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# Bivariate dependence, stochastic orders and conditional tails of the recurrence times in a renewal process

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**Abstract** The structure of dependence between the forward and the backward recurrence times in a renewal process is considered. Monotonicity properties, as a function of time, for the tail of the bivariate distribution for the recurrence times are discussed, as well as their link with aging properties of the interarrival distribution $F$. A necessary and sufficient condition for the renewal function to be concave is also obtained. Finally, some properties of the conditional tail for one of the two recurrence times, given some information on the other, are studied. The results are illustrated by some numerical examples.

**Keywords** Renewal process, forward recurrence time, backward recurrence time, aging classes, dependence, positive regression dependence, negative regression dependence, upper orthant stochastic order

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## 1 Introduction

Two of the quantities of primary interest in renewal theory are the forward and backward recurrence times, denoted by $\gamma_t$ and $\delta_t$, respectively. On one hand, their asymptotic distribution is intimately related to the renewal theorem; see, for instance, Resnick

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[19] and Omey and Teugels [17]. On the other hand, and from a more practical viewpoint, if at time $t$ in a renewal process one is interested in the probability that there will be no renewals for $y$ time units ahead, this is simply $\mathbb{P}(\gamma_t > y)$. Moreover, if we know the current age of the item in operation at time $t$ (that is, the backward recurrence time) one expects that, in 'most cases,' we have a better view of the chance that there will be a gap with no renewals for $y$ time units from now. For, if the current age is $x$, the required probability is now $\mathbb{P}(\gamma_t > y \mid \delta_t = x)$.

To be more precise, it is well-known that the two recurrence times are independent if and only if the underlying stochastic model is that of a Poisson process; for the more general case of a renewal process, one expects that the dependence structure of the pair $(\gamma_t, \delta_t)$ reflects in some way the aging pattern of the interarrival distribution $F$. Our first aim in the present paper is to explore this link and study conditions under which a certain dependence form of the pair $(\gamma_t, \delta_t)$ implies an aging property for $F$ and vice versa. Moreover, aging properties for $F$ are well-known to be related to the concavity of the renewal function and the stochastic monotonicity of $\gamma_t$ and $\delta_t$; see Brown [4] and Shaked and Zhu [21]. Note that, when we say, e.g., that the forward recurrence times are stochastically increasing, this means that $\gamma_t$ is smaller, in the usual univariate stochastic order, than $\gamma_s$ for any $0 \le t < s$; that is, for any $y \ge 0$, it holds that $\mathbb{P}(\gamma_t > y) \le \mathbb{P}(\gamma_s > y)$.

In Section 4 of the present paper we obtain results that link the stochastic monotonicity of the *pair* $(\gamma_t, \delta_t)$ with aging properties for the interarrival distribution $F$ and the concavity of the renewal function. Further, in Section 5 we discuss how similar properties of the conditional tail for one of the recurrence times, given the other, provide information for $F$ and vice versa. Our results extend and generalize those obtained earlier by Brown [4], Chen [5] and Losidis et al. [15]. Further, aging properties and monotonicity results for the forward recurrence times were studied by Polatioglu and Sahin [18]; for diverse areas of potential application of the recurrence times in practice, see Losidis [12, 13].

To begin with, a renewal counting process $\{N(t) : t \ge 0\}$ is a stochastic process that registers the successive occurrences of an event through time. We assume that the time between the $(i-1)$th and the $i$th such occurrence (interarrival time) is denoted by a random variable $X_i$ and that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random variables with distribution function $F$. For each $t$, the variable $N(t)$ then represents the number of events until time $t$; note that, in contrast to some books in this area, we do not assume the occurrence of an event at time zero. If we set $S_0 = 0$ and for $n = 1, 2, \ldots$, $S_n = X_1 + X_2 + \cdots + X_n$, the recurrence times are defined as follows: the forward recurrence time $\gamma_t = S_{N(t)+1} - t$ denotes the time, from time $t$, until the next event (renewal), while $\delta_t = t - S_{N(t)}$ is the time elapsed since the last renewal. Another quantity of key interest is the renewal function $U(t) = E[N(t)]$. We assume throughout that the interarrival distribution $F$ is absolutely continuous with a density $f$, so that $U$ also has a density which is denoted by $u(t)$, and is called the renewal density.

For a distribution function $F$, we write $\overline{F} = 1 - F$ for the associated tail (or survival) function. We shall use the following well-known aging classes of distributions on the nonnegative half-line. A distribution $F$ is said to have an increasing failure rate (IFR) if $\log \overline{F}$ is concave, that is, for all $x > 0$ the function $\overline{F}(x+t)/\overline{F}(t)$ is decreasing in $t$ for all $t$ such that $\overline{F}(t) > 0$. If $F$ has a density $f$, this is equivalent to the condition that $f(t)/\overline{F}(t)$ is increasing in $t$, for $t$ such that $\overline{F}(t) > 0$. Similarly, if $\log \overline{F}$ is convex, then $F$ is called a decreasing failure rate (DFR) distribution. Next, a distribution $F$ is called new better than used (NBU) if $\overline{F}(x+t) \le \overline{F}(x)\,\overline{F}(t)$ for all $x, t \ge 0$, while $F$ is called new worse than used (NWU) if $\overline{F}(x+t) \ge \overline{F}(x)\,\overline{F}(t)$ for all $x, t \ge 0$. It is known that the DFR (IFR) is a subclass of the NWU (resp., NBU) class of distributions. Further details on these and other aging classes of distributions can be found in the books of Shaked and Shanthikumar [20] and Willmot and Lin [22].

