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A group action on increasing sequences of set-indexed Brownian motions

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Abstract We prove that a square-integrable set-indexed stochastic process is a set-indexed Brownian motion if and only if its projection on all the strictly increasing continuous sequences are one-parameter G-time-changed Brownian motions. In addition, we study the "sequence-independent variation" property for group stationary-increment stochastic processes in general and for a set-indexed Brownian motion in particular. We present some applications.

Keywords Set indexed process, Brownian motion, increasing path
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1 Introduction

The set-indexed Brownian motion $\{X_A : A \in \mathbf{A}\}$ is well defined and well studied (see [6]). We will mention that the indexing collection $\mathbf{A}$ is a compact set collection on a topological space T. The choice of the collection $\mathbf{A}$ is crucial: it must be sufficiently rich in order to generate the Borel sets of T, but small enough to ensure the existence of a continuous Gaussian process defined on $\mathbf{A}$.

In this paper, we define a group action on the indexing collection $\mathbf{A}$, and from that we characterize the set-indexed Brownian motion by using the notion of an increasing path introduced in [2]. The characterization of a set-indexed Brownian motion by group action (Theorem 1 in this article, which says that a square-integrable set-indexed stochastic process is a set-indexed Brownian motion if and only if its projection on all the strictly increasing continuous sequences are one-parameter G-time-changed Brownian motions) is the key to most of the proofs in this article. It is of great importance since it allows us to “divide and conquer.” Therefore, many of the proofs for a set-indexed Brownian motion can be recovered by reducing to a (classical) one-dimensional Brownian motion. The results that we have extended from a classical Brownian motion to a set-indexed Brownian motion involve the following issues: hitting time, maximum value, reflection principle, exiting from an interval, time inversion, iterated logarithms, strong law of large numbers, unboundedness, zero crossing, zero set, nondifferentiability, path-independent variation, martingale in Brownian motion, and the like.

The frame of a set-indexed Brownian motion is not only a new step of generalization of a classical Brownian motion, but it proved a new look upon a Brownian motion. In recent years, there have been many new results related to the dynamical properties of random processes indexed by a class of sets. Set-indexed processes have many potential areas of applications. For example: environment (increased occurrence of polluted wells in a rural area could indicate a geographic region that has been subjected to industrial waste), astronomy (a cluster of black holes could be a result of an unobservable phenomenon affecting a region in space), quality control (an increased rate of breakdowns in a certain type of equipment might follow the failure of one or more components), population health (unusually frequent outbreaks of a disease such as leukemia near a nuclear power plant could signal a region of possible air or ground contamination), and the like.

Cairoli and Walsh [2] introduced the notion of path-independent variation (p.i.v) for two-parameter processes. They proved (under some assumptions) that any strong martingale has the path-independent variation property. We extend their results to set-indexed strong martingales.

In the last section, we present some results concerning the compensators of a set-indexed strong martingale and analyze the concept of path-independent variation in connection with independent increments in set-indexed process. We introduce compensators and demonstrate that the path-independent variation property permits a better understanding of the Doob–Meyer decomposition.

2 Preliminaries

We recall the definitions and notation from [6].

Definition 0. Let (T, τ) be a nonvoid sigma-compact connected topological space. A nonempty class $\mathbf{A}$ of compact connected subsets of T is called an indexing collection if it satisfies the following:

(1) $\varnothing \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T = \bigcup_{n=1}^{\infty} B_n^{\circ}$.

(2) $\mathbf{A}$ is closed under arbitrary intersections, and if $A, B \in \mathbf{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i \subseteq B_n$ for every i, then $\overline{\bigcup_i A_i} \in \mathbf{A}$.

(3) $\sigma(\mathbf{A}) = \mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.

Remarks. (a) Note that any collection of sets closed under intersections is a semilattice with respect to the partial order of the inclusion.

(b) Definition 0 implies that a space T cannot be discrete and that $\mathbf{A}$ is at least a continuum.

Examples. (a) The classical example is $T = \Re_+^2$ and $\mathbf{A} = \mathbf{A}(\Re_+^2) = \{[0, x] : x \in \Re_+^2\}$ (This example can be extended to $T = \Re_+^d$ and $\mathbf{A}(\Re_+^d) = \{[0, x] : x \in \Re_+^d\}$, which will give rise to a sort of 2^d-sides process).

(b) The example (a) may be generalized as follows. Let $T = \Re_+^2$ and take $\mathbf{A}$ (or $\mathbf{A}(Ls)$) to be the class of compact *lower sets*, i.e. the class of compact subsets A of T satisfying $t \in A$ implies $[0, t] \subseteq A$.

Definition 1. Let (Ω, F, P) be a complete probability space equipped with an $\mathbf{A}$-indexed filtration $\{F_A : A \in \mathbf{A}\}$ that satisfies the following conditions:

(i) for all $A \in \mathbf{A}$, we have $F_A \subseteq F$, and F_A contains the P-null sets.

(ii) for all $A, B \in \mathbf{A}$, if $A \subseteq B$, then $F_A \subseteq F_B$.

(iii) $F_{\bigcap A_i} = \bigcap F_{A_i}$ for any decreasing sequence $\{A_i\}$ in $\mathbf{A}$ (for consistency, in what follows, if $T \notin \mathbf{A}$, we define $F_T = F$).

We will need other classes of sets generated by $\mathbf{A}$. The first is $\mathbf{A}(\mathbf{u})$, which is the class of finite unions of sets in $\mathbf{A}$. We remark that $\mathbf{A}(\mathbf{u})$ is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$ consist of all the subsets of T of the form $C = A\backslash B$, $A \in \mathbf{A}$, $B \in \mathbf{A}(\mathbf{u})$. If $C \in \mathbf{C}(\mathbf{u})\backslash\mathbf{A}$ ($\mathbf{C}(\mathbf{u})$ is the class of finite unions of sets in $\mathbf{C}$), then we denote

$$\mathbf{G}_C^* = \bigvee_{A \in \mathbf{A}(\mathbf{u}),\, A \cap C = \varnothing} F_A.$$

In addition, let A' be any finite subsemilattice of $\mathbf{A}$ closed under intersection. For $A \in A'$, define the left neighborhood of A in A' to be the set $C_A = A \backslash \bigcup_{B \in A', B \subset A} B$. We note that $\bigcup_{A \in A'} A = \bigcup_{A \in A'} C_A$ and that the latter union is disjoint. The sets in A' can always be numbered in the following way: $A_0 = \varnothing'$ ($\varnothing' = \bigcap_n \bigcap_{A \in \mathbf{A}_n, A \neq \varnothing} A$; note that $\varnothing' \neq \varnothing$), and given $A_0, \ldots, A_{i-1}$, we choose A_i to be any set in A' such that $A \subset A_i$ implies that $A = A_j$ for some $j = 1, \ldots, i-1$. Any such numbering $A' = \{A_0, \ldots, A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left neighborhood of A_i in A', then $C_i = \bigcup_{j=0}^{i} A_j \backslash \bigcup_{j=0}^{i-1} A_j$ and $A_j \cap C_i = \varnothing$ for all $j = 0, \ldots, i-1$, $i = 1, 2, \ldots, k$).

