Modern Stochastics: Theory and Applications 12 (2) (2025) 123–134
https://doi.org/10.15559/24-VMSTA264

A note on optimal liquidation with linear price impact

Yan Dolinsky*, **Doron Greenstein**

Hebrew University, Department of Statistics, Mount Scopus, Jerusalem, Israel

yan.dolinsky@mail.huji.ac.il (Y. Dolinsky), doron.greenshtein@mail.huji.ac.il (D. Greenstein)

Received: 14 April 2024, Revised: 4 August 2024, Accepted: 4 August 2024,
Published online: 20 August 2024

Abstract In this note the maximization of the expected terminal wealth for the setup of quadratic transaction costs is considered. First, a very simple probabilistic solution to the problem is provided. Although the problem was largely studied, as far as authors know up to date this simple and probabilistic form of the solution has not appeared in the literature. Next, the general result is applied for the numerical study of the case where the risky asset is given by a fractional Brownian motion and the information flow of the investor can be diversified.

Keywords Linear price impact, optimal liquidation, fractional Brownian motion
2020 MSC 91G10, 60G44

1 Preliminaries and the general result

Consider a model with one risky asset which we denote by $S = (S_t)_{0\le t\le T}$, where $T < \infty$ is the time horizon. We assume that the investor has a bank account that, for simplicity, bears no interest. The risky asset S is RCLL (right continuous with left limits) and adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfies the usual assumptions (right continuity and completeness). Let us emphasize that we do not assume that the σ-algebra $\mathcal{F}_0$ is the trivial σ-algebra.

In financial markets, trading moves prices against the trader: buying faster increases execution prices, and selling faster decreases them. This aspect of liquidity,

*Corresponding author.

known as market depth (see [2]) or price-impact, has received large attention in optimal liquidation problems, see, for instance, [1, 8, 4, 7] and the references therein.

Following [1], we model the investor's market impact in a temporary linear form and thus, when at time t the investor turns over her position Φ_t at the rate $\phi_t = \dot{\Phi}_t$ the execution price is $S_t + \frac{\Lambda}{2}\phi_t$ for some constant $\Lambda > 0$. In our setup the investor has to liquidate his position, namely $\Phi_T = \Phi_0 + \int_0^T \phi_t dt = 0$. For a given initial number (deterministic) of shares Φ_0, denote by $\mathcal{A}_{\Phi_0}$ the set of all progressively measurable processes $\phi = (\phi_t)_{0\le t\le T}$ which satisfy $\int_0^T \phi_t^2 dt < \infty$ and $\Phi_0 + \int_0^T \phi_t dt = 0$. As usual, all the equalities and the inequalities are understood in the almost surely sense.

The profits and losses from trading are given by

$$V_T^{\Phi_0,\phi} := -\Phi_0 S_0 - \int_0^T \phi_t S_t dt - \frac{\Lambda}{2}\int_0^T \phi_t^2 dt. \tag{1}$$

Observe that for $\phi \in \mathcal{A}_{\Phi_0}$ the right-hand side of (1) is well defined if $\int_0^T S_t^2 dt < \infty$. This inequality follows from the integrability condition given by (3). In particular, we do not assume that S is a semimartingale.

Let us explain formula (1) in more detail. At time 0 the investor has Φ_0 stocks and the sum $-\Phi_0 S_0$ on her savings account. At time $t \in [0, T)$ the investor buys $\phi_t dt$, an infinitesimal number of stocks or, more intuitively, sell $-\phi_t dt$ number of shares and so the (infinitesimal) change in the savings account is expressed by $-\phi_t \left(S_t + \frac{\Lambda}{2}\phi_t\right) dt$. Since we liquidate the portfolio at the maturity date, the terminal portfolio value is equal to the terminal amount on the savings account and expressed by $-\Phi_0 S_0 - \int_0^T \phi_t \left(S_t + \frac{\Lambda}{2}\phi_t\right) dt$. We arrive at the right-hand side of (1). For the case where S is a semimartingale, by applying the integration by parts formula $\int_0^T \Phi_t dS_t = \Phi_T S_T - \Phi_0 S_0 - \int_0^T S_t d\Phi_t$ and using the fact that $\Phi_T = 0$ (liquidation) we get that the right-hand side of (1) is equal to $\int_0^T \Phi_t dS_t - \frac{\Lambda}{2}\int_0^T \phi_t^2 dt$.

