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Regularity of paths of stochastic measures

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Abstract Random functions $\mu(x)$, generated by values of stochastic measures are considered. The Besov regularity of the continuous paths of $\mu(x)$, $x \in [0, 1]^d$, is proved. Fourier series expansion of $\mu(x)$, $x \in [0, 2\pi]$, is obtained. These results are proved under weaker conditions than similar results in previous papers.

Keywords Stochastic measure, trajectories of random functions, Besov space, random Fourier series

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1 Introduction

In this paper, we consider the properties of paths of processes generated by values of stochastic measures (SMs). SM is a stochastic set function that is σ-additive in probability, see the exact definition and examples in Subsection 2.1.

The Besov regularity of paths of SMs was studied in [17] and [18], Fourier series expansions of paths of SMs were obtained in [19]. Some important results in these papers were obtained under the following condition for SM μ defined on measurable space $(\mathsf{X}, \mathcal{B})$.

Assumption A 1. There exists a real-valued finite measure m on $(\mathsf{X}, \mathcal{B})$ with the following property: if a measurable function $g : \mathsf{X} \to \mathbb{R}$ is such that $\int_{\mathsf{X}} g^2 \, d\mathsf{m} < +\infty$, then g is integrable with respect to (w.r.t.) μ on X.

In this paper, we prove that some statements from [18] and [19] remain valid if we assume the following condition instead of A1.

Assumption A 2. There exists a real-valued finite measure m on $(\mathsf{X}, \mathcal{B})$ with the following property: if a measurable function $g : \mathsf{X} \to \mathbb{R}$ is such that for all $\lambda \in \mathbb{R}$, $\lambda > 0$, it holds that $\int_{\mathsf{X}} 2^{\lambda|g|}\, d\mathsf{m} < +\infty$, then g is integrable with respect to (w.r.t.) μ on X.

Obviously, condition A2 is weaker than A1.

Our proofs are very similar to respective proofs in [18] and [19] with one essential change – now we use Lemma 2 from the present paper. In the previous publications, the statement of Lemma 2 was obtained under condition A1, in our paper, we assume A2. Note that in many of our results, we assume that the paths of our processes are continuous.

The rest of the paper is organised as follows. In Section 2 we recall the basic facts concerning SMs and Besov spaces and give a short literature review. In Section 3 we prove Lemma 2 and other auxiliary lemmas. Section 4 contains the result about the Besov regularity of SMs on $[0, 1]^d$. In Section 5 we study the Fourier series defined by process $\mu(t) = \mu((0, t])$, $t \in [0, 2\pi]$.

2 Preliminaries

2.1 Stochastic measures

In this subsection, we give basic information concerning stochastic measures in a general setting. In statements of Sections 4 and 5, this set function is defined on Borel subsets of $[0, 1]^d$ or $[0, 2\pi]$.

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and $\mathcal{B}$ a σ-algebra of subsets of X.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

We do not assume the moment existence or martingale properties for SM. We can say that μ is an L_0-valued measure.

We note the following examples of SMs. The orthogonally scattered stochastic measures are SMs with values in $\mathsf{L}_2(\Omega, \mathcal{F}, \mathsf{P})$. The α-stable random measures defined on a σ-algebra for $\alpha \in (0, 1) \cup (1, 2]$ are independently scattered SMs, see [22, Chapter 3], Definition 1 holds by [22, Proposition 3.5.1].

Many examples of the SMs on the Borel subsets of $[0, T]$ may be given by the Wiener type integral $\mu(\mathsf{A}) = \int_{[0,T]} \mathbf{1}_{\mathsf{A}}(t)\, dX_t$. For example, this holds if X_t is any square integrable martingale or fractional Brownian motion with the Hurst index $H > 1/2$. Other examples may be found in [15, Section 2.1] or [21, Section 1.2.1].

For deterministic measurable functions $f : \mathsf{X} \to \mathbb{R}$, an integral of the form $\int_{\mathsf{X}} f\, d\mu$ is studied in [13, Chapter 7], [21, Chapter 1]. In particular, every bounded measurable f is integrable w.r.t. any μ, and for any integrable g holds

$$\lim_{c\to+\infty} \sup_{|h|\le|g|} \mathsf{P}\left\{\left|\int_{\mathsf{X}} h\, d\mu\right| \ge c\right\} = 0 \tag{1}$$

(this follows from Corollary 1.1, Lemma 1.8, and Theorem 1.2 [21]). An analogue of the Lebesgue dominated convergence theorem holds for this integral (see [13, Proposition 7.1.1] or [21, Theorem 1.5]).

