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Exponential utility maximization in small/large financial markets

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Abstract Obtaining a utility-maximizing optimal portfolio in a closed form is a challenging issue when the return vector follows a more general distribution than the normal one. In this paper, for markets based on finitely many assets, a closed-form expression is given for optimal portfolios that maximize an exponential utility function when the return vector follows normal mean-variance mixture models. Especially, the used approach expresses the closed-form solution in terms of the Laplace transformation of the mixing distribution of the normal mean-variance mixture model and no distributional assumptions on the mixing distribution are made.

Also considered are large financial markets based on normal mean-variance mixture models, and it is shown that the optimal exponential utilities in small markets converge to the optimal exponential utility in the large financial market. This shows, in particular, that to reach the best utility level investors need to diversify their investments to include infinitely many assets into their portfolio, and with portfolios based on only finitely many assets they will never be able to reach the optimum level of utility.

Keywords Expected utility, mean-variance mixtures, Hara utility functions, large financial markets, martingale measures
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1 Introduction

We consider a frictionless financial market with $d + 1$ assets. We assume the first asset is a risk-free asset with risk-free interest rate r_f and the remaining d assets are risky assets with returns modeled by a d-dimensional random vector X. In this note, we assume that X follows a normal mean-variance mixture (NMVM) distribution,

$$X \stackrel{d}{=} \mu + \gamma Z + \sqrt{Z} A N, \quad (1)$$

where $\mu \in \mathbb{R}^d$ is location parameter, $\gamma \in \mathbb{R}^d$ controls the skewness, $Z \sim G$ is a nonnegative random variable with distribution function G, $A \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite $d \times d$ matrix of real numbers, $N \sim N(0, I)$ is a d-dimensional Gaussian random vector with identity covariance matrix I in $\mathbb{R}^d \times \mathbb{R}^d$, and N is independent of the mixing distribution Z.

In this paper we use the following notations. For any vectors $x = (x_1, x_2, \ldots, x_d)^T$ and $y = (y_1, y_2, \ldots, y_d)^T$ in $\mathbb{R}^d$, where the superscript T stands for the transpose of a vector, $<x, y> = x^T y = \sum_{i=1}^d x_i y_i$ denotes the scalar product of the vectors x and y, and $|x| = \sqrt{\sum_{i=1}^d x_i^2}$ denotes the Euclidean norm of the vector x. We sometimes use the short-hand notation $X \sim N(\mu + \gamma z, z\Sigma) \circ G$ for (1), where $\Sigma = A^T A$. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$ denotes the set of nonnegative real numbers. Following the notations of [13], $\mathcal{J}$ denotes the family of infinitely divisible random variables on $\mathbb{R}_+$, $\mathcal{S}$ denotes the set of self-decomposable random variables on $\mathbb{R}_+$, and $\mathcal{G}$ denotes the class of generalized gamma convolutions (GGCs) on $\mathbb{R}_+$ that will be introduced later. The Laplace transformation of any distribution G is denoted by $\mathcal{L}_G(s) = \int e^{-sy} G(dy)$. A gamma random variable with density function $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ is denoted by $G = G(\alpha, \beta)$.

A prominent example of the NMVM models is generalized hyperbolic (GH) distributions, where the mixing distribution Z follows a generalized inverse Gaussian (GIG) distribution denoted as $GIG(\lambda, a, b)$. The probability density function of a GIG distribution, denoted by $f_{GIG}(\lambda, a, b)$, takes the form

$$f_{GIG}(x; \lambda, a, b) = \left(\frac{b}{a}\right)^\lambda \frac{1}{K_\lambda(ab)} x^{\lambda-1} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)} 1_{(0,+\infty)}(x), \quad (2)$$

where $K_\lambda(x)$ denotes the modified Bessel function of third kind with index λ and the allowed parameter ranges for λ, a, b in (2) are (i) $a \geq 0, b > 0$ if $\lambda > 0$, (ii) $a > 0$, $b \geq 0$ if $\lambda < 0$, (iii) $a > 0, b > 0$ if $\lambda = 0$. Here the case $a = 0$ in (i) or the case $b = 0$ in (ii) above need to be understood in limiting cases of (2) and in these special cases we have

$$\begin{aligned} f_{GIG}(x; \lambda, 0, b) &= \left(\frac{b^2}{2}\right)^\lambda \frac{x^{\lambda-1}}{\Gamma(\lambda)} e^{-\frac{b^2}{2}x} 1_{(0,+\infty)}(x), \quad \lambda > 0, \\ f_{GIG}(x; \lambda, a, 0) &= \left(\frac{2}{a^2}\right)^\lambda \frac{x^{\lambda-1}}{\Gamma(-\lambda)} e^{-\frac{a^2}{2x}} 1_{(0,+\infty)}(x), \quad \lambda < 0, \end{aligned} \quad (3)$$

where $\Gamma(x)$ denotes the Gamma function. Here $f_{GIG}(x; \lambda, 0, b)$ is the density function of a Gamma distribution $G(\lambda, \frac{2}{b^2})$ and $f_{GIG}(x; \lambda, a, 0)$ is the density function of an inverse Gamma distribution $iG(\lambda, \frac{a^2}{2})$.

The GH distribution in dimension d is denoted by $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma)$ and it satisfies $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma) \sim N(\mu + z\Sigma\beta, z\Sigma) \circ GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^T\Sigma\beta})$. The parameter ranges of this distribution are $\lambda \in \mathbb{R}$, $\alpha, \delta \in \mathbb{R}_+$, $\beta, \mu \in \mathbb{R}^d$ and (i′) $\delta \geq 0$, $0 \leq \sqrt{\beta^T\Sigma\beta} < \alpha$ if $\lambda > 0$, (ii′) $\delta > 0$, $0 \leq \sqrt{\beta^T\Sigma\beta} < \alpha$ if $\lambda = 0$, (iii′) $\delta > 0$, $0 \leq \sqrt{\beta^T\Sigma\beta} \leq \alpha$ if $\lambda < 0$. The class of GH distributions includes two popular models in finance: if $\lambda = -\frac{1}{2}$ we have a normal inverse Gaussian distribution which is denoted by $NIG_d(\alpha, \beta, \delta, \mu, \Sigma)$, and when $\lambda = \frac{1+d}{2}$ we have the class of hyperbolic distributions denoted by $HYP_d(\alpha, \beta, \delta, \mu, \Sigma)$. As in the case of the GIG distributions, the case $\delta = 0$ in (i′) above and the case $\sqrt{\beta^T\Sigma\beta} = \alpha$ or $\alpha = 0$ in (iii′) above need to be understood as limiting cases of the GH distributions. If $\lambda > 0$, $\delta \to 0$ in case (i′) above then

$$\begin{aligned} GH_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma) &\xrightarrow{w} N_d(\mu + z\Sigma\beta, z\Sigma) \circ G\left(\lambda, \frac{\alpha^2 - \beta^T\Sigma\beta}{2}\right) \\ &=: VG_d(\lambda, \alpha, \beta, \mu, \Sigma), \end{aligned} \tag{4}$$

where $\overset{w}{=}$ denotes weak convergence of distributions and VG_d represents the class of variance gamma distributions. If $\lambda < 0$ and $\alpha \to 0$ as well as $\beta \to 0$ in case (iii′) above we have the shifted t distributions with degrees of freedom -2λ

$$GH_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma) \xrightarrow{w} N(\mu, z\Sigma) \circ iG\left(\lambda, \frac{\delta^2}{2}\right) =: t_d(\lambda, \delta, \mu, \Sigma). \tag{5}$$

If $\alpha \to \infty$, $\delta \to \infty$ and $\frac{\delta}{\alpha} \to \sigma^2 < \infty$, we have the following relation that shows that the normal random vectors are limiting cases of the GH distributions,

$$GH_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma) \xrightarrow{w} N(\mu + z\Sigma\beta, z\Sigma) \circ \epsilon_{\sigma^2} =: N\big(\mu + \sigma^2\Sigma\beta, \sigma^2\Sigma\big), \tag{6}$$

where ϵ_{σ^2} is the Dirac function that equals to 1 when $z = \sigma^2$ and equals to zero otherwise, see Chapter 2 of [10] for the details. All of normal inverse Gaussian, hyperbolic, variance gamma, and Student t distributions are very popular models in finance, see [12], [1], [3], [8], [11], [21], [20], [14], [22] for this.

The class of GIG distributions belongs to the class of GGCs. A positive random variable Z is a GGC, without translation term, if there exists a positive Radon measure ν on $\mathbb{R}_+$ such that

$$\mathcal{L}_Z(s) = Ee^{-sZ} = e^{-\int_0^\infty \ln(1+\frac{s}{z})\nu(dz)}, \tag{7}$$

with

$$\int_0^1 |lnx|\nu(dx) < \infty, \qquad \int_1^\infty \frac{1}{x}\nu(dx) < \infty. \tag{8}$$

The measure ν is called Thorin's measure associated with Z. For the definition of the GGCs, see the survey paper [13]. In Proposition 1.1 of [13], it was shown that any GGC random variable can be written as the Wiener-Gamma integral

$$Z = \int_0^\infty h(s)d\gamma_s, \tag{9}$$

where $h(s) : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic function with $\int_0^\infty \ln(1 + h(s))ds < \infty$ and $\{\gamma_s\}$ is a standard Gamma process with the Lévy measure $e^{-x}\frac{dx}{x},\ x > 0$.

Proposition 1.23 of [10] shows that the class of GIG random variables belongs to the class GGC. It provides the description of the corresponding Thorin's measures (in terms of the functions U_{GIG} in the proposition) for all the cases of parameters of GIG. The class of GGC distributions is rich as stated in the introduction of [13] and we have the relation $\mathcal{G} \subset \mathcal{S} \subset \mathcal{J}$. In our model (1) the mixing distribution Z can be any distribution in $\mathcal{J}$. In fact, Z can be any nonnegative random variable.

Given an initial endowment $W_0 > 0$, the investor must determine the portfolio weights x on the d risky assets to maximize the expected utility of the next period wealth. The wealth that corresponds to the portfolio weight x on the risky assets is given by

$$\begin{aligned} W(x) &= W_0\big[1 + \big(1 - x^T 1\big)r_f + x^T X\big] \\ &= W_0(1 + r_f) + W_0\big[x^T(X - \mathbf{1}r_f)\big] \end{aligned} \tag{10}$$

and the investor's problem is

$$\max_{x \in D} \ EU\big(W(x)\big), \tag{11}$$

for some domain D of the portfolio set D. Note here that x represents the portfolio weights on the risky assets and $1 - x^T\mathbf{1}$ is the proportion of the initial wealth invested on the risk-free asset. The portfolio weights x on risky assets are allowed to be any vector in D.

The main goal of this paper is to discuss the solution to the problem (11) for an exponential utility function U when the returns of the risky assets have an NMVM distribution as in (1). This type of utility maximization problems in one period models were studied in many papers in the past, see [17], [18], [15], [29], [2]. Especially, the recent paper [3] made an interesting observation that, with generalized hyperbolic models and with exponential utility, the optimal portfolios of the corresponding expected utility maximization problems can be written as a sum of two portfolios that are determined by the location and skewness parameters of the model (1) separately. The present paper extends their result to a more general class of NMVM models as a compliment.

The paper is organized as follows. In Section 2 below we present a closed-form solution for an optimal portfolio when the utility function U is exponential. In Section 3 we show that the optimal expected utilities in small financial markets converge to an overall best-expected utility in a large financial market. In Section 4 we present examples as applications of our results.