We are interested in the following optimal liquidation problem:

$$\text{Maximize } \mathbb{E}\left[V_T^{\Phi_0,\phi}\right] \quad \text{over } \phi \in \mathcal{A}_{\Phi_0}, \tag{2}$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

The following theorem provides a completely probabilistic solution to the optimization problem (2).

Theorem 1. *Assume that*

$$\mathbb{E}\left[\int_0^T S_t^2 dt\right] < \infty. \tag{3}$$

Introduce the martingale

$$M_t := \mathbb{E}\left[\int_0^T S_u du \mid \mathcal{F}_t\right], \quad t \in [0, T]. \tag{4}$$

The unique ($dt \otimes \mathbb{P}$ a.s) solution to the optimization problem (2) is given by

$$\hat{\phi}_t := -\frac{\Phi_0}{T} + \frac{M_0}{T\Lambda} + \frac{1}{\Lambda}\left(\int_0^t \frac{dM_u}{T-u} - S_t\right), \quad t \in [0, T), \tag{5}$$

and the corresponding value is equal to

$$\max_{\phi\in\mathcal{A}_{\Phi_0}} \mathbb{E}\left[V_T^{\Phi_0,\phi}\right] = \mathbb{E}\left[V_T^{\Phi_0,\hat{\phi}}\right]$$
$$= -\frac{\Phi_0^2\Lambda}{2T} + \Phi_0\mathbb{E}\left[\frac{M_0}{T} - S_0\right] + \frac{1}{2\Lambda}\mathbb{E}\left[\int_0^T \left(S_t - \frac{M_0}{T} - \int_0^t \frac{dM_u}{T-u}\right)^2 dt\right]. \quad (6)$$

A slightly more general form of the linear-quadratic optimization problem (2) has been considered in [3], however, for the relatively simple setup of optimal liquidation Theorem 1 provides a much simpler solution than [3]. As far as we know, up to date this simple and probabilistic form of the solution has not appeared in the literature.

Before we prove Theorem 1, let us briefly collect some observations from this result. First, let us notice that it is sufficient to define the optimal portfolio on the half-open interval $[0, T)$ (as we do in (5)). We can just set $\phi_T := 0$.

Next, observe that the optimal value given by the right-hand side of (6) can be decomposed into three terms, the first term $-\frac{\Phi_0^2\Lambda}{2T}$ does not depend on the risky asset, the second term is a product of the initial number of shares Φ_0 and the term $\mathbb{E}\left[\frac{M_0}{T} - S_0\right]$ which can be interpreted as the average drift of the risky asset S (recall that we do not assume that S is a semimartingale). The last term $\frac{1}{2\Lambda}\mathbb{E}\left[\int_0^T \left(S_t - \frac{M_0}{T} - \int_0^t \frac{dM_u}{T-u}\right)^2 dt\right]$ is a product of the market depth $\frac{1}{2\Lambda}$ and the distance of the risky asset S from a martingale. In particular, if S is a martingale then the last term is zero. Indeed, if S is a martingale then (4) implies $M_t = \int_0^t S_u du + (T-t)S_t$, $t \in [0, T]$. From the (stochastic) Leibniz rule we get $dM_t = S_t dt + (T-t)dS_t - S_t dt = (T-t)dS_t$. This together with the equality $\frac{M_0}{T} = S_0$ gives $S_t = \frac{M_0}{T} + \int_0^t \frac{dM_u}{T-u}$ for all t.

Next, we prove Theorem 1.

Proof. The proof will be done in three steps.

Step I. Introduce the process $N_t := \int_0^t \frac{dM_u}{T-u}$, $t \in [0, T)$. In this step we show that

$$\mathbb{E}\left[\int_0^T S_t N_t dt\right] = \mathbb{E}\left[\int_0^T N_t^2 dt\right] \le \mathbb{E}\left[\int_0^T S_t^2 dt\right]. \quad (7)$$

Fix $n \in \mathbb{N}$ and define the process $N^n = (N_t^n)_{0\le t\le T}$ by $N_t^n := N_{t\wedge(T-1/n)}$, $t \in [0, T]$. From (3) it follows that M and N^n are square integrable martingales.

