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Skorokhod M_1 convergence of maxima of multivariate linear processes with heavy-tailed innovations and random coefficients

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Abstract In this paper, functional convergence is derived for the partial maxima stochastic processes of multivariate linear processes with weakly dependent heavy-tailed innovations and random coefficients. The convergence takes place in the space of $\mathbb{R}^d$-valued càdlàg functions on [0, 1] endowed with the weak Skorokhod M_1 topology.

Keywords Functional limit theorem, multivariate linear process, regular variation, extremal process, M_1 topology
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1 Introduction

Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables, and denote by $M_n = \max\{X_1, X_2, \dots, X_n\}$, $n \geq 1$, its partial maxima. The asymptotic distributional behavior of M_n is one of the main objects of interest of classical extreme value theory. When (X_i) is an i.i.d. sequence and there exist constants $a_n > 0$ and b_n such that

$$\mathrm{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \to G(x) \quad \text{as } n \to \infty, \tag{1}$$

with nondegenerated limit G, the limit belongs to the class of extreme value distributions, see [13]. It is known that generalizations of this result to weak convergence of partial maxima processes in the space of càdlàg functions hold. More precisely, relation (1) implies

$$a_n^{-1}(M_n(\,\cdot\,) - b_n) := a_n^{-1}\bigg(\bigvee_{i=1}^{\lfloor n\,\cdot\rfloor} X_i - b_n\bigg) \xrightarrow{d} Y(\,\cdot\,) \tag{2}$$

in the space $D([0,1],\mathbb{R})$ of real-valued càdlàg functions on $[0,1]$ endowed with the Skorokhod J_1 topology, with Y being an extremal process generated by G (see [11], and Proposition 4.20 in [13]). Simplifying notation, we sometimes omit brackets and write $a_n^{-1}(M_n - b_n) \xrightarrow{d} Y$. The convergence in relation (2) also holds for a special class of weakly dependent random variables, the linear or moving averages processes with i.i.d. heavy-tailed innovations and deterministic coefficients (see Proposition 4.28 in [13]).

Recently, it was shown in [9] that the functional convergence in (2) holds for linear processes with i.i.d. heavy-tailed innovations and random coefficients. In this paper we aim to generalize this result in two directions, the first one by studying linear processes with weakly dependent innovations (and random coefficients), and the second one by extending this theory to the multivariate setting. Due to possible clustering of large values, the J_1 topology becomes inappropriate, and therefore we will use the weaker Skorokhod M_1 topology. This topology works well if all extremes within each cluster of large values have the same sign.

The paper is organized as follows. In Section 2 we introduce basic notions about regular variation, linear processes, point processes and Skorokhod topologies. In Section 3 we derive the weak M_1 convergence of the partial maxima stochastic process for finite order multivariate linear processes with weakly dependent heavy-tailed innovations and random coefficients. In Section 4 we extend this result to infinite order multivariate linear processes, and give an example which shows that the convergence in the weak M_1 topology in general cannot be replaced by the standard M_1 convergence.

2 Preliminaries

We use superscripts in parentheses to designate vector components and coordinate functions, i.e. $a = (a^{(1)}, \ldots, a^{(d)}) \in \mathbb{R}^d$ and $x = (x^{(1)}, \ldots, x^{(d)}) \colon [0,1] \to \mathbb{R}^d$. For two vectors $a = (a^{(1)}, \ldots, a^{(d)})$, $b = (b^{(1)}, \ldots, b^{(d)}) \in \mathbb{R}^d$, $a \le b$ means $a^{(k)} \le b^{(k)}$ for all $k = 1, \ldots, d$. The vector $(a^{(1)}, \ldots, a^{(d)}, b^{(1)}, \ldots, b^{(d)})$ will be denoted by (a, b), and the vector $(a^{(1)}, b^{(1)}, a^{(2)}, b^{(2)}, \ldots, a^{(d)}, b^{(d)})$ will be denoted by $(a^{(i)}, b^{(i)})^*_{i=1,\ldots,d}$. Denote by $a \vee b$ the vector $(a^{(1)} \vee b^{(1)}, \ldots, a^{(d)} \vee b^{(d)})$, where for $c, d \in \mathbb{R}$ we put $c \vee d = \max\{c, d\}$. Sometimes for convenience we will denote the vector a by $(a^{(i)})_{i=1,\ldots,d}$. For a real number c we write $ca = (ca^{(1)}, \ldots, ca^{(d)})$.

2.1 Regular variation

The $\mathbb{R}^d$-valued random vector ξ is (multivariate) regularly varying if there exist $\alpha > 0$ and a random vector Θ on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ in $\mathbb{R}^d$, such that for every $u > 0$,

$$\frac{\mathrm{P}(\|\xi\| > ux,\ \xi/\|\xi\| \in \cdot\,)}{\mathrm{P}(\|\xi\| > x)} \xrightarrow{w} u^{-\alpha}\,\mathrm{P}(\Theta \in \cdot\,) \quad \text{as } x \to \infty, \tag{3}$$

where the arrow "$\xrightarrow{w}$" denotes the weak convergence of finite measures and $\|\cdot\|$ denotes the max-norm on $\mathbb{R}^d$. This definition does not depend on the choice of the norm, since if (3) holds for some norm on $\mathbb{R}^d$, it holds for all norms (of course, with different distributions of Θ). The number α is called the index of regular variation of ξ, and the probability measure $\mathrm{P}(\Theta \in \cdot\,)$ is called the spectral measure of ξ with respect to the norm $\|\cdot\|$. In the one-dimensional case regular variation is characterized by $\mathrm{P}(|\xi| > x) = x^{-\alpha}L(x)$, $x > 0$, for some slowly varying function L and the tail balance condition

$$\lim_{x\to\infty} \frac{\mathrm{P}(\xi > x)}{\mathrm{P}(|\xi| > x)} = p, \qquad \lim_{x\to\infty} \frac{\mathrm{P}(\xi < -x)}{\mathrm{P}(|\xi| > x)} = q,$$

where $p \in [0, 1]$ and $p + q = 1$.

A strictly stationary $\mathbb{R}^d$-valued random process $(\xi_n)_{n\in\mathbb{Z}}$ is regularly varying with index $\alpha > 0$ if for any nonnegative integer k the kd-dimensional random vector $\xi = (\xi_1, \ldots, \xi_k)$ is multivariate regularly varying with index α. According to [2] the regular variation property of the sequence (ξ_n) is equivalent to the existence of a process $(Y_n)_{n\in\mathbb{Z}}$ which satisfies $\mathrm{P}(\|Y_0\| > y) = y^{-\alpha}$ for $y \geq 1$, and

$$\big((x^{-1}\,\xi_n)_{n\in\mathbb{Z}}\,\big|\,\|\xi_0\| > x\big) \xrightarrow{\text{fidi}} (Y_n)_{n\in\mathbb{Z}} \quad \text{as } x \to \infty, \tag{4}$$

where "$\xrightarrow{\text{fidi}}$" denotes convergence of finite-dimensional distributions. The process (Y_n) is called the tail process of (ξ_n).