The theory of SMs in detail is considered in [21]. Equations driven by SMs are studied, for example, in [3, 14, 23]. SMs may be used for the study of stochastic dynamical systems (see [1, 2]).

In the sequel, $\mathcal{B}(\mathsf{X})$ denotes the Borel σ-algebra of subsets of X, m_L denotes the Lebesgue measure.

2.2 Besov spaces

These classical functional Banach spaces found many applications in mathematical physics, function theory, and functional analysis. We recall the definition of Besov spaces following [12].

For functions $f \in \mathsf{L}_p([0,1]^d) = \mathsf{L}_p([0,1]^d, \mathsf{m}_L)$ we set

$$\|f\|_{B^\alpha_{p,q}([0,1]^d)} = \|f\|_{\mathsf{L}_p([0,1]^d)} + \Big(\int_0^1 (\omega_p(f,r))^q r^{-\alpha q-1}\,dr\Big)^{1/q},$$

where ω_p denotes the L_p-modulus of continuity,

$$\omega_p(f,r) = \sup_{|h|\le r}\Big(\int_{I_h} |f(x+h)-f(x)|^p\,dx\Big)^{1/p},$$
$$I_h = \{x \in [0,1]^d : x+h \in [0,1]^d\},$$

$|h|$ denotes the Euclidean norm in $\mathbb{R}^d$. Then

$$B^\alpha_{p,q}([0,1]^d) = \{f \in \mathsf{L}_p([0,1]^d) : \|f\|_{B^\alpha_{p,q}([0,1]^d)} < +\infty\},$$

and $\|\cdot\|_{B^\alpha_{p,q}([0,1]^d)}$ is a norm in this space.

The Besov regularity of trajectories was studied for different types of random processes. For example, Gaussian processes were considered in [5], Lévy processes in [10], generalized periodic Lévy processes in [9]. In [16], it was proved that paths of the mild solution of a parabolic equation driven by the cylinder Wiener process belong to some Besov–Orlicz spaces. Also, the Besov–Orlicz regularity of Hermite processes was proved in [6]. The Besov regularity and continuity of paths of multidimensional integral w.r.t. SMs are considered in [15].

3 Auxiliary lemmas

In the first statement, we recall the well-known Paley–Zygmund inequality (see, for example, Lemma 4.3(a) [24] or Lemma 2.1 [21]).

Lemma 1. *Let ε_k, $1 \le k \le m$, be independent random variables with the distribution*

$$\mathsf{P}[\varepsilon_k = 1] = \mathsf{P}[\varepsilon_k = -1] = 1/2.$$

Then for each $\lambda_k \in \mathbb{R}$

$$\mathsf{P}\Big[\Big(\sum_{k=1}^m \lambda_k\varepsilon_k\Big)^2 \ge \frac{1}{4}\sum_{k=1}^m \lambda_k^2\Big] \ge \frac{1}{8}. \tag{2}$$

The following lemma is the main result of this section. Assuming A1, a similar statement was obtained in Lemma 3.3 [18].

Lemma 2. *Assume that μ is an SM on $(\mathsf{X}, \mathcal{B})$ and Assumption A2 holds.*
Let the measurable functions $f_k : \mathsf{X} \to \mathbb{R}$, $k \geq 1$, be such that

$$\sup_{x\in\mathsf{X}} \sum_{k=1}^{\infty} (f_k(x))^2 < \infty. \tag{3}$$

Then

$$\sum_{k=1}^{\infty} \Big(\int_{\mathsf{X}} f_k \, d\mu \Big)^2 < +\infty \quad a.s. \tag{4}$$

Proof. Consider independent random variables ε_k, $k \geq 1$, defined on some other probability space $(\Omega', \mathcal{F}', \mathsf{P}')$, such that

$$\mathsf{P}'[\varepsilon_k(\omega') = 1] = \mathsf{P}'[\varepsilon_k(\omega') = -1] = 1/2.$$

Consider the sum

$$\eta(x, \omega') = \sum_{k=1}^{\infty} \varepsilon_k(\omega') f_k(x). \tag{5}$$

The well-known two-series theorem (see, for example, Theorem 2.5.6 [7]) and condition (3) imply that for each such x series (5) converges P'-a.s. on Ω'.