Next, for any square integrable martingales X, Y we denote by $[X]$ the quadratic variation of X and by $[X, Y]$ the covariation of X and Y. Also, denote by $\mathbb{I}_{\cdot}$ the indicator function.

Observe that

$$\mathbb{E}\left[\int_0^T S_t N_t^n dt\right] = \mathbb{E}\left[N_T^n \int_0^T S_t dt\right] = \mathbb{E}\left[M_T N_T^n\right]$$
$$= \mathbb{E}\left[[M, N^n]_T\right] = \mathbb{E}\left[\int_0^T \frac{\mathbb{I}_{s<T-1/n}}{T-s} d[M]_s\right]$$
$$= \mathbb{E}\left[\int_0^T \int_s^T \frac{\mathbb{I}_{s<T-1/n}}{(T-s)^2} dt\, d[M]_s\right]$$
$$= \mathbb{E}\left[\int_0^T \int_0^t \frac{\mathbb{I}_{s<T-1/n}}{(T-s)^2} d[M]_s dt\right] = \mathbb{E}\left[\int_0^T |N_t^n|^2 dt\right].$$

Indeed, the first equality follows from the fact that N^n is a square integrable martingale. The second equality is due to (4). The third equality follows from Theorem 6.28 in [12] (we note that $N_0^n = 0$). The fourth equality follows from Theorem 9.15 in [12] where the integral with respect to $d[M]$ is the (pathwise) Stieltjes integral with respect to the nondecreasing process $[M]$. The fifth equality is obvious. The sixth equality is due to the Fubini theorem. Finally, the last equality is due to the (generalized) Itô isometry (see Chapter IX in [12]) which says that for any bounded and predictable process $\mathcal{H}$ and a square integrable martingale X we have $\mathbb{E}\left[\left(\int_0^T H_t dX_t\right)^2\right] = \mathbb{E}\left[\int_0^T H_t^2 d[X]_t\right]$.

We conclude

$$\mathbb{E}\left[\int_0^T S_t N_t^n dt\right] = \mathbb{E}\left[\int_0^T |N_t^n|^2 dt\right]. \tag{8}$$

Hence,

$$0 \le \mathbb{E}\left[\int_0^T |S_t - N_t^n|^2 dt\right] = \mathbb{E}\left[\int_0^T S_t^2 dt\right] - \mathbb{E}\left[\int_0^T |N_t^n|^2 dt\right]. \tag{9}$$

From (3) and (8)–(9) we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T S_t N_t dt\right] &= \lim_{n\to\infty} \mathbb{E}\left[\int_0^T S_t N_t^n dt\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\int_0^T |N_t^n|^2 dt\right] \\ &= \mathbb{E}\left[\int_0^T N_t^2 dt\right] \le \mathbb{E}\left[\int_0^T S_t^2 dt\right]\end{aligned}$$

and (7) follows.

Step II. Let $\phi \in \mathcal{A}_{\Phi_0}$. In this step we prove that $\mathbb{E}\left[V_T^{\Phi_0,\phi}\right]$ is not bigger than the right-hand side of (6). Without loss of generality we assume that $\mathbb{E}\left[V_T^{\Phi_0,\phi}\right] > -\infty$.

From (1) and the Cauchy–Schwarz inequality it follows that

$$\sqrt{\int_0^T S_t^2 dt}\sqrt{\int_0^T \phi_t^2 dt} - \frac{\Lambda}{2}\int_0^T \phi_t^2 dt \ge V_T^{\Phi_0,\phi} + \Phi_0 S_0.$$

Thus,

$$\frac{\Lambda}{2}\left(\sqrt{\int_0^T \phi_t^2 dt} - \frac{1}{\Lambda}\sqrt{\int_0^T S_t^2 dt}\right)^2 \le \frac{1}{2\Lambda}\int_0^T S_t^2 dt - V_T^{\Phi_0,\phi} - \Phi_0 S_0.$$

This together with the integrability condition (3) and the inequality $\mathbb{E}\left[V_T^{\Phi_0,\phi}\right] > -\infty$ gives that $\sqrt{\int_0^T \phi_t^2 dt} - \frac{1}{\Lambda}\sqrt{\int_0^T S_t^2 dt} \in L^2(\mathbb{P})$. Clearly, (due to (3)) $\sqrt{\int_0^T S_t^2 dt} \in L^2(\mathbb{P})$, and so we conclude that $\sqrt{\int_0^T \phi_t^2 dt} \in L^2(\mathbb{P})$, i.e. $\mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty$.