For a nonnegative random variable $X$, which in our context typically represents lifetime, the variable $X_t = X - t \mid X > t$ denotes the *residual lifetime* at age $t$. We write $F_t$ for the distribution of $X_t$.

Further, for a distribution function $F$ with finite first moment $\mu$, the associated equilibrium (or stationary renewal) distribution, denoted by $F_e$ is defined for $x \ge 0$ by $F_e(x) = \mu^{-1} \int_0^x \overline{F}(y)dy$. Finally, we note that throughout the paper the term increasing (decreasing) stands for nondecreasing (nonincreasing).

The outline of the paper is as follows: in the next section we recall the main types of dependence for pairs of random variables. In Section 3, we discuss how a dependence structure for the pair $(\gamma_t, \delta_t)$ is linked with an aging property for the interarrival distribution $F$. Next, our results in Sections 4 and 5 demonstrate the interplay between an aging pattern for $F$ and various monotonicity properties for the joint and the conditional tail of the distribution for the pair $(\gamma_t, \delta_t)$. We also show that the renewal function is concave if and only if the pair $(\gamma_t, \delta_t)$ is increasing with respect to the upper orthant stochastic order. Our results are illustrated by two numerical examples in Section 6.

## 2 Bivariate dependence and stochastic orders

In this section we recall some well-known dependence concepts and stochastic orders of two-dimensional random variables that will be needed in the sequel. For further details the reader is referred to the books by Gupta et al. [9] and Denuit et al. [6].

The pair $(X, Y)$ of nonnegative random variables is called

(i) positive quadrant dependent (PQD) if for any $x, y \ge 0$,

$$\mathbb{P}(X \le x,\ Y \le y) \ge \mathbb{P}(X \le x)\,\mathbb{P}(Y \le y); \tag{1}$$

(ii) left tail decreasing (LTD) if for any $y \ge 0$ and $v < z$,

$$\mathbb{P}(Y \le y \mid X \le v) \ge \mathbb{P}(Y \le y \mid X \le z),$$

i.e. if the function $\mathbb{P}(Y \le y \mid X \le x)$ is decreasing in $x$ for any $y$;

(iii) right tail increasing (RTI) if for any $y \ge 0$ and $v < z$,

$$\mathbb{P}(Y > y \mid X > v) \le \mathbb{P}(Y > y \mid X > z), \tag{2}$$

i.e. if the function $\mathbb{P}(Y > y \mid X > x)$ is increasing in $x$ for any $y$;

(iv) positive regression dependent (PRD) if for any $y \geq 0$ and $v < z$,

$$\mathbb{P}(Y \leq y \mid X = v) \geq \mathbb{P}(Y \leq y \mid X = z), \tag{3}$$

i.e. if the function $\mathbb{P}(Y \leq y \mid X = x)$ is decreasing in $x$ for any $y$.

For future reference, we mention that if the distribution of $X$ has finite support, then (3) is only required to hold for $v, z$ in the support of $F_X$. Further, we note that in the bivariate case that we consider here, the condition in (1) is equivalent to the condition that for $x, y \geq 0$,

$$\mathbb{P}(X > x,\ Y > y) \geq \mathbb{P}(X > x)\,\mathbb{P}(Y > y).$$

The dual concepts of dependence are defined by reversing the sign of the inequalities above. For example, $X, Y$ are negatively quadrant dependent (NQD) if for any $x, y \geq 0$,

$$\mathbb{P}(X \leq x, Y \leq y) \leq \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq y).$$

Another related notion is that of *association*, introduced by Esary et al. [7]. We say that the random variables $X$ and $Y$ are (positively) associated if for any bivariate increasing (in both dimensions) functions $g(x, y)$ and $h(x, y)$,

$$Cov(g(X, Y), h(X, Y)) \geq 0,$$

provided of course that all the relevant expectations exist. If the sign of the above inequality is reversed, we say that the variables are negatively associated.

Among the various types of dependence mentioned above, positive regression dependence is the strongest, as every other dependence structure that we have defined is implied by it. In particular, we have the following implications [9]:

1. PRD implies LTD and RTI;
2. if two random variables $X, Y$ are PRD, then they are also associated;
3. if two random variables satisfy one of the following conditions: (i) LTD; (ii) RTI; (iii) they are associated; then they are also PQD.

It is important to note in our context that, as pointed out by Lehmann [11] and in contrast, e.g., to PQD or association, PRD is not symmetric in $X$ and $Y$; that is, the fact that the pair $(X, Y)$ is PRD does not necessarily imply that the same is true for $(Y, X)$. However, it is argued in Lehmann [11] that if one of these two pairs is PRD, this suffices to yield the other types of dependence for the variables $X$ and $Y$.

In the following sections we shall compare the pairs $(\gamma_t, \delta_t)$ for different values of $t$. This leads us to the notion of a bivariate stochastic order. Univariate stochastic comparisons of the excess lifetime at different times of a renewal process have been studied by Belzunce et al. [2]. In fact, there are various ways to generalize the usual (univariate) stochastic ordering between two random variables $X$ and $Y$. A standard reference for multivariate stochastic orders is Shaked and Shanthikumar [20, Chap. 6]; see also Belzunce et al. [1, Chap. 3]. Here, we shall confine ourselves to the notion of *upper orthant stochastic order* for two pairs of random variables.