Denote

$$C_f = \sup_{x\in\mathsf{X}} \sum_{k=1}^{\infty} (f_k(x))^2.$$

Then, applying the Chebyshev inequality, for any $\lambda > 0$ we obtain

$$\mathsf{P}'\{\lambda|\eta(x, \omega')| \geq 1\} \leq \mathrm{Var}\,(\lambda|\eta(x, \omega')|) = \lambda^2 \sum_{k=1}^{\infty} f_k^2(x) \leq \lambda^2 C_f.$$

Take $\lambda_0 > 0$ such that $\lambda_0^2 C_f \leq 1/64$.

In Theorem 2.3 [11] some properties of a series of the form $\sum_{k=1}^{\infty} u_k \varepsilon_k(\omega')$, $u_k \in \mathbb{R}$, are established. By this theorem, if, for some $\lambda, r, a > 0$, it holds that

$$\mathsf{P}'\{\lambda|\eta(x, \omega')| \geq r\} \leq a/2,$$

then

$$\mathsf{P}'\{\lambda|\eta(x, \omega')| \geq 2r\} \leq a^2.$$

We have that

$$\mathsf{P}'\{\lambda_0|\eta(x, \omega')| \geq 1\} \leq 2^{-6},$$

therefore

$$\mathsf{P}'\{\lambda_0|\eta(x, \omega')| \geq 2^k\} \leq 2^{-2^{k+2}-2}.$$

The Lévy inequality for symmetric variables (see, for example, [11, Lemma 2.3]) implies that for any $x \in \mathsf{X}$, $c > 0$, for

$$\zeta(x,\omega') = \sup_{n\ge 1}\Big|\sum_{k=1}^{n} \varepsilon_k(\omega') f_k(x)\Big|,$$

it holds that

$$\mathsf{P}'[\zeta(x,\omega') > c] \le 2\mathsf{P}'[|\eta(x,\omega')| > c].$$

Therefore,

$$\mathsf{P}'\{\lambda_0\zeta(x,\omega') \ge 2^k\} \le 2^{-2^{k+2}-1}.$$

We get

$$\begin{aligned}
\mathsf{E}_{\mathsf{P}'} 2^{\lambda_0\zeta(x,\omega')} &\le 2\mathsf{P}'\{\lambda_0\zeta(x,\omega') \le 2\} \\
&\quad + \sum_{k=1}^{\infty} 2^{2^{k+1}} \mathsf{P}'\{2^k \le \lambda_0\zeta(x,\omega') \le 2^{k+1}\} \\
&\le 2 + \sum_{k=1}^{\infty} 2^{2^{k+1}} \mathsf{P}'\{\lambda_0\zeta(x,\omega') \ge 2^k\} \le 2 + \sum_{k=1}^{\infty} 2^{2^{k+1}} 2^{-2^{k+2}-1} < 3.
\end{aligned}$$

By the Fubini–Tonelli theorem,

$$\mathsf{E}_{\mathsf{P}'} \int_{\mathsf{X}} 2^{\lambda_0\zeta(x,\omega')}\, d\mathsf{m}(x) = \int_{\mathsf{X}} \mathsf{E}_{\mathsf{P}'} 2^{\lambda_0\zeta(x,\omega')}\, d\mathsf{m}(x) \le 3\mathsf{m}(\mathsf{X}) < +\infty.$$

Therefore,

$$\int_{\mathsf{X}} 2^{\lambda_0\zeta(x,\omega')}\, d\mathsf{m}(x) < +\infty \quad \mathsf{P}'\text{-a.s.}$$

and Assumption A2 implies that $\zeta(x,\omega')$ is integrable w.r.t. μ P'-a.s.

For all $j \ge 1$

$$\Big|\sum_{k=1}^{j} \varepsilon_k(\omega') f_k(x)\Big| \le \zeta(x,\omega').$$

For each ω' (excluding a set of zero P'-measure) we use the dominated convergence theorem [21, Theorem 1.5] for the integral w.r.t. μ and obtain

$$\int_{\mathsf{X}} \eta(x,\omega')\, d\mu(x) = \sum_{k=1}^{\infty} \varepsilon_k(\omega') \int_{\mathsf{X}} f_k(x)\, d\mu(x),$$

where the last series converges in probability P. Using this, it is easy to obtain that this series converges in probability $\mathsf{P}\times\mathsf{P}'$ on $\Omega\times\Omega'$, and its partial sums are bounded in probability $\mathsf{P}\times\mathsf{P}'$.