Any $\mathbf{A}$-indexed function that has a (finitely) additive extension to $\mathbf{C}$ will be called additive (and is easily seen to be additive on $\mathbf{C}(\mathbf{u})$ as well). For stochastic processes, we do not necessarily require that each sample path be additive, but the additivity will be imposed in an almost sure sense:

Definition 2. A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if it has an (almost sure) additive extension to $\mathbf{C}$: $X_\varnothing = 0$, and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \varnothing$, then almost surely $X_{C_1} + X_{C_2} = X_C$. In particular, if $C \in \mathbf{C}$ and $C = A \backslash \bigcup_{i=1}^n A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$, then almost surely $X_C = X_A - \sum_{i=1}^n X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \cdots + (-1)^n X_{A \cap \bigcap_{i=1}^n A_i}$.

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to $\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C}(\mathbf{u})$.

Definition 3. Let $(G, \cdot)$ be a group. The group G will be called a permutation group on $[a, b]$ if $G = \{\pi : [a, b] \to [a, b] \mid \pi$ is a one-to-one and onto function$\}$, and we denote this group by $S_{[a,b]}$ (i.e., $S_{[a,b]}$ is the class of all the bijection functions from $[a, b]$ to $[a, b]$).

Definition 4. A positive measure σ on $(T, \mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if $\sigma_{\emptyset'} = 0$ and $\sigma_A < \sigma_B$ for all $A, B \in \mathbf{A}$ such that $A \subsetneq B$. The collection of these measures is denoted by $M(\mathbf{A})$.

The classical examples for this definition are the Lebesgue measure or Radon measure when $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d)$.

Definition 5. Let $\sigma \in M(\mathbf{A})$, and let $(G, \cdot)$ be a group. A group action $*$ of $(G, \cdot)$ on $\mathbf{A}$ is defined by $g * (A \cup B) = g * A \cup g * B$, $g * (A \backslash B) = g * A \backslash g * B$ for all $A, B \in \mathbf{A}$ and $g \in G$, and there exists $\eta : G \to \Re_+$ such that $\sigma(g * A) = \eta(g)\sigma(A)$ for all $A \in \mathbf{A}$ and $g \in G$.

The classical examples are the following:

(a) Let $G = (\Re_+^d, \cdot)$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0, x] : x \in \Re_+^d\}$. Then a group action is defined by $g * [0, t] = [0, g \cdot t] = [0, g_1 t_1] \times [0, g_2 t_2] \times \cdots \times [0, g_d t_d]$, $\sigma(g * [0, t]) = g_1 g_2 \ldots g_n \sigma([0, t])$ for all $g = (g_1, \ldots, g_d) \in G$ and $t = (t_1, \ldots, t_d) \in \Re_+^d$.

(b) Let $G = (S_{[0,\infty)}, \circ)$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0, x] : x \in \Re_+^d\}$. Then a group action is defined by $\pi * [0, t] = [0, \pi \circ t_1] \times \cdots \times [0, \pi \circ t_d]$ for all $\pi \in S_{[0,\infty)}$ and $t = (t_1, \ldots, t_d) \in \Re_+^d$.

Definition 6. Let $I \subseteq \Re$, and let $A = \{A_\alpha\}_{\alpha \in I}$ be increasing sequence in $\mathbf{A}(\mathbf{u})$.

(a) The sequence A is called "strictly increasing" if $A_\alpha \subsetneq A_\beta$ for all $\alpha, \beta \in I$ such that $\alpha < \beta$.

(b) If $I = [a, b]$, then the sequence A is called a "continuous sequence" if $A_s = \overline{\bigcup_{u<s} A_u} = \bigcap_{v>s} A_v$ for all $s \in (a, b)$ and $A_a = \bigcap_{v>a} A_v$, $A_b = \overline{\bigcup_{u<b} A_u}$.

Given a set-indexed stochastic process X and increasing sequence $\{A_\alpha\}_{\alpha \in [a,b]}$ in $\mathbf{A}(\mathbf{u})$, we define a process Y indexed by $[a, b]$ as follows: $Y_s = X_{A_s} = X_s^A$ for all $s \in [a, b]$.

A set-indexed stochastic process X is called outer-continuous if X is finitely additive on $\mathbf{C}$ and for any decreasing sequence $\{A_n\} \in \mathbf{A}$, $X_{\bigcap_n A_n} = \lim_n X_{A_n}$ and is called inner-continuous if for any increasing sequence $\{A_n\} \in \mathbf{A}$ such that $\overline{\bigcup_n A_n} = A \in \mathbf{A}$, $X_A = \lim_n X_{A_n}$.

Lemma 1 ([6]). *Let $A' = \{\emptyset' = A_0, \ldots, A_k\}$ be any finite subsemilattice of* $\mathbf{A}$ *equipped with a numbering consistent with the strong past. Then there exists a continuous (strictly) flow $f : [0, k] \to \mathbf{A}(\mathbf{u})$ such that the following are satisfied:*

(i) $f(0) = \varnothing'$, $f(k) = \bigcup_{j=0}^{k} A_j$.

(ii) *Each left neighborhood* C *generated by* A' *is of the form* $C = f(i) \backslash f(i-1)$, $1 \leq i \leq k$.