More precisely, let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two pairs of random variables. Then we say that $(X_1, X_2)$ is smaller than $(Y_1, Y_2)$ in the upper orthant order (denoted by $(X_1, X_2) \leq_{uo} (Y_1, Y_2)$) if for any $x$, $y$,

$$\mathbb{P}(X_1 > x, X_2 > y) \leq \mathbb{P}(Y_1 > x, Y_2 > y).$$

It is clear that the upper orthant order implies the univariate stochastic order of the component variables, so that from the last expression we obtain $\mathbb{P}(X_1 > x) \leq \mathbb{P}(Y_1 > x)$ and $\mathbb{P}(X_2 > y) \leq \mathbb{P}(Y_2 > y)$ for any $x, y \geq 0$.

## 3 Dependence relations for the recurrence times

Here we study the dependence structure of the recurrence times $\gamma_t$, $\delta_t$ in a renewal process at the same time point $t$. More explicitly, let $X$ denote an interarrival time in the process, having distribution function $F$, and let $(\gamma_t, \delta_t)$ be the pair of the forward and backward recurrence times, respectively, at time $t$. Then we have the following result. Here, NRD stands for negatively regression dependent, the dual dependence relation of PRD.

**Proposition 3.1.** $(i)$ *The distribution $F$ has increasing failure rate if and only if for any $t > 0$ the variables $\delta_t$ and $\gamma_t$ are NRD.*

$(ii)$ *The distribution $F$ has decreasing failure rate if and only if for any $t > 0$ the variables $\delta_t$ and $\gamma_t$ are PRD.*

**Proof.** (i) Let $y > 0$ be fixed. If $F$ has increasing failure rate, then

$$\frac{\overline{F}(v + y)}{\overline{F}(v)} \geq \frac{\overline{F}(z + y)}{\overline{F}(z)} \tag{4}$$

for any $v < z \leq t$. But this is equivalent to the condition

$$1 - \mathbb{P}(\gamma_t > y \mid \delta_t = v) \leq 1 - \mathbb{P}(\gamma_t > y \mid \delta_t = z), \tag{5}$$

so that the variables $\delta_t$ and $\gamma_t$ are NRD. Conversely, assuming that $\delta_t$ and $\gamma_t$ are NRD, then (5) holds and this in turn implies (4), which means that $F$ is IFR.

(ii) The proof for the DFR case is similar by reversing the sign of the inequalities above. □

Because of the last proposition, and bearing in mind the discussion in the previous section, we see that if $F$ has a monotone failure rate, then the pair $(\gamma_t, \delta_t)$ satisfies all other (weaker) types of dependence discussed there. For instance, if $F$ is DFR, then $\gamma_t$ and $\delta_t$ are associated, so that in particular $Cov(\gamma_t, \delta_t) \geq 0$ for all $t$. Similarly, if $F$ is IFR, the recurrence times are negatively associated, and thus $Cov(\gamma_t, \delta_t) \leq 0$ for all $t$. In fact, much less is required for the recurrence times to be (negatively) correlated. For, using Hoeffding's identity (see Lehmann [11]), we get

$$Cov(\gamma_t, \delta_t) = \int_0^\infty \int_0^\infty [\mathbb{P}(\gamma_t > x, \delta_t > y) - \mathbb{P}(\gamma_t > x)\,\mathbb{P}(\delta_t > y)]\,dx\,dy,$$

so that the covariance between $\gamma_t$ and $\delta_t$ is greater than (less than) zero provided that $(\gamma_t, \delta_t)$ are positively (resp., negatively) quadrant dependent.

## 4 Joint tail of the backward and forward recurrence times

By Brown [4], if $F$ is DFR, then

$$\gamma_t \text{ and } \delta_t \text{ are stochastically increasing in } t \geq 0 \tag{6}$$

and

$$\text{the renewal function } U(t) \text{ is concave.} \tag{7}$$

Recently, Losidis et al. [15] proved that if $F$ is DFR, then

$$(\gamma_t, \delta_t) \leq_{uo} (\gamma_s, \delta_s) \quad \text{for } 0 \leq t \leq s.$$

In view of (6) and (7), Shaked and Zhu [21] proved the equivalence

$$\gamma_t \text{ is stochastically increasing } \quad \Leftrightarrow \quad U(t) \text{ is concave.} \tag{8}$$

Motivated by the above discussion, in the next theorem we prove that the pair $(\gamma_t, \delta_t)$ is increasing in the upper orthant order if and only if the renewal function $U(t)$ is concave.

**Theorem 4.1.** *In a renewal process, it holds that* $(\gamma_t, \delta_t) \leq_{uo} (\gamma_s, \delta_s)$ *for any* $0 \leq t \leq s$ *if and only if* $U(t)$ *is a concave function.*

**Proof.** ($\Rightarrow$) For $0 \leq t \leq s$, assume that $(\gamma_t, \delta_t) \leq_{uo} (\gamma_s, \delta_s)$. This means that $\mathbb{P}(\gamma_t > y, \delta_t > x) \leq \mathbb{P}(\gamma_s > y, \delta_s > x)$ for any $x, y$. If we put $x = 0^-$, the result follows by (8).