Further, define $\xi_k(\omega) = \int_{\mathsf{X}} f_k\, d\mu$. Suppose (4) fails. Then

$$\exists\delta_0 > 0\ \forall c > 0\ \exists m_c \ge 1 :\ \mathsf{P}\Big[\sum_{k=1}^{m_c} \xi_k^2(\omega) \ge c\Big] \ge \delta_0. \tag{6}$$

Applying (2) for $\lambda_k = \xi_k(\omega)$, and (6), we obtain

$$\mathsf{P}'\Big[\omega' : \Big(\sum_{k=1}^{m_c} \varepsilon_k(\omega')\xi_k(\omega)\Big)^2 \geq \frac{c}{4}\Big] \geq \frac{1}{8}$$

for each fixed $\omega \in \Omega_c = \big\{\sum_{k=1}^{m_c} \xi_k^2 \geq c\big\}$, and $\mathsf{P}(\Omega_c) \geq \delta_0$. Integrating over the set Ω_c, we get

$$\mathsf{P} \times \mathsf{P}'\Big[(\omega, \omega') : \Big(\sum_{k=1}^{m_c} \varepsilon_k(\omega')\xi_k(\omega)\Big)^2 \geq \frac{c}{4}\Big] \geq \frac{\delta_0}{8}.$$

Thus, there exists ω'_0 such that

$$\mathsf{P}\Big[\omega : \Big(\sum_{k=1}^{m_c} \varepsilon_k(\omega'_0)\xi_k(\omega)\Big)^2 \geq \frac{c}{4}\Big] \geq \frac{\delta_0}{8}$$

and $\zeta(x, \omega'_0)$ is integrable w.r.t. μ. For the function $g(x) = \sum_{k=1}^{m_c} \varepsilon_k(\omega'_0) f_k(x)$ we have

$$|g(x)| \leq \zeta(x, \omega'_0), \quad \mathsf{P}\Big[\Big|\int_{\mathsf{X}} g\, d\mu\Big| \geq \frac{\sqrt{c}}{2}\Big] \geq \frac{\delta_0}{8}.$$

Recall that $\delta_0 > 0$ is fixed and c is arbitrary. Therefore, we obtain a contradiction to (1). □

Example 1. We give an example, where Assumption A2 is valid, and A1 may be not.

Let ε_k, $k \geq 1$, be independent Bernoulli random variables. Consider an SM on Borel subsets on [0, 1],

$$\mu(\mathsf{A}) = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{k^{4/3}} \int_{\mathsf{A}} x^{k^{-1/3}-1}\, dx.$$

This series converges a.s. for each $\mathsf{A} \in \mathcal{B}((0, 1])$ because

$$\sum_{k=1}^{\infty}\Big(\frac{1}{k^{4/3}} \int_{\mathsf{A}} x^{k^{-1/3}-1}\, dx\Big)^2 < \infty,$$

and μ is an SM with values in $\mathsf{L}_2(\Omega, \mathcal{F}, \mathsf{P})$.

For any measurable function $f : [0, 1] \to \mathbb{R}$ such that

$$\sum_{k=1}^{\infty}\Big(\frac{1}{k^{4/3}} \int_{[0,1]} |f(x)| x^{k^{-1/3}-1}\, dx\Big)^2 < \infty$$

f is integrable w.r.t. μ, and it holds that

$$\int_{\mathsf{A}} f\, d\mu = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{k^{4/3}} \int_{\mathsf{A}} f(x) x^{k^{-1/3}-1}\, dx. \tag{7}$$

This fact is obvious for a simple function f. For an arbitrary measurable f, we take simple f_n that converge to f pointwise, $|f_n(x)| \le |f(x)|$. Then

$$\sum_{k=1}^{\infty}\Big(\frac{1}{k^{4/3}}\int_{\mathsf{A}}(f(x)-f_n(x))x^{k^{-1/3}-1}\,dx\Big)^2 \to 0, \quad n\to\infty,$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{\varepsilon_k}{k^{4/3}}\int_{\mathsf{A}} f_n(x)x^{k^{-1/3}-1}\,dx \overset{\mathsf{L}_2}{\to} \sum_{k=1}^{\infty}\frac{\varepsilon_k}{k^{4/3}}\int_{\mathsf{A}} f(x)x^{k^{-1/3}-1}\,dx, \quad n\to\infty,$$

for each $\mathsf{A}\in\mathcal{B}([0,1])$. Theorem 1.8 [21] implies that f is integrable w.r.t. μ, and (7) holds.