Next, set $Z := -\frac{\Phi_0 \Lambda}{T} + \frac{M_0}{T}$ and choose $n \in \mathbb{N}$. From the estimate $\mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty$ and the fact that N^n is a square integrable martingale we obtain

$$\mathbb{E}\left[\int_0^T \phi_t N_t^n dt\right] = \mathbb{E}\left[N_T^n \int_0^T \phi_t dt\right] = -\Phi_0 \mathbb{E}\left[N_T^n\right] = 0.$$

This together with (1) and the simple inequality $xy - \frac{\Lambda}{2}x^2 \le \frac{y^2}{2\Lambda}$, $x, y \in \mathbb{R}$, yields

$$\begin{aligned}\mathbb{E}\left[V_T^{\Phi_0,\phi}\right] &= \mathbb{E}\left[-\Phi_0(S_0 - Z) - \int_0^T \phi_t (S_t - Z - N_t^n) dt - \frac{\Lambda}{2}\int_0^T \phi_t^2 dt\right] \\ &\le \mathbb{E}\left[-\Phi_0(S_0 - Z) + \frac{1}{2\Lambda}\int_0^T |S_t - Z - N_t^n|^2 dt\right].\end{aligned}$$

By taking $n \to \infty$ in the above inequality and applying (7) we obtain

$$\mathbb{E}\left[V_T^{\Phi_0,\phi}\right] \le -\frac{\Phi_0^2 \Lambda}{2T} + \Phi_0 \mathbb{E}\left[\frac{M_0}{T} - S_0\right] + \frac{1}{2\Lambda}\mathbb{E}\left[\int_0^T \left(S_t - \frac{M_0}{T} - N_t\right)^2 dt\right] \tag{10}$$

as required.

Step III. In this step we complete the proof. Consider the trading strategy given by (5). From the Fubini theorem it follows that

$$\int_0^T \hat{\phi}_t dt = -\Phi_0 + \frac{1}{\Lambda}\left(M_0 + M_T - M_0 - \int_0^T S_t dt\right) = -\Phi_0.$$

Moreover, from (7) it follows that $\mathbb{E}\left[\int_0^T \hat{\phi}_t^2 dt\right] < \infty$. Thus, $\hat{\phi} \in \mathcal{A}_{\Phi_0}$.

Next, choose $n \in \mathbb{N}$. By using the same arguments as in Step II we get $\mathbb{E}\left[\int_0^T \hat{\phi}_t N_t^n dt\right] = 0$. Observe that for $t \le T - 1/n$ we have $\hat{\phi}_t = \frac{Z + N_t^n - S_t}{\Lambda}$, where (recall) $Z = -\frac{\Phi_0 \Lambda}{T} + \frac{M_0}{T}$. Hence,

$$\begin{aligned}\mathbb{E}\left[V_T^{\Phi_0,\hat{\phi}}\right] &= \mathbb{E}\left[-\Phi_0(S_0 - Z) - \int_0^T \hat{\phi}_t (S_t - Z - N_t^n) dt - \frac{\Lambda}{2}\int_0^T \hat{\phi}_t^2 dt\right] \\ &= \mathbb{E}\left[-\Phi_0(S_0 - Z) + \frac{1}{2\Lambda}\int_0^{T-1/n} |S_t - Z - N_t|^2 dt\right] \\ &\quad - \mathbb{E}\left[\int_{T-1/n}^T \hat{\phi}_t (S_t - Z - N_t^n) dt + \frac{\Lambda}{2}\int_{T-1/n}^T \hat{\phi}_t^2 dt\right].\end{aligned}$$