($\Leftarrow$) We assume that $U(t)$ is concave. For any $0 \leq x \leq t$ and $y \geq 0$ the joint tail of forward and backward recurrence times satisfies the equation, see, for example, Gakis and Sivazlian [8],

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \overline{F}(t+y) + \int_0^{t-x} \overline{F}(t+y-z)\, u(z)\, dz. \tag{9}$$

For $x = 0$ the above equation gives,

$$\mathbb{P}(\gamma_t > y) = \overline{F}(t+y) + \int_0^{t} \overline{F}(t+y-z)\, u(z)\, dz,$$

and then equation (9) can be written as

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \mathbb{P}(\gamma_t > y) - \int_{t-x}^{t} \overline{F}(t+y-z)\, u(z)\, dz, \tag{10}$$

or equivalently,

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \mathbb{P}(\gamma_t > y) - \int_{y}^{x+y} \overline{F}(z)\, u(t+y-z)\, dz. \tag{11}$$

Under the assumption that the renewal function is concave (thus $u(t)$ is a decreasing function) for any $0 \leq t \leq s$ we obtain that

$$\int_{y}^{x+y} \overline{F}(z)\, u(t+y-z)\, dz \geq \int_{y}^{x+y} \overline{F}(z)\, u(s+y-z)\, dz,$$

and, by using (8) we have $\mathbb{P}(\gamma_t > y) \leq \mathbb{P}(\gamma_s > y)$. Thus, by (11), for $0 \leq t \leq s$ and $x \leq t$ it follows that

$$\mathbb{P}(\gamma_t > y, \delta_t > x) \leq \mathbb{P}(\gamma_s > y, \delta_s > x),$$

namely $(\gamma_t, \delta_t) \leq_{uo} (\gamma_s, \delta_s)$. For $0 \leq t < x$, the left-hand side above is zero and this completes the proof. □

Apart from the forward and backward recurrence times, another quantity of interest is their sum, $\alpha_t = \gamma_t + \delta_t$, known as the total lifetime (or the spread), which is simply the length of the interarrival time including the current time $t$. Our next result concerns the stochastic monotonicity of the total lifetime. Recall that a variable $X$ is smaller than $Y$ in the increasing convex order (icx order), denoted by $X \leq_{icx} Y$, if

$$E[\phi(X)] \leq E[\phi(Y)]$$

for any increasing, convex function $\phi$ (see, e.g., Shaked and Shanthikumar [20]).

**Corollary 4.2.** *If the renewal function $U(t)$ is concave, then $\{\alpha_t : t \geq 0\}$ is increasing in $t$ with respect to the increasing convex order; that is, if $U$ is concave, $\alpha_t \leq_{icx} \alpha_s$ for $0 \leq t \leq s$.*

**Proof.** Assuming that $U$ is concave, we have from Theorem 4.1 that $(\gamma_t, \delta_t) \leq_{uo} (\gamma_s, \delta_s)$ for $0 \leq t \leq s$. The result now follows from Theorem 3 in Boutsikas and Vaggelatou [3]. □

Thus, for example, if the interarrival distribution $F$ is DFR, then $U$ is concave and from the corollary we have that $\alpha_t$ is increasing with respect to the icx order.

In the rest of this section, we focus on the bivariate tail of the joint distribution for the recurrence times, $\gamma_t$ and $\delta_t$. Extending recent work by Losidis and Politis [14] and Losidis et al. [15], who studied the covariance and the joint tail of these variables, we discuss monotonicity properties of this joint tail and their link with aging properties of the interarrival distribution $F$.

The next result generalizes Brown's result in (6). The result has also been obtained, by different arguments, in Losidis et al. [15].

**Corollary 4.3.** *If the distribution $F$ of the interarrival times is DFR, then the probability $\mathbb{P}(\gamma_t > y, \delta_t > x)$ is increasing in $t$ for any $x, y \geq 0$.*

**Proof.** Let $t \geq s$. By (11) and recalling the results in (6) and (7), for any $y \geq 0$ and $0 \leq x \leq t$ we have that

$$\begin{aligned}\mathbb{P}(\gamma_t > y, \delta_t > x) &\leq \mathbb{P}(\gamma_s > y) - \int_y^{x+y} \overline{F}(z)\, u(s+y-z)\, dz \\ &= \mathbb{P}(\gamma_s > y, \delta_s > x),\end{aligned}$$

and the result follows; for $x > t$, we have that $\mathbb{P}(\gamma_t > y, \delta_t > x) = 0$ so that the result is trivially true is this case, too. □

An interesting question is whether the renewal density $u(t)$ is monotone, decreasing or increasing, in $t \geq 0$. By (7), we see that if the distribution $F$ is DFR, then $u(t)$

is a decreasing function in $t \geq 0$. However, if $F$ is IFR, this does not imply that $u(t)$ is an increasing function (see, e.g., Shaked and Zhu [21]). Assuming that the renewal density $u(t)$ is an increasing function, we can see that $\mathbb{P}(\gamma_t > y, \delta_t > x)$ is decreasing in $t$.