For a measurable f such that $\int_{[0,1]}2^{\lambda|f(x)|}\,dx < \infty$, we will obtain that (7) fulfills. Applying the Hölder inequality, we get

$$\begin{aligned}
&\int_{[0,1]}|f(x)|x^{k^{-1/3}-1}\,dx\\
&\quad\le\Big(\int_{[0,1]}|f(x)|^{2k^{1/3}-1}\,dx\Big)^{\frac{1}{2k^{1/3}-1}}\Big(\int_{[0,1]}x^{(1/2)k^{-1/3}-1}\,dx\Big)^{\frac{2k^{1/3}-2}{2k^{1/3}-1}}\\
&\quad\le 2k^{1/3}\Big(\int_{[0,1]}|f(x)|^{2k^{1/3}-1}\,dx\Big)^{\frac{1}{2k^{1/3}-1}}.
\end{aligned}\tag{8}$$

It is easy to calculate that

$$|f(x)|^{2k^{1/3}-1}\le C_{k,\lambda}2^{\lambda|f|},\quad \text{where } C_{k,\lambda}=\Big(\frac{2k^{1/3}-1}{\lambda\ln 2}\Big)^{2k^{1/3}-1}2^{-\frac{2k^{1/3}-1}{\ln 2}}.\tag{9}$$

We get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\Big(\frac{1}{k^{4/3}}\int_{\mathsf{A}}|f(x)|x^{k^{-1/3}-1}\,dx\Big)^2\\
&\quad\overset{(8)}{\le} 4\sum_{k=1}^{\infty}\frac{1}{k^2}\Big(\int_{[0,1]}|f(x)|^{2k^{1/3}-1}\,dx\Big)^{\frac{2}{2k^{1/3}-1}}\\
&\quad\overset{(9)}{\le} C\sum_{k=1}^{\infty}\frac{1}{k^2}\Big(\int_{[0,1]}\Big(\frac{2k^{1/3}-1}{\lambda\ln 2}\Big)^{2k^{1/3}-1}2^{-\frac{2k^{1/3}-1}{\ln 2}}2^{\lambda|f|}\,dx\Big)^{\frac{2}{2k^{1/3}-1}}\\
&\quad\le C\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}\Big(\int_{[0,1]}2^{\lambda|f|}\,dx\Big)^{\frac{2}{2k^{1/3}-1}}<\infty,
\end{aligned}$$

where C denotes a positive constant. Therefore, f is integrable w. r. t. μ.

Thus, Assumption A2 holds for $\mathsf{m}=\mathsf{m}_L$, where m_L denotes the Lebesgue measure. At the same time, it is not clear how we can check A1 for our μ. For m_L or $\mathsf{m}(\mathsf{A})=\int_{\mathsf{A}}x^{\alpha}\,dx$, $-1<\alpha<0$, we can find $f(x)=x^{\beta}$, $\beta<0$, such that f^2 is integrable to m but some elements of sum in (7) are not defined. Investigation of all finite m on $\mathcal{B}([0,1])$ looks difficult. □

The following statements will be used in Section 5 for the series expansion of SMs.

Lemma 3. *Let Assumption A2 hold. Then the set of random variables*

$$\Big\{\sum_{k=1}^{j}\Big(\int_{\mathsf{X}} f_k\,d\mu\Big)^2 \;\Big|\; f_k:\mathsf{X}\to\mathbb{R} \textit{ is measurable},\ \sum_{k=1}^{j} f_k^2(x)\le 1,\ j\ge 1\Big\}$$

is bounded in probability.