By taking $n \to \infty$ in the above equality and applying (7) we obtain (notice that $\mathbb{E}\left[\int_0^T \hat{\phi}_t^2 dt\right] < \infty$)

$$\begin{aligned}\mathbb{E}\left[V_T^{\Phi_0,\hat{\phi}}\right] &= \mathbb{E}\left[-\Phi_0(S_0 - Z) + \frac{1}{2\Lambda}\int_0^T |S_t - Z - N_t|^2 dt\right] \\ &= -\frac{\Phi_0^2 \Lambda}{2T} + \Phi_0 \mathbb{E}\left[\frac{M_0}{T} - S_0\right] + \frac{1}{2\Lambda}\mathbb{E}\left[\int_0^T \left(S_t - \frac{M_0}{T} - N_t\right)^2 dt\right].\end{aligned} \tag{11}$$

By combining (10)–(11) we conclude (6).

Finally, the uniqueness of the optimal trading strategy follows from the strict convexity of the map $\phi \to V_T^{\Phi_0,\phi}$. □

We end this section with the following example.

Example 1. Assume that S is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$. By applying the same arguments as in the paragraph before the proof of Theorem 1, we obtain that $S_t = \frac{M_0}{T} + \int_0^t \frac{dM_u}{T-u}$, $t \in [0,T]$. This together with (5) gives that the optimal strategy is purely deterministic and equals to $\hat{\phi}_t \equiv -\frac{\Phi_0}{T}$. Namely, we liquidate our initial position Φ_0 at a constant rate. From (6) we obtain that the corresponding value is equal to $-\frac{\Phi_0^2 \Lambda}{2T}$. Since $\hat{\phi}$ is deterministic, then in the case of partial information, i.e. where the investor's filtration is smaller than $(\mathcal{F}_t)_{0\le t\le T}$, the solution to the optimization problem (2) will be the same.

A more interesting case is where the filtration is larger than $(\mathcal{F}_t)_{0\le t\le T}$. More precisely, fix $\Delta \in (0,T]$ and consider the case where the investor can peek Δ time units into the future, and so her information flow is given by the filtration $(\mathcal{F}_{t+\Delta})_{t\ge 0}$.

From (4) we obtain that

$$M_t = \int_0^{(t+\Delta)\wedge T} S_u du + (T-t-\Delta)^+ S_{(t+\Delta)\wedge T}, \quad t \in [0,T].$$

Thus, $M_0 = \int_0^\Delta S_u du + (T-\Delta)S_\Delta$ and from the Leibniz rule we get

$$\begin{aligned} dM_t &= \mathbb{I}_{t<T-\Delta}\left(S_{t+\Delta}dt + (T-t-\Delta)\,dS_{t+\Delta} - S_{t+\Delta}dt\right) \\ &= \mathbb{I}_{t<T-\Delta}\,(T-t-\Delta)\,dS_{t+\Delta}, \quad t \in [0,T]. \end{aligned}$$

Hence,

$$\begin{aligned} &\tfrac{M_0}{T} + \int_0^t \tfrac{dM_u}{T-u} - S_t \\ &= \tfrac{1}{T}\left(\int_0^\Delta S_u du + (T-\Delta)S_\Delta\right) + \int_\Delta^{(t+\Delta)\wedge T} \tfrac{T-u}{T+\Delta-u} dS_u - S_t \\ &= \tfrac{\int_0^\Delta (S_u - S_\Delta)du}{T} + S_{(t+\Delta)\wedge T} - S_t - \Delta \int_\Delta^{(t+\Delta)\wedge T} \tfrac{dS_u}{T+\Delta-u}, \quad t \in [0,T]. \end{aligned}$$

This together with (5), (6) yields that the optimal strategy is given by

$$\hat{\phi}_t = -\frac{\Phi_0}{T} + \frac{\int_0^\Delta (S_u - S_\Delta)du}{T\Lambda} + \frac{S_{(t+\Delta)\wedge T} - S_t}{\Lambda} - \frac{\Delta}{\Lambda}\int_\Delta^{(t+\Delta)\wedge T} \frac{dS_u}{T+\Delta-u}$$

and the corresponding value (notice that $\mathbb{E}[M_0] = S_0 T$) is equal to

$$\mathbb{E}\left[V_T^{\Phi_0,\hat{\phi}}\right] = -\frac{\Phi_0^2 \Lambda}{2T} + \frac{I}{2\Lambda}$$

where

$$I := \mathbb{E}\left[\int_0^T \left(\frac{\int_0^\Delta (S_u - S_\Delta)du}{T} + S_{(t+\Delta)\wedge T} - S_t - \Delta \int_\Delta^{(t+\Delta)\wedge T} \frac{dS_u}{T+\Delta-u}\right)^2 dt\right]$$

can be viewed as the premium of being able to peek ahead by Δ units of time.