(iii) *If* $C = f(t) \backslash f(s)$, *then* $C \in \mathbf{C}(\mathbf{u})$ *and* $F_{f(s)} \subseteq \mathbf{G}_C^*$ *(for the definition of a continuous flow, see [6]).*

Lemma 2. *Let* $A' = \{\varnothing' = A_0, \ldots, A_k\}$ *be any finite subsemilattice of* $\mathbf{A}$ *equipped with a numbering consistent with the strong past.*

(a) *Then there exists a strictly increasing and continuous sequence*

$$B^{(k)} = \left\{B_\alpha^{(k)}\right\}_{\alpha \in [0,k]}$$

in $\mathbf{A}(\mathbf{u})$ *such that the following are satisfied:*

(i) $B_0^{(k)} = \varnothing'$, $B_k^{(k)} = \bigcup_{j=0}^{k} A_j$.

(ii) *Each left neighborhood* C *generated by* A' *is of the form* $C = B_i^{(k)} \backslash B_{i-1}^{(k)}$, $1 \leq i \leq k$.

(iii) *If* $C = B_t^{(k)} \backslash B_s^{(k)}$, *then* $C \in \mathbf{C}(\mathbf{u})$ *and* $F_{B_s^{(k)}} \subseteq \mathbf{G}_C^*$.

(b) *Then there exists a strictly increasing and continuous sequence*

$$B = \{B_\alpha\}_{\alpha \in [0,\infty)}$$

in $\mathbf{A}(\mathbf{u})$ *such that the following are satisfied:*

(i) $B_0 = \varnothing'$, $B_k = \bigcup_{j=0}^{k} A_j$.

(ii) *Each left neighborhood* C *generated by* A' *is of the form* $C = B_i \backslash B_{i-1}$, $1 \leq i \leq k$.

(iii) *If* $C = B_t \backslash B_s$, *then* $C \in \mathbf{C}(\mathbf{u})$ *and* $F_{B_s} \subseteq \mathbf{G}_C^*$.

Proof. (a) It is clear from Lemma 1 by setting $B_i^{(k)} = f(i)$, $1 \leq i \leq k$.

(b) Notice that for each k, $B^{(k)} = B^{(k+1)}$ on $[0, k]$. Then we can define the sequence $B = \{B_\alpha\}_{\alpha \in [0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ by $B_\alpha = B_\alpha^{([\alpha]+1)}$ for all α. □

Remark 1. Similarly to the construction performed in Lemma 2, we can prove that for all increasing sequences $\{B_n\}_{n=1}^{\infty} \in \mathbf{A}(\mathbf{u})$, there exists a strictly increasing and continuous sequence $\{A_\alpha\}_{\alpha \in [0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ such that $A_n = B_n$.

3 A characterization of a set-indexed Brownian motion by sequences

Definition 7. Let $\sigma \in M(\mathbf{A})$. We say that an $\mathbf{A}$-indexed process X is a Brownian motion with variance σ if X can be extended to a finitely additive process on $\mathbf{C}(\mathbf{u})$ and if for disjoint sets $C_1, \ldots, C_n \in \mathbf{C}$, $X_{C_1}, \ldots, X_{C_n}$ are independent zero-mean Gaussian random variables with variances $\sigma_{C_1}, \ldots, \sigma_{C_n}$, respectively.

For any $\sigma \in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [6].

Definition 8. (a) Let $X = \{X_t : t \geq 0\}$ be a stochastic process, and let $*$ be a group action of $(G, \cdot)$ on $\Re_+$. The process X is called a G-time-changed Brownian motion if there exists $g \in G$ such that $X^g = \{X_{g*t} : t \geq 0\}$ is a Brownian motion.

(b) Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed process, $\{A_\alpha\}_{\alpha\in[0,\infty)}$ be an increasing sequence in $\mathbf{A}(\mathbf{u})$, and $*$ be a group action of $(S_{[0,\infty)}, \circ)$ on $\Re_+$. The process X^A (see Definition 6) is called a G-time-changed Brownian motion if there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{\pi*A_\alpha} : \alpha \in [0,\infty)\} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ is a Brownian motion.

The characterization of a set-indexed Brownian motion by a group action on a sequence (Theorem 1) is very important and is the key to most of the proofs in this part of the paper. It is of great importance since it allows us to "divide and conquer." Therefore, many properties of a set-indexed Brownian motion can be recovered by reducing them to a (classical) one-dimensional Brownian motion. Theorem 1 further says that a square-integrable set-indexed stochastic process is a set-indexed Brownian motion if and only if its projections on all the strictly increasing continuous sequences by a group action are one-parameter time-changed Brownian motions.

Theorem 1 (Characterization of a set-indexed Brownian motion by a group action on sequences)**.** *Let $X = \{X_A : A \in \mathbf{A}\}$ be a square-integrable set-indexed stochastic process. Suppose that there exists a group action $*$ of $(S_{[0,\infty)}, \circ)$ on $\mathbf{A}$. Let $\sigma \in M(\mathbf{A})$. Then X is a set-indexed Brownian motion with variance σ if and only if the process $X^A = \{X_{A_\alpha} : \alpha \in [0,\infty)\}$ is an $S_{[0,\infty)}$-time-changed Brownian motion for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. In other words (by Definition 8), for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{\pi*A_\alpha} : \alpha \in [0,\infty)\} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ is a Brownian motion.*

Proof. (*if*) Suppose that X is a set-indexed Brownian motion with variance σ. Define $\theta : \Re_+ \to \Re_+$by $\theta(\alpha) = \sigma(A_\alpha)$,$\alpha \in [0,\infty)$. The function θ is strictly increasing and continuous because A is strictly increasing and continuous. Since $\sigma \in M(\mathbf{A})$, θ is invertible. Let $\pi(\alpha) = \theta^{-1}(\alpha)$; π is continuous, and $\sigma(A_{\pi(\alpha)}) = \alpha$. Then $\pi \in S_{[0,\infty)}$, and $X^{\pi,A} = \{X_{\pi*A_\alpha} : \alpha \in [0,\infty)\} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ is a Brownian motion.