Proof. If the statement fails, for some $\delta_0 > 0$ and all $n \ge 1$ we can find functions f_{kn}, $1 \le k \le j_n$, such that

$$\sum_{k=1}^{j_n} f_{kn}^2(x)\le 1,\quad \mathsf{P}\Big\{\sum_{k=1}^{j_n}\Big(\int_{\mathsf{X}} f_{kn}\,d\mu\Big)^2 > 2^n\Big\} > \delta_0.$$

Then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{j_n}\big(2^{-n/2} f_{kn}(x)\big)^2 \le 1,\quad \sum_{n=1}^{\infty}\sum_{k=1}^{j_n}\Big(\int_{\mathsf{X}} 2^{-n/2} f_{kn}\,d\mu\Big)^2 \text{ does not converge,}$$

which contradicts Lemma 2. □

Lemma 4. *Let Assumption A2 hold, μ be an SM on $\mathcal{B}([0,T])$, and the process $\mu(t) = \mu((0,t])$, $0 \le t \le T$, have continuous paths. Then for any T_1, $0 < T_1 < T$, we have*

$$\int_{[0,T_1]} \frac{\big|\mu(s+\varepsilon)-\mu(s)\big|^3}{\varepsilon}\,ds \xrightarrow{\mathsf{P}} 0,\quad \varepsilon\to 0+.$$

Proof. We have that

$$\int_{[0,T_1]} \frac{\big|\mu(s+\varepsilon)-\mu(s)\big|^3}{\varepsilon}\,ds \le \sup_s\big|\mu(s+\varepsilon)-\mu(s)\big|\int_{[0,T_1]} \frac{\big|\mu(s+\varepsilon)-\mu(s)\big|^2}{\varepsilon}\,ds. \tag{10}$$

For any $n \ge 1$ take the partition of $[0, T_1]$ by points $s_{kn} = \Big(\frac{k}{n}T_1\varepsilon\Big) \wedge T_1$, $0 \le k \le j_n$, and consider the Riemann integral sum for the last integral in (10)

$$\sum_{k=1}^{j_n} \frac{\big|\mu(s_{kn}+\varepsilon)-\mu(s_{kn})\big|^2}{\varepsilon}\,\frac{T_1\varepsilon}{n} = T_1\sum_{k=1}^{j_n}\Big(\int_{[0,T_1]} f_{kn}\,d\mu\Big)^2, \quad \text{where}\quad f_{kn}(x) = \frac{1}{\sqrt{n}}\mathbf{1}_{(s_{kn},s_{kn}+\varepsilon]}(x). \tag{11}$$

We have $\sum_{k=1}^{j_n} f_{kn}^2(x) \le 1$, by Lemma 3 set of sums (11) is bounded in probability. For continuous $\mu(s)$, $\sup_s\big|\mu(s+\varepsilon)-\mu(s)\big| \to 0$ as $\varepsilon \to 0$, and this implies the statement of Lemma 4. □

4 Besov regularity of SM in $[0, 1]^d$

Now we will consider SM μ defined on the Borel σ-algebra of $[0, 1]^d$, $d \geq 1$, and obtain the Besov regularity of μ with continuous realizations.

For $x = (x_1, x_2, \dots, x_d) \in [0, 1]^d$ set

$$\mu(x) = \mu\Big(\prod_{i=1}^{d}[0, x_i]\Big).$$

By $e^{(i)}$ we denote the ith coordinate unit vector in $\mathbb{R}^d$, and consider discrete sets

$$U(n, i) = \Big\{y = \Big(\frac{k_1}{2^n}, \frac{k_2}{2^n}, \dots, \frac{k_d}{2^n}\Big) \,\Big|\, k_j = 0,\ 1,\ \dots,\ 2^n,\ 1 \leq j \leq d;\ y + 2^{-n}e^{(i)} \in [0, 1]^d\Big\}.$$

By Corollary 3.3 [12], if for each i, $1 \leq i \leq d$, for respective fixed $\omega \in \Omega$,

$$\sum_{n=1}^{\infty} 2^{n(\alpha p-d)} \sum_{y\in U(n,i)} |\mu(y + 2^{-n}e^{(i)}) - \mu(y)|^p < +\infty, \tag{12}$$

then continuous paths of $\mu(x)$ belong to $B^{\alpha}_{p,p}([0, 1]^d)$. Applying this result we prove the following statement.