2 The case of fractional Brownian motion

The fractional Brownian motion $B^H = (B_t^H)_{t=0}^{\infty}$ with the Hurst parameter $H \in (0, 1)$ is a continuous, zero-mean Gaussian process such that

$$cov\left(B_t^H, B_u^H\right) = \frac{t^{2H} + u^{2H} - |t-u|^{2H}}{2}, \quad t, u \geq 0.$$

The process B^H is self similar, $B_{at}^H \sim a^H B_t^H$, and has stationary increments. Moreover, the successive increments of B^H are positively correlated for $H > 1/2$, negatively correlated for $H < 1/2$, while $H = 1/2$ recovers the usual Brownian motion with independent increments.

A fractional Brownian motion which displays the long-range dependence observed in empirical data (see [6, 16, 18] and the references therein) is not a semimartingale when $H \neq \frac{1}{2}$ and so, in the frictionless case it leads to arbitrage opportunities (see, for instance, [17, 5]). In the presence of market price impact arbitrage opportunities disappear and the expected profits are finite (see [10, 11]). In [11] the authors studied the asymptotic behavior (as the maturity date goes to infinity) of the optimal liquidation problem with temporary price impact, for the case where the risky asset is given by a fractional Brownian motion. It is also important to mention the recent paper [9] which is closely related.

In this section, for the financial model where the risky asset is given by a fractional Brownian motion, we study the dependence of the optimal liquidation problem as a function of the investor's information. We deal with three types of investors. The first one is the "usual" investor with information flow which is given by the filtration generated by the risky asset. The second type is an investor which receives the information with a delay. The last type is a "frontrunner" which is able to peek some time units into the future. Of course the "frontrunner" cannot freely take an advantage of her extra knowledge due to the linear price impact which leads to quadratic transaction costs. For the above three cases we solve the corresponding optimal liquidation problem and derive numerical results for the value (see Figure 1) and for the optimal strategy (see Figure 2).

Let $H \in (0, 1)$ and consider the optimization problem (2) for the case where the risky asset is of the form $S_t = S_0 + \sigma B_t^H + \mu t$ where $\sigma > 0$ and $\mu \in \mathbb{R}$ are constants. From Theorem 1 and the discussion afterwards it follows that (for simplicity) we can take $\mu = S_0 = 0$ and $\sigma = \Lambda = 1$. Thus, $S = B^H$ for some $H \in (0, 1)$ and $\Lambda = 1$.

For $H \in (0, 1)$ introduce the Volterra kernel

$$Z_H(t,s) = c_H\left(\left(\tfrac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - \left(H-\tfrac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right), \quad 0 < s < t,$$

where $c_H := \left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}$. Then, taking an ordinary Brownian motion $W = (W_t)_{t=0}^{\infty}$, the formula

$$B_t^H = \int_0^t Z_H(t,s)dW_s, \quad t \geq 0, \tag{12}$$

defines a fractional Brownian motion with the Hurst parameter H, which generates the same filtration as W (see [15]). Moreover, given B^H, the Wiener process W can be recovered by the relations

$$W_t := \frac{2H}{c_H}\int_0^t s^{H-\frac{1}{2}} d\mathcal{M}_s, \quad t \ge 0,$$

where

$$\mathcal{M}_t := \frac{1}{2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right)}\int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB^H_s, \quad t \ge 0.$$

Denote by $(\mathcal{F}^W_t)_{t\ge 0}$ the augmented filtration which is generated by W.

