/(2 - \beta(\sigma)), 1)$, we get that

$$|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \leq C(t_2 - t_1)^{\rho_1}(t_1 - s - h)^{\rho_2}, \tag{38}$$

where

$$\rho_1 = 1 - \lambda + \lambda\beta(\sigma) > \beta(\sigma), \quad \rho_2 = -1 + \lambda - \lambda\beta(\sigma)/2 > -1/2.$$

Raising (37) to the power λ and (36) to the power $1 - \lambda$, we obtain that

$$|\bar{Q}(z, s + h) - \bar{Q}(z, s)| \leq C(h^{\beta(\sigma)}(t_2 - t_1)^{1-\lambda} + h^{\rho_1})(t_1 - s - h)^{\rho_2}. \tag{39}$$

We choose m_0 which satisfies a condition

$$2^{-m_0} T < t_1 \leq 2^{-m_0+1} T.$$

The function $\bar{Q}(z, s)$ was already defined on $[0, t_1]$, let $\bar{Q}(z, s) = \bar{Q}(z, t_1)$ for $s > t_1$. Now function $\bar{Q}$ is continuous on $[0, t_2]$ and we can use Lemma 2:

$$|J_2| \leq |\bar{Q}(z, 0)\mu((0, t_1])|$$

$$+\sum_{n\ge 1}\sum_{k=1}^{2^n}|\bar{Q}(z,d_{(k-1)n}^{(T)})-\bar{Q}(z,d_{(k-2)n}^{(T)})||\mu(\Delta_{kn}^{(T)}\cap(0,t_1])|$$

$$\le|\bar{Q}(z,0)\mu((0,t_1])|+\sum_{n\ge m_0}\sum_{k=2}^{k_{n1}}|\bar{Q}(z,d_{(k-1)n}^{(T)})-\bar{Q}(z,d_{(k-2)n}^{(T)})||\mu(\Delta_{kn}^{(T)}\cap(0,t_1])|$$

$$\le|\bar{Q}(z,0)\mu((0,t_1])|+\sum_{n\ge m_0}|\bar{Q}(z,d_{(k_{n1}-1)n}^{(T)})-\bar{Q}(z,d_{(k_{n2}-2)n}^{(T)})||\mu(d_{(k_{n1}-1)n}^{(T)},t_1])|$$

$$+\sum_{n=m_0}^{n_0-1}\sum_{k=2}^{k_{n1}-1}|\bar{Q}(z,d_{(k-1)n}^{(T)})-\bar{Q}(z,d_{(k-2)n}^{(T)})||\mu(\Delta_{kn}^{(T)})|$$

$$+\sum_{n=n_0}^{\infty}\sum_{k=2}^{k_{n1}-1}|\bar{Q}(z,d_{(k-1)n}^{(T)})-\bar{Q}(z,d_{(k-2)n}^{(T)})||\mu(\Delta_{kn}^{(T)})|=U_1+U_2+U_3+U_4.$$

Using (33), we easily obtain that

$$U_1\le C(\omega)(t_2-t_1)^{\beta(\sigma)/2},\tag{40}$$

$$U_2\le C(\omega)(t_2-t_1)^{\beta(\sigma)/2}\sum_{n\ge m_0}2^{-n\beta(\mu)}=C(\omega)(t_2-t_1)^{\beta(\sigma)/2}.\tag{41}$$

In order to estimate U_3, we use (38):

$$U_3\le C\Big(\sum_{n\ge 1}2^{-n\beta}\sum_{k=1}^{2^n}|\mu(\Delta_{kn}^{(T)})|^2\Big)^{1/2}$$

$$\times\Big(\sum_{n=m_0}^{n_0-1}2^{n\beta}\sum_{k=2}^{k_{n1}-1}|\bar{Q}(z,d_{(k-1)n}^{(T)})-\bar{Q}(z,d_{(k-2)n}^{(T)})|^2\Big)^{1/2}$$

$$\le C(\omega)(t_2-t_1)^{\rho_1}\Big(\sum_{n=m_0}^{n_0-1}2^{n\beta}\sum_{k=2}^{k_{n1}-1}(t_1-d_{(k-1)n}^{(T)})^{2\rho_2}\Big)^{1/2}$$