2 Closed-form solution for optimal portfolios under an exponential utility

In this section, we study the solution to the problem (11) when the utility function of the investor is exponential,

$$U(W) = -e^{-aW}, \quad a > 0, \tag{12}$$

and when the investment opportunity set consists of the above-stated $d+1$ assets. Below we obtain an expression that relates $EU(W)$ to the Laplace transformation of the mixing distribution Z as in (14) below. First, observe that we have

$$W(x) \stackrel{d}{=} W_0(1+r_f) + W_0\big[x^T(\mu - \mathbf{1}r_f) + x^T\gamma Z + \sqrt{x^T\Sigma x}\sqrt{Z}N(0,1)\big]. \quad (13)$$

Lemma 2.1. *For any portfolio $x \in \mathbb{R}^d$ such that $EU(W(x))$ is finite, we have*

$$EU\big(W(x)\big) = -e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\left(aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x\right), \quad (14)$$

where $\mathcal{L}_Z(s) = Ee^{-sZ}$ is the Laplace transformation of Z.

Proof. From (13), we have

$$\begin{aligned}
EU\big(W(x)\big) &= -\, Ee^{-aW_0(1+r_f)-aW_0[x^T(\mu-\mathbf{1}r_f)+x^T\gamma Z+\sqrt{x^T\Sigma x}\sqrt{Z}N(0,1)]} \\
&= -\, e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)} \\
&\quad \times \int_0^{+\infty} Ee^{-aW_0x^T\gamma z-aW_0\sqrt{x^T\Sigma x}\sqrt{z}N(0,1)}f_Z(z)dz \\
&= -\, e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)} \\
&\quad \times \int_0^{+\infty} e^{-aW_0x^T\gamma z}Ee^{-aW_0\sqrt{x^T\Sigma x}\sqrt{z}N(0,1)}f_Z(z)dz \\
&= -\, e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\int_0^{+\infty} e^{-aW_0x^T\gamma z}e^{\frac{a^2W_0^2}{2}x^T\Sigma xz}f_Z(z)dz \\
&= -\, e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\int_0^{+\infty} e^{-(aW_0x^T\gamma-\frac{a^2W_0^2}{2}x^T\Sigma x)z}f_Z(z)dz \\
&= -\, e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\left(aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x\right).
\end{aligned}$$

□

Remark 2.2. If $\mu - \mathbf{1}r_f = 0$ in our model (1), from (14) we have

$$EU\big(W(x)\big) = -e^{-aW_0(1+r_f)}\mathcal{L}_Z\left(aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x\right).$$

Since $\mathcal{L}_Z(s)$ is a strictly decreasing function, the expected utility maximization problem becomes the maximization problem of the quadratic function $aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x$ in this case. Especially, if the risk-free interest rate r_f is zero and our model (1) is such that the location parameter μ is zero, then the utility optimizing portfolio can be found by optimizing a quadratic function. Therefore for the rest of the paper, we assume that our model (1) is such that $\mu - \mathbf{1}r_f \neq 0$. Also we assume that $Z \neq 0$ with positive probability.

Remark 2.3. By using the relation (11) and by checking the first order condition for optimality, it is easy to see that the optimal portfolio $x^\star$ satisfies the relation

$$x^\star = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - \frac{\mathcal{L}_Z(g(x^\star))}{\mathcal{L}'_Z(g(x^\star))}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right], \tag{15}$$

where $g(x)$ is given in the expression (16) below. There are several questions that one needs to address when applying the direct approach (15) in obtaining the optimal portfolio $x^\star$: (i) if the function $x \to EU(W(x))$ is continuously differentiable; (ii) if the optimal portfolio is the interior point of the corresponding domain; (iii) if the equation (15) has a unique solution. After these questions are addressed the next challenge becomes how to compute $x^\star$ numerically. This problem is not trivial if the dimension d is a large number, i.e. $x \in \mathbb{R}^d$ for large d. To overcome these problems, in this paper we take different approach and obtain $x^\star$ in near closed form: to calculate $x^\star$ we only need to find the minimizing point of a convex function on the real line.

Lemma 2.1 expresses the expected utility in terms of the linear function $x^T(\mu - \mathbf{1}r_f)$ and the quadratic function $aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x$ of the portfolio $x \in \mathbb{R}^n$. For convenience, we introduce the notations

$$\begin{aligned}
g(x) &=: aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x,\\
G(x) &=: e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\Big(aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x\Big),\\
&= e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\big(g(x)\big).
\end{aligned} \tag{16}$$

Then the relation (14) becomes

$$EU(W) = -e^{aW_0(1+r_f)}G(x) = -e^{aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\big(g(x)\big). \tag{17}$$

Therefore we have the obvious relation

$$\arg\max_{x\in D} EU(W) = \arg\min_{x\in D} G(x) \tag{18}$$

for any domain $D \in \mathbb{R}^d$ of the portfolio set. Note here that the equality in (18) means the equality of two sets if there is more than one optimizing point.

Our goal in this section is to give a closed-form solution to the problem (11) for some domains of the portfolio set. Before we start our analysis, we first present the following example.

Example 2.4. Consider the model (1) with $\gamma = 0$ and with the mixing distribution $Z \sim e^{N(0,1)}$. Then for any $x \neq 0$ we have

$$EU\big(W(x)\big) = -\infty.$$

To see this, assume that there is $x \neq 0$ such that $EU(W(x))$ is finite. Then by Lemma 2.1 we have

$$EU\big(W(x)\big) = -e^{-aW_0(1+r_f)}e^{-aW_0x^T(\mu-\mathbf{1}r_f)}\mathcal{L}_Z\Big(-\frac{a^2W_0^2}{2}x^T\Sigma x\Big).$$

For any $x \neq 0$ we have $x^T \Sigma x > 0$ as Σ is positive definite by the assumption of the model (1). Now it is well known that when $Z \sim e^{N(0,1)}$ we have $\mathcal{L}_Z(s) = +\infty$ whenever $s < 0$. Therefore $\mathcal{L}_Z(-\frac{a^2 W_0^2}{2} x^T \Sigma x) = +\infty$ whenever $x \neq 0$ and this contradicts the finiteness assumption of $EU(W(x))$ made above. Thus we have $EU(W(x)) = -\infty$ whenever $x \neq 0$. Therefore the problem (11) does not have a solution when the domain D does not include the zero vector. But if $0 \in D$, then $x = 0$ is the optimal portfolio and $\max_{x \in D} \; EU(W(x)) = -e^{-aW_0(1+r_f)}$. This case corresponds to investing all the initial wealth W_0 on the risk-free asset as an optimal portfolio. We remark here that since $\gamma = 0$ by Jensen's inequality we have

$$EU\big(W(x)\big) \le U\big(EW(x)\big) = U\big(W_0(1 + r_f) + W_0 x^T(\mu - \mathbf{1} r_f)\big).$$

From this relation it is difficult to see that 0 is the expected utility optimizing portfolio when $Z \sim e^{N(0,1)}$. But with the assistance of Lemma 2.1 it becomes trivial to determine that 0 is the optimal portfolio as discussed earlier.

Example 2.4 shows that when the model (1) satisfies the conditions in the example and when $0 \in D$, the zero portfolio $x = 0$ is an optimal portfolio as when $x \neq 0$ one has $EU(W(x)) = -\infty$ always. It is obvious that, in this case, the function $x \to EU(W(x))$ is not differentiable at $x = 0$. Therefore we call $x = 0$ an irregular solution to the optimization problem (18). Before we give the formal definition of irregularity, we first introduce the following definition.

Definition 2.5. For any mixing distribution Z, if $\mathcal{L}_Z(s) < \infty$ for all $s \in \mathbb{R}$, we set $\hat{s} = -\infty$ and if $\mathcal{L}_Z(s) < \infty$ for some $s \in \mathbb{R}$ and $\mathcal{L}_Z(s) = +\infty$ for some $s \in \mathbb{R}$, we let $\hat{s}$ be the real number such that

$$\mathcal{L}_Z(s) = Ee^{-sZ} < \infty, \; \forall s > \hat{s} \quad \text{and} \quad \mathcal{L}_Z(s) = Ee^{-sZ} = +\infty, \; \forall s < \hat{s}. \tag{19}$$

We call $\hat{s}$ the critical value (CV) of Z under the Laplace transformation. We use the acronym CV-L from now on, where L means that CV is in the context of the Laplace transformation. One can also define this CV in the context of moment-generating functions and in this case an acronym CV-M can be used. Observe that since Z is a nonnegative random variable we always have $\hat{s} \le 0$.

Remark 2.6. In Definition 2.5, the value of $\mathcal{L}_Z(s)$ at $s = \hat{s}$ is not specified. Both the cases $\mathcal{L}_Z(\hat{s}) < \infty$ and $\mathcal{L}_Z(\hat{s}) = +\infty$ are possible. For example, if $Z \sim e^{N(0,1)}$, then $\hat{s} = 0$ and clearly $\mathcal{L}_Z(0) = 1 < \infty$. If $Z \sim x^{\alpha-1}e^{-x/\beta}/[\Gamma(\alpha)\beta^\alpha]$ is a Gamma distribution, then $\mathcal{L}_Z(s) = 1/[(1+\beta s)^\alpha]$. In this case $\hat{s} = -1/\beta$ and we have $\mathcal{L}_Z(\hat{s}) = +\infty$.

Below we define some domains for the portfolio set.

$$\begin{aligned}
S_a &=: \left\{ x \in \mathbb{R}^d : aW_0 x^T \gamma - \frac{a^2 W_0^2}{2} x^T \Sigma x > \hat{s} \right\}, \\
\partial S_a &=: \left\{ x \in \mathbb{R}^d : aW_0 x^T \gamma - \frac{a^2 W_0^2}{2} x^T \Sigma x = \hat{s} \right\}, \\
\bar{S}_a &=: S_a \cup \partial S_a.
\end{aligned} \tag{20}$$

Remark 2.7. Our main objective in this section is to find a closed-form solution for the optimal portfolio for the problem

$$\max_{x\in\mathbb{R}^d} EU\big(W(x)\big). \tag{21}$$

The following relations are easy to see:

$$\max_{x\in\mathbb{R}^d} EU\big(W(x)\big) = \max_{x\in S_a} EU\big(W(x)\big), \tag{22}$$

if $\mathcal{L}_Z(\hat{s}) = +\infty$, and

$$\max_{x\in\mathbb{R}^d} EU\big(W(x)\big) = \max_{x\in \bar{S}_a} EU\big(W(x)\big), \tag{23}$$

if $\mathcal{L}_Z(\hat{s}) < +\infty$. Observe here that if $\hat{s} < 0$, then S_a is a nonempty set as the zero vector $x = 0$ is in it. If $\hat{s} = 0$, then the set $\bar{S}_a$ is nonempty as $x = 0$ is in it.

In this section we attempt to give closed-form solutions to the problems (22) and (23) above. Our approach for this is based on the following idea: we fix the term $x^T(\mu - \mathbf{1}r_f)$ at some constant level c and optimize the quadratic term $aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x$ in (14). More specifically, we solve the optimization problem

$$\begin{aligned} &\max_x aW_0x^T\gamma - \frac{a^2W_0^2}{2}x^T\Sigma x, \\ &\text{s.t. } x^T(\mu - r_f\mathbf{1}) = c \end{aligned} \tag{24}$$

first, and plug in the solution, which we denote by x_c, into the expression (14) so that the utility maximization problem becomes an optimization problem of a function of one variable c.