2.2 Linear processes

Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random vectors in $\mathbb{R}^d$, and assume Z_1 is multivariate regularly varying with index $\alpha > 0$. We study multivariate linear processes with random coefficients, defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z}, \tag{5}$$

where $(C_j)_{j\geq 0}$ is a sequence of $d \times d$ matrices (with real-valued random variables as entries) independent of (Z_i) such that the above series is a.s. convergent. One sufficient condition for that is $\sum_{j=0}^{\infty} \mathrm{E}\|C_j\|^{\delta} < \infty$ for some $\delta < \alpha$, $0 < \delta \leq 1$ (see Section 4.5 in [13]), where for a $d \times d$ matrix $C = (C_{i,j})$, $\|C\|$ denotes the operator norm

$$\|C\| = \sup\{\|Cx\| : x \in \mathbb{R}^d, \|x\| = 1\} = \max_{i=1,\ldots,d} \sum_{j=1}^{d} |C_{i,j}|.$$

2.3 Skorokhod topologies

Denote by $D^d \equiv D([0, 1], \mathbb{R}^d)$ the space of all right-continuous $\mathbb{R}^d$-valued functions on $[0, 1]$ with left limits. For $x \in D^d$ the completed (thick) graph of x is defined as

$$G_x = \{(t, z) \in [0, 1] \times \mathbb{R}^d : z \in [[x(t-), x(t)]]\},$$

where $x(t-)$ is the left limit of x at t and $[[a, b]]$ is the product segment, i.e. $[[a, b]] = [a^{(1)}, b^{(1)}] \times \cdots \times [a^{(d)}, b^{(d)}]$ for $a = (a^{(1)}, \ldots, a^{(d)})$, $b = (b^{(1)}, \ldots, b^{(d)}) \in \mathbb{R}^d$, and $[a^{(i)}, b^{(i)}]$ coincides with the closed interval $[a^{(i)} \wedge b^{(i)}, a^{(i)} \vee b^{(i)}]$, with $c \wedge d = \min\{c, d\}$ for $c, d \in \mathbb{R}$. On the graph G_x we define an order by saying that $(t_1, z_1) \le (t_2, z_2)$ if either (i) $t_1 < t_2$, or (ii) $|x_j(t_1-) - z_1^{(j)}| \le |x_j(t_2-) - z_2^{(j)}|$ for all $j = 1, 2, \ldots, d$. A weak parametric representation of the graph G_x is a continuous nondecreasing function (r, u) mapping $[0, 1]$ into G_x, with r being the time component and u the spatial component, such that $r(0) = 0$, $r(1) = 1$ and $u(1) = x(1)$. Let $\Pi_w(x)$ denote the set of weak parametric representations of G_x. For $x_1, x_2 \in D^d$ define

$$d_w(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi_w(x_i), i = 1, 2\},$$

where $\|x\|_{[0,1]} = \sup\{\|x(t)\| : t \in [0, 1]\}$. Now we say that a sequence $(x_n)_n$ converges to x in D^d in the weak Skorokhod M_1 topology if $d_w(x_n, x) \to 0$ as $n \to \infty$. If we replace the graph G_x with the completed (thin) graph

$$\Gamma_x = \{(t, z) \in [0, 1] \times \mathbb{R}^d : z = \lambda x(t-) + (1 - \lambda)x(t) \text{ for some } \lambda \in [0, 1]\},$$

and weak parametric representations with strong parametric representations, that is continuous nondecreasing functions (r, u) mapping $[0, 1]$ onto Γ_x, then we obtain the standard (or strong) Skorokhod M_1 topology. This topology is induced by the metric

$$d_{M_1}(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi_s(x_i), i = 1, 2\},$$

where $\Pi_s(x)$ is the set of strong parametric representations of the graph Γ_x. Since $\Pi_s(x) \subseteq \Pi_w(x)$ for all $x \in D^d$, the weak M_1 topology is weaker than the standard M_1 topology on D^d, but they coincide for $d = 1$. The weak M_1 topology coincides with the topology induced by the metric

$$d_p(x_1, x_2) = \max\{d_{M_1}(x_1^{(j)}, x_2^{(j)}) : j = 1, \ldots, d\} \tag{6}$$

for $x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \in D^d$ and $i = 1, 2$. The metric d_p induces the product topology on D^d.

By using parametric representations in which only the time component r is nondecreasing instead of (r, u) we obtain Skorokhod's weak and strong M_2 topologies. The metric

$$d_{M_2}(x_1, x_2) = \left(\sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a, b) \right) \vee \left(\sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a, b) \right),$$

where $d(a, b) = \max\{|a^{(i)} - b^{(i)}| : i = 1, \ldots, d+1\}$ for $a = (a^{(1)}, \ldots, a^{(d+1)})$, $b = (b^{(1)}, \ldots, b^{(d+1)}) \in \mathbb{R}^{d+1}$, induces the strong M_2 topology, which is weaker than the M_1 topology. For more details and discussion on the M_1 and M_2 topologies we refer to Sections 12.3-5 and 12.10-11 in [17]. Since the sample paths of the partial maxima processes in (2) are nondecreasing, we will restrict our attention to the subspace $D_{\uparrow}^d$ of functions x in D^d for which the coordinate functions $x^{(i)}$ are nondecreasing for all $i = 1, \ldots, d$.

2.4 Point processes

Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$. Assume the elements of this sequence are pairwise asymptotically (or extremally) independent in the sense that

$$\lim_{x\to\infty} \frac{\mathrm{P}(\|Z_i\| > x, \|Z_j\| > x)}{\mathrm{P}(\|Z_1\| > x)} = 0 \quad \text{for all } i \neq j. \tag{7}$$

We also assume asymptotical independence of the components of each Z_i:

$$\lim_{x\to\infty} \mathrm{P}(|Z_i^{(j)}| > x \,\big|\, |Z_i^{(k)}| > x) = 0 \quad \text{for all } j, k \in \{1, \dots, d\},\ j \neq k. \tag{8}$$

Condition (7) implies the sequence (Z_i) is regularly varying with index α (Proposition 2.1.8 in [10]) with the tail process as in the i.i.d. case, that is $Y_i = 0$ for $i \neq 0$, and $\mathrm{P}(\|Y_0\| > y) = y^{-\alpha}$ for $y \geq 1$. Relation (8) implies that Y_0 a.s. has no two nonzero components.