$$\le C(\omega)(t_2-t_1)^{\rho_1}\Big(\sum_{n=m_0}^{n_0-1}2^{n\beta}\sum_{i=1}^{k_{n1}-1}(i2^{-n}T)^{2\rho_2}\Big)^{1/2}$$

$$\le C(\omega)(t_2-t_1)^{\rho_1}\Big(\sum_{n=m_0}^{n_0-1}2^{n(\beta-2\rho_2)}(k_{n1}-1)^{2\rho_2+1}\Big)^{1/2}$$

$$\le C(\omega)(t_2-t_1)^{\rho_1}\Big(\sum_{n=m_0}^{n_0-1}2^{n(\beta-2\rho_2)}2^{n(2\rho_2+1)}\Big)^{1/2}$$

$$\le C(\omega)(t_2-t_1)^{\rho_1}2^{n_0(\beta+1)/2}\le C(t_2-t_1)^{\rho_1-(1+\beta)/2}.\tag{42}$$

Now we estimate U_4, applying (39):

$$U_4\le C\Big(\sum_{n\ge 1}2^{-n\beta}\sum_{k=1}^{2^n}|\mu(\Delta_{kn}^{(T)})|^2\Big)^{1/2}$$

$$
\begin{aligned}
&\times\Big(\sum_{n=n_0}^{\infty} 2^{n\beta} \sum_{k=2}^{k_{n1}-1} |\bar{Q}(z, d^{(T)}_{(k-1)n}) - \bar{Q}(z, d^{(T)}_{(k-2)n})|^2\Big)^{1/2} \\
&\le C(\omega)\Big(\sum_{n=n_0}^{\infty} 2^{n\beta} \sum_{k=2}^{k_{n1}-1} ((t_2-t_1)^{2-2\lambda}(2^{-n}T)^{2\beta(\sigma)} + (2^{-n}T)^{2\rho_1})(t_1 - d^{(T)}_{(k-1)n})^{2\rho_2}\Big)^{1/2} \\
&\le C(\omega)\Big(\sum_{n=n_0}^{\infty} 2^{n\beta}((t_2-t_1)^{2-2\lambda}2^{-2n\beta(\sigma)} + 2^{-2n\rho_1}) \sum_{j=1}^{k_{n1}-1} |j2^{-n}T|^{2\rho_2}\Big)^{1/2} \\
&\le C(\omega)\Big(\sum_{n=n_0}^{\infty} 2^{n(\beta-2\rho_2)}((t_2-t_1)^{2-2\lambda}2^{-2n\beta(\sigma)} + 2^{-2n\rho_1})(k_{n1}-1)^{2\rho_2+1}\Big)^{1/2} \\
&\le C(\omega)\Big(\sum_{n=n_0}^{\infty} 2^{n(\beta-2\rho_2)}((t_2-t_1)^{2-2\lambda}2^{-2n\beta(\sigma)} + 2^{-2n\rho_1})2^{n(2\rho_2+1)}\Big)^{1/2} \\
&= C(\omega)\Big(\sum_{n=n_0}^{\infty} 2^{n(\beta-2\beta(\sigma)+1)}(t_2-t_1)^{2-2\lambda} + \sum_{n=n_0}^{\infty} 2^{n(\beta-2\rho_1+1)}\Big)^{1/2} \\
&\le C(\omega)\Big(2^{n_0(\beta-2\beta(\sigma)+1)}(t_2-t_1)^{2-2\lambda} + 2^{n_0(\beta-2\rho_1+1)}\Big)^{1/2} \\
&\le C(\omega)\Big((t_2-t_1)^{-\beta+2\beta(\sigma)-1}(t_2-t_1)^{2-2\lambda} + (t_2-t_1)^{-\beta+2\rho_1-1}\Big)^{1/2} \\
&\le C(t_2-t_1)^{\rho_1-(1+\beta)/2}. \qquad (43)
\end{aligned}
$$

The estimates (42) and (43) hold for each $\beta > 0$. For each fixed $\gamma_2 < \beta(\sigma)/(2(2-\beta(\sigma)))$ we take

$$
\lambda = \frac{1-\beta-2\gamma_2}{2(1-\beta(\sigma))} \Rightarrow \rho_1 - (1+\beta)/2 = \gamma_2.
$$