2.1 Standard information

Consider the case where the filtration $(\mathcal{F}_t)_{0\le t\le T}$ (which represent the investor's flow of information) is equal to $(\mathcal{F}^W_t)_{0\le t\le T}$. From the Fubini theorem and (12) it follows that the martingale defined in (4) is equal to

$$M^H_t = \int_0^t \left(\int_s^T Z_H(u,s)du\right) dW_s, \quad t \in [0,T].$$

Hence, (5) and (12) yield that the optimal strategy is given by

$$\hat{\phi}^H_t := \int_0^t \left(\frac{\left(\int_s^T Z_H(u,s)du\right)}{T-s} - Z_H(t,s)\right) dW_s, \quad t \in [0,T].$$

From the Itô isometry and (6) we obtain that the corresponding value is given by

$$\begin{aligned}\mathbb{E}\left[V_T^{0,\hat{\phi}^H}\right] &= \int_0^T \int_0^t Z_H^2(t,s)dsdt - \int_0^T \frac{\left(\int_s^T Z_H(u,s)du\right)^2}{T-s}ds \\ &= \frac{T^{2H+1}}{2H+1} - \int_0^T \frac{\left(\int_s^T Z_H(u,s)du\right)^2}{T-s}ds.\end{aligned}$$

2.2 Delayed information

We fix a positive number $\Delta \in (0,T]$ and consider a situation where the risky asset S is observed with a delay $\Delta > 0$. Namely, the filtration is $\mathcal{F}_t = \mathcal{F}^W_{(t-\Delta)^+}$, $t \in [0,T]$. In particular the underlying process $S = B^H$ is no longer adapted to the above filtration.

For the continuous filtration $\mathcal{F}^W_{(t-\Delta)^+}$, $t \in [0,T]$, consider the corresponding optional projection (see Chapter V in [12]) of B^H

$$\hat{S}_t := \mathbb{E}\left[B^H_t | \mathcal{F}^W_{(t-\Delta)^+}\right] = \int_0^{(t-\Delta)^+} Z_H(t,s)dW_s, \quad t \in [0,T].$$

The Fubini theorem gives that for any process $\gamma \in L^2(dt \otimes \mathbb{P})$ which is progressively measurable with respect to $\mathcal{F}^W_{(t-\Delta)^+}$, $t \in [0,T]$, we have $\mathbb{E}\left[\int_0^T \gamma_t B^H_t dt\right] = \mathbb{E}\left[\int_0^T \gamma_t \hat{S}_t dt\right]$. Hence, we can apply Theorem 1 for the optional projection $\hat{S}$.

From the Fubini theorem

$$\int_0^T \hat{S}_t dt = \int_0^{T-\Delta} \left(\int_{s+\Delta}^T Z_H(u,s)du \right) dW_s.$$

Thus, the martingale M defined in (4) is equal to

$$M_t^{H,\Delta,-} = \int_0^{(t-\Delta)^+} \left(\int_{s+\Delta}^T Z_H(u,s)du \right) dW_s, \quad t \in [0,T],$$

and so, the optimal strategy is given by

$$\hat{\phi}_t^{H,\Delta,-} = \int_0^{(t-\Delta)^+} \left(\frac{\int_{s+\Delta}^T Z_H(u,s)du}{T-\Delta-s} - Z_H(t,s) \right) dW_s, \quad t \in [0,T].$$

Finally, the corresponding value is given by

$$\mathbb{E}\left[V_T^{0,\hat{\phi}^{H,\Delta,-}} \right] = \int_0^T \int_0^{(t-\Delta)^+} Z_H^2(t,s)dsdt - \int_0^{T-\Delta} \frac{\left(\int_{s+\Delta}^T Z_H(u,s)\,du \right)^2}{T-\Delta-s} ds.$$

2.3 Insider information

Rather than having access to just the natural augmented filtration $(\mathcal{F}_t^W)_{t\geq 0}$ for making decisions, the investor can peek $\Delta \in (0,T]$ time units into the future, and so, her information flow is given by the filtration $(\mathcal{F}_{t+\Delta}^W)_{t\geq 0}$.