Define the time-space point processes

$$N_n = \sum_{i=1}^{n} \delta_{(i/n,\, Z_i/a_n)} \quad \text{for all } n \in \mathbb{N},$$

with (a_n) being a sequence of positive real numbers such that

$$n\,\mathrm{P}(\|Z_1\| > a_n) \to 1 \quad \text{as } n \to \infty. \tag{9}$$

The point process convergence for the sequence (N_n) on the space $[0, 1] \times \mathbb{E}^d$, where $\mathbb{E}^d = [-\infty, \infty]^d \setminus \{0\}$, was obtained in [3] under the following two weak dependence conditions.

Condition 2.1 (Mixing condition)**.** *There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every nonnegative continuous function f on $[0, 1] \times \mathbb{E}^d$ with compact support, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \to \infty$,*

$$\mathrm{E}\left[\exp\left\{-\sum_{i=1}^{n} f\left(\frac{i}{n}, \frac{Z_i}{a_n}\right)\right\}\right] - \prod_{k=1}^{k_n} \mathrm{E}\left[\exp\left\{-\sum_{i=1}^{r_n} f\left(\frac{kr_n}{n}, \frac{Z_i}{a_n}\right)\right\}\right] \to 0.$$

Condition 2.2 (Anticlustering condition)**.** *There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every $u > 0$,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\max_{m \leq |i| \leq r_n} \|Z_i\| > ua_n \,\middle|\, \|Z_0\| > ua_n \right) = 0.$$

The sequences (r_n) in these two conditions are assumed to be the same. It can be shown that Condition 2.1 holds for strongly mixing random sequences (see [5, 7]). Now we show that Condition 2.2 holds under condition (7). Let

$$x_{k,n} := \sum_{|i|=1}^{k} \frac{\mathrm{P}(\|Z_i\| > ua_n, \|Z_0\| > ua_n)}{\mathrm{P}(\|Z_0\| > ua_n)}$$

for $k, n \in \mathbb{N}$. For every $k \in \mathbb{N}$ condition (7) implies that $x_{k,n} \to 0$ as $n \to \infty$. An application of the triangular argument (Lemma A.1.3 in [10]) yields that there exists a nondecreasing sequence of positive integers $(s_n)_n$ such that $s_n \to \infty$ and $x_{s_n,n} \to 0$ as $n \to \infty$. Denote

$$r_n := \min\{s_n, \lfloor \sqrt{n} \rfloor\}, \quad n \in \mathbb{N}. \tag{10}$$

Then $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$. For a fixed $m \in \mathbb{N}$ and large r_n (such that $r_n \geq m$) it holds that

$$\mathrm{P}\bigg(\max_{m \leq |i| \leq r_n} \|Z_i\| > ua_n \,\bigg|\, \|Z_0\| > ua_n \bigg) \leq \sum_{|i|=m}^{r_n} \frac{\mathrm{P}(\|Z_i\| > ua_n, \|Z_0\| > ua_n)}{\mathrm{P}(\|Z_0\| > ua_n)}$$
$$\leq x_{s_n,n},$$

and letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \mathrm{P}\bigg(\max_{m \leq |i| \leq r_n} \|Z_i\| > ua_n \,\bigg|\, \|Z_0\| > ua_n \bigg) = 0$$

for every $m \in \mathbb{N}$. Hence, letting $m \to \infty$, we see that Condition 2.2 holds. The latter condition holds also under Leadbetter's condition D':

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{i=1}^{\lfloor n/k \rfloor} \mathrm{P}(\|Z_0\| > xa_n, \|Z_i\| > xa_n) = 0 \quad \text{for all } x > 0. \tag{11}$$

The asymptotical independence condition (7) also holds under condition D'. For more discussion of the above weak dependence conditions, in the context of partial sums, we refer to Section 9.1 in [12] and [16].

In the sequel whenever we assume that Condition 2.1 holds we suppose that the sequence (r_n) that appears in this condition is the same as in (10), and this will ensure that Conditions 2.1 and 2.2 are satisfied by the same sequence (r_n). Under Condition 2.1 by Theorem 3.1 in [3], as $n \to \infty$,

$$N_n \xrightarrow{d} N = \sum_i \sum_j \delta_{(T_i, P_i \eta_{ij})} \tag{12}$$

in $[0, 1] \times \mathbb{E}^d$, where

(i) $\sum_{i=1}^{\infty} \delta_{(T_i, P_i)}$ is a Poisson process on $[0, 1] \times (0, \infty)$ with intensity measure $Leb \times \nu$, with $\nu(\mathrm{d}x) = \theta\alpha x^{-\alpha-1}\,\mathrm{d}x$ and $\theta = \mathrm{P}(\sup_{i\leq -1} \|Y_i\| \leq 1)$.

(ii) $(\sum_{j=1}^{\infty} \delta_{\eta_{ij}})_i$ is an i.i.d. sequence of point processes in $\mathbb{E}^d$ independent of $\sum_i \delta_{(T_i, P_i)}$ and with common distribution equal to the distribution of the point process $\sum_j \delta_{\widetilde{Y}_j / L(\widetilde{Y})}$, where $L(\widetilde{Y}) = \sup_{j\in\mathbb{Z}} \|\widetilde{Y}_j\|$ and $\sum_j \delta_{\widetilde{Y}_j}$ is distributed as $(\sum_{j\in\mathbb{Z}} \delta_{Y_j} \mid \sup_{i\leq -1} \|Y_i\| \leq 1)$.

Taking into account the form of the tail process (Y_i), it holds that $\theta = 1$ and $N = \sum_i \delta_{(T_i, P_i \eta_{i0})}$ with $\|\eta_{i0}\| = 1$. Hence, denoting $Q_i = \eta_{i0}$, the limiting point process in relation (12) reduces to

$$N = \sum_i \delta_{(T_i, P_i Q_i)}. \tag{13}$$

Since the sequence (Q_i) is independent of the Poisson process $\sum_{i=1}^{\infty} \delta_{(T_i,P_i)}$, an application of Proposition 5.3 in [14] yields that $\sum_i \delta_{(T_i,P_i,Q_i)}$ is a Poisson process on $[0,1] \times (0,\infty) \times \mathbb{E}^d$ with intensity measure $Leb \times \nu \times F$, where F is the common probability distribution of Q_i.