Choose β such that $\beta+2\gamma_2 < \beta(\sigma)/(2-\beta(\sigma))$; then $\lambda > 1/(2-\beta(\sigma))$. Taking into consideration that $\beta(\sigma)/2 > \beta(\sigma)/(2(2-\beta(\sigma))) > \gamma_2$ and estimates (40), (41), we finally obtain

$$
|J_2| \le C(\omega)(t_2-t_1)^{\gamma_2}. \qquad (44)
$$

The substitution of (30) and (44) into (28) leads to inequality

$$
|\hat{\zeta}(t_2) - \hat{\zeta}(t_1)| \le |J_1| + |J_2| \le C(\omega)(t_2-t_1)^{\gamma_2}.
$$

That completes the proof of the lemma. □

Now we can return to the proof of the Theorem 1.

Proof. The item (1) is proved in the same way as item (i) in [16], using the following iteration process: $u^{(0)}(t,x) = 0$,

$$
u^{(n)}(t,x) = \int_B G(t,x;0,y)u_0(y)dy + \int_0^t ds \int_B G(t,x;s,y) f(s,y,u^{(n-1)}(s,y))dy + \int_{(0,t]} d\mu(s) \int_B G(t,x;s,y)\sigma(s,y)dy; \qquad (45)
$$

consequently, we give only a brief version of the proof. Denote

$$g_n(t) = \sup_{x\in\bar{B}} |u^{(n+1)}(t,x) - u^{(n)}(t,x)|, \quad n \geq 1.$$

Then for each $\omega \in \Omega$ the following estimates hold:

$$\left|u^{(2)}(t,x) - u^{(1)}(t,x)\right| \leq C\int_0^t ds \int_B |G(t,x;s,y)|dy \overset{(7)}{\leq} C_1 t \Rightarrow g_1(t) \leq C_1 t,$$

$$\left|u^{(n+1)}(t,x) - u^{(n)}(t,x)\right| \leq L_f \int_0^t ds \int_B |G(t,x;s,y)|\left|u^{(n)}(s,y) - u^{(n-1)}(s,y)\right|dy$$

$$\leq C_2 \int_0^t g_{n-1}(s)ds \Rightarrow g_n(t) \leq C_2 \int_0^t g_{n-1}(s)ds,\ n \geq 2; \tag{46}$$

and we can prove by induction that

$$g_n(t) \leq C_1 C_2^{n-1}\frac{t^n}{n!},$$

and the series $\sum_{n=0}^{\infty} g_n(t)$ converges uniformly in $[0,T]$. Hence there exists a limit function $u(t,x) = \lim_{n\to\infty} u^{(n)}(t,x)$, which is the solution of (6). Prove that it is unigue. Let $w(t,x)$ be another solution of (6); then, using the same arguments as in the proof of (46), we obtain that for a function $g(t) = \sup_{x\in\bar{B}} |u(t,x) - w(t,x)|$,

$$g(t) \leq C_1 t, \quad g(t) \leq C_2 \int_0^t g(s)ds,$$

and

$$g(t) \leq C_1 C_2^{n-1}\frac{t^n}{n!}$$

for each $n \geq 1$. Sending n to infinity, we obtain that $u = w$.

In order to prove item (2), we represent (45) as

$$u^{(n)}(t,x) = u_1(t,x) + u_2^{(n)}(t,x) + \int_{(0,t]} d\mu(s) \int_B G(t,x;s,y)\sigma(s,y)dy,$$

where

$$u_1(t,x) = \int_B G(t,x;0,y)u_0(y)dy,$$

$$u_2^{(n)}(t,x) = \int_0^t ds \int_B G(t,x;s,y)f(s,y,u^{(n-1)}(s,y))dy.$$

We will prove that function $u^{(n)}$ is Hölder continuous in $[\delta,T]\times\bar{B}'$ for each fixed $\omega\in\Omega$ with the exponent γ_1 by induction on n; if $n=0$, the statement is obvious. The function $u_1(t,x)$ satisfies the equation $\mathcal{L}u_1 = 0$ in $(0,T]\times B$ (see, for example, the proof of Theorem 4.3 in [8]), and, consequently, in $[\delta,T]\times\bar{B}'$. On the other hand, [8, Theorem 4.3] implies that function $u_2^{(n)}$ is a solution of the problem

$$\begin{cases} \mathcal{L}u_2^{(n)}(t,x) = -f(t,x,u^{(n-1)}(t,x))\,, \\ u_2^{(n)}|_S = 0, \quad u_2^{(n)}|_{t=0} = 0\,. \end{cases}$$