(*only if*) Suppose that for all strictly continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A}$ is a Brownian motion. It must be shown that if $\{C_1, \ldots, C_k\} \in \mathbf{C}$ are disjoint, then $X_{C_1}, \ldots, X_{C_k}$ are independent normal random with variances $\sigma(C_1), \ldots, \sigma(C_k)$, respectively. Without loss of generality, we may assume that the sets $\{C_1, \ldots, C_k\}$ are the left neighborhoods of the subsemilattice A' of $\mathbf{A}$ equipped with a numbering consistent with the strong past. By Lemma 2 there exists a strictly increasing and continuous sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ such that each left neighborhood generated by A' is of the form $C_i = A_i \backslash A_{i-1}$, $1 \leq i \leq k$. Thus, X^A is an $S_{[0,\infty]}$-time-changed Brownian motion such that $X_{C_i} = X_{A_i} - X_{A_{i-1}}$ and $\sigma(C_i) = \sigma(A_i) - \sigma(A_{i-1})$; therefore, $X_{C_1}, \ldots, X_{C_k}$ are independent normal random with variances $\sigma(C_1), \ldots, \sigma(C_k)$, respectively. □

Corollary 1. *Let $X = \{X_A : A \in \mathbf{A}\}$ be a square-integrable set-indexed stochastic process with $X_{\emptyset'} = 0$ that is inner- and outer-continuous. Let $\sigma \in M(\mathbf{A})$. Then X is a set-indexed Brownian motion with variance σ if and only if for all strictly continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, the process X^A has independent increments and* *there exists* $\pi \in S_{[0,\infty)}$ *such that* $X^{\pi,A} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ *has stationary increments. (The definition and more details about independent increments and stationary increments can be found in [6].)*

Proof. (*if*) Obvious.

(*only if*) Suppose that for all strictly continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, the process X^A has independent increments and there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ has stationary increments. Since X is inner- and outer-continuous, X^A is continuous (see [6]). The process X^A has independent increments, and there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A}$ has stationary increments; therefore, X^A is an $S_{[0,\infty)}$-time-changed Brownian motion for all strictly continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. Thus, from Theorem 1 we conclude that X is a set-indexed Brownian motion with variance σ. □

Definition 9 ([6])**.** Let $X = \{X_A : A \in \mathbf{A}\}$ be an integrable additive set-indexed stochastic process adapted with respect to filtration $F = \{F_A : A \in \mathbf{A}\}$. The process X is said to be:

1. A $\mathbf{C}$-strong martingale (or in short notation, a strong martingale) if for all $C \in \mathbf{C}$, we have $E[X_C|\mathbf{G}^*_C] = 0$;

2. A martingale if for any $A, B \in \mathbf{A}$ such that $A \subseteq B$, we have $E[X_B|F_A] = X_A$.

For studies of different kinds of martingales, see [7, 8, 11].

In particular, if $T = \Re^2_+$ and $\mathbf{A} = \mathbf{A}(\Re^2_+)$ then X is said to be a strong martingale-$\Re^2_+$ if X is adapted, vanishes on the axes, and $E[X((z,z'])|F^1_z \vee F^2_z] = 0$ for all $z \le z'$, where $[z,z'] = [s,s'] \times [t,t']$, $F^1_z = \bigvee_v F_{sv}$, $F^2_z = \bigvee_u F_{ut}$, $z = (s,t)$, $z' = (s',t')$. (This definition and additional explanation can be found in [2]).

Remark. Under some hypotheses, we can define $\langle X \rangle$ to be the compensator associated with the submartingale X^2. The definition and more details regarding $\langle X \rangle$ can be found in [6, 2].

From the well-known Lévy martingale characterization of the Brownian motion (see [3] or [10]) we get the following corollary.

Corollary 2. *Let* $X = \{X_A : A \in \mathbf{A}\}$ *be a square-integrable set-indexed martingale with* $X_{\emptyset'} = 0$ *that is inner- and outer-continuous. Let* $\sigma \in M(\mathbf{A})$*. Then* X *is a set-indexed Brownian motion with variance* σ *if and only if* $\langle X^A \rangle$ *is deterministic for all strictly increasing and continuous sequence* $\{A_\alpha\}_{\alpha\in[0,\infty)}$ *in* $\mathbf{A}(\mathbf{u})$.

Proof. (*if*) Suppose that X is a set-indexed Brownian motion with variance σ. By Theorem 1 the process X^A is an $S_{[0,\infty)}$-time-changed Brownian motion for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. Then from the Lévy characterization we conclude that $\langle X^A \rangle$ is deterministic for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$.

(*only if*) Suppose that $\langle X^A \rangle$ is deterministic for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. Since X is inner- and outer-continuous, X^A is continuous (see [6]). Since X is a set-indexed martingale, $X^A$is a martingale. But if the process X^A is a martingale and $\langle X^A \rangle$ is deterministic, then based on the Lévy characterization we get that X^A is an $S_{[0,\infty)}$-time-changed Brownian motion for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. Thus, from Theorem 1 we conclude that X is a set-indexed Brownian motion with variance σ. $\square$

In addition, Theorem 1 is an important "bridge" from a set-indexed Brownian motion to a Brownian motion, and from that we extend many theorems, such as hitting time (Corollary 3), reflection principle (Corollary 4), exiting from an interval (Corollary 5), unboundedness (Corollary 6), strong law of large numbers (Corollary 7), law of iterated logarithm (Corollary 8), the zero set (Corollary 9), and the like.

Let L be a decreasing continuous line in $\Re^2_+$. If $A \in \mathbf{A}(\Re^2_+)$, then we

(a) write $A \prec L$ ($A \succ L$) if there exist $(x, y) \in L$ such that $A \subset [0, x] \times [0, y]$ ($A \supset [0, x] \times [0, y]$);

(b) write $A \preceq L$ ($A \succeq L$) if there exist $(x, y) \in L$ such that $A \subseteq [0, x] \times [0, y]$ ($A \supseteq [0, x] \times [0, y]$).

(c) write $A \in L$ if there exist $(x, y) \in L$ such that $A = [0, x] \times [0, y]$.

Let $X = \{X_A : A \in \mathbf{A}(\Re^2_+)\}$ be a set-indexed Brownian motion. For $a > 0$, we define L_a to be a decreasing continuous line in $\Re^2_+$ such that

(a) if $A \prec L_a$, then $X_A < a$, and

(b) if $A \in L_a$, then $X_A \geq a$ for the first time on A.

(In other words, L_a is the collection of points (x, y) when X reaches the value a for the first time.)

Corollary 3 (Hitting time)**.** *Let $X = \{X_A : A \in \mathbf{A}(\Re^2_+)\}$ be a set-indexed Brownian motion with variance σ (Lebesgue measure). Then*

$$P[L_a \preceq A] = 2 - 2\Phi\left(\frac{a}{\sqrt{\sigma_A}}\right) \quad \textit{for all } A \in \mathbf{A}\left(\Re^2_+\right)$$

(Φ is the standard Gaussian distribution function).