For $x \in \mathbb{R}$ let $x^+ = |x| 1_{\{x>0\}}$ and $x^- = |x| 1_{\{x<0\}}$. Define the maximum functional $\Phi \colon \mathbf{M}_p([0,1] \times \mathbb{E}^d) \to D_{\uparrow}^{2d^2}$ by

$$\Phi\Big(\sum_i \delta_{(t_i,(x_i^{(1)},\ldots,x_i^{(d)}))}\Big)(t) = \bigg(\Big(\bigvee_{t_i \le t} x_i^{(j)+}, \bigvee_{t_i \le t} x_i^{(j)-}\Big)^*_{j=1,\ldots,d}\bigg)_{k=1,\ldots,d} \tag{14}$$

for $t \in [0,1]$ (with the convention $\vee\emptyset = 0$), where the space $\mathbf{M}_p([0,1] \times \mathbb{E}^d)$ of Radon point measures on $[0,1]\times\mathbb{E}^d$ is equipped with the vague topology (see Chapter 3 in [13]). Note that on the right-hand side in (14) we repeat the $2d$ coordinates of the vector $\Big(\bigvee_{t_i \le t} x_i^{(j)+}, \bigvee_{t_i \le t} x_i^{(j)-}\Big)^*_{j=1,\ldots,d}$ consecutively d times. Let

$$\begin{aligned}\Lambda = \{\eta \in \mathbf{M}_p([0,1] \times \mathbb{E}^d) : \eta(\{0,1\} \times \mathbb{E}^d) = 0 \text{ and}\\ \eta([0,1] \times \{(x^{(1)},\ldots,x^{(d)}) : |x^{(i)}| = \infty \text{ for some } i\}) = 0\}.\end{aligned}$$

Then Proposition 3.1 in [9] and the definition of the metric d_p in (6) yield the continuity of the maximum functional Φ on the set Λ in the weak M_1 topology.

3 Finite order linear processes

Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$. Fix $m \in \mathbb{N}$, and let

$$X_i = \sum_{j=0}^{m} C_j Z_{i-j}, \quad i \in \mathbb{Z}, \tag{15}$$

be a finite order linear process, where $C_0, C_1, \ldots, C_m$ are random $d \times d$ matrices independent of (Z_i). Define the corresponding partial maxima process by

$$M_n(t) = \begin{cases} a_n^{-1} \bigvee_{i=1}^{\lfloor nt \rfloor} X_i = \Big(a_n^{-1} \bigvee_{i=1}^{\lfloor nt \rfloor} X_i^{(k)}\Big)_{k=1,\ldots,d}, & t \ge \dfrac{1}{n}, \\ a_n^{-1} X_1 = a_n^{-1}(X_1^{(1)},\ldots,X_1^{(d)}), & t < \dfrac{1}{n}, \end{cases} \tag{16}$$

for $t \in [0,1]$, with the normalizing sequence (a_n) as in (9). For $k, j \in \{1,\ldots,d\}$ let

$$D_+^{k,j} = \bigvee_{i=0}^{m} C_{i;k,j}^+ \quad \text{and} \quad D_-^{k,j} = \bigvee_{i=0}^{m} C_{i;k,j}^-, \tag{17}$$

where $C_{i;k,j}$ is the (k,j)th entry of the matrix C_i, $C_{i;k,j}^+ = |C_{i;k,j}| 1_{\{C_{i;k,j}>0\}}$ and $C_{i;k,j}^- = |C_{i;k,j}| 1_{\{C_{i;k,j}<0\}}$.

First, we show in the proposition below that a particular process W_n, constructed from the sequence (Z_i), converges in $D_{\uparrow}^d$ with the weak M_1 topology. Later, in the main result of this section, we show that the weak M_1 distance between processes M_n and W_n is asymptotically negligible (as $n \to \infty$), which will imply the functional convergence of the maxima process M_n. The limiting process will be described in terms of certain extremal processes derived from the point process $N = \sum_i \delta_{(T_i, P_i Q_i)}$ in relation (13). Extremal processes can be derived from Poisson processes in the following way. Let $\xi = \sum_k \delta_{(t_k, j_k)}$ be a Poisson process on $[0,\infty) \times [0,\infty)^d$ with mean measure $Leb \times \mu$, where μ is a measure on $[0,\infty)^d$ satisfying $\mu(\{x \in [0,\infty)^d : \|x\| > \delta\}) < \infty$ for any $\delta > 0$. The extremal process $G(\,\cdot\,)$ generated by ξ is defined by $G(t) = \bigvee_{t_k \le t} j_k$ for $t > 0$. Then for $x \in [0,\infty)^d$, $x \neq 0$, and $t > 0$ it holds that $\mathrm{P}(G(t) \le x) = \exp(-t\mu([[0,x]]^c))$ (cf. Section 5.6 in [14]). The measure μ is called the exponent measure.

Proposition 3.1. *Let (X_i) be a linear process defined in (15), where $(Z_i)_{i \in \mathbb{Z}}$ is a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$ that satisfy (7) and (8), and $C_0, C_1, \ldots, C_m$ are random $d \times d$ matrices independent of (Z_i). Assume Condition 2.1 holds. Let*

$$W_n(t) := \left(\bigvee_{i=1}^{\lfloor nt \rfloor} \bigvee_{j=1}^{d} a_n^{-1} \Big(D_+^{k,j} Z_i^{(j)+} \vee D_-^{k,j} Z_i^{(j)-} \Big) \right)_{k=1,\ldots,d}, \qquad t \in [0,1],$$

with $D_+^{k,j}$ and $D_-^{k,j}$ defined in (17). Then, as $n \to \infty$,

$$W_n(\,\cdot\,) \xrightarrow{d} M(\,\cdot\,) := \left(\bigvee_{j=1}^{d} \Big(\widetilde{D}_+^{k,j} M^{(j+)}(\,\cdot\,) \vee \widetilde{D}_-^{k,j} M^{(j-)}(\,\cdot\,) \Big) \right)_{k=1,\ldots,d} \tag{18}$$

in $D_{\uparrow}^d$ with the weak M_1 topology, where $M^{(j+)}$ and $M^{(j-)}$ are extremal processes with exponent measures ν_{j+} and ν_{j-} respectively, with

$$\nu_{j+}(\mathrm{d}x) = \mathrm{E}(Q_1^{(j)+})^{\alpha}\, \alpha x^{-\alpha-1}\, \mathrm{d}x \quad \textit{and} \quad \nu_{j-}(\mathrm{d}x) = \mathrm{E}(Q_1^{(j)-})^{\alpha}\, \alpha x^{-\alpha-1}\, \mathrm{d}x$$

for $x > 0$ ($j = 1, \ldots, d$), and $((\widetilde{D}_+^{k,j}, \widetilde{D}_-^{k,j})^*_{j=1,\ldots,d})_{k=1,\ldots,d}$ is a $2d^2$-dimensional random vector, independent of $(M^{(j+)}, M^{(j-)})_{j=1,\ldots,d}$, such that

$$((\widetilde{D}_+^{k,j}, \widetilde{D}_-^{k,j})^*_{j=1,\ldots,d})_{k=1,\ldots,d} \stackrel{d}{=} ((D_+^{k,j}, D_-^{k,j})^*_{j=1,\ldots,d})_{k=1,\ldots,d}.$$

Remark 3.2. In Proposition 3.1, as well as in the sequel of this paper, we suppose $M^{(j+)}$ is an extremal process if $\mathrm{E}(Q_1^{(j)+})^{\alpha} > 0$, and a zero process if this quantity is equal to zero. Analogously for $M^{(j-)}$.