The martingale M defined in (4) is equal to

$$M_t^{H,\Delta,+} = \int_0^{(t+\Delta)\wedge T} \left(\int_s^T Z_H(u,s)du \right) dW_s, \quad t \in [0,T].$$

Hence, the optimal strategy is given by

$$\hat{\phi}_t^{H,\Delta,+} = \frac{1}{T} \int_0^\Delta \left(\int_s^T Z_H(u,s)du \right) dW_s$$
$$+ \int_\Delta^{(t+\Delta)\wedge T} \frac{\int_s^T Z_H(u,s)du}{T+\Delta-s} dW_s - \int_0^t Z_H(t,s)dW_s, \quad t \in [0,T],$$

and the corresponding value is given by

$$\mathbb{E}\left[V_T^{0,\hat{\phi}^{H,\Delta,+}} \right] = \int_0^T \int_0^t Z_H^2(t,s)dsdt - \frac{\left| M_0^{H,\Delta,+} \right|^2}{T} - \int_\Delta^T \frac{\left(\int_s^T Z_H(u,s)du \right)^2}{T+\Delta-s} ds$$
$$= \frac{T^{2H+1}}{2H+1} - \frac{1}{T} \int_0^\Delta \left(\int_s^T Z_H(u,s)du \right)^2 ds - \int_\Delta^T \frac{\left(\int_s^T Z_H(u,s)du \right)^2}{T+\Delta-s} ds.$$

Remark 1. Observe that the calculations of this section can be done in a similar way for any square integrable Gaussian–Volterra process with RCLL paths and the following property: the process generates the same filtration as the underlying Brownian motion. This property was studied in details in [13, 14]. In this paper we focus on the case where the risky asset is given by a fractional Brownian motion. In particular, we apply the obtained formulas in order to study numerically the value of the liquidation problem (for different flows of information) as a function of the Hurst parameter.

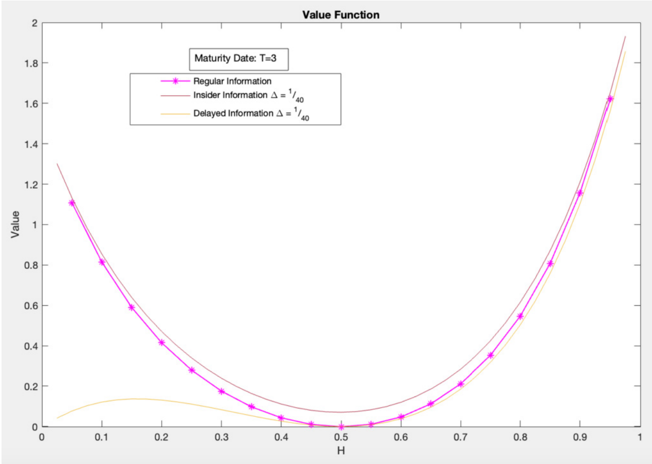

Fig. 1. The value of the liquidation problem for different flows of information (shown in different colors) as a function of the Hurst parameter H. Observe that for delayed information the value function is no longer decreasing for $H < 0.5$. The reason is that for very low H values the correlation between the increments decays faster to 0 with their time distance, hence a delay results in almost complete loss of information regarding the current price

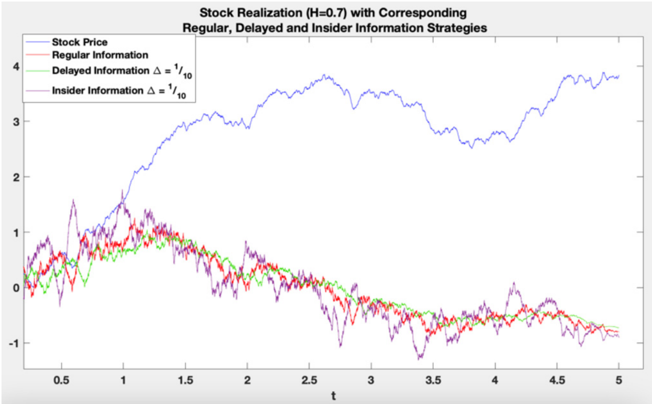

Fig. 2. In this figure we simulate a sample path of a fractional Brownian motion with the Hurst parameter $H = 0.7$ and the corresponding optimal trading strategies (we take maturity date $T = 5$). We observe that the Regular Information graph, is a “lagged version” of the Insider Information graph and the Delayed Information graph is a “lagged version” of the Regular Information graph

Funding

The work is supported in part by the ISF grant 230/21.

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