Theorem 1. *Let Assumption A2 hold, and the random function $\mu(x)$, $x \in [0, 1]^d$, have continuous realizations. Then for any $1 \leq p < +\infty$, $0 < \alpha < \min\{1/p, 1/2\}$, the realization $\mu(x)$, $x \in [0, 1]^d$, with probability 1 belongs to the Besov space $B^{\alpha}_{p,p}([0, 1]^d)$.*

Proof. Firstly, consider the case $2 \leq p < +\infty$, then $0 < \alpha < 1/p$. To obtain (12), it is sufficient, for each i, to prove the convergence of the series

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{n(\alpha p-d)} &\sum_{y\in U(n,i)} |\mu(y + 2^{-n}e^{(i)}) - \mu(y)|^2 \\ &= \sum_{n\geq 1, y\in U(n,i)} \Big(2^{n(\alpha p-d)/2} \int_{[0,1]^d} h_{n,y}(x)\, d\mu(x)\Big)^2,\end{aligned} \tag{13}$$

where

$$h_{n,y}(x) = \mathbf{1}_{\{x_j \leq k_j/2^n,\ j\neq i,\ k_i/2^n < x_i \leq (k_i+1)/2^n\}}(x), \quad y = \Big(\frac{k_1}{2^n}, \frac{k_2}{2^n}, \dots, \frac{k_d}{2^n}\Big).$$

Here, for each n, each fixed x belongs to at most $2^{n(d-1)}$ sets from the indicators in the functions $h_{n,y}$. Therefore,

$$\sum_{n\geq 1, y\in U(n,i)} (2^{n(\alpha p-d)/2} h_{n,y}(x))^2 \leq \sum_{n=1}^{\infty} 2^{n(\alpha p-d)} 2^{n(d-1)} = \sum_{n=1}^{\infty} 2^{n(\alpha p-1)} < +\infty.$$

Lemma 2 implies that series (13) converges a.s.

For $1 \leq p < 2$, we can repeat the proof of this case from Theorem 5.1 [18] (or Theorem 2.2 [21]), and obtain that (13) coverges for $\alpha = 1/2$. ◻

Note, that for paths of multidimensional integral w.r.t. SM, sufficient conditions of the Besov regularity are given in [15, Theorem 5]. Now we see that we can change A1 to A2 in that statement because we can refer to our Theorem 1 instead of Theorem 5.1 [18] in the proof of Theorem 5 [15].

5 Fourier expansion of SM

Let μ be an SM on $\mathcal{B}([0, 2\pi])$. Consider the random process $\mu(t) = \mu((0, t])$, $0 \le t \le 2\pi$.

Assume that paths of the process $\mu(t)$ are Riemann integrable on $[0, 2\pi]$. Consider the Fourier series defined by $\mu(t)$ for each fixed $\omega \in \Omega$:

$$\xi_k = \frac{1}{\pi}\int_{[0,2\pi]} \mu(s)\cos ks\, ds, \quad \eta_k = \frac{1}{\pi}\int_{[0,2\pi]} \mu(s)\sin ks\, ds, \tag{14}$$

$$\mu(t) \sim \frac{\xi_0}{2} + \sum_{k=1}^{\infty}(\xi_k \cos kt + \eta_k \sin kt). \tag{15}$$

Set

$$S_n(t) = \frac{\xi_0}{2} + \sum_{k=1}^{n}(\xi_k \cos kt + \eta_k \sin kt).$$

Applying the integration by parts formula for integrals in (14) (see [19, Lemma 1] or [21, Lemma 2.7]), we get

$$\begin{gathered}\xi_k = -\frac{1}{k\pi}\int_{(0,2\pi]} \sin ks\, d\mu, \quad \eta_k = \frac{1}{k\pi}\int_{(0,2\pi]} (\cos ks - 1)\, d\mu, \quad k \ge 1,\\ \xi_0 = 2\mu((0, 2\pi]) - \frac{1}{\pi}\int_{(0,2\pi]} s\, d\mu,\end{gathered} \tag{16}$$

and these integrals are defined for any SM μ without the assuming Riemann integrability.

In the sequel, we will assume that ξ_k and η_k are defined by (16), an consider the Fourier series for arbitrary SM μ on $\mathcal{B}([0, 2\pi])$. First, we prove that series (15) converges.