Proof of Proposition 3.1. As noted in Subsection 2.4, condition (7) implies that the sequence (Z_i) is regularly varying with index α and that Condition 2.2 holds. This, with Condition 2.1, implies the point process convergence in (12) with the limiting point process N described in (13). Since N is a Poisson process, it almost surely belongs to the set Λ. Therefore, since the maximum functional Φ is continuous on Λ, the continuous mapping theorem (see, for instance, Theorem 3.1 in [14]) applied to the convergence in (12) yields $\Phi(N_n) \xrightarrow{d} \Phi(N)$ in $D_{\uparrow}^{2d^2}$ under the weak M_1 topology, i.e.

$$W_n^{\star}(\,\cdot\,) := \left(\left(a_n^{-1} \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} Z_i^{(j)+},\, a_n^{-1} \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} Z_i^{(j)-} \right)^{*}_{j=1,\dots,d} \right)_{k=1,\dots,d}$$
$$\xrightarrow{d} W(\,\cdot\,) := \left(\left(\bigvee_{T_i \le \,\cdot} P_i Q_i^{(j)+},\, \bigvee_{T_i \le \,\cdot} P_i Q_i^{(j)-} \right)^{*}_{j=1,\dots,d} \right)_{k=1,\dots,d}. \tag{19}$$

By the same arguments as in the proof of Proposition 3.2 in [9] we obtain that $D_{\uparrow}^{2d^2}$ with the weak M_1 topology is a Polish space, and hence by Corollary 5.18 in [4] we can find a random vector $((\widetilde{D}_{+}^{k,j}, \widetilde{D}_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d}$, independent of W, such that

$$((\widetilde{D}_{+}^{k,j}, \widetilde{D}_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d} \stackrel{d}{=} ((D_{+}^{k,j}, D_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d}.$$

This, relation (19) and the fact that $((D_{+}^{k,j}, D_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d}$ is independent of $W_n^{\star}$, by an application of Theorem 3.29 in [4], imply that

$$(B, W_n^{\star}) \xrightarrow{d} (\widetilde{B}, W) \quad \text{as } n \to \infty \tag{20}$$

in $D_{\uparrow}^{4d^2}$ with the product M_1 topology, where $B = ((B_{+}^{k,j}, B_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d}$ and $\widetilde{B} = ((\widetilde{B}_{+}^{k,j}, \widetilde{B}_{-}^{k,j})^{*}_{j=1,\dots,d})_{k=1,\dots,d}$ are random elements in $D_{\uparrow}^{2d^2}$ such that $B_{+}^{k,j}(t) = D_{+}^{k,j}$, $B_{-}^{k,j}(t) = D_{-}^{k,j}$, $\widetilde{B}_{+}^{k,j}(t) = \widetilde{D}_{+}^{k,j}$ and $\widetilde{B}_{-}^{k,j}(t) = \widetilde{D}_{+}^{k,j}$ for $t \in [0, 1]$.

A multivariate version of Lemma 2.1 in [9] implies that the function $g \colon D_{\uparrow}^{4d^2} \to D_{\uparrow}^{2d^2}$ defined by

$$g(x) = (x^{(1)} x^{(2d^2+1)}, x^{(2)} x^{(2d^2+2)}, \dots, x^{(2d^2)} x^{(4d^2)})$$

for $x = (x^{(1)}, \dots, x^{(4d^2)}) \in D_{\uparrow}^{4d^2}$, is continuous in the weak M_1 topology on the set of all functions in $D_{\uparrow}^{4d^2}$ for which the first $2d^2$ component functions have no discontinuity points, and this yields $\mathrm{P}[(\widetilde{B}, W) \in \mathrm{Disc}(g)] = 0$, where $\mathrm{Disc}(g)$ denotes the set of discontinuity points of g. A multivariate version of Lemma 2.2 in [9] shows that the function $h \colon D_{\uparrow}^{2d^2} \to D_{\uparrow}^{d}$, defined by

$$h(x) = \left(\bigvee_{i=1}^{2d} x^{(i)}, \bigvee_{i=2d+1}^{4d} x^{(i)}, \dots, \bigvee_{i=2(d-1)d+1}^{2d^2} x^{(i)} \right)$$

for $x = (x^{(i)})_{i=1,\dots,2d^2} \in D_{\uparrow}^{2d^2}$, is continuous when both spaces $D_{\uparrow}^{2d^2}$ and $D_{\uparrow}^{d}$ are endowed with the weak M_1 topology. Therefore, the continuous mapping theorem applied to the convergence in (20) yields $(h \circ g)(B, W_n^{\star}) \xrightarrow{d} (h \circ g)(\widetilde{B}, W)$ as $n \to \infty$, i.e.

$$\Big(\bigvee_{i=1}^{\lfloor n \cdot \rfloor} \bigvee_{j=1}^{d} \frac{D_{+}^{k,j} Z_i^{(j)+} \vee D_{-}^{k,j} Z_i^{(j)-}}{a_n} \Big)_{k=1,\ldots,d}$$

$$\xrightarrow{d} \Big(\bigvee_{T_i \leq \cdot} \bigvee_{j=1}^{d} (\widetilde{D}_{+}^{k,j} P_i Q_i^{(j)+} \vee \widetilde{D}_{-}^{k,j} P_i Q_i^{(j)-}) \Big)_{k=1,\ldots,d}$$

in $D_{\uparrow}^d$ with the weak M_1 topology. Note that $(h \circ g)(B, W_n^{\star})$ is equal to W_n.

To finish the proof, it remains to show that $(h \circ g)(\widetilde{B}, W)$ is equal to the limiting process in relation (18). By an application of Propositions 5.2 and 5.3 in [14] we obtain that for every $j = 1, \ldots, d$ the point process $\sum_i \delta_{(T_i, P_i Q_i^{(j)+})}$ is a Poisson process with intensity measures $Leb \times \nu_{j+}$, and hence $M^{(j+)}(t) := \bigvee_{T_i \leq t} P_i Q_i^{(j)+}$ is an extremal processes with exponent measures ν_{j+} (see Section 4.3 in [13]; and [14], p. 161). Analogously, $M^{(j-)}(t) := \bigvee_{T_i \leq t} P_i Q_i^{(j)-}$ is an extremal processes with exponent measures ν_{j-}, and hence

$$(h \circ g)(\widetilde{B}, W) = \Big(\bigvee_{j=1}^{d} \Big(\widetilde{D}_{+}^{k,j} M^{(j+)} \vee \widetilde{D}_{-}^{k,j} M^{(j-)} \Big) \Big)_{k=1,\ldots,d}, \quad t \in [0, 1].$$

□

The proof of the next theorem relies on the proof of Theorem 3.3 in [9] where the functional convergence of the partial maxima process is established for univariate linear processes with i.i.d. innovations and random coefficients. We will omit some details of those parts of the proof that remain the same in our case, but we will show how to handle those parts that differ due to the multivariate setting and weak dependence of innovations.