Lemma 2.8. *Consider the optimization problem (21). Let $\bar{x} \in \mathbb{R}^d$ be a solution to this problem. Then $\bar{x}$ solves (24) for some c.*

Proof. Define $\bar{c} =: \bar{x}^T(\mu - \mathbf{1}r_f)$. Let $\tilde{x}$ be the solution to the problem (24) with c replaced by $\bar{c}$ (here the solution is unique as Σ is positive definite by assumption). By the optimality of $\tilde{x}$, we have $g(\bar{x}) \le g(\tilde{x})$. Since $\mathcal{L}_Z(s)$ is a decreasing function, we have $\mathcal{L}_Z(g(\tilde{x})) \le \mathcal{L}_Z(g(\bar{x}))$. Since $\bar{c} = \bar{x}^T(\mu - \mathbf{1}r_f) = \tilde{x}^T(\mu - \mathbf{1}r_f)$, we have $G(\tilde{x}) \le G(\bar{x})$. This shows that $EU(W(\tilde{x})) \ge EU(W(\bar{x}))$. But $\bar{x}$ is optimal for (11) with $D = \mathbb{R}^d$. Therefore we should have $EU(W(\tilde{x})) = EU(W(\bar{x}))$. This implies $G(\tilde{x}) = G(\bar{x})$ and this in turn implies $g(\bar{x}) = g(\tilde{x})$ again due to $\bar{c} = \bar{x}^T(\mu - \mathbf{1}r_f) = \tilde{x}^T(\mu - \mathbf{1}r_f)$. The uniqueness of the optimization point for (24) then implies $\bar{x} = \tilde{x}$. □

Remark 2.9. Lemma 2.8 gives a characterization of the optimal portfolios for the problem (11). But it doesn't tell us if the optimal portfolio for the problem (2.8) is unique. It shows only that any optimal portfolio for the problem (11) solves a quadratic optimization problem (24) for some appropriate c. Now consider the case of Example 2.4. In the setting of this example, consider the utility maximization problem (11). Since $0 \in \mathbb{R}^d$, as explained in Example 2.4, the vector $\hat{x} = 0$ is the solution to the optimization problem (11). Now let $x^\star$ be the optimal solution to the problem (24) with $c = 0$ (which means $(x^\star)^T(\mu - r_f\mathbf{1})) = 0$). Then we should have $g(x^\star) \geq g(\hat{x})$. But if $g(x^\star) > g(\hat{x})$, then $\hat{x} = 0$ cannot be an optimal solution to (11). Therefore we should have $g(x^\star) = g(\hat{x})$. The uniqueness of the optimal solution to (24) with $c = 0$ then implies $x^\star = \hat{x} = 0$.

Definition 2.10. Consider the optimization problem (11) for some given model (1) and for some domain $D \subset \mathbb{R}^d$. Let $\hat{s}$ denote the CV-L of the mixing distribution Z. Let $x^\star \in D$ be a solution to (11). We say that $x^\star$ is irregular if $g(x^\star) = \hat{s}$. If $g(x^\star) > \hat{s}$, we call the solution $x^\star$ regular.

Remark 2.11. Clearly, the definition of irregular and regular solutions depends on the CV-L number $\hat{s}$ of the mixing distribution Z in (1). If $\mathcal{L}_Z(\hat{s}) = +\infty$, then the solution to (11) cannot be irregular. Therefore, the irregularity can happen only when $\mathcal{L}_Z(\hat{s}) < +\infty$. Observe that the solution $x = 0$ in Example 2.4 is an irregular solution.

Remark 2.12. Consider the optimization problem (11). From Lemma 2.8, any optimal portfolio $x^\star$ is a solution to the quadratic optimization problem (24) with $x^T(\mu - r_f\mathbf{1}) = c^\star$ for some fixed $c^\star$. If $x^\star$ is irregular, then $g(x^\star) = \hat{s}$. The optimality and uniqueness (on the hyperplane $x^T(\mu - r_f\mathbf{1}) = c^\star$) of $x^\star$ implies that we have $g(x) < g(x^\star) = \hat{s}$ for all $x \neq x^\star$ on the hyperplane $x^T(\mu - r_f\mathbf{1}) = c^\star$. Therefore we have $EU(W(x)) = -\infty$ for all $x \neq x^\star$ on the hyperplane $x^T(\mu - r_f\mathbf{1}) = c^\star$. From this we conclude that if the optimal portfolio for the problem (24) is irregular, then any small neighborhood of this portfolio contains some portfolios with infinite expected utility. In comparison, if the optimal portfolio is regular, then it has a small ball around it with finite expected value for each portfolio in this small ball.

As it was shown in Lemma 2.8, the solutions to the utility maximization problem (11) can be obtained by solving the quadratic optimization problem (24). For a given optimization problem (11), if we know the corresponding c in (24) such that the solution to (24) is the solution to (11), then we just need to solve the optimization problem (24) to obtain the optimal portfolio. But figuring out such an c is not a trivial issue. We first prove following lemma.

Lemma 2.13. *For any real number c, when $x^T(\mu - \mathbf{1}r_f) = c$, the maximizing point x_c of $g(x)$ is given by*

$$x_c = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - q_c\Sigma^{-1}(\mu - \mathbf{1}r_f)\right], \tag{25}$$

and we have

$$g(x_c) = \frac{1}{2}\gamma^T\Sigma^{-1}\gamma - \frac{q_c^2}{2}(\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f), \tag{26}$$

where

$$q_c = \frac{\gamma^T\Sigma^{-1}(\mu - \mathbf{1}r_f) - aW_0c}{(\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f)}. \tag{27}$$

Proof. We form the Lagrangian $L = g(x) + \lambda(c - x^T(\mu - \mathbf{1}r_f))$ with the Lagrangian parameter λ. Denoting the maximizing point by x_c, the first order condition gives

$$x_c = \frac{1}{aW_0}\Sigma^{-1}\gamma - \frac{\lambda}{a^2W_0^2}\Sigma^{-1}(\mu - \mathbf{1}r_f). \tag{28}$$

We plug x_c into $x_c^T(\mu - \mathbf{1}r_f) = c$ and obtain

$$c = \frac{1}{aW_0}\gamma^T\Sigma^{-1}(\mu - \mathbf{1}r_f) - \frac{\lambda}{a^2W_0^2}(\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f). \tag{29}$$

From this we find λ as

$$\lambda = \frac{aW_0\gamma^T\Sigma^{-1}(\mu - \mathbf{1}r_f) - ca^2W_0^2}{(\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f)}. \tag{30}$$

Then we plug λ into the expression (28) of x_c above and obtain (25). To obtain (26), we plug x_c into $g(x)$ in (16). After doing some algebra, we obtain

$$g(x_c) = \frac{1}{2}\gamma^T\Sigma^{-1}\gamma - \frac{1}{2}q_c^2(\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f), \tag{31}$$

with q_c given as in (27). This completes the proof. □

For the rest of the paper, as in [3], for convenience, we use the notations

$$\mathcal{A} = \gamma^T\Sigma^{-1}\gamma, \quad \mathcal{C} = (\mu - \mathbf{1}r_f)^T\Sigma^{-1}(\mu - \mathbf{1}r_f), \quad \mathcal{B} = \gamma^T\Sigma^{-1}(\mu - \mathbf{1}r_f). \tag{32}$$

We first observe that $\mathcal{C} > 0$ due to the assumption in Remark 2.2 and the assumption on positive definiteness of Σ. With these notations we have

$$g(x_c) = \frac{\mathcal{A}}{2} - \frac{q_c^2}{2}\mathcal{C}, \qquad q_c = \frac{\mathcal{B}}{\mathcal{C}} - \frac{aW_0}{\mathcal{C}}c. \tag{33}$$

From the relation (33), we express c as a function of q_c as

$$c = \frac{1}{aW_0}[\mathcal{B} - \mathcal{C}q_c]. \tag{34}$$

We define the function

$$Q(\theta) = e^{\mathcal{C}\theta}\mathcal{L}_Z\left[\frac{1}{2}\mathcal{A} - \frac{\theta^2}{2}\mathcal{C}\right], \tag{35}$$

and we define $\hat{\theta} =: \sqrt{\frac{\mathcal{A}-2\hat{s}}{\mathcal{C}}}$, where $\hat{s}$ is the IN of Z. If $\hat{s} = -\infty$, the $\hat{\theta}$ is understood to be equal to $+\infty$. Note here that $\hat{s} \le 0$ as Z is a nonnegative random variable. Therefore $\hat{\theta}$ is well defined. If $\mathcal{L}_Z(\hat{s}) < +\infty$, $Q(\theta)$ is finite iff $\frac{1}{2}\mathcal{A} - \frac{\theta^2}{2}\mathcal{C} \ge \hat{s}$ and this translates into: $Q(\theta)$ is finite iff $\theta \in [-\hat{\theta}, \hat{\theta}]$. If $\mathcal{L}_Z(\hat{s}) = +\infty$, $Q(\theta)$ is finite iff $\frac{1}{2}\mathcal{A} - \frac{\theta^2}{2}\mathcal{C} > \hat{s}$ and this translates into: $Q(\theta)$ is finite iff $\theta \in (-\hat{\theta}, \hat{\theta})$.

Next we prove the following lemma that relates Q to G.

Lemma 2.14. *Let x_c be the solution to the problem (24) for a given c. Assume $x_c \in S_a$ if $\mathcal{L}_Z(\hat{s}) = +\infty$ and $x_c \in \bar{S}_a$ if $\mathcal{L}_Z(\hat{s}) < +\infty$. Then, for any x with $x^T(\mu - \mathbf{1}r_f) = c$, we have*

$$e^{-\mathcal{B}}Q(q_c) \le G(x), \tag{36}$$

where q_c is given by (27) and $\mathcal{B}$ is given by (32). We also have $e^{-\mathcal{B}}Q(q_c) = G(x_c)$.

Proof. Note that $G(x) = e^{-aW_0 x^T(\mu - \mathbf{1}r_f)}\mathcal{L}_Z(g(x))$. The conditions stated on x_c in the lemma ensure that $G(x_c) = e^{-aW_0 c}\mathcal{L}_Z(g(x_c))$ is finite. Since $g(x) \leq g(x_c)$ for any x with $x^T(\mu - \mathbf{1}r_f) = c$ by the definition of x_c (the optimizing point) and also since $\mathcal{L}_Z(s)$ is a decreasing function of s, we have

$$G(x_c) \leq G(x) \tag{37}$$

for any x with $x^T(\mu - \mathbf{1}r_f) = c$. We plug c in (34) into the expression of $G(x_c)$ and obtain

$$G(x_c) = e^{-\mathcal{B}} e^{\mathcal{C}q_c} \mathcal{L}_Z\left[\frac{1}{2}\mathcal{A} - \frac{q_c^2}{2}\mathcal{C}\right] = e^{-\mathcal{B}} Q(q_c). \tag{38}$$

$\square$

Remark 2.15. Lemma 2.14 shows that the function $G(x)$ achieves its unique (as the solution to (24) is unique in a hyperplane) minimum value on the hyperplane $x^T(\mu - r_f\mathbf{1}) = c$ at x_c and its minimum value is given by $e^{-\mathcal{B}}Q(q_c)$ with q_c in (33). For any $\theta_0 \in [-\hat{\theta}, \hat{\theta}]$, we can let c_0 be such that $q_{c_0} = \theta_0$. Let x_0 be the optimal solution to (24) with c replaced by c_0. From Lemma 2.13, we have $g(x_0) = \frac{1}{2}\mathcal{A} - \frac{q_{c_0}^2}{2}\mathcal{C}$. If $|q_{c_0}| = \hat{\theta}$, then $g(x_0) = \hat{s}$. If $|q_{c_0}| < \hat{\theta}$, then $g(x_0) > \hat{s}$.

Theorem 2.16. *Consider the optimization problem (21). A portfolio $x^\star$ is a solution to (21) if and only if*

$$x^\star = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - q_{min}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right] \tag{39}$$

for some

$$q_{min} \in \arg\min_{\theta \in \Theta} Q(\theta), \tag{40}$$

where $\Theta = [-\hat{\theta}, \hat{\theta}]$ if $\hat{\theta} = \sqrt{\frac{\mathcal{A} - 2\hat{s}}{\mathcal{C}}} < \infty$ and $\Theta = (-\infty, +\infty)$ if $\hat{\theta} = +\infty$. Here $\hat{s}$ is the CV-L of the mixing distribution Z.