In the following result, we present a formula for the joint tail $\mathbb{P}(\gamma_t > y, \delta_t > x)$. Using that formula we will be able to present conditions under which $Cov(\gamma_t, \delta_t) \geq (\leq)\, 0$. More specifically:

**Proposition 4.4.** *For $y \geq 0$ and $0 \leq x \leq t$, the joint probability $\mathbb{P}(\gamma_t > y, \delta_t > x)$ satisfies the equation*

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \mathbb{P}(\gamma_t > y)\,\mathbb{P}(\delta_t > x) + \int_{t-x}^{t} [\mathbb{P}(\gamma_t > y) - \overline{F}_{t-s}(y)]\,\overline{F}(t-s)\,u(s)\,ds.$$

**Proof.** For $y = 0$ formula (10) yields

$$\mathbb{P}(\delta_t > x) + \int_{t-x}^{t} \overline{F}(t-s)\,u(s)\,ds = 1.$$

Then, by (10), we have

$$\begin{aligned}\mathbb{P}(\gamma_t > y, \delta_t > x) &= \mathbb{P}(\gamma_t > y)\left[\mathbb{P}(\delta_t > x) + \int_{t-x}^{t} \overline{F}(t-s)\,u(s)\,ds\right] \\ &\quad - \int_{t-x}^{t} \overline{F}(t+y-s)\,u(s)\,ds \\ &= \mathbb{P}(\gamma_t > y)\,\mathbb{P}(\delta_t > x) \\ &\quad + \int_{t-x}^{t} [\mathbb{P}(\gamma_t > y)\overline{F}(t-s) - \overline{F}(t+y-s)]\,u(s)\,ds,\end{aligned}$$

and the result follows. $\square$

Proposition 4.4 could be used to determine conditions under which we have $\mathbb{P}(\gamma_t > y, \delta_t > x) \geq (\leq)\, \mathbb{P}(\gamma_t > y)\,\mathbb{P}(\delta_t > x)$ which, in view of the definitions in Section 2, means that the pair $(\gamma_t, \delta_t)$ is PQD (NQD). More specifically, from Proposition 4.4 we obtain immediately the following two corollaries.

**Corollary 4.5.** *For $y \geq 0$ and $0 \leq x \leq t$, it holds that $\mathbb{P}(\gamma_t > y, \delta_t > x) \geq (\leq)\, \mathbb{P}(\gamma_t > y)\,\mathbb{P}(\delta_t > x)$ if and only if*

$$\mathbb{P}(\gamma_t > y) \geq (\leq)\, \frac{\int_{t-x}^{t} \overline{F}_{t-s}(y)\,\overline{F}(t-s)\,u(s)\,ds}{\int_{t-x}^{t} \overline{F}(t-s)\,u(s)\,ds}.$$

**Corollary 4.6.** *For $y \geq 0$ and $0 \leq x \leq t$, if $\mathbb{P}(\gamma_t > y) \geq (\leq)\, \overline{F}_{t-s}(y)$ for any $t-x \leq s \leq t$, then*

$$\mathbb{P}(\gamma_t > y, \delta_t > x) \geq (\leq)\, \mathbb{P}(\gamma_t > y)\,\mathbb{P}(\delta_t > x),$$

*so that $(\gamma_t, \delta_t)$ is PQD (NQD).*

## 5 Conditional tails

In the literature, the joint distribution of the backward and forward recurrence times and the right tails of those quantities have been studied extensively. However, relatively little attention has been paid to the conditional distribution of one of these quantities, given some information on the other. This gives us, in particular, additional insight into the dependence structure and the covariance between the two recurrence times. In this section we are focusing on the following two types of events.

1. At least $x$ time units have passed since the last renewal, given that the next renewal will occur after $y$ time units. In this case, we study the conditional probability $\mathbb{P}(\delta_t > x \mid \gamma_t > y)$.

2. No renewal will occur during the next $y$ time units, given that more than $x$ time units have passed since the last revival. Here we look at the probability $\mathbb{P}(\gamma_t > y \mid \delta_t > x)$.

Next, we present a formula for the two conditional probabilities above. Using these expressions we will be able to present conditions under which $Cov(\gamma_t, \delta_t) \geq (\leq) 0$. More specifically:

**Proposition 5.1.** *For $y \geq 0$ and $0 \leq x \leq t$ the conditional probabilities $\mathbb{P}(\delta_t > x \mid \gamma_t > y)$ and $\mathbb{P}(\gamma_t > y \mid \delta_t > x)$ are obtained as follows:*

$$\mathbb{P}(\delta_t > x \mid \gamma_t > y) = \mathbb{P}(\delta_t > x) + \frac{1}{\mathbb{P}(\gamma_t > y)} \int_{t-x}^{t} [\mathbb{P}(\gamma_t > y) - \overline{F}_{t-s}(y)]\, \overline{F}(t-s)\, u(s)\, ds$$

*and*

$$\mathbb{P}(\gamma_t > y \mid \delta_t > x) = \mathbb{P}(\gamma_t > y) + \frac{1}{\mathbb{P}(\delta_t > x)} \int_{t-x}^{t} [\mathbb{P}(\gamma_t > y) - \overline{F}_{t-s}(y)]\, \overline{F}(t-s)\, u(s)\, ds.$$

**Proof.** Both results follow directly from Proposition 4.4. □

**Theorem 5.2.** *For $y \geq 0$ and $0 \leq x \leq t$, if $\mathbb{P}(\gamma_t > y)/\overline{F}(t + y - s)$ is increasing in $y$ for any $s \in [t - x, t]$, then the pair $(\gamma_t, \delta_t)$ is RTI.*

**Proof.** For $y \geq 0$ and $0 \leq x \leq t$ dividing by $\mathbb{P}(\gamma_t > y)$ each term of equation (10) we obtain

$$\mathbb{P}(\delta_t > x \mid \gamma_t > y) = 1 - \int_{t-x}^{t} \frac{\overline{F}(t + y - s)\, u(s)}{\mathbb{P}(\gamma_t > y)}\, ds,$$

and thus, the probability $\mathbb{P}(\delta_t > x \mid \gamma_t > y)$ is increasing in $y$. The result follows directly from (2), on noting that for $0 \leq t < x$ we have $\mathbb{P}(\delta_t > x \mid \gamma_t > y) = 0$. □