Theorem 3.3. *Let $(Z_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$ that satisfy (7) and (8), and let $C_0, C_1, \ldots, C_m$ be random $d \times d$ matrices independent of (Z_i). Assume Condition 2.1 holds. Then $M_n \xrightarrow{d} M$ as $n \to \infty$ in $D_{\uparrow}^d$ endowed with the weak M_1 topology.*

Proof. Let W_n be as defined in Proposition 3.1. If we show that for every $\delta > 0$,

$$\lim_{n \to \infty} \mathrm{P}[d_p(W_n, M_n) > \delta] = 0,$$

then from Proposition 3.1 by an application of Slutsky's theorem (see Theorem 3.4 in [14]) it will follow that $M_n \xrightarrow{d} M$ in $D_{\uparrow}^d$ with the weak M_1 topology. Taking into account (6) we need to show

$$\lim_{n \to \infty} \mathrm{P}[d_{M_1}(W_n^{(j)}, M_n^{(j)}) > \delta] = 0,$$

for every $j = 1, \ldots, d$, but it is enough to prove the last relation only for $j = 1$ (since the proof is analogous for all coordinates j). In fact, it suffices to show

$$\lim_{n \to \infty} \mathrm{P}[d_{M_2}(W_n^{(1)}, M_n^{(1)}) > \delta] = 0, \tag{21}$$

since for $x, y \in D_{\uparrow}^1$ it holds that $d_{M_2}(x,y) = d_{M_1}^*(x,y)$, where $d_{M_1}^*$ is a complete metric topologically equivalent to d_{M_1} (see Remark 12.8.1 in [17]; and [9], page 247).

In order to show (21), fix $\delta > 0$ and let $n \in \mathbb{N}$ be large enough, i.e. $n > \max\{2m, 2m/\delta\}$. By the definition of the metric d_{M_2} we have

$$d_{M_2}(W_n^{(1)}, M_n^{(1)}) = \bigg(\sup_{v \in \Gamma_{W_n^{(1)}}} \inf_{z \in \Gamma_{M_n^{(1)}}} d(v,z) \bigg) \vee \bigg(\sup_{v \in \Gamma_{M_n^{(1)}}} \inf_{z \in \Gamma_{W_n^{(1)}}} d(v,z) \bigg)$$
$$=: R_n \vee T_n.$$

Hence
$$\mathrm{P}[d_{M_2}(W_n^{(1)}, M_n^{(1)}) > \delta] \le \mathrm{P}(R_n > \delta) + \mathrm{P}(T_n > \delta). \tag{22}$$
To estimate the first term on the right-hand side of (22), define

$$D_n = \{\exists\, v \in \Gamma_{W_n^{(1)}} \text{ such that } d(v,z) > \delta \text{ for every } z \in \Gamma_{M_n^{(1)}}\}.$$

Note that $\{R_n > \delta\} \subseteq D_n$. On the event D_n it holds that $d(v, \Gamma_{M_n^{(1)}}) > \delta$. Let $v = (t_v, x_v)$. Then as in the proof of Theorem 3.3 in [9], for all $l = 0, 1, \ldots, m$ it holds that
$$\bigg| W_n^{(1)}\Big(\frac{i^*}{n}\Big) - M_n^{(1)}\Big(\frac{i^*+l}{n}\Big) \bigg| \ge d(v, \Gamma_{M_n^{(1)}}) > \delta \tag{23}$$
with $i^* = \lfloor nt_v \rfloor$ or $i^* = \lfloor nt_v \rfloor - 1$. Note that i^* is a random index. Let $D = \bigvee_{k,j=1,\ldots,d}(D_+^{k,j} \vee D_-^{k,j})$. This implies $|C_{i:k,j}| \le D$ for all $i \in \{0,\ldots,m\}$ and $k, j \in \{1,\ldots,d\}$. Denote $\delta^* = \delta/[8(m+1)d]$. We claim that
$$D_n \subseteq H_{n,1} \cup H_{n,2} \cup H_{n,3}, \tag{24}$$

where
$$H_{n,1} = \bigg\{ \exists\, l \in \{-m,\ldots,m\} \cup \{n-m+1,\ldots,n\} \text{ s.t. } \frac{D\|Z_l\|}{a_n} > \delta^* \bigg\},$$
$$H_{n,2} = \bigg\{ \exists\, k \in \{1,\ldots,n\} \text{ and } \exists\, l \in \{k-m,\ldots,k+m\} \setminus \{k\}$$
$$\text{such that } \frac{D\|Z_k\|}{a_n} > \delta^* \text{ and } \frac{D\|Z_l\|}{a_n} > \delta^* \bigg\},$$
$$H_{n,3} = \bigg\{ \exists\, k \in \{1,\ldots,n\},\ \exists\, j_0 \in \{1,\ldots,d\} \text{ and } \exists\, p \in \{1,\ldots,d\} \setminus \{j_0\}$$
$$\text{such that } \frac{D|Z_k^{(j_0)}|}{a_n} > \delta^* \text{ and } \frac{D|Z_k^{(p)}|}{a_n} > \delta^* \bigg\}.$$
Note that relation (24) will be proven if we show that

$$\widehat{D}_n := D_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c = \emptyset.$$

Assume the event $\widehat{D}_n$ occurs. Then necessarily $W_n^{(1)}(i^*/n) > \delta^*$. Indeed, if $W_n^{(1)}(i^*/n) \le \delta^*$, that is

$$\bigvee_{i=1}^{i^*} \bigvee_{j=1}^{d} a_n^{-1} \Big(D_+^{1,j} Z_i^{(j)+} \vee D_-^{1,j} Z_i^{(j)-} \Big) = W_n^{(1)}\Big(\frac{i^*}{n}\Big) \le \delta^*,$$

then for every $s \in \{m+1, \dots, i^*\}$ it holds that

$$\frac{X_s^{(1)}}{a_n} = \sum_{r=0}^{m} \sum_{j=1}^{d} \frac{C_{r;1,j} Z_{s-r}^{(j)}}{a_n} \le \sum_{r=0}^{m} \sum_{j=1}^{d} \frac{D_+^{1,j} Z_{s-r}^{(j)+} \vee D_-^{1,j} Z_{s-r}^{(j)-}}{a_n}$$
$$\le \sum_{r=0}^{m} \sum_{j=1}^{d} \frac{\delta}{8(m+1)d} = \frac{\delta}{8}, \tag{25}$$

since by the definition of $D_+^{1,j}$ and $D_-^{1,j}$ we have $D_+^{1,j} Z_{s-r}^{(j)+} \ge 0$, $D_-^{1,j} Z_{s-r}^{(j)-} \ge 0$ and

$$C_{r;1,j} Z_{s-r}^{(j)} \le \begin{cases} D_+^{1,j} Z_{s-r}^{(j)+}, & \text{if } C_{r;1,j} > 0 \text{ and } Z_{s-r}^{(j)} > 0, \\ D_-^{1,j} Z_{s-r}^{(j)-}, & \text{if } C_{r;1,j} < 0 \text{ and } Z_{s-r}^{(j)} < 0, \\ 0, & \text{if } C_{r;1,j} \cdot Z_{s-r}^{(j)} \le 0. \end{cases}$$