Proof. First we show that if $\hat{x}$ is a solution to (21), then $\hat{x}$ is given by (39). By Lemma 2.8, $\hat{x}$ is a solution to the optimization problem (24) with some $c = \hat{c}$. By Lemma 2.13, $\hat{x}$ takes the form

$$\hat{x} = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - \hat{q}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right],$$

with $\hat{q} = \mathcal{B}/\mathcal{C} - (aW_0/\mathcal{C})\hat{c}$. Again by Lemma 2.13 we have (see (33))

$$g(\hat{x}) = \frac{\mathcal{A}}{2} - \frac{(\hat{q})^2}{2}\mathcal{C}.$$

Since $\hat{x}$ is a solution to (21) we have $G(\hat{x}) < \infty$ and this implies $g(\hat{x}) \geq \hat{s}$ if $\hat{s}$ is finite and $g(\hat{x}) > \hat{s}$ if $\hat{s} = -\infty$ (note that $g(\hat{x}) = -\infty$ implies $G(\hat{x}) = +\infty$ due to the assumption $Z \neq 0$ in Remark 2.2 and $G(\hat{x}) = e^{-aW_0\hat{x}^T(\mu - r_f\mathbf{1})}\mathcal{L}_Z(g(\hat{x}))$). The expression of $g(\hat{x})$ above then implies $\hat{q} \in \Theta$ (note here that for the case $\hat{\theta} = +\infty$, we can't have $\hat{q}^2 = +\infty$ as $g(\hat{x})$ is finite as explained above).

Now we need to show $\hat{q} \in \arg\min_{\theta\in\Theta} Q(\theta)$. From Lemma 2.14, we have $G(\hat{x}) = e^{-\mathcal{B}}Q(\hat{q})$. Take any $\theta_0 \in \Theta$ (including the case $\Theta = (-\infty, +\infty)$). Let c_0 be such that $\theta_0 = q_{c_0}$ (see Remark 2.15). Let x_0 be the solution to (24) with c replaced by c_0. By Lemma 2.13 we have $g(x_0) = \frac{\mathcal{A}}{2} - \frac{(q_{c_0})^2}{2}\mathcal{C}$. Since $\theta_0 = q_{c_0} \in \Theta$, we have $g(x_0) \geq \hat{s}$ if $\hat{s}$ is finite and $g(x_0) > \hat{s}$ if $\hat{s} = -\infty$. Therefore, either $x_0 \in S_a$ or $x_0 \in \bar{S}_a$. Then by Lemma 2.14 we have $G(x_0) = e^{-\mathcal{B}}Q(q_{c_0})$. Since $\hat{x}$ is the optimal portfolio, it is the minimizing point for the function $G(x)$ (see (18) for this). Therefore we have $G(\hat{x}) \leq G(x_0)$. This implies $Q(\hat{q}) \leq Q(q_{c_0}) = Q(\theta_0)$. Since θ_0 is arbitrary, we conclude that $\hat{q} \in \arg\min_{\theta\in\Theta} Q(\theta)$.

Next we show that any portfolio of the form (39) is an optimal portfolio for (21). Fix an arbitrary $q_m \in \arg\min_{\theta\in\Theta} Q(\theta)$. Then $q_m \in [-\hat{\theta}, \hat{\theta}]$ if $\hat{\theta}$ is finite and $q_m \in (-\infty, +\infty)$ if $\hat{\theta} = +\infty$. Let c_m be such that $q_m = q_{c_m}$ and let x_m be the solution to (24) with c replaced by c_m. By Lemma 2.13, we have

$$x_m = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - q_m\Sigma^{-1}(\mu - \mathbf{1}r_f)\right],$$

and $g(x_m) = \frac{\mathcal{A}}{2} - \frac{q_m^2}{2}\mathcal{C}$. The condition on q_m above implies $g(x_m) \geq \hat{s}$ if $\hat{s}$ is finite and $g(x_m) > -\infty$ if $\hat{s} = -\infty$. Therefore, either $x_m \in S_a$ or $x_m \in \bar{S}_a$. By Lemma 2.14 we have $G(x_m) = e^{-\mathcal{B}}Q(q_m)$ which is a finite number. To show x_m is an optimal portfolio we need to show $G(x_m) \leq G(x)$ for any x that $G(x)$ is finite (note that either $G(x) = +\infty$ or it is finite). Fix an arbitrary $\bar{x}$ with $G(\bar{x}) < +\infty$. Let $\bar{c} = \bar{x}^T(\mu - r_f\mathbf{1})$. Let $x_{\bar{c}}$ be the solution to (24) with c replaced by $\bar{c}$. Since $G(x) < \infty$, we either have $x \in \bar{S}_a$ or $x \in S_a$. This means that $x_{\bar{c}} \in \bar{S}_a$. By Lemma 2.13 we have $g(x_{\bar{c}}) = \frac{\mathcal{A}}{2} - \frac{q_{\bar{c}}^2}{2}\mathcal{C}$, where $q_{\bar{c}}$ is given by (33) with c replaced by $\bar{c}$. Therefore, we have $q_{\bar{c}} \in [-\hat{\theta}, \hat{\theta}]$ if $\hat{\theta}$ is finite and $q_{\bar{c}} \in (-\infty, +\infty)$ if $\hat{\theta} = +\infty$. By the definition of q_m, we have $Q(q_m) \leq Q(q_{\bar{c}})$. Therefore, we have $G(x_m) = e^{-\mathcal{B}}Q(q_m) \leq e^{-\mathcal{B}}Q(q_{\bar{c}}) = G(\bar{x})$. □

Proposition 2.17. *Consider the optimization problem (21). If $x^\star$ is a regular solution to (21) then*

$$x^\star = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - q_{min}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right], \tag{41}$$

for some

$$q_{min} \in \arg\min_{\theta\in(-\hat{\theta},\hat{\theta})} Q(\theta), \tag{42}$$

where $\hat{\theta} =: \sqrt{\frac{\mathcal{A}-2\hat{s}}{\mathcal{C}}}$ and $\hat{s}$ is the CV-L of the mixing distribution Z.

Proof. Let $\hat{x}$ be a regular solution. By Lemma 2.8, $\hat{x}$ is a solution to the optimization problem (24) with some $c = \hat{c}$. By Lemma 2.13, $\hat{x}$ takes the form

$$\hat{x} = \frac{1}{aW_0}\left[\Sigma^{-1}\gamma - \hat{q}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right]$$

with $\hat{q} = \mathcal{B}/\mathcal{C} - (aW_0/\mathcal{C})\hat{c}$. Again by Lemma 2.13 we have (see (33))

$$g(\hat{x}) = \frac{\mathcal{A}}{2} - \frac{(\hat{q})^2}{2}\mathcal{C}.$$

Since $\hat{x}$ is regular, we have $g(\hat{x}) > \hat{s}$. From this we conclude $\hat{q} \in (-\hat{\theta}, \hat{\theta})$. From Lemma 2.14, we have $G(\hat{x}) = e^{-\mathcal{B}} Q(\hat{q})$. Note that $\hat{q} = q_{\hat{c}}$. Now we show that $\hat{q} \in \arg\min_{\theta \in (-\hat{\theta}, \hat{\theta})} Q(\theta)$. Take any $\theta_0 \in (-\hat{\theta}, \hat{\theta})$. Let c_0 be such that $\theta_0 = q_{c_0}$ (see Remark 2.15). Let x_0 be the solution to (24) with c replaced by c_0. By Lemma 2.13 we have $g(x_0) = \frac{\mathcal{A}}{2} - \frac{(q_{c_0})^2}{2}\mathcal{C}$. Since $\theta_0 = q_{c_0} \in (-\hat{\theta}, \hat{\theta})$, we have $g(x_0) > \hat{s}$. Therefore $x_0 \in S_a$. Then by Lemma 2.14 we have $G(x_0) = e^{-\mathcal{B}} Q(q_{c_0})$. Since $\hat{x}$ is the optimal portfolio, it is the minimizing point for the function $G(x)$ (see (18) for this). Therefore, we have $G(\hat{x}) \le G(x_0)$. This implies $Q(\hat{q}) \le Q(q_{c_0}) = Q(\theta_0)$. Since θ_0 is arbitrary, we conclude that $\hat{q} \in \arg\min_{\theta \in (-\hat{\theta}, \hat{\theta})} Q(\theta)$. $\square$

Remark 2.18. Let us look at the case of Example 2.4. From the analysis in this example the optimal solution to the problem (21) is $x^\star = 0$ and it is unique. Here we would like to check that this optimal portfolio $x^\star = 0$ can also be derived from (39). To see this, note that in this example $\gamma = 0$. Therefore we have $Q(\theta) = e^{\mathcal{C}\theta} \mathcal{L}_Z(-\frac{\theta^2}{2}\mathcal{C})$ and $q_c = -\frac{aW_0}{\mathcal{C}} c$. Observe that $0 \in \{x^T(\mu - \mathbf{1}r_f) : x \in \mathbb{R}^n\}$. Also for any $\theta \neq 0$ we have $Q(\theta) = +\infty$ as the CV-L of $Z \sim e^{N(0,1)}$ is $\hat{s} = 0$. Therefore $\arg\min_{\theta \in \Theta} Q(\theta)$ has only one element $q_{min} = 0$. Then (39) gives $\bar{x}^\star = 0$ as the only optimal solution. Observe that in fact in this example we have $\mathcal{A} = 0$ and therefore $\hat{\theta} = 0$. Thus $\bar{q}_{min} = \arg\min_{\theta \in \{0\}} Q(\theta) = 0$.

Remark 2.19. We remark here that our closed-form formula (39) expresses the optimal portfolio in terms of the critical value (see Definition 2.10) of the mixing distribution Z and its Laplace transformation which is hidden in the function $Q(\theta)$. This has some advantage in determining the optimal portfolio for some cases of models (1), see our Corollary 4.5 below for this.

3 Large financial markets

In the previous section we gave a closed-form solution for the optimal portfolio for an exponential utility maximizer in a market that contains one risk-free asset and finitely many risky assets with return vector that follow (1). Our Theorem 2.16 gives the complete characterization of the optimal portfolio in such small markets.

The next natural question to ask is what happens if the consumer with an exponential utility wants to increase her expected utility as much as possible by adding as many as necessary assets into her portfolio. We can best investigate this possibility by working in mathematical models with countably infinitely many assets.

In this section we consider a sequence of economies with increasing number of assets. In the nth economy, there are n risky assets and one riskless asset. The return vector of the risky assets in the nth economy satisfies (1). A consumer with an exponential utility maximizes her expected utility based on the $n + 1$ assets in each nth economy. Our main concern in this section is to investigate if the optimal expected utility of the consumer converges to a limit as $n \to \infty$, and we would like to identify this limit as the optimizer in the market with infinitely many assets.

Such “stability” of optimal investment problems was proved in [7] for a wide range of models. The methods of [7], however, cannot deal with exponential utilities. So we need to apply somewhat different, new arguments.

Our main result in this section shows that the consumer can achieve the maximum possible (in a market where she can trade on countably infinitely many risky assets) expected utility by following the sequence of optimal trading strategies in each nth economy, which are shown to converge to a limit (see our Lemma 3.6 below). We call this limit portfolio the "overall best optimal portfolio" in this paper.

An economy that allows to trade on countably infinitely many risky assets is called a large financial market in the literature. They serve well to describe, e.g., bonds of various maturities. The first model of this type, the "Arbitrage Pricing Model" (APM), goes back to [26]. We consider a slight extension of that model in the present section. As the main result of this section, we will show that the exponential utiliy maximization problem in a large financial market can be approximated by similar problems for finitely many assets (and the latter can be solved by the results of the previous sections).

Before we state and prove our main result of this section, we first specify the structure of our nth economy for all n. The return on the bank account is $R_0 := r_f$ where $r_f \geq 0$ is the risk-free interest rate. For simplicity we assume $r_f = 0$ henceforth. For $i = 1$, $R_1 := \gamma_1 Z + \mu_1 + \bar{\beta}_1 \sqrt{Z}\varepsilon_1$ is the return on the "market portfolio", which may be thought of as an investment into an index. For $i \geq 2$, let the return on risky asset i be given by

$$R_i = \gamma_i Z + \mu_i + \beta_i \sqrt{Z}\varepsilon_1 + \bar{\beta}_i \sqrt{Z}\varepsilon_i. \tag{43}$$

Here $(\varepsilon_i)_{i\geq 1}$ are assumed to be independent standard Gaussian variables, Z is a positive random variable, independent of ε_i, $\beta_i, i \geq 2, \bar{\beta}_i \neq 0, \gamma_i, \mu_i, i \geq 1$ are constants. The classical APM corresponds to $Z \equiv 1$. We refer to [26] for further discussions on that model.

We consider investment strategies in finite market segments. A strategy investing in the first n assets is a sequence of numbers $\phi_0, \phi_1, \ldots, \phi_n$. For simplicity, we assume 0 to be initial capital and also that every asset has price 1 at time 0. Self-financing imposes $\sum_{i=0}^{n} \phi_i = 0$, so a strategy is, in fact, described by $\phi_1, \ldots, \phi_n$ which can be *arbitrary* real numbers. The return on the portfolio ϕ is thus

$$V(\phi) = \sum_{i=1}^{n} \phi_i R_i,$$

noting also that $R_0 = 0$ is assumed.

For a utility maximization problem to be well-posed, one should assume a certain arbitrage-free property for the market. Notice that a probability $Q_n \sim P$ is a martingale measure for the first n assets (that is, $E_{Q_n}[R_i] = 0$ for all $1 \leq i \leq n$) provided that

$$E_{Q_n}[\varepsilon_1 | Z = z] = b_1(z) := -\frac{\gamma_1 \sqrt{z}}{\bar{\beta}_1} - \frac{\mu_1}{\sqrt{z}\bar{\beta}_1}, \quad z \in (0, \infty), \tag{44}$$

and, for each $i \geq 2$,

$$E_{Q_n}[\varepsilon_i | Z = z] = b_i(z) := -\frac{\gamma_i \sqrt{z}}{\bar{\beta}_i} - \frac{\mu_i}{\sqrt{z}\bar{\beta}_i} - \frac{\beta_i b_1(z)\sqrt{z}}{\bar{\beta}_i}, \quad z \in (0, \infty). \tag{45}$$

Now notice that, in fact, the set of such $V(\phi)$ coincides with the set of

$$V(h) := \sum_{i=1}^{n} h_i \sqrt{Z}\big(\varepsilon_i - b_i(Z)\big)$$

where $h_1, \dots, h_n$ are arbitrary real numbers. We denote by H_n the set of all n-tuples $(h_1, \dots, h_n)$. It is more convenient to use this "h-parametrization" in the sequel.

Assumption 3.1. There are finite real numbers $0 < c < C$, such that $c \le Z \le C$.

Let us define $d_i := \sup_{z\in[c,C]} |b_i(z)|$, $i \ge 1$. The next assumption is similar in spirit to the no-arbitrage condition derived in [26], see also [25].

Assumption 3.2. We stipulate $\sum_{i=1}^{\infty} d_i^2 < \infty$.

Fact. If X is standard normal then $E[e^{-\theta X-\theta^2/2}] = 1$ and $E[Xe^{-\theta X-\theta^2/2}] = \theta$, for all $\theta \in \mathbb{R}$. Notice also that, for all $p \ge 1$,

$$E\big[e^{-p\theta X-p\theta^2/2}\big] = e^{(p^2-p)\theta^2/2}. \tag{46}$$

Let us now define

$$f_n(z) := \exp\bigg(-\sum_{i=1}^{n}\big[b_i(z)\varepsilon_i + b_i(z)^2\big]\bigg).$$

Clearly, $E[f_n(z)] = 1$ and $E[f_n(z)\varepsilon_i] = b_i(z)$ for $i = 1, \dots, n$. Then Q_n defined by $dQ_n/dP := f_n(Z)$ will be a martingale measure for the first n assets. Indeed,

$$E\big[f_n(Z)\big] = \int_{[c,C]} E\big[f_n(z)\big]\,\mathrm{Law}(Z)(dz) = 1$$

and

$$E\big[f_n(Z)\varepsilon_i | Z = z\big] = E\big[\big(\varepsilon_i - b_i(z)\big)e^{-b_i(z)\varepsilon_i - b_i(z)^2/2}\big] = 0, \quad 1 \le i \le n.$$

It follows from (46) and from Assumption 3.2 that $\sup_n E[(dQ_n/dP)^2] < \infty$ hence $dQ/dP := \lim_{n\to\infty} dQ_n/dP$ exists almost surely and in L^2, and this is a martingale measure for *all* the assets, that is, $E_Q[R_i] = 0$ for all $i \ge 1$. Note also that $E[(dQ/dP)^2] < \infty$.

Using the previous sections, we may find $h_n^* \in H_n$ such that

$$U_n := E\big[e^{-V(h_n^*)}\big] = \min_{h\in H_n} E\big[e^{-V(h)}\big].$$

If we wish to find (asymptotically) optimal strategies for this large financial market, then we also need to verify that $U_n \to U := \inf_{h\in\cup_{n\ge1}H_n} E[e^{-V(h)}]$ as $n \to \infty$.

Let us introduce

$$\ell_2 := \bigg\{(h_i)_{i\ge1},\ h_i \in \mathbb{R},\ i \ge 1,\ \sum_{i=1}^{\infty} h_i^2 < \infty\bigg\}$$

which is a Hilbert space with the norm $||h||_{\ell_2} := \sqrt{\sum_{i=1}^{\infty} h_i^2}$. We may and will identify each $(h_1, \dots, h_n) \in H_n$ with $(h_1, h_2, \dots) \in \ell_2$ for all $n \ge 1$. Also define $d := (d_1, d_2, \dots) \in \ell_2$.

Theorem 3.3. *Under Assumptions 3.1 and 3.2, one has $U_n \to U$, $n \to \infty$.*

Proof. It follows from Lemma 3.6 below that there is $\bar{h}^* \in \ell_2$ such that $U = E[e^{-V(\bar{h}^*)}]$. Define now $\tilde{h}_n := (\bar{h}^*_1, \dots, \bar{h}^*_n) \in H_n$. It is clear that $U_n \ge U$ and $E[e^{-V(\tilde{h}_n)}] \ge U_n$ for all $n \ge 1$. Hence it remains to establish $E[e^{-V(\tilde{h}_n)}] \to U$.

Noting that $V(\tilde{h}_n) \to V(h^*)$ almost surely, it suffices to show that $\sup_{n\in\mathbb{N}} E[e^{-2V(\tilde{h}_n)}] < \infty$. This follows from

$$E\big[e^{-2V(\tilde{h}_n)}\big] \le e^{2\sqrt{C}||\tilde{h}_n||_2||d||_2} E\big[e^{2\sqrt{C}||\tilde{h}_n||_2|N|}\big] \le e^{2\sqrt{C}||h^*||_2||d||_2} E\big[e^{2\sqrt{C}||h^*||_2|N|}\big],$$

where N is a standard normal random variable. □

Remark 3.4. The main message of Theorem 3.3 is that the sequence of optimal expected utilities in the small markets defined above is a convergent sequence, the limit being a finite number. This means that after the consumer increases the number of assets in her/his portfolio to a certain level, a further increase of the number of assets will not bring significant increments of the expected utility. It is not trivial to have some estimations on the number of assets needed for the optimal expected utility to be sufficiently close to the overall best utility level. It would be interesting to see how fast this sequence converges to the overall best utility level U. We leave this for further discussions.

Lemma 3.5. *There exists $\alpha > 0$ such that, for all $h \in \ell_2$ with $\|h\|_{\ell_2} = 1$, $P(V(h) \le -\alpha) \ge \alpha$ holds.*

Proof. We follow closely the proof of Proposition 3.2 in [7], see also [6]. We argue by contradiction. Assume that for all $n \ge 1$, there is $g_n = (g_n(1), g_n(2), \dots) \in \cup_{n\ge 1} H_n$ with $\|g_n\|_{\ell_2} = 1$ and $P(V(g_n) \le -1/n) \le 1/n$.

Clearly, $V(g_n)^- \to 0$ in probability as $n \to \infty$. We claim that $E_Q[V(g_n)^-] \to 0$. By the Cauchy–Schwarz inequality

$$E_Q\big[V(g_n)^-\big] \le \|dQ/dP\|_{L^2(P)}\big(E\big[\big(V(g_n)^-\big)^2\big]\big)^{1/2}.$$

However,

$$V(g_n)^- \le |V(g_n)| \le \sqrt{C}[|N| + ||d||_2] \tag{47}$$

for some standard normal N. This implies $E[(V(g_n)^-)^2]$, $n \to \infty$, and hence our claim.

Since $E_Q[V(g_n)] = 0$ by the martingale measure property of Q, we also get that $E_Q[V(g_n)^+] \to 0$. It follows that $E_Q[|V(g_n)|] \to 0$, hence $V(g_n)$ goes to zero Q-a.s. (along a subsequence) and, as Q is equivalent to P, P-a.s. Using that $|V(g_n)|^2$, $n \in \mathbb{N}$, is uniformly P-integrable by (47), we get $E[V(g_n)^2] \to 0$. An auxiliary calculation gives

$$E\big[V(g_n)^2\big] = \|g_n\|^2_{\ell_2} E[Z] + \sum_{i=1}^{\infty} g_n^2(i) E\big[b_i^2(Z)Z\big] \ge E[Z] > 0,$$

a contradiction proving our lemma. □

Lemma 3.6. *There is $h^* \in \ell_2$ such that $U = E[e^{-V(h^*)}]$.*

Proof. There are $h_n \in \cup_{j\in\mathbb{N}} H_j$, $n \in \mathbb{N}$, such that $E[e^{-V(h_n)}] \to U$. If we had $\sup_n ||h_n||_{\ell_2} = \infty$, then (taking a subsequence still denoted by n), $||h_n||_{\ell_2} \to \infty$, $n \to \infty$. By Lemma 3.5,

$$P\big(V(h_n) \le -\alpha||h_n||_{\ell_2}\big) \ge \alpha$$

and this implies $E[e^{-V(h_n)}] \to \infty$, which contradicts $E[e^{-V(h_n)}] \to U \le E[e^0] = 1$.

Then necessarily $\sup_n ||h_n||_{\ell_2} < \infty$ and the Banach–Saks theorem implies that convex combinations $\bar{h}_n$ of h_n converge to some $h^* \in \ell_2$ (in the norm of ℓ_2). By Fatou's lemma,

$$E\big[e^{-V(h^*)}\big] \le \liminf_{n\to\infty} E\big[e^{-V(\bar{h}_n)}\big] \le \liminf_{n\to\infty} E\big[e^{-V(h_n)}\big] = U,$$

using also convexity of the exponential function. This proves the statement. □

4 Applications and examples

Our Theorem 2.16 gives a closed-form expression for the optimal portfolios for the problem (21) by using the function $Q(\theta)$ defined in (35). In this section, we first study some properties of this function. Then we present some examples.

Let $\mathcal{M}_Z(s) = Ee^{sZ}$ and $\mathcal{K}_Z(s) = \ln \mathcal{M}_Z(s)$ denote the moment generating function (MGF) and the cumulant generating function (CGF) of the mixing distribution Z, respectively. We have the obvious relation

$$Q(\theta) = e^{\mathcal{C}\theta}\mathcal{M}_Z\Big(\frac{\mathcal{C}}{2}\theta^2 - \frac{\mathcal{A}}{2}\Big), \qquad \ln Q(\theta) = \mathcal{C}\theta + \mathcal{K}_Z\Big(\frac{\mathcal{C}}{2}\theta^2 - \frac{\mathcal{A}}{2}\Big).$$

Therefore the minimizing points of $Q(\theta)$ in (40) can also be found by using the MGF or KGF of Z. In the following lemma we state some properties of the function $Q(\theta)$.

Lemma 4.1. *Consider the model (1) with a nontrivial mixing distribution Z. Let $\hat{s}$ denote the CV-L of Z and $\hat{\theta}$ be defined as in Section 2. Let the function $Q(\theta)$ be defined by (35). Assume our model (1) is such that either $\mathcal{A} \neq 0$ or $\hat{s} \neq 0$ which ensures $\hat{\theta} = \sqrt{(\mathcal{A} - 2\hat{s})/\mathcal{C}} \neq 0$ and hence $(-\hat{\theta}, \hat{\theta})$ is a nonempty open interval. Then we have the following.*

a) The function $Q(\theta)$ is infinitely differentiable on $(-\hat{\theta}, \hat{\theta})$. If $\hat{s}$ is finite and $\mathcal{L}_Z(\hat{s}) = +\infty$ or if $\hat{s} = -\infty$, we have

$$\lim_{\theta\to\hat{\theta}^-} Q(\theta) = +\infty, \qquad \lim_{\theta\to-\hat{\theta}^+} Q(\theta) = +\infty. \tag{48}$$

When $\hat{s}$ is finite and $\mathcal{L}_Z(\hat{s}) < \infty$ we have $Q(\hat{\theta}) < \infty$ and $Q(-\hat{\theta}) < \infty$. When $\hat{s}$ is finite and $\theta \notin [-\hat{\theta}, \hat{\theta}]$ we have $Q(\theta) = +\infty$.

b) The function $Q(\theta)$ is strictly increasing on $[0, \hat{\theta}]$ when $\hat{s}$ is finite. It is strictly increasing on $[0, +\infty)$ when $\hat{s} = -\infty$. We have $Q'(0) \neq 0$ which implies q_{min} in (39) cannot be zero under the stated conditions.

c) *The function $Q(\theta)$ is strictly convex on the open interval $(-\hat{\theta}, \hat{\theta})$ when $\hat{s}$ is finite and $\mathcal{L}(\hat{s}) = +\infty$ or when $\hat{s} = -\infty$. $Q(\theta)$ is strictly convex on $[-\hat{\theta}, \hat{\theta}]$ when $\hat{s}$ is finite and $\mathcal{L}(\hat{s}) < \infty$.*

Proof. a) It is sufficient to prove that the function $\theta \to \mathcal{L}_Z(\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2)$ is infinitely differentiable when $\theta \in (-\hat{\theta}, \hat{\theta})$. This function is a composition of two functions $s \to \mathcal{L}_Z(s)$ and $\theta \to \frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2$. So it is sufficient to prove the infinite differentiability of $s \to \mathcal{L}_Z(s)$ in the corresponding domain. If $\mathcal{L}_Z(s)$ is k-times differentiable then we will have $\mathcal{L}_Z^{(k)}(s) = (-s)^k E[Z^k e^{-sZ}]$. To justify the change of the order of derivative with expectation for this we need to show $E[Z^k e^{-sZ}] < \infty$. Let us look at the case $\hat{s} \neq 0$ first. In this case we have $Ee^{sZ} < \infty$ in $(-\infty, |\hat{s}|)$. Thus all the moments of Z are finite. This implies $E[Z^k e^{-sZ}] < \infty$ for any positive integer k and all $s \in (\hat{s}, +\infty)$. If $\theta \in (-\hat{\theta}, \hat{\theta})$, then $\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2 \in (\hat{s}, \frac{\mathcal{A}}{2})$. Therefore, when $\hat{s} \neq 0$, the infinite differentiability of $Q(\theta)$ follows. Now let us look at the case $\hat{s} = 0$. In this case $\hat{\theta} = \sqrt{\frac{\mathcal{A}}{\mathcal{C}}}$ and for any $\theta \in (-\hat{\theta}, \hat{\theta})$ we have $\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2 \in (0, \frac{\mathcal{A}}{2})$. Therefore, it is sufficient to prove infinite differentiability of $\mathcal{L}_Z(s)$ on $(0, \frac{\mathcal{A}}{2})$. Fix an arbitrary positive integer k. When $s \in (0, \frac{\mathcal{A}}{2})$ we have $Z^k/e^{sZ} = (Z^k/e^{sZ})1_{\{Z \leq M\}} + (Z^k/e^{sZ})1_{\{Z > M\}}$ for any positive number M. For sufficiently large $M = M_0$, we have $(Z^k/e^{sZ})1_{\{Z > M_0\}} \leq 1$ and $Z^k/e^{sZ} = (Z^k/e^{sZ})1_{\{Z \leq M_0\}}$ is a bounded random variable. Thus $E(Z^k e^{-sZ}) < \infty$ for any positive integer k when $s \in (0, \frac{\mathcal{A}}{2})$. This shows that $\theta \to \mathcal{L}_Z(\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2)$ is infinitely differentiable when $\hat{s} = 0$ also.

When $\hat{s}$ is finite and when $\theta \to \hat{\theta}$ from the left-hand side or when $\theta \to -\hat{\theta}$ from the right-hand side, the function $\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2$ decreasingly converges to $\hat{s}$ (in some neighborhood of $\hat{s}$). Then the monotone convergence theorem gives the claim (48). Now assume $\hat{s} = -\infty$ which happens when the mixing distribution Z is a bounded nontrivial random variable. The result $\lim_{\theta \to +\infty} Q(\theta) = +\infty$ is clear as both $e^{\mathcal{C}\theta}$ and $\mathcal{L}_Z(\frac{\mathcal{A}}{2} - \frac{\theta^2}{2}\mathcal{C})$ go to $+\infty$. The limit $\lim_{\theta \to -\infty} Q(\theta) = +\infty$ is less clear as $e^{\mathcal{C}\theta} \to 0$ and $\mathcal{L}_Z(\frac{\mathcal{A}}{2} - \frac{\theta^2}{2}\mathcal{C}) \to +\infty$ in this case. But since $Z \neq 0$ with positive probability, we have a positive number $\delta > 0$ with $P(Z \geq \delta) > 0$. We have

$$Q(\theta) = Ee^{[\frac{\mathcal{C}}{2}\theta^2 - \frac{\mathcal{A}}{2}]Z + \mathcal{C}\theta} \geq e^{[\frac{\mathcal{C}}{2}\theta^2 - \frac{\mathcal{A}}{2}]\delta + \mathcal{C}\theta} P(Z \geq \delta) \tag{49}$$

for all θ with $\frac{\mathcal{C}}{2}\theta^2 - \frac{\mathcal{A}}{2} > 0$. Then, since the right-hand side of (49) goes to $+\infty$ when $\theta \to -\infty$, the claim follows. The remaining property of Q in part a) above is obvious by the definition of $\hat{\theta}$.

b) For any $\theta \in (-\hat{\theta}, \hat{\theta})$ we have

$$Q'(\theta) = \mathcal{C}e^{\mathcal{C}\theta}\mathcal{L}_Z\left[\frac{\mathcal{A}}{2} - \frac{\theta^2}{2}\mathcal{C}\right] - \theta\mathcal{C}e^{\mathcal{C}\theta}\mathcal{L}'_Z\left[\frac{\mathcal{A}}{2} - \frac{\theta^2}{2}\mathcal{C}\right]. \tag{50}$$

Observe that $0 \in (-\hat{\theta}, \hat{\theta})$ always (in both cases $\hat{s} \neq 0$ and $\hat{s} = 0$). Therefore, $Q'(0)$ always exists and from (50) we see that $Q'(0) \neq 0$. Now since $\mathcal{L}_Z(s)$ is a strictly decreasing function, we have $\mathcal{L}'_Z(s) < 0$. Therefore, $Q'(\theta)$ is finite and $Q'(\theta) > 0$ when $\theta \in (0, \hat{\theta})$. At $\theta = 0$, we have $Q(0) = \mathcal{C}\mathcal{L}_Z(\mathcal{A}/2)$ and clearly we have $Q(0) < Q(\theta)$ for all $\theta \in (0, \hat{\theta})$. At $\theta = \hat{\theta}$, we have $Q(\theta) = \mathcal{L}_Z(\hat{s})$ which is either $+\infty$ or finite. When it is finite we have $Q(\theta) < Q(\hat{\theta})$ for all $\theta \in [0, \hat{\theta})$ also.

c) Define $f_z(\theta) =: e^{\frac{\mathcal{C}}{2}z\theta^2+\mathcal{C}\theta-\frac{\mathcal{A}}{2}z}$ for any real number $z \geq 0$ and for all $\theta \in \mathbb{R}$. We have $f_z'(\theta) = (\mathcal{C}z\theta + \mathcal{C})e^{\frac{\mathcal{C}}{2}z\theta^2+\mathcal{C}\theta-\frac{\mathcal{A}}{2}z}$ and $f_z''(\theta) = \mathcal{C}ze^{\frac{\mathcal{C}}{2}z\theta^2+\mathcal{C}\theta-\frac{\mathcal{A}}{2}z} + (\mathcal{C}z\theta + \mathcal{C})^2e^{\frac{\mathcal{C}}{2}z\theta^2+\mathcal{C}\theta-\frac{\mathcal{A}}{2}z} > 0$ for any $z \geq 0$. Therefore, $f_z(\theta)$ is a strictly convex function for any fixed $z \geq 0$. Therefore, we have

$$f_z\big(\lambda\theta_1 + (1-\lambda)\theta_2\big) < \lambda f_z(\theta_1) + (1-\lambda)f_z(\theta_2)$$

for any $\lambda \in [0, 1]$ and for all $\theta_1, \theta_2 \in \mathbb{R}$ for each fixed $z \geq 0$. This strict inequality also holds when $z = Z$. Also, observe that when $\hat{s}$ is finite and $\mathcal{L}_Z(\hat{s}) = +\infty$ or when $\hat{s} = -\infty$, for $\theta_1, \theta_2 \in (-\hat{\theta}, \hat{\theta})$ we have $Ef_Z(\theta_1) < \infty$ and $Ef_Z(\theta_2) < \infty$. When $\hat{s}$ is finite and $\mathcal{L}_Z(\hat{s}) < \infty$, for all $\theta_1, \theta_2 \in [-\hat{\theta}, \hat{\theta}]$ we have $Ef_Z(\theta_1) < \infty$ and $Ef_Z(\theta_2) < \infty$. We take expectation to the above inequality when $z = Z$ and obtain $Q(\lambda\theta_1 + (1-\lambda)\theta_2) < \lambda_1 Q(\theta_1) + (1-\lambda)Q(\theta_2)$. This shows the strict convexity of $Q(\theta)$ stated in the lemma. □

Remark 4.2. The main message of Lemma 4.1 is that the optimal solution to the problem (21) is always unique. Now assume $\mathcal{L}_Z(\hat{s}) < \infty$. In this case, if the optimal portfolio $x^\star$ for the problem (21) is irregular then q_{min} in (39) satisfy $q_{min} = -\hat{\theta}$. This means that $-\hat{\theta}$ is the minimizing point of $Q(\theta)$ in $[-\hat{\theta}, \hat{\theta}]$. As $Q(\theta)$ is a strictly convex function on $[-\hat{\theta}, \hat{\theta}]$ as shown in Lemma 4.1, we conclude that $Q(\theta)$ is a strictly increasing, strictly convex function on $[-\hat{\theta}, \hat{\theta}]$. In comparison, when the solution to (21) is regular, then the corresponding $Q(\theta)$ is strictly convex but not strictly increasing on $[-\hat{\theta}, \hat{\theta}]$.

Example 4.3. Assume the mixing distribution Z in our model (1) takes finitely many values $\{z_i\}_{1\leq i\leq m}$ with corresponding probabilities $(p_i)_{1\leq i\leq m}$. Then X in (1) is a mixture of normal random vectors

$$X \sim \sum_{i=1}^{m} p_i N_d(\mu + \gamma z_i, z_i\Sigma). \tag{51}$$

In this case, the function $Q(\theta)$ takes the form

$$Q(\theta) = \sum_{i=1}^{m} p_i e^{(\frac{\theta^2}{2}C-\frac{1}{2}A)z_i+\theta C}. \tag{52}$$

From part c) of the above Lemma 4.1 we know that the function $Q(\theta)$ is strictly convex on $(-\infty, +\infty)$. Thus the solution to the optimization problem (21) is unique and it is given by (41) with $q_{min} = \arg\min_{\theta\in(-\infty,0)} Q(\theta)$. Now, assume $Z = 1$ with probability one instead. Then $\mathcal{L}_Z(s) = e^{-s}$ and in this case it is easy to see that

$$Q(\theta) = e^{\frac{C}{2}(\theta^2+2\theta)-\frac{A}{2}}.$$

The minimizing point of this function is $\theta = -1$ and so $q_{min} = -1$. Then, from (39), the optimal portfolio is given by

$$x^\star = \frac{1}{aW_0}\Sigma^{-1}(\gamma + \mu - \mathbf{1}r_f).$$

Note here that since we assumed $Z = 1$, X in (1) is a Gaussian random vector and therefore one can obtain the above optimal portfolio by direct calculation as our utility function is exponential. However, our above approach seems more convenient.

In the next example, we look at the case of GH models.

Example 4.4. Let us look at the case of the model (1) when the mixing distribution Z is given by GIG models. First assume $Z \sim iG(\lambda, \frac{a^2}{2})$, the inverse Gaussian distribution. In this case, we have $\lambda < 0$ by the definition of inverse Gaussian random variable. From Proposition 9 of [10] we have $\mathcal{L}_Z(s) = (\frac{2}{a\sqrt{2s}})^{\lambda}\frac{2K_\lambda(a\sqrt{2s})}{\Gamma(-\lambda)}$ and therefore $Q(\theta) = e^{\mathcal{C}\theta}(\frac{2}{a\sqrt{\mathcal{A}-\mathcal{C}\theta^2}})^{\lambda}\frac{2K_\lambda(a\sqrt{\mathcal{A}-\mathcal{C}\theta^2})}{\Gamma(-\lambda)}$. In this case, the CV-L is $\hat{s} = 0$ and $\hat{\theta} = \sqrt{\mathcal{A}/\mathcal{C}}$. If $\gamma = 0$, as discussed in Example 2.4, the optimal solution to (21) is $x^\star = 0$. In this case, the solution $x^\star = 0$ is irregular. Note that in this case $\mathcal{A} = 0$ and therefore $\hat{\theta} = 0$. If $\gamma \neq 0$, then $\hat{\theta} > 0$ and in this case q_{min} in (39) is given by $q_{min} = \arg\min_{\theta\in[-\sqrt{\mathcal{A}/\mathcal{C}},0)} Q(\theta)$ (due to Lemma 4.1). Note that either by using the fact $\hat{s} = 0$ or by using the property (A. 8) in [10] directly, one can easily check that $(\frac{2}{a\sqrt{\mathcal{A}-\mathcal{C}\theta^2}})^{\lambda}\frac{2K_\lambda(a\sqrt{\mathcal{A}-\mathcal{C}\theta^2})}{\Gamma(-\lambda)} \to 1$ when $\theta^2 \to \mathcal{A}/\mathcal{C}$. Therefore $Q(-\sqrt{\frac{\mathcal{A}}{\mathcal{C}}}) = e^{-\sqrt{\mathcal{A}\mathcal{C}}}$. In this case, it is not clear if $q_{min} = -\sqrt{\frac{\mathcal{A}}{\mathcal{C}}}$ (the solution $x^\star$ is irregular) or $q_{min} \in (-\sqrt{\frac{\mathcal{A}}{\mathcal{C}}}, 0)$ (the solution $x^\star$ is regular).

Now let us look at the case $Z \sim GIG(\lambda, a, b)$ when $a > 0$, $b > 0$. Again from Proposition 9 of [10] we have $\mathcal{L}_Z(s) = (\frac{b}{\sqrt{b^2+2s}})^{\lambda}\frac{K_\lambda(a\sqrt{b^2+2s})}{K_\lambda(ab)}$ and $Q(\theta) = e^{\mathcal{C}\theta}(\frac{b}{\sqrt{b^2+\mathcal{A}-\mathcal{C}\theta^2}})^{\lambda}\frac{K_\lambda(a\sqrt{b^2+\mathcal{A}-\mathcal{C}\theta^2})}{K_\lambda(ab)}$. In this case $\hat{s} = -b^2/2$ and $\hat{\theta} = \sqrt{\frac{\mathcal{A}+b^2}{\mathcal{C}}}$. One can easily check that $\mathcal{L}_Z(\hat{s}) = +\infty$ in this case. Therefore the unique optimal solution to (21) is given by (41) and it is regular.

Corollary 4.5. *Consider the model (1) with $\gamma = 0$. In this case the distribution of X is Elliptical distribution. Assume the CV-L of the mixing distribution Z is $\hat{s} = 0$. Then the corresponding optimization problem (21) has a unique solution $x^\star = 0$. The CV-L of Z is $\hat{s} = 0$ if $EZ^n = +\infty$ for some positive integer n.*

Proof. Observe that in this case $\mathcal{A} = 0$ and therefore $\hat{\theta} = 0$. Then $[-\hat{\theta}, \hat{\theta}] = \{0\}$. Therefore q_{min} in (39) is $q_{min} = 0$. As $\gamma = 0$ also by assumption, we have $x^\star = 0$ by (39). It is clear that this solution is unique. If $\hat{s} \neq 0$, then the Laplace transformation of Z is finite in $(-\infty, |\hat{s}|)$ and this would imply that all the moments of Z are finite. Therefore infiniteness of one of the moments of Z implies $\hat{s} = 0$. $\square$

Example 4.6 (Stable distributions). Let us look at the case of α-stable distributions. Here we look at the 1-parametrization of the stable distributions (see Definition 1.5 of [24]). For other parameterizations, see [24]. A distribution W follows α-stable distribution with parameters $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$, $u \in \mathbb{R}$, and we write $W \sim S(\alpha, \beta, \sigma, u)$ if its characteristic function is given by

$$\phi(t) = Ee^{itW} = \begin{cases} e^{-\sigma^\alpha|t|^\alpha[1-i\beta\operatorname{sign}(t)\tan(\frac{\pi\alpha}{2})]+itu}, & \alpha \neq 1, \\ e^{-\sigma|t|[1+i\beta\frac{2}{\pi}\operatorname{sign}(t)\ln|t|]+itu}, & \alpha = 1. \end{cases} \tag{53}$$

When $\alpha = 2$, a stable distribution is a normal distribution. When $\alpha \in (0, 2)$, $EW^2 = +\infty$ for all $\beta \in [-1, 1]$, $\sigma > 0$, $u \in \mathbb{R}$. Therefore, for the mixing distributions $Z = |W|$, $\alpha \in (0, 2)$, $\beta \in [-1, 1]$, $\sigma > 0$, $u \in \mathbb{R}$, the corresponding CV-L is $\hat{s} = 0$. Thus when $\gamma = 0$ and when $Z = |W|$, $\alpha \in (0, 2)$, $\beta \in [-1, 1]$, $\sigma > 0$, $u \in \mathbb{R}$, in the model (1), the optimization problem (21) has a unique solution $x^\star = 0$. This means that when the mixing distribution Z in (1) is equal to the absolute value of a stable distribution with $\alpha \in (0, 2)$ and when $\gamma = 0$, then the optimal portfolio for an exponential utility maximizer is to invest all her/his wealth into the risk-free asset.

Remark 4.7. Stable distributions are infinitely divisible. The characteristic functions (53) of the stable laws can be obtained directly from their Lévy–Khintchine representations. The generelized central limit theorem states that stable laws are the only nontrivial limits of normalized sums of independent identically distributed random variables. As such they were proposed to model many empirical (heavy tails, skewness, etc.) financial phenomena in the past. The heavy-tailedness of them is related with the CV-L of them being $\hat{s} = 0$. Example 4.6 shows that time-changed Brownian motion models with stable subordinators (the ones with Elliptical marginal distributions) always give the trivial portfolio, investing everything on the risk-free asset, as the optimal portfolio for an exponential utility maximizer.

As pointed out in Remark 4.2, our Lemma 4.1 shows that the solution to the problem (21) is unique. Part b) of this lemma shows that $\theta = 0$ is not the minimizing point of the function $Q(\theta)$ under the condition that $\mathcal{A} \neq 0$ or $\hat{s} \neq 0$. For this unique minimizing point $\theta \neq 0$ of $Q(\theta)$ the first order condition (50) can equivalently be written as

$$\frac{\mathcal{L}'_Z(\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2)}{\mathcal{L}_Z(\frac{\mathcal{A}}{2} - \frac{\mathcal{C}}{2}\theta^2)} = \frac{1}{\theta}. \tag{54}$$

A change of variable $\eta = \mathcal{A}/2 - (\mathcal{C}/2)\theta^2$, which gives $\theta = -\sqrt{(\mathcal{A} - 2\beta)/\mathcal{C}}$ due to $\theta < 0$ by Lemma 4.1, then gives

$$\frac{\mathcal{L}'_Z(\beta)}{\mathcal{L}_Z(\beta)} = -\sqrt{\mathcal{C}/(\mathcal{A} - 2\beta)}, \quad \hat{s} < \beta < \mathcal{A}/2. \tag{55}$$

From this we can conclude that if $x^\star$ is a regular solution to (21), then $\beta_{min} =: \mathcal{A}/2 - (\mathcal{C}/2)q_{min}^2$ with q_{min} in (41) satisfies the relation (55). This observation is useful if it can be confirmed that the solution to the equation (55) is unique. Then this unique solution equals to β_{min}. Consider, for example, the case $Z = 1$ in the model (1). As discussed in Example 4.3, in this case we have $\mathcal{L}_Z(s) = e^{-s}$. Then $\mathcal{L}'_Z(\beta)/\mathcal{L}_Z(\beta) = -1$ and it is clear that the equation $1 = \sqrt{\mathcal{C}/(\mathcal{A} - 2\beta)}$ has a unique solution $\beta = \mathcal{A}/2 - \mathcal{C}/2$. This implies $q_{min}^2 = 1$ which then shows that $q_{min} = -1$ is the minimizing point of $Q(\theta)$.

A positive random variable Z is a GGC with a generating pair (τ, ν) if

$$\mathcal{L}_Z(s) = Ee^{-sZ} = e^{-\tau - \int_0^\infty \ln(1+\frac{s}{z})\nu(dz)}. \tag{56}$$

If Z is a GGC with a generating pair (τ, ν), then $\frac{\mathcal{L}'_Z(\beta)}{\mathcal{L}_Z(\beta)} = -\tau - \int_0^{+\infty} \frac{1}{t-\beta}\nu(dt)$. So if the solution to (21) is regular, then β_{min} defined above satisfies the equation

$$-\tau - \int_{|\hat{s}|}^{+\infty} \frac{1}{t-\beta}\nu(dt) = -\sqrt{\mathcal{C}/(\mathcal{A}-2\beta)},$$

where $\hat{s}$ is the CV-L of the GGC random variable Z.

Now consider the case of positive α-stable random variables $Z = S(\alpha, 1, \sigma, u)$, $0 < \alpha < 1, u > 0$. Here we took $\beta = 1$ (see Lemma 1.1 of [24]). After normalization these mixing distributions have the Laplace transformation $\mathcal{L}_Z(s) = e^{-s^{\alpha}}$ (see Proposition 1 of [4] and also see [28]). Thus we have $\mathcal{L}'_Z(s)/\mathcal{L}_Z(s) = -s^{\alpha}\ln s$. Assume the problem (21) has a regular solution (a necessary condition for this is $\gamma \neq 0$, see Corollary 4.5). Let $\beta_{min} = \mathcal{A}/2-(\mathcal{C}/2)q_{min}^2$ with q_{min} in (41). Then $0 < \beta_{min} < \mathcal{A}/2$ and due to (55) it satisfies the equation

$$\beta^{\alpha}\ln\beta = \sqrt{\mathcal{C}/(\mathcal{A}-2\beta)}.$$

We square both sides of this equation and obtain

$$\mathcal{A}\beta^{2\alpha}(\ln\beta)^2 - 2\beta^{2\alpha+1}(\ln\beta)^2 = \mathcal{C}.$$

As discussed earlier, if this equation has a unique solution β then it is β_{min}.

Remark 4.8. We should mention here that the formula (39) for the optimal portfolio for the problem (21) is related to the Laplace transformation of the mixing distribution Z in the model (1) only. Namely, we don't need to know the probability density function of Z to find the optimal portfolio for the optimization problem (21). The relation (55) gives a convenient approach to locating the unique optimal portfolio as discussed earlier.

Next, we discuss the applications of our results in continuous time financial modeling. First, we recall Lemma 2.6 of [10] here. According to this lemma, for each model $F = N_d(\mu+\gamma z, z\Sigma)\circ G$ in (1) there is a corresponding Lévy process

$$Y_t = \mu t + \gamma\tau_t + \bar{B}_{\tau_t}, \tag{57}$$

with $Law(Y_1) = F$ and $Law(\tau_1) = G$ as long as $G \in \mathcal{J}$ (note that if $G \in \mathcal{J}$ then $X \in \mathcal{J}$ also from Lemma 2.5 of [10]). In the model (57), $(\bar{B}_t)_{t\geq 0} = (AB_t)_{t\geq 0}$ where B_t is an n-dimensional standard Brownian motion independent from $(\tau_t)_{t\geq 0}$ and $(\tau_t)_{t\geq 0}$ is a subordinator (a nonnegative Lévy process with increasing sample paths). We denote the Lévy measure of this subordinator by ρ and its Laplace transformation by

$$\mathcal{L}_{\tau_t}(s) = e^{-t\Psi(s)}, \tag{58}$$

where $\Psi(s) = bs + \int_0^{\infty}(1-e^{-sy})\rho(dy)$ with a constant $b \geq 0$. As stated in Proposition 2.3 of [16], the function $\Psi(s)$ is continuous, nondecreasing, nonnegative, and convex. At each time point $t > 0$ we have

$$Y_t \stackrel{d}{=} \mu t + \gamma\tau_t + \sqrt{\tau_t}AN_d. \tag{59}$$

Now consider a market with n risky assets with the price process $S_t \in \mathbb{R}^d$ and one risk-free asset with price process $B_t = e^{tr_f}$. Assume the log-return process $Y_t =$

$(Y_t^{(1)}, Y_t^{(2)}, \dots, Y_t^{(d)})$, where $Y_t^{(i)} = \ln(S_t^{(i)}/S_0^{(i)})$, has the dynamics as in (57). The log-return in the risk-free asset is $\ln(B_t/B_0) = r_f t$. An exponential utility maximizer wants to determine the optimal portfolio at each time point t based on the log-return vector of risky assets $R \in \mathbb{R}^d$ with components $R^{(i)} = \ln(S_{t+\triangle}^{(i)}/S_t^{(i)})$ and the log-return of the risk-free asset $R^{(0)} = \ln(B_{t+\triangle}/B_t) = \triangle r_f$ in the time horizon $[t, t+\triangle]$. Assume the time increment is $\triangle = 1$. Then we have

$$R \stackrel{d}{=} \mu + \gamma\tau_1 + \sqrt{\tau_1} A N_d, \tag{60}$$

and from our Theorem 2.16 the exponential utility maximizer's optimal portfolio at time t is

$$x_t^\star = \frac{1}{aW_0^{(t)}}\left[\Sigma^{-1}\gamma - q_{min}^{(t)}\Sigma^{-1}(\mu - \mathbf{1}r_f)\right], \tag{61}$$

where $W_0^{(t)}$ is its (initial) wealth that it invests in the $n+1$ assets for the period $[t, t+\triangle]$ and $q_{min}^{(t)}$ in (61) is given by $q_{min}^{(t)} = \arg\min_{\theta\in\Theta} Q(\theta)$ in the corresponding domain θ. Here

$$Q(\theta) = e^{C\theta - \Psi(\frac{1}{2}A - \frac{\theta^2}{2}C)}, \tag{62}$$

due to (58).

Example 4.9 (Variance-gamma model)**.** Consider the financial market that was discussed in the paper [21]. The stock price is given by $S(t) = S(0)e^{mt + X(t;\, \sigma_S,\, \nu_S,\, \theta_S) + \omega_S t}$ in their equation (21), where m is the mean-rate of return on the stock under the statistical probability measure, $\omega_S = \frac{1}{\nu_S}\ln(1 - \theta_S\nu_S - \sigma_S^2\nu_S/2)$, and $X(t; \sigma_S, \nu_S, \theta_S) = b(\gamma(t; 1, \nu_S); \theta_S, \sigma_S)$ with $b(t; \theta, \sigma) = \theta t + \sigma W(t)$ being a Brownian motion with drift θ and volatility σ. Here the gamma process $\gamma(t; \mu, \nu)$ has mean rate μ and variance rate ν (note here that $\gamma(t; \mu, \nu) \sim G(\mu^2/\nu, \nu/\mu)$ with our notation for gamma random variables in this paper). The increment $g_0 =: \gamma(t+1; 1, \nu_S) - \gamma(t; 1, \nu_S) \stackrel{d}{=} \gamma(1; 1, \nu_S)$ of this process has the Laplace transformation

$$\mathcal{L}_{g_0}(s) = \left(\frac{1}{1 + s\nu_S}\right)^{\frac{1}{\nu_S}}, \tag{63}$$

which can be seen also from the characteristic function expression in (3) of [21] for gamma processes. The risk-free asset in this financial market is given by $B_t = B_0 e^{tr_f}$. The log-returns of these two assets in the time horizon $[t, t+1]$ are given by

$$\begin{aligned} R &=: \ln\big(S(t+1)/S(t)\big) \stackrel{d}{=} m + \omega_S + \theta_S\gamma(1; 1, \nu) + \sigma_S\sqrt{\gamma(1; 1, \nu_S)}N(0, 1), \\ R^0 &=: \ln(B_{t+1}/B_t) = r_f. \end{aligned}$$

An exponential utility maximizer with the utility function $u(x) = -e^{-ax}$, $a > 0$, and wealth $W_0^{(t)}$ at time t wants to decide on the optimal proportion $x^\star$ on the risky asset of his wealth for the period $[t, t+1]$. His acceptable set for $x^\star$ is given by

$$S_a = \left\{x \in \mathbb{R} : aW_0^{(t)}\theta_S x - \frac{a^2(W_0^{(t)})^2}{2}\sigma_S^2 x^2 > -\frac{1}{\nu_S}\right\}, \tag{64}$$

as $\hat{s} = -\frac{1}{\nu_S}$ in this case. The corresponding expressions for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in (32) are given by

$$\mathcal{A} = \left(\frac{\theta_S}{\sigma_S}\right)^2, \qquad \mathcal{C} = \left(\frac{m + \omega_S - r_f}{\sigma_S}\right)^2, \mathcal{B} = \frac{\theta_S(m + \omega_S - r_f)}{\sigma_S^2}.$$

Since the mixing distribution is of a gamma random variable, the solution to the corresponding problem (21) is regular. Our Theorem 2.16 shows that the optimal portfolio is given by

$$x^\star = \frac{1}{aW_0}\left[\frac{1}{\sigma_S^2}\theta_S - q_{min}\frac{1}{\sigma_S^2}(m + \omega_S - r_f)\right]. \tag{65}$$

where $q_{min} = \arg\min_{\theta \in (-\hat{\theta}, \hat{\theta})} Q(\theta)$ with $Q(\theta)$ given by (35). Here $\hat{\theta} = \sqrt{\frac{\mathcal{A}+2/\nu_S}{\mathcal{C}}}$. Next, we calculate q_{min} explicitly. We have $Q(\theta) = e^{\mathcal{C}\theta}\mathcal{L}_{g_0}(\mathcal{A}/2 - (\mathcal{C}/2)\theta^2)$ and from this we get $\ln Q(\theta) = \mathcal{C}\theta - \frac{1}{\nu_S}\ln(1 + \frac{\mathcal{A}}{2}\nu_S - \frac{\mathcal{C}}{2}\nu_S\theta^2)$. The first order condition for the minimizing point of $\ln Q(\theta)$ gives $(\theta + \frac{1}{\mathcal{C}\nu_S})^2 = \frac{1+\mathcal{C}\nu_S(2+\mathcal{A}\nu_S)}{\mathcal{C}^2\nu_S^2}$. This gives two solutions $\theta = -\frac{1}{\mathcal{C}\nu_S} \pm \frac{1}{\mathcal{C}\nu_S}\sqrt{1 + \mathcal{C}\nu_S(2 + \mathcal{A}\nu_S)}$. But since θ needs to be negative due to Lemma 4.1, we take $q_{min} = \theta = -\frac{1}{\mathcal{C}\nu_S} - \frac{1}{\mathcal{C}\nu_S}\sqrt{1 + \mathcal{C}\nu_S(2 + \mathcal{A}\nu_S)}$. We then plug this into (39) and obtain

$$x^\star = \frac{1}{aW_0^{(t)}\sigma_S^2}\left[\theta_S + \frac{m + \omega_S - r_f}{\mathcal{C}\nu_S} + \frac{m + \omega_S - r_f}{\mathcal{C}\nu_S}\sqrt{1 + \mathcal{C}\nu_S(2 + \mathcal{A}\nu_S)}\right]. \tag{66}$$

Therefore in this case we have a closed-form expression for the optimal portfolio. We should mention that one can use similar calculations to obtain a closed-form expression for optimal portfolio in a market where risky assets are modeled by multidimensional variance gamma (MVG) model, see [20] for the details of MVG models.

Remark 4.10. Price processes with log-returns of the type (57) have been quite popular in financial literature in the past. Such models include inverse Gaussian Lévy processes, hyperbolic Lévy motions, variance gamma models, and CGYM models, and all of these models were shown to fit empirical data quite well, see [5, 9, 27, 8, 19] and the references therein for this. In fact, every semimartingale can be written as a time-change of Brownian motion, see [23] for this. This means that all the Lévy processes are time-changed Brownian motions. In all these cases, if the time-changing subordinator is independent of the Brownian motion, then our Theorem 2.16 is applicable in principle. However, it is not easy to find the time-change used for general semimartingales. Recently, the paper [19] obtained the time-change used for the CGMY model and Meixner processes. Our results in this paper can be applied to such processes to determine optimal portfolios for an exponential utility maximizer in a market where single or multiple risky asset dynamics follow such models.

5 Conclusion

The main result of this paper is Theorem 2.16 where we show that the problem of locating the optimal portfolio for (11) when the utility function is exponential boils down to finding the minimum point of a real-valued function on the real line, improving Theorem 1 of [3] for the case of GH models and in the meantime extending it from the class of GH models to the general class of NMVM models. Our Theorem 3.3 shows that an optimal exponential utility in small markets converge to the overall best exponential utility in the large financial market. While optimal portfolio problems under expected utility criteria for exponential utility functions have been discussed extensively in the past financial literature, an explicit solution of the optimal portfolio as in Theorem 2.16 above seems to be new. This is partly due to the condition we impose on the return vector X of being an NMVM model. However, despite this restrictive condition on X, asset price dynamics with NMVM distributions in their log-returns often show up in financial literature like exponential variance gamma and exponential generalized hyperbolic Lévy motions.

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