Proof. Let $A \in \mathbf{A}(\Re^2_+)$. Then $P[X_A \geq a] = P[X_A \geq a|A \prec L_a]P[A \prec L_a] + P[X_A \geq a|A \succeq L_a]P[A \succeq L_a]$. From the definition of L_a we conclude that if $A \prec L_a$, then $X_A < a$, and thus $P[X_A \geq a|A \prec L_a] = 0$. It is clear that there exist a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ and $\alpha_0 \geq 0$ such that $B_{\alpha_0} = A$. The sequence $B = \{B_\alpha\}_{\alpha\in[0,\infty)}$ is strictly continuous; therefore, from Theorem 1 we conclude that X^B is a G-time-changed Brownian motion. (In other words, there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,B}$ is a Brownian motion.) By the symmetry of $X^{\pi,B}$ it is clear that $P[X_A \geq a|A \succeq L_a] = P[X^{\pi,B}_{\alpha_0} \geq a|A \succeq L_a] = P[X_{B_{\pi(\alpha_0)}} \geq a|A \succeq L_a] = \frac{1}{2}$, and thus $P[L_a \preceq A] = 2P[X_A \geq a] = 2 - 2\Phi(\frac{a}{\sqrt{\sigma_A}})$. $\square$

Corollary 4 (Reflection principle)**.** *Let* $X = \{X_A : A \in \mathbf{A}(\Re^2_+)\}$ *be a set-indexed Brownian motion with variance* σ *(Lebesgue measure). Then, for* $A \in \mathbf{A}$*,*

$$W_A = \left\{ \begin{array}{cc} X_A, & A \prec L_a \\ 2a - X_A, & A \succeq L_a \end{array} \right\}$$

is a set-indexed Brownian motion with variance σ*.*

Proof. We must show that if $\{C_1, \ldots, C_k\}$ are disjoint, then $W_{C_1}, \ldots, W_{C_k}$ are independent normal random variables with variances $\sigma_{C_1}, \ldots, \sigma_{C_k}$, respectively. Similarly to the construction done in Theorem 1, we can get that there exists a strictly increasing and continuous sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ such that $C_i = A_i \backslash A_{i-1}$. The sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ is strictly continuous; therefore, from Theorem 1 we conclude that there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A}$ is a Brownian motion. Clearly, there exists $\alpha_a \geq 0$ such that $A_{\pi(\alpha_a)} \in L_a$. We recall that if $X = \{X_t : t \geq 0\}$ is be a classical Brownian motion and $T_a = \inf\{t \geq 0 : X_t = a\}$, then

$$Z_t = \left\{ \begin{array}{cc} X_t, & t < T_a \\ 2a - X_t, & t \geq T_a \end{array} \right\} \quad (t \geq 0)$$

is a Brownian motion [4]. Thus, if we define

$$W_\alpha = \left\{ \begin{array}{cc} X^{\pi,A}_\alpha, & \alpha < \alpha_a \\ 2a - X^{\pi,A}_\alpha, & \alpha \geq \alpha_a \end{array} \right\},$$

then W_α turns out to be a Brownian motion, and so $W_{C_1}, \ldots, W_{C_k}$ are independent normal random variables with variances $\sigma_{C_1}, \ldots, \sigma_{C_k}$, respectively. □

Let $X = \{X_A : A \in \mathbf{A}(\Re^2_+)\}$ be a set-indexed Brownian motion. For $a < 0 < b$, define $D_{(a,b)} = \{A \in \mathbf{A} : X_A \notin (a, b)$ for the first time$\}$. (In other words, $D_{(a,b)}$ is the collection of sets $A \in \mathbf{A}$ such that if $A \in D_{(a,b)}$, then $X_A \notin (a, b)$ for the first time on A).

Corollary 5 (Reflection principle)**.** *Let* $X = \{X_A : A \in \mathbf{A}(\Re^2_+)\}$ *be a set-indexed Brownian motion with variance* σ *(Lebesgue measure). If* $T(a, b) = \bigcap_{A\in D(a,b)} A$*, then* $P[X_{T(a,b)} = b] = \frac{|a|}{b+|a|}$*.*

Proof. It is clear that $T(a, b) \in \mathbf{A}(\Re^2_+)$. It is easy to see that there exists a strictly increasing and continuous sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ and there exists $0 \leq \beta$ such that $A_\beta = T(a, b)$. The sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ is strictly increasing and continuous; therefore, from Theorem 1 we conclude that X^A is an $S_{[0,\infty)}$-time-changed Brownian motion. (In other words, there exists a $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0, \infty)\}$ is a Brownian motion, and there exists $0 \leq \alpha_{(a,b)}$ such that $A_\beta = A_{\pi(\alpha_{(a,b)})} = T(a, b)$). We recall that if $X = \{X_t : t \geq 0\}$ is a Brownian motion and $t(a, b) := \inf_{t\geq 0}\{X_t \notin (a, b)\}$ for $a < 0 < b$, then $P[X_{t(a,b)} = b] = \frac{|a|}{b+|a|}$. Thus, $P[X_{T(a,b)} = b] = P[X^{\pi,A}_{\alpha_{(a,b)}} = b] = \frac{|a|}{b+|a|}$. □

Corollary 6 (Unboundedness)**.** *Let* $\sigma \in M(\mathbf{A})$*, and let* $X = \{X_A : A \in \mathbf{A}\}$ *be a set-indexed Brownian motion with variance* σ*. Then* $\sup_{A\in\mathbf{A}} X_A(\omega) = +\infty$ *and* $\inf_{A\in\mathbf{A}} X_A(\omega) = -\infty$ *for almost all* ω*.*

Proof. Clearly, we have $\sup_{\alpha\geq 0} X_{A_\alpha}(\omega) \leq \sup_{A\in\mathbf{A}} X_A(\omega)$ ($\inf_{\alpha\geq 0} X_{A_\alpha}(\omega) \geq \inf_{A\in\mathbf{A}} X_A(\omega)$) for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. By Theorem 1 the process X^A is an $S_{[0,\infty)}$-time-changed Brownian motion for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$; therefore, $+\infty = \sup_{\alpha\geq 0} X_{A_{\pi(\alpha)}}(\omega) \leq \sup_{A\in\mathbf{A}} X_A(\omega)$ for almost all ω ($-\infty \inf_{\alpha\geq 0} X_{A_{\pi(\alpha)}}(\omega) \geq \inf_{A\in\mathbf{A}} X_A(\omega)$ for almost all ω). □

Let $B \in \mathbf{A}(\mathbf{u})$.