Since the event $H_{n,1}^c$ occurs for every $s \in \{1, \dots, m\}$, we also have

$$\frac{|X_s^{(1)}|}{a_n} \le \sum_{r=0}^{m} \sum_{j=1}^{d} |C_{r;1,j}| \frac{|Z_{s-r}^{(j)}|}{a_n} \le \sum_{r=0}^{m} \sum_{j=1}^{d} \frac{D\|Z_{s-r}\|}{a_n} \le (m+1)d\,\delta^* = \frac{\delta}{8}. \tag{26}$$

Combining (25) and (26) we obtain

$$-\frac{\delta}{8} \le \frac{X_1^{(1)}}{a_n} \le M_n^{(1)}\Big(\frac{i^*}{n}\Big) = \bigvee_{s=1}^{i^*} \frac{X_s^{(1)}}{a_n} \le \frac{\delta}{8},$$

and thus

$$\Big|W_n^{(1)}\Big(\frac{i^*}{n}\Big) - M_n^{(1)}\Big(\frac{i^*}{n}\Big)\Big| \le \Big|W_n^{(1)}\Big(\frac{i^*}{n}\Big)\Big| + \Big|M_n^{(1)}\Big(\frac{i^*}{n}\Big)\Big| \le \frac{\delta}{8(m+1)d} + \frac{\delta}{8} \le \frac{\delta}{4},$$

which is in contradiction to (23).

Therefore $W_n^{(1)}(i^*/n) > \delta^*$, and hence there exist $k \in \{1, \dots, i^*\}$ and $j_0 \in \{1, \dots, d\}$ such that

$$W_n^{(1)}\Big(\frac{i^*}{n}\Big) = a_n^{-1}\Big(D_+^{1,j_0} Z_k^{(j_0)+} \vee D_-^{1,j_0} Z_k^{(j_0)-}\Big) > \delta^*.$$

This implies

$$\frac{D\|Z_k\|}{a_n} = \frac{D}{a_n} \bigvee_{j=1}^{d} |Z_k^{(j)}| \ge \frac{D}{a_n} |Z_k^{(j_0)}| \ge \frac{1}{a_n}\Big(D_+^{1,j_0} Z_k^{(j_0)+} \vee D_-^{1,j_0} Z_k^{(j_0)-}\Big) > \delta^*.$$

From this, since $H_{n,1}^c \cap H_{n,2}^c \cap H_{n,3}^c$ occurs, it follows that $m+1 \le k \le n-m$,

$$\frac{D\|Z_l\|}{a_n} \le \delta^* \quad \text{for all } l \in \{k-m, \dots, k+m\} \setminus \{k\}, \tag{27}$$

and

$$\frac{D|Z_k^{(p)}|}{a_n} \leq \delta^* \quad \text{for all } p \in \{1,\dots,d\} \setminus \{j_0\}. \tag{28}$$

Similarly as in the proof of Theorem 3.3 in [9] one can show that $M_n^{(1)}(i^*/n) = X_j^{(1)}/a_n$ for some $j \in \{1,\dots,i^*\} \setminus \{k,\dots,k+m\}$. Now we have four cases:

(A1) all random vectors $Z_{j-m},\dots,Z_j$ are "small",

(A2) exactly one is "large" with exactly one "large" component,

(A3) exactly one is "large" with at least two "large" components,

(A4) at least two of them are "large",

where we say Z is "large" if $D\|Z\|/a_n > \delta^*$, otherwise it is "small", and similarly the component $Z^{(s)}$ is "large" if $D|Z^{(s)}|/a_n > \delta^*$.

Following the arguments from [9], adjusted to the multivariate setting, it can be shown that the cases (A1) and (A2) are not possible (see the arXiv preprint [6] for details). The case (A3) is not possible on the event $H_{n,3}^c$, and the case (A4) is not possible on the event $H_{n,2}^c$. Since neither of the four cases (A1)–(A4) is possible, we conclude that $\widehat{D}_n = \emptyset$, and hence (24) holds.

The next step is to show that $\mathrm{P}(H_{n,k}) \to 0$ as $n \to \infty$ for $k = 1,2,3$. By stationarity we have $\mathrm{P}(H_{n,1}) \leq (3m+1)\,\mathrm{P}(D\|Z_1\| > \delta^* a_n)$, and therefore

$$\lim_{n\to\infty} \mathrm{P}(H_{n,1}) = 0. \tag{29}$$

As for $H_{n,2}$ we have

$$\begin{aligned}
\mathrm{P}(H_{n,2} \cap \{D \leq c\}) &= \sum_{k=1}^{n} \sum_{\substack{l=k-m \\ l \neq k}}^{k+m} \mathrm{P}\left(\frac{D\|Z_k\|}{a_n} > \delta^*,\ \frac{D\|Z_l\|}{a_n} > \delta^*,\ D \leq c\right) \\
&\leq 2n \sum_{i=1}^{m} \mathrm{P}\left(\frac{\|Z_0\|}{a_n} > \frac{\delta^*}{c},\ \frac{\|Z_i\|}{a_n} > \frac{\delta^*}{c}\right) \\
&\leq 2 \sum_{i=1}^{m} n\,\mathrm{P}\left(\frac{\|Z_0\|}{a_n} > \frac{\delta^*}{c}\right) \frac{\mathrm{P}\left(\frac{\|Z_0\|}{a_n} > \frac{\delta^*}{c},\ \frac{\|Z_i\|}{a_n} > \frac{\delta^*}{c}\right)}{\mathrm{P}\left(\frac{\|Z_0\|}{a_n} > \frac{\delta^*}{c}\right)}
\end{aligned}$$

for an arbitrary $c > 0$. Therefore regular variation and the asymptotical independence condition (7) yield $\lim_{n\to\infty} \mathrm{P}(H_{n,2} \cap \{D \leq c\}) = 0$, and this implies

$$\limsup_{n\to\infty} \mathrm{P}(H_{n,2}) \leq \limsup_{n\to\infty} \mathrm{P}(H_{n,2} \cap \{D > c\}) \leq \mathrm{P}(D > c).$$