Next, we note that for any $x, y, t \geq 0$,

$$\mathbb{P}(\gamma_t > y + x) = \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x), \tag{12}$$

see, e.g., Janssen and Manca [10, p. 82]. By using this, we see that

$$\mathbb{P}(\delta_{t+x} > x \mid \gamma_{t+x} > y) = \frac{\mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x)}{\mathbb{P}(\gamma_{t+x} > y)} = \frac{\mathbb{P}(\gamma_t > y + x)}{\mathbb{P}(\gamma_{t+x} > y)} \tag{13}$$

and

$$\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x) = \frac{\mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x)}{\mathbb{P}(\delta_{t+x} > x)} = \frac{\mathbb{P}(\gamma_t > y + x)}{\mathbb{P}(\gamma_t > x)}, \tag{14}$$

where in the last equality we use that $\mathbb{P}(\delta_{t+x} > x) = \mathbb{P}(\gamma_t > x)$; this follows by formula (12) or formula (13), for $y = 0$ (see also Janssen and Manca [10, p. 79]). It is worth mentioning that the conditional probability in (14), is (in some sense) the residual distribution of $\gamma_t$. In the next theorem we prove that the probability $\mathbb{P}(\delta_{t+x} > x \mid \gamma_{t+x} > y)$ is increasing in $t$, under the assumption that the interarrival times are DFR, generalizing Brown's result in (6).

**Theorem 5.3.** *If the distribution $F$ of the interarrival times is DFR, then the probability $\mathbb{P}(\delta_{t+x} > x \mid \gamma_{t+x} > y)$ is increasing in $t \geq 0$ for any $x, y \geq 0$.*

**Proof.** By (11) it follows that

$$\mathbb{P}(\delta_{t+x} > x \mid \gamma_{t+x} > y) = 1 - \int_y^{x+y} \frac{\overline{F}(z)\, u(t + x + y - z)}{\mathbb{P}(\gamma_{t+x} > y)}\, dz. \tag{15}$$

Under the hypothesis that $F$ is DFR, by formula (15) and using the results in (6) and (7), for any $0 \leq t \leq s$ we have

$$\begin{aligned} \mathbb{P}(\delta_{t+x} > x \mid \gamma_{t+x} > y) &\leq 1 - \int_y^{x+y} \frac{\overline{F}(z)\, u(s + x + y - z)}{\mathbb{P}(\gamma_{s+x} > y)}\, dz \\ &= \mathbb{P}(\delta_{s+x} > x \mid \gamma_{s+x} > y), \end{aligned}$$

and this completes the proof. □

We now look at the other direction; the converse of Brown's result in (7), namely that the concavity of $U$ implies DFR interarrival times, is known as Brown's conjecture and was disproved by Yu [23]. However, Chen [5] proved a weaker result: if $U$ is concave, then the interarrival distribution $F$ is NWU. He also showed that if $U$ is convex, then $F$ is NBU. To this end, we offer the following generalization of Chen's result.

**Theorem 5.4.** *For a fixed $x \geq 0$, define a function $g_x(t, y)$ by*

$$g_x(t, y) = \mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x), \quad t \geq 0, y \geq 0.$$

*Then, provided that $g_x(t, y)$ is increasing (decreasing) in $t$ for all $y \geq 0$, the residual distribution $F_x$ is NWU (NBU).*

**Proof.** First we mention that the function $\mathbb{P}(\gamma_t > y)$ satisfies the renewal equation (see, for example, Chen [5])

$$\mathbb{P}(\gamma_t > y) = \overline{F}(t + y) + \int_0^t \mathbb{P}(\gamma_{t-z} > y)\, dF(z) \tag{16}$$

for $t, y \geq 0$. Then, by (12), (14) and (16) we obtain that

$$g_x(t, y) = \frac{\overline{F}(t + x + y)}{\mathbb{P}(\delta_{t+x} > x)} + \frac{1}{\mathbb{P}(\delta_{t+x} > x)} \int_0^t \mathbb{P}(\gamma_{t+x-z} > y, \delta_{t+x-z} > x)\, dF(z)$$

$$= \frac{\overline{F}(t+x+y)}{\mathbb{P}(\delta_{t+x} > x)} + \int_0^t g_x(t-z, y)\,\frac{\mathbb{P}(\delta_{t+x-z} > x)}{\mathbb{P}(\delta_{t+x} > x)}\,dF(z).$$

Assume now that $g_x(t, y)$ is increasing in $t$. We then have

$$g_x(t, y) \le \frac{\overline{F}(t+x+y)}{\mathbb{P}(\delta_{t+x} > x)} + g_x(t, y) \int_0^t \frac{\mathbb{P}(\delta_{t+x-z} > x)}{\mathbb{P}(\delta_{t+x} > x)} dF(z). \tag{17}$$

Recalling that $\mathbb{P}(\delta_{t+x} > x) = \mathbb{P}(\gamma_t > x)$ and using (16), we get

$$\begin{aligned} \mathbb{P}(\delta_{t+x} > x) &= \overline{F}(t+x) + \int_0^t \mathbb{P}(\gamma_{t-s} > x)\,dF(s) \\ &= \overline{F}(t+x) + \int_0^t \mathbb{P}(\delta_{t+x-s} > x)\,dF(s), \end{aligned}$$

or equivalently,

$$\int_0^t \mathbb{P}(\delta_{t+x-s} > x)\,dF(s) = \mathbb{P}(\delta_{t+x} > x) - \overline{F}(t+x).$$