Theorem 2. *If Assumption A2 holds then*

$$\mathsf{P}\big[S_n(t) \text{ converges } \mathsf{m}_L\text{-a.e. on } [0, 2\pi]\big] = 1.$$

Proof. Applying Lemma 2 for

$$f_k = \frac{\sin kt}{\pi k} \quad \text{and} \quad f_k = \frac{\cos kt - 1}{\pi k},$$

using (16), we get $\sum_k(\xi_k^2 + \eta_k^2) < \infty$ a.s. Applying the famous Carleson's theorem (see [4]) we obtain our statement. □

Further, we prove that, under some assumptions, S_n converges to some value of μ.

Representations of processes in the form of random series began with the well-known Paley–Wiener expansion of the Wiener process. A similar representation of the fractional Brownian motion was obtained in [8]. We give such a result for continuous paths of SMs.

The following statement gives a generalization of the Dirichlet–Jordan theorem about Fourier series expansion of functions of bounded variation, see, for example, [25, Theorem II.8.1].

Theorem 3. *Let Assumption A2 hold, and paths of the process $\mu(t)$, $0 \le t \le 2\pi$, be continuous. Then for any $t \in (0, 2\pi)$ it holds that $S_n(t) \xrightarrow{\mathsf{P}} \mu(t)$ and $S_n(0) = S_n(2\pi) \xrightarrow{\mathsf{P}} \mu(2\pi)/2$ as $n \to \infty$.*

Proof. Without loss of generality, we may assume that $\mu((0, 2\pi]) = 0$, and the periodic continuation of $\mu(t)$ to $\mathbb{R}$ is continuous. Otherwise, we can consider the SM

$$\widetilde{\mu}(\mathsf{A}) = \mu(\mathsf{A}) - \mathsf{m}_L(\mathsf{A}) \frac{\mu((0, 2\pi])}{2\pi}.$$

Set

$$S_n^*(t) = \frac{1}{2}(S_{n-1}(t) + S_n(t)) = S_{n-1}(t) + \frac{1}{2}(\xi_n \cos nt + \eta_n \sin nt),$$

$$\varphi_t(s) = \frac{1}{2}(\mu(t+s) + \mu(t-s) - 2\mu(t)) = \frac{1}{2}\big(\mu((t, t+s]) - \mu((t-s, t])\big).$$

From Theorem (II.10.1) [25], it follows that for $h = \pi/n$ it holds

$$|S_n^*(t) - \mu(t)| \le \frac{1}{\pi} \int_h^\pi \frac{|\varphi_t(s) - \varphi_t(s+h)|}{s} ds + h \int_h^\pi \frac{|\varphi_t(s)|}{s^2} ds$$
$$+ \frac{2}{h} \int_0^{2h} |\varphi_t(s)| ds + o(1) := I_1 + I_2 + I_3 + o(1),$$

where $o(1)$ is uniform in t as $n \to \infty$. Continuity of $\mu(t)$ and L'Hôpital's rule give that $I_2, I_3 \to 0$ as $h \to 0$ for each $t \in \mathbb{R}$ and $\omega \in \Omega$. Using the Hölder inequality, we obtain

$$2\pi I_1 = \int_h^\pi \frac{|\mu((t+s, t+s+h]) - \mu((t-s-h, t-s])|}{s} ds$$
$$\le \left(\int_h^\pi \frac{h^{1/2}}{s^{3/2}} ds \right)^{\frac{2}{3}} \left(\int_h^\pi \frac{|\mu((t+s, t+s+h]) - \mu((t-s-h, t-s])|^3}{h} ds \right)^{\frac{1}{3}}.$$

The last value tends to zero in probability as $h \to 0$ by Lemma 4.

Thus, $S_n^*(t) \xrightarrow{\mathsf{P}} \mu(t)$. By the analogue of the Lebesgue dominated convergence theorem (see [13, Proposition 7.1.1] or [21, Theorem 1.5]), $\xi_n, \eta_n \xrightarrow{\mathsf{P}} 0$, and therefore $S_n(t) \xrightarrow{\mathsf{P}} \mu(t)$. □

Remark. Also, Lemma 3 of our paper allows to replace Assupmtion A1 by Assumption A2 in Theorem 3.1 2) [20], where, for an equation driven by SM, the rate of the convergence in the averaging principle is established. This is obvious from the proof of that theorem.

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