Letting $c \to \infty$ we conclude

$$\lim_{n\to\infty} \mathrm{P}(H_{n,2}) = 0. \tag{30}$$

By the definition of the set $H_{n,3}$ and stationarity it holds that

$$\mathrm{P}(H_{n,3} \cap \{D \le c\}) = \sum_{k=1}^{n} \sum_{\substack{l,s=1 \\ l \ne s}}^{d} \mathrm{P}\bigg(\frac{D|Z_k^{(l)}|}{a_n} > \delta^*, \ \frac{D|Z_k^{(s)}|}{a_n} > \delta^*, \ D \le c \bigg)$$
$$\le \sum_{\substack{l,s=1 \\ l \ne s}}^{d} n\,\mathrm{P}\bigg(\frac{|Z_1^{(s)}|}{a_n} > \frac{\delta^*}{c} \bigg) \mathrm{P}\bigg(\frac{|Z_1^{(l)}|}{a_n} > \frac{\delta^*}{c} \,\bigg|\, \frac{|Z_1^{(s)}|}{a_n} > \frac{\delta^*}{c} \bigg)$$
$$\le \sum_{\substack{l,s=1 \\ l \ne s}}^{d} n\,\mathrm{P}\bigg(\frac{\|Z_1\|}{a_n} > \frac{\delta^*}{c} \bigg) \mathrm{P}\bigg(\frac{|Z_1^{(l)}|}{a_n} > \frac{\delta^*}{c} \,\bigg|\, \frac{|Z_1^{(s)}|}{a_n} > \frac{\delta^*}{c} \bigg),$$

and hence regular variation and condition (8) yield

$$\lim_{n\to\infty} \mathrm{P}(H_{n,3}) = 0. \tag{31}$$

Now from relations (24) and (29)–(31) we obtain $\lim_{n\to\infty} \mathrm{P}(D_n) = 0$, and since $\{R_n > \delta\} \subseteq D_n$, we conclude that

$$\lim_{n\to\infty} \mathrm{P}(R_n > \delta) = 0. \tag{32}$$

Interchanging the roles of $M_n^{(1)}$ and $W_n^{(1)}$ in handling the event D_n, and using the arguments from the proof of Theorem 3.3 in [9], adjusted to the multivariate setting, we can show

$$\lim_{n\to\infty} \mathrm{P}(T_n > \delta) = 0 \tag{33}$$

(for details see the arXiv preprint [6]). Now from (22), (32) and (33) we obtain (21), which means that $M_n \xrightarrow{d} M$ in $D_{\uparrow}^d$ with the weak M_1 topology. □

4 Infinite order linear processes

Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$, and $(C_i)_{i\ge0}$ a sequence of random $d \times d$ matrices independent of (Z_i) such that the series defining the linear process

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z}, \tag{34}$$

is a.s. convergent. For $k, j \in \{1, \dots, d\}$ let

$$D_+^{k,j} = \max\{C_{i;k,j}^+ : i \ge 0\} \quad \text{and} \quad D_-^{k,j} = \max\{C_{i;k,j}^- : i \ge 0\},$$

where $C_{i;k,j}$ is the (k, j)th entry of the matrix C_i. Let M_n be the partial maxima process as defined in (16), and M the limiting process from Proposition 3.1.

To obtain functional convergence of the partial maxima process for infinite order linear processes, we first approximate them by a sequence of finite order linear processes, for which Theorem 3.3 holds, and then show that the error of approximation is negligible in the limit with respect to the weak M_1 topology. In this case, besides the conditions from Theorem 3.3 for finite order linear processes, we will need also some moment conditions on the sequence of coefficients.

Theorem 4.1. *Let $(Z_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of regularly varying $\mathbb{R}^d$-valued random vectors with index $\alpha > 0$ that satisfy (7) and (8), and let $(C_i)_{i \geq 0}$ be a sequence of random $d \times d$ matrices independent of (Z_i). Assume Condition 2.1 holds and suppose*

$$\begin{cases} \sum_{j=0}^{\infty} \mathrm{E}(\|C_j\|^{\delta} + \|C_j\|^{\gamma}) < \infty, & \text{if } \alpha \in (0,1), \\ \sum_{j=0}^{\infty} \mathrm{E}(\|C_j\|^{\delta} + \|C_j\|) < \infty, & \text{if } \alpha = 1, \\ \sum_{j=0}^{\infty} \mathrm{E}\|C_j\| < \infty, & \text{if } \alpha > 1, \end{cases} \tag{35}$$

for some $\delta \in (0, \alpha)$ and $\gamma \in (\alpha, 1)$. Then $M_n \xrightarrow{d} M$ as $n \to \infty$ in $D_{\uparrow}^d$ endowed with the weak M_1 topology.

Proof. For $m \in \mathbb{N}$, $m \geq 2$, define

$$X_i^m = \sum_{j=0}^{m-2} C_j Z_{i-j} + C^{(m,\vee)} Z_{i-m+1} + C^{(m,\wedge)} Z_{i-m}, \quad i \in \mathbb{Z},$$

and

$$M_{n,m}(t) = \begin{cases} a_n^{-1} \bigvee_{i=1}^{\lfloor nt \rfloor} X_i^m, & t \in \left[\frac{1}{n}, 1\right], \\ a_n^{-1} X_1^m, & t \in \left[0, \frac{1}{n}\right), \end{cases}$$

where $C^{(m,\vee)} = \max\{C_i : i \geq m-1\}$ and $C^{(m,\wedge)} = \min\{C_i : i \geq m-1\}$, with the maximum and minimum of matrices interpreted componentwise, i.e. the (k, j)th entry of the matrix $C^{(m,\vee)}$ is $C_{k,j}^{(m,\vee)} = \max\{C_{i;k,j} : i \geq m-1\}$, and the (k, j)th entry of the matrix $C^{(m,\wedge)}$ is $C_{k,j}^{(m,\wedge)} = \min\{C_{i;k,j} : i \geq m-1\}$.

For $k, j \in \{1, \ldots, d\}$ define

$$D_+^{m,k,j} = \left(\bigvee_{i=0}^{m-2} C_{i;k,j}^+ \right) \vee C_{k,j}^{(m,\vee)+} \vee C_{k,j}^{(m,\wedge)+}$$

and

$$D_-^{m,k,j} = \left(\bigvee_{i=0}^{m-2} C_{i;k,j}^- \right) \vee C_{k,j}^{(m,\vee)-} \vee C_{k,j}^{(m,\wedge)-}.$$

Then $D_+^{m,k,j} = D_+^{k,j}$ and $D_-^{m,k,j} = D_-^{k,j}$, and therefore for the sequence of finite order linear processes $(X_i^m)_i$ by Theorem 3.3 we obtain $M_{n,m} \xrightarrow{d} M$ as $n \to \infty$ in $D_{\uparrow}^d$ with the weak M_1 topology.

If we show that for every $\epsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}[d_p(M_n, M_{n,m}) > \epsilon] = 0,$$

then by a generalization of Slutsky's theorem (see Theorem 3.5 in [14]) it will follow that $M_n \xrightarrow{d} M$ in $D_{\uparrow}^d$ with the weak M_1 topology. Taking into account (6) and the fact that the metric d_{M_1} on $D_{\