Substituting the latter equality into (17) we obtain

$$g_x(t, y)\,\overline{F}(t+x) \le \overline{F}(t+x+y).$$

By the assumption that $\mathbb{P}(\gamma_t > y \mid \delta_t > x)$ is increasing in $t$ we have

$$g_x(0, y)\,\overline{F}(t+x) = \frac{\overline{F}(x+y)}{\overline{F}(x)}\,\overline{F}(t+x) \le g_x(t, y)\,\overline{F}(t+x) \le \overline{F}(t+x+y).$$

Thus, for any $t, y \ge 0$, we conclude that

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} \le \frac{\overline{F}(t+x+y)}{\overline{F}(t+x)},$$

or equivalently,

$$\overline{F}_x(t)\,\overline{F}_x(y) \le \overline{F}_x(t+y),$$

which shows that $F_x$ is NWU. The proof for the NBU case is similar with the sign of the inequalities reversed. □

**Remark 5.5.** (i) Applying Theorem 5.4 for $x = 0$, we obtain that if $\mathbb{P}(\gamma_t > y)$ is increasing (decreasing) in $t \ge 0$ for all $y \ge 0$, then the distribution $F$ is NWU (NBU), a result obtained by Chen [5].

(ii) In general, the residual life distributions of NWU distributions need not be NWU; see, for example, Marshall and Olkin [16, p. 166]. If the assumption of monotonicity in Theorem 5.4 holds for any $x \ge 0$, then $F_x$ is NWU for any $x$. But, $F_x$ is NWU for any $x$ if and only if $F$ is DFR, and we thus conclude that the hypothesis that $\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x)$ is increasing in $t$ for any $x, y \ge 0$, yields that $F$ is DFR. This shows in particular that the converse of Theorem 5.3 is also true, so that the two conditions there are in fact equivalent. It is clear that the assumption that

$\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x)$ is increasing in $t$ for any $x, y \geq 0$ is more strict compared with the assumption that $\mathbb{P}(\gamma_t > y)$ is increasing in $t$ for any $y \geq 0$ (see also part (i) of this remark), while DFR is a subclass of NWU distributions. A similar remark applies in the NBU case.

(iii) In view of (14), Theorem 5.4 can be read as: if the conditional tail of $\gamma_t$, given that $\gamma_t > x$, is increasing (decreasing) in $t$, then the residual lifetime of $F$ at $x$ is NWU (NBU) for every $x \geq 0$.

## 6 Examples

To illustrate the findings of the previous sections, we now give two examples; in the first one, the distribution $F$ is DFR, in the second one it is IFR.

**Example 6.1.** Suppose that the interarrival distribution $F$ is absolutely continuous with tail function

$$\overline{F}(t) = \frac{1}{2}\, e^{-3t} + \frac{1}{2}\, e^{-7t}, \quad t \geq 0,$$

so that $F$ is a mixture of two exponential distributions with parameters 3 and 7 with respective weights 1/2 and 1/2. After some manipulation, we obtain that the renewal function is $U(t) = (4 - 4e^{-5t} + 105t)/25$, and the renewal density is $u(t) = (21 + 4e^{-5t})/5$. Moreover, by using formula (9), we get for $x, y, t \geq 0$ that

$$\begin{aligned}\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x) &= \frac{\overline{F}(t+x+y) + \int_0^t \overline{F}(t+x+y-z)\, u(z) dz}{\overline{F}(t+x) + \int_0^t \overline{F}(t+x-z)\, u(z)\, dz} \\ &= q_{t,x}\, e^{-3y} + (1-q_{t,x})\, e^{-7y},\end{aligned}$$

where

$$q_{t,x} = \frac{e^{4x}(7e^{5t}-2)}{e^{4x}(7e^{5t}-2) + 3e^{5t} + 2}.$$

It is easy to check that $q_{t,x} \in [0, 1]$ for any $t$, $x$. We observe that the conditional probability $\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x)$ is also a tail of a mixture of two exponential distributions with parameters 3 and 7 and weights $q_{t,x}$ and $1 - q_{t,x}$. Furthermore, after some computation we get

$$\frac{d}{dt}\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x) = \frac{100 e^{5t+4x-7y}\,(e^{4y}-1)}{(e^{4x}\,(7e^{5t}-2) + 3e^{5t} + 2)^2},$$

which is nonnegative for all $t, x, y \geq 0$. Finally, since $F$ is a mixture of exponentials, it is DFR (see, e.g., Willmot and Lin [22]) and, in view of Theorem 4.3, the function $\mathbb{P}(\gamma_t > y, \delta_t > x)$ is increasing in $t$. In fact, for $y \geq 0$ and $0 \leq x \leq t$ we have

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \frac{7}{10}\, e^{-3(x+y)} + \frac{3}{10}\, e^{-7(x+y)} - \frac{1}{5}\, e^{-5t}\,(e^{2x-3y} - e^{-2x-7y}),$$

and it is immediately checked that this is an increasing function of $t$. We also observe that

$$\lim_{t\to x+} \mathbb{P}(\gamma_t > y, \delta_t > x) = \frac{1}{2}\, e^{-3(x+y)} + \frac{1}{2}\, e^{-7(x+y)} = \overline{F}(x+y)$$

and

$$\lim_{t\to\infty} \mathbb{P}(\gamma_t > y, \delta_t > x) = \frac{7}{10}\, e^{-3(x+y)} + \frac{3}{10}\, e^{-7(x+y)} = \overline{F}_e(x+y).$$