1. Let $A = \{A_n\}$ be an increasing sequence in $\mathbf{A}$. We write $A_n \uparrow B$ (or, in short notation, $A \uparrow B$) if $A_n \neq B$ for all n and $\overline{\bigcup_n A_n} = B$.
2. Let $A = \{A_n\}$ be a decreasing sequence in $\mathbf{A}$. We write $A_n \downarrow B$ (or, in short notation, $A \downarrow T$) if $A_n \neq B$ for all n and $\bigcap_n A_n = B$.

Corollary 7 (Strong law of large numbers and unboundedness)**.** *Let $\sigma \in M(\mathbf{A})$, and let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed Brownian motion with variance σ. Then*

(a) $\underline{\lim}_{A\uparrow T} \frac{X_A}{\sigma_A} = 0$ *for almost all ω, and for all sequences $A_n \uparrow T$, $\{A_n\} \in \mathbf{A}$.*

(b) $P[\underline{\lim}_{A\uparrow T} X_A = -\infty] = P[\overline{\lim}_{A\uparrow T} X_A = \infty] = 1.$

Proof. Let $A_n \uparrow T$. By Theorem 1 and Remark 1 there exists a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ such that X^B is an $S_{[0,\infty)}$-time-changed Brownian motion when $A_n = B_n$. (In other words, there exist a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ and $\{\alpha_n\} \in [0,\infty)$ such that $X^{B,\pi}$ is a Brownian motion when $A_n = B_n = B_{\pi(\alpha_n)}$ and $\pi \in S_{[0,\infty)}$). Then

(a) $\underline{\lim}_{A\uparrow T} \frac{X_A}{\sigma_A} = \underline{\lim}_{n\to\infty} \frac{X_n^{B,\pi}}{\sigma(B_{\pi(n)})} = \underline{\lim}_{\alpha_n\to\infty} \frac{X^B_{\alpha_n}}{\alpha_n} \overset{*}{=} 0$ for almost all ω (for the equality $\overset{*}{=}$, see [1]).

(b) $P[\underline{\lim}_{A\uparrow T} X_A = -\infty] = P[\underline{\lim}_{\alpha_n\to\infty} X^B_{\alpha_n} = -\infty] \overset{**}{=} 1$ and $P[\overline{\lim}_{A\uparrow T} X_A = \infty] = P[\overline{\lim}_{\alpha_n\to\infty} X^B_{\alpha_n} = \infty] \overset{**}{=} 1$ (for the equality $\overset{**}{=}$, see [4]). □

Corollary 8 (Law of iterated logarithm)**.** *Let $\sigma \in M(\mathbf{A})$, and let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed Brownian motion with variance σ. Then*

$$\underline{\lim}_{A\uparrow T} \frac{X_A}{\sqrt{2\sigma_A \ln\ln(\sigma_A)}} = -1$$

and

$$\overline{\lim}_{A\uparrow T} \frac{X_A}{\sqrt{2\sigma_A \ln\ln(\sigma_A)}} = 1$$

for almost all ω and for all $A \uparrow T$.

Proof. Let $A_n \uparrow T$. By Theorem 1 and Remark 1 there exists a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ such that X^B is an $S_{[0,\infty)}$-time-changed Brownian motion when $A_n = B_n$. (In other words, there exist a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ and $\{\alpha_n\} \in [0,\infty)$ such that $X^{B,\pi}$ is a Brownian motion when $A_n = B_n = B_{\pi(\alpha_n)}$ and $\pi \in S_{[0,\infty)}$). Then

$$\underline{\lim}_{A\uparrow T} \frac{X_A}{\sqrt{2\sigma_A \ln\ln(\sigma_A)}} = \underline{\lim}_{\alpha_n\to\infty} \frac{X^{B,\pi}_{\alpha_n}}{\sqrt{2\alpha_n \ln\ln(\alpha_n)}} \stackrel{*}{=} -1$$

and

$$\overline{\lim}_{A\uparrow T} \frac{X_A}{\sqrt{2\sigma_A \ln\ln(\sigma_A)}} = \overline{\lim}_{\alpha_n\to\infty} \frac{X^{B,\pi}_{\alpha_n}}{\sqrt{2\alpha_n \ln\ln(\alpha_n)}} \stackrel{*}{=} 1$$

for almost all ω (for the equality $\stackrel{*}{=}$, see [1]). □

Corollary 9 (The zero set)**.** *Let $\sigma \in M(\mathbf{A})$, and let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed Brownian motion with variance σ. Let $\omega \in \Omega$ and set $Z_\omega = \{A \in \mathbf{A} : X_A(\omega) = 0\}$; then the set Z_ω is uncountable and without monotone accumulation sets.*

(The set $A \in \mathbf{A}$ is said to be a monotone accumulation set if there exists an increasing (decreasing) sequence $\{A_n\} \in Z_\omega$ such that $A_n \neq A$ and $A_n \uparrow A$ ($A_n \downarrow A$).)

Proof. From Theorem 1 we conclude that the process X^A is an $S_{[0,\infty)}$-time-changed Brownian motion for all strict increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ (in other words, for all strict increasing and continuous sequences $\{A_\alpha\}$ in $\mathbf{A}(\mathbf{u})$, there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{\pi * A_\alpha} : \alpha \in [0,\infty)\} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ is a Brownian motion). Then (see [1]) the set $Z^A_\omega = \{\alpha \geq 0 : X^{A,\pi}_\alpha(\omega) = 0\}$ is uncountable and without monotone accumulation sets. Thus, Z_ω is uncountable. The set Z_ω is without monotone accumulation sets; if not, then there exists an increasing sequence $\{A_n\} \in Z_\omega$ such that $A_n \neq A$ and $A_n \uparrow A$, so that based on Theorem 1, it is easy to see that there exists a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ such that X^B is an $S_{[0,\infty)}$-time-changed Brownian motion when $A_n = B_n$. (In other words, there exist a strictly increasing and continuous sequence $\{B_\alpha\}_{\alpha\in[0,\infty)} \in \mathbf{A}(\mathbf{u})$ and $\{\alpha_n\} \in [0,\infty)$ such that $X^{B,\pi}$ is a Brownian motion when $A_n = B_n = B_{\pi(\alpha_n)}$ and $\pi \in S_{[0,\infty)}$). Then the set $Z^{B,\pi}_\omega = \{\alpha \geq 0 : X^{B,\pi}_{\alpha_n}(\omega) = 0\}$ has a monotone accumulation set, which is a contradiction (see [1]). (In the same way, we can proceed in the case where $A_n \downarrow A$). □