The Hölder continuity of $f(s, y, u^{(n-1)}(s, y))$ by y follows from the inequalities

$$
\begin{aligned}
&|f(s, y_1, u^{(n-1)}(s, y_1)) - f(s, y_2, u^{(n-1)}(s, y_2))| \\
&\qquad \le L_f(|y_1 - y_2|^{\beta(f)} + |u^{(n-1)}(s, y_1) - u^{(n-1)}(s, y_2)|) \le L_2|y_1 - y_2|^{\beta_1},
\end{aligned}
$$

where $\beta_1 = \min\{\beta(f), \gamma_1\}$. Theorem 1 in [6] implies that for each $\epsilon \in (0, 1)$

$$
\|u_2^{(n)}\|_{1+\epsilon}^{Q} \le C_2 \sup_{Q} |f(\cdot, \cdot, u^{(n-1)}(\cdot, \cdot))| \le C_2\|f\|_0^{\bar{Q}},
$$

where constant C_2 depends only on ϵ and the operator $\mathcal{L}$. Applying Lemma 3, we obtain that there exist the versions $\tilde{u}_n^{(x)}$ of the functions $u^{(n)}$ such that

$$
|\tilde{u}_n^{(x)}(t, x_1) - \tilde{u}_n^{(x)}(t, x_2)| \le L_{\tilde{u}^{(x)}}|x_1 - x_2|^{\gamma_1}, \quad \forall t \in [\delta, T],\ x_1,\ x_2 \in \bar{B}',
$$

where constant $L_{\tilde{u}^{(x)}}$ does not depend on n. Sending n to infinity, we obtain the statement of the item.

The beginning of the proof of the item (3) is similar to the proof of the item (2), we just use Lemma 4 instead of Lemma 3 and get that

$$
|\tilde{u}_n^{(t)}(t_1, x) - \tilde{u}_n^{(t)}(t_2, x)| \le L_{\tilde{u}^{(t)}}|t_1 - t_2|^{\gamma_2}, \quad \forall t \in [\delta, T],\ x_1,\ x_2 \in \bar{B}',
$$

where constant $L_{\tilde{u}^{(t)}}$ does not depend on n. Therefore, there exists a version $\tilde{u}^{(t)}$ of a function u such that

$$
|\tilde{u}^{(t)}(t_1, x) - \tilde{u}^{(t)}(t_2, x)| \le L_{\tilde{u}^{(t)}}|t_1 - t_2|^{\gamma_2}, \quad \forall t \in [\delta, T],\ x_1,\ x_2 \in \bar{B}'.
$$

On the other hand, we have already built a version $\tilde{u}^{(x)}$, which satisfies (11). We exclude all $\omega \in \Omega$ such that $\tilde{u}^{(x)}(t, x) \ne \tilde{u}^{(t)}(t, x)$ for at least one pair of rational $(t, x) \in [\delta, T] \times \bar{B}'$. For other $\omega \in \Omega$ we take $\tilde{u} = \tilde{u}^{(t)} = \tilde{u}^{(x)}$ for rational (t, x) and define $\tilde{u}$ for other pairs $(t, x) \in [\delta, T] \times \bar{B}'$ by continuity. The function $\tilde{u}$ which is built in such way is Hölder continuous on $[\delta, T] \times \bar{B}'$. $\square$

Now we compare Theorem 1 with the results of the paper [3], where the heat equation was considered in the unbounded multidimensional domain. We obtained the existence and uniqueness of the solution in the same sense as in [3], also the Hölder regularity with the same exponents was obtained. However, considering of bounded domains allowed us to weaken conditions on the functions u_0 and f; the Hölder continuity of u_0 is not required, and function f is not necessary Lipschitz continuous on x.

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