**Example 6.2.** We assume that the interarrival distribution $F$ has density

$$f(t) = \frac{1}{2}(1+t)e^{-t}, \quad t \geq 0,$$

so that $F$ is a Lindley distribution with parameter $\theta = 1$. The right tail is $\overline{F}(t) = e^{-t}(2+t)/2$ and the equilibrium distribution is given by $\overline{F}_e(t) = e^{-t}(3+t)/3$. The failure rate is $\lambda(t) = f(t)/\overline{F}(t) = (1+t)/(2+t)$, with first derivative

$$\frac{d}{dt}\lambda(t) = -\frac{1+t}{(2+t)^2} + \frac{1}{2+t} \geq 0, \quad \text{for } t \geq 0.$$

After some manipulation, we obtain that the renewal function is $U(t) = (-1 + e^{-3t/2} + 6t)/9$, and the renewal density is $u(t) = (4 - e^{-3t/2})/6$. The first moment is $\mu = 3/2$. Figure 1 presents the renewal density $u(t)$ (solid line) and the quantity $\mu^{-1}$ (dashed line).

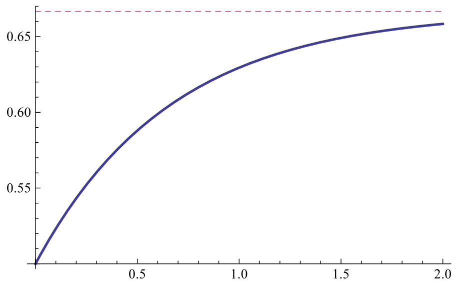


**Fig. 1.** Renewal density $u(t)$ and $\mu^{-1}$

It is clear that $u(t)$ is increasing and tends to $\mu^{-1}$ as $t \to \infty$. Further, using (9), the joint tail of the forward and backward times $\mathbb{P}(\gamma_t > y, \delta_t > x)$ can be found to be

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \frac{1}{6}e^{-3t/2-x-y}\left(e^{3x/2}(x+y) + 2e^{3t/2}(3+x+y)\right).$$

Figure 2 presents the probability $\mathbb{P}(\gamma_t > y, \delta_t > x)$, which is decreasing in $t$ for any $y \geq 0$ and $0 \leq x \leq t$ (see the discussion after Corollary 4.3). For our calculation, we set $y = 0.6$ and $x = 0.1$.

Next, using formula (9), we get for $x, y, t \geq 0$ that

$$\mathbb{P}(\gamma_{t+x} > y \,|\, \delta_{t+x} > x) = \frac{0.813284 + 8.59757e^{3t/2}}{0.2117 + 13.1254e^{3t/2}}.$$

Figure 3 shows the conditional probability $\mathbb{P}(\gamma_{t+x} > y \,|\, \delta_{t+x} > x)$, which is a decreasing function of $t \geq 0$ and tends to $\overline{F}_e(x+y)/\overline{F}_e(x) = 0.655033$ when $t \to \infty$ (dashed line in the graph). Finally, employing Theorem 5.4 we expect that $\overline{F}_x(t)\,\overline{F}_x(y) - \overline{F}_x(t+y) \geq 0$; this is illustrated in Figure 4.

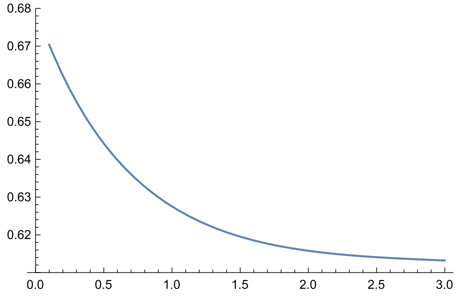


**Fig. 2.** Monotonicity of $\mathbb{P}(\gamma_t > y, \delta_t > x)$ for $y = 0.6$ and $x = 0.1$

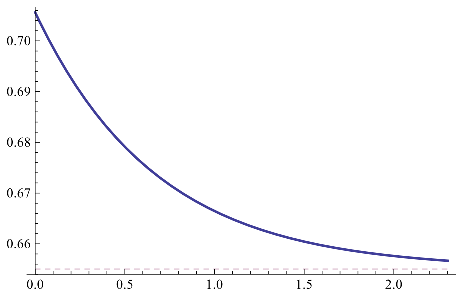


**Fig. 3.** Monotonicity of $\mathbb{P}(\gamma_{t+x} > y \mid \delta_{t+x} > x)$ for $y = 0.6$ and $x = 0.1$

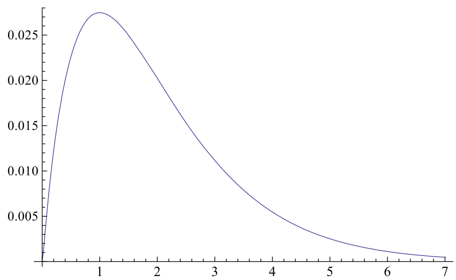


**Fig. 4.** The function $\overline{F}_X(t)\,\overline{F}_X(y) - \overline{F}_X(t+y)$ for $y = 0.6$ and $x = 0.1$

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