4 Sequence-independent variation

Definition 10. Let σ be a positive and continuous measure in $\mathbf{A}$ (Radon measure). For $A \in \mathbf{A}$ and $\varepsilon > 0$, we define $D^\varepsilon_A = \{B \in \mathbf{A} : A \subseteq B, \sigma(B\backslash A) = \varepsilon\}$. Assume that $D^\varepsilon_A \neq \emptyset$ and let A^ε be an element in D^ε_A.

Hereafter, we assume that the space T has a positive and continuous measure σ in $\mathbf{A}$ such that for all $A \in \mathbf{A}$, there exists A^ε such that $\sigma(A^\varepsilon\backslash A) = \varepsilon$ for all $\varepsilon > 0$.

The classical example for definition is $T = \Re^2_+$ and $\mathbf{A} = \mathbf{A}(\Re^2_+)$ when σ is a Lebesgue or Radon measure.

Definition 11. Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed stochastic process.

(a) X is said to have σ-stationary increments if

$$X_{A_1^\varepsilon} - X_{A_1} \stackrel{d}{=} \cdots \stackrel{d}{=} X_{A_n^\varepsilon} - X_{A_n}$$

for all $\{A_i\}_{i=1}^n \in \mathbf{A}$, all $\varepsilon > 0$, and all $A_i^\varepsilon \in D_{A_i}^\varepsilon$ (the notation $\stackrel{d}{=}$ means the equality in distribution).

(b) X is said to have independent increments if $X_{C_1}, \ldots, X_{C_n}$ are independent random variables whenever $C_1, \ldots, C_n$ are disjoint sets in $\mathbf{C}$.

Let $X = \{X_t : t \geq 0\}$ be a square-integrable martingale. It is known that we can associate with X a unique predictable process, denoted $\langle X \rangle$, such that $X^2 - \langle X \rangle$ is a martingale. Little is known in the set-indexed case. However, the concept of increasing path allows us to study such processes.

Definition 12. A square-integrable set-indexed martingale $X = \{X_A : A \in \mathbf{A}\}$ is said to have sequence-independent variation (s.i.v.) on $\mathbf{A}(\mathbf{u})$ (or path-independent variation (p.i.v.)) if for any strict increasing continuous sequences $\{A_\alpha\}_{\alpha \in [a,b]}$ and $\{B_\beta\}_{\beta \in [a,b]}$ in $\mathbf{A}(\mathbf{u})$ with $A_a = B_a$ and $A_b = B_b$, we have $\langle X^A \rangle(b) = \langle X^B \rangle(b)$.

Remarks. (a) This definition of s.i.v. on $\mathbf{A}(\mathbf{u})$ was introduced by Cairoli and Walsh in the plane [2]. Here we extend it to the set-indexed framework.

(b) The definition and more details about $\langle X \rangle$, can be found in [6].

Theorem 2. *Let $X = \{X_A : A \in \mathbf{A}(\Re_+^2)\}$ be a set-indexed strong martingale-$\Re_+^2$. If $\sup_{A \in \mathbf{A}} E[X_A^4] < \infty$ or the filtration F is generated by a Brownian motion, then X has s.i.v. on $\mathbf{A}(\mathbf{u})$.*

Proof. It suffices to prove that for all increasing continuous sequences $\{A_\alpha\}_{\alpha \in [a,b]}$ in $\mathbf{A}(\mathbf{u})$ and all $a \leq s \leq t \leq b$, $E[X_{A_t}^2 - X_{A_s}^2 | F_{A_s}] = E[\langle X \rangle_{A_t} - \langle X \rangle_{A_s} | F_{A_s}]$. If $\{A_\alpha\}_{\alpha \in [a,b]}$ is an increasing continuous sequence, then $A_s \subseteq A_t$. The set $A_t \backslash A_s$ can be divided to n disjoint rectangles $\{R_i\}_{i=1}^n$ such that $A_t \backslash A_s = \bigcup_{i=1}^n R_i$. If X is a strong martingale-$\Re_+^2$, then $E[X_{A_t}^2 - X_{A_s}^2 | F_{A_s}] = E[(X_{A_t} - X_{A_s})^2 | F_{A_s}]$ and, by the division, $E[(X_{A_t} - X_{A_s})^2 | F_{A_s}] = E[(X_{R_1} + \cdots + X_{R_n})^2 | F_{A_s}]$. It is clear that $F_{A_s} \subseteq \mathbf{G}_{R_i}^*$ and for all $i \neq j$, $E[X_{R_i} X_{R_j} | F_{A_s}] = E[E[X_{R_i} X_{R_j} | \mathbf{G}_{R_i}^*] | F_{A_s}] = E[X_{R_i} E[X_{R_j} | \mathbf{G}_{R_i}^*] | F_{A_s}] = 0$; therefore, $E[X_{A_t}^2 - X_{A_s}^2 | F_{A_s}] = E[X_{R_1}^2 + \cdots + X_{R_n}^2 | F_{A_s}] \stackrel{\#}{=} E[\langle X \rangle_{R_1} + \cdots + \langle X \rangle_{R_n} | F_{A_s}] = E[\langle X \rangle_{A_t} - \langle X \rangle_{A_s} | F_{A_s}]$ (for the equality $\stackrel{\#}{=}$, see [2]). □

Theorem 3. *Let $\sigma \in M(\mathbf{A})$. If $X = \{X_A : A \in \mathbf{A}\}$ is a set-indexed square-integrable outer-continuous process with independent and σ-stationary increments, then X has s.i.v.*

For the proof, we need two auxiliary propositions.

Proposition 1. *If $\{A_\alpha\}_{\alpha \in [0,\infty)}$ is a strictly increasing continuous sequence in $\mathbf{A}(\mathbf{u})$, then X^A is a G-time-changed right-continuous martingale with independent and stationary increments. (In other words, for all strictly increasing continuous sequences*

$\{A_\alpha\}_{\alpha\in[0,\infty)}$ *in* $\mathbf{A}(\mathbf{u})$*, there exists* $\pi \in G$ *such that* $X^{\pi,A}$ *is a right-continuous martingale with independent and stationary increments).*

Proof. The process X is outer-continuous; therefore, X^A is right-continuous. Since X has independent increments, it is a strong martingale. In particular, X is a martingale [6]. It is easy to see that X^A is a martingale for all strict continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. Moreover, X has σ-stationary increments; therefore, X^A is a G-time-changed right-continuous martingale with independent and stationary increments for all strict continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$. □

Now, for any increasing continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ and $\{B_\beta\}_{\beta\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, $X^{\pi_1,A}$ and $X^{\pi_2,B}$ are right-continuous martingales with independent and stationary increments. We recall that if $X = \{X_t : t \geq 0\}$ is a right-continuous martingale with independent and stationary increments such that $E[X_t^2] < \infty$ for all t, then $\langle X\rangle_t = tE[X_1^2]$ for all t. Thus, $\langle X^{\pi_1,A}\rangle$, $\langle X^{\pi_2,B}\rangle$ are deterministic; in particular, $\langle X^{\pi_1,A}\rangle(c) = \langle X^{\pi_2,B}\rangle(c)$ when $\pi_1 * A_c = \pi_2 * B_c$, and for all $0 \leq t$, $\langle X^{\pi_1,A}\rangle(c) = \sigma(A_{\pi_1(t)})$.

Proposition 2. *If* $\sigma(A_c) = k$*, then there exists a unique* $s \in \Re_+$ *such that* $A_{\pi_1(s)} = A_c$ *and* $s = k$ *for all a strict continuous sequences* $\{A_\alpha\}_{\alpha\in[0,\infty)}$ *in* $\mathbf{A}(\mathbf{u})$.

Proof. It is clear that $\sigma(A_{\pi_1(k)}) = k$ from the definition of π_1 ($\pi_1(\alpha) = \theta^{-1}(\alpha)$ when $\theta : \Re_+ \to \Re_+, \theta(\alpha) = \sigma(A_\alpha)$); therefore, $A_{\pi_1(k)} = A_c$ or there exists $r \neq k$ when $A_{\pi_1(r)} = A_c$ and $A_{\pi_1(r)} \neq A_{\pi_1(k)}$ because of strictly increasing continuous sequences. Without loss of generality, we may assume that $A_{\pi_1(r)} \subset A_{\pi_1(k)}$ when $\sigma(A_{\pi_1(r)}) = \sigma(A_{\pi_1(k)})$, which is a contradiction to $\sigma \in M(\mathbf{A})$. □

Proof of Theorem 3. Let $\{A_\alpha\}_{\alpha\in[0,\infty)}$ and $\{B_\beta\}_{\beta\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ be two strictly increasing continuous sequences. If $A_0 = B_0$ and $A_c = B_c$, then $\sigma(A_c) = \sigma(B_c) = k$. From Proposition 2 we get that if $A_c = A_{\pi_1(k)}, B_c = B_{\pi_2(k)}$, then $\langle X^A\rangle(c) = \langle X^{\pi_1,A}\rangle(k) = \langle X^{\pi_2,B}\rangle(k) = \langle X^B\rangle(c)$. □

Theorem 4. *Let* $\sigma \in M(\mathbf{A})$*, and let* $X = \{X_A : A \in \mathbf{A}\}$ *be a set-indexed Brownian motion with variance* σ*. Then* X *has s.i.v, and* $\langle X\rangle_A = \sigma_A$ *for all* $A \in \mathbf{A}$.

Proof. Since X is a set-indexed Brownian motion, by Theorem 1 X^A is a $G_{[0,\infty)}$-time-changed Brownian motion for all strictly continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ (in other words, for all strictly increasing and continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, there exists $\pi \in S_{[0,\infty)}$ such that $X^{\pi,A} = \{X_{\pi * A_\alpha} : \alpha \in [0,\infty)\} = \{X_{A_{\pi(\alpha)}} : \alpha \in [0,\infty)\}$ is a Brownian motion). For any increasing continuous sequences $\{A_\alpha\}_{\alpha\in[0,\infty)}$ and $\{B_\beta\}_{\beta\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$, $X^{\pi_1,A}$ and $X^{\pi_2,B}$ are Brownian motions; therefore, $\langle X^{\pi_1,A}\rangle$ and $\langle X^{\pi_2,B}\rangle$ are deterministic; in particular, $\langle X^{\pi_1,A}\rangle(c) = \langle X^{\pi_2,B}\rangle(c)$ when $A_{\pi_1(c)} = B_{\pi_2(c)}$, and for all $0 \leq t$, $\langle X^{\pi_1,A}\rangle(c) = \sigma(A_{\pi_1(t)})$.

Let $\{A_\alpha\}_{\alpha\in[0,\infty)}$ and $\{B_\beta\}_{\beta\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ be two increasing continuous sequences with $A_0 = B_0$ and $A_c = B_c$ then $\sigma(A_{\pi_1(c)}) = \sigma(B_{\pi_2(c)}) = k$

Returning to the proof of Theorem 4, from Proposition 2 in Theorem 3 we get that if $A_c = A_{\pi_1(k)}$ and $B_c = B_{\pi_2(k)}$, then $\langle X^A\rangle(c) = \langle X^{\pi_1,A}\rangle(k) = \langle X^{\pi_2,B}\rangle(k) = \langle X^B\rangle(c)$ and, in particular, $\langle X^A\rangle(t) = \sigma(A_t)$ for all $0 \leq t$.

Now, if $D \in \mathbf{A}$, then there exist a continuous sequence $\{A_\alpha\}_{\alpha\in[0,\infty)}$ in $\mathbf{A}(\mathbf{u})$ and $d \in \Re_+$ such that $A_d = D$. Then, by Theorem 1, $X^{\pi,A}$ is a $G_{[0,\infty)}$-time-changed Brownian motion, and $\langle X\rangle_D = \langle X^{\pi,A}\rangle_d = \langle X^\pi\rangle_{A_d} = \sigma_{A_d} = \sigma_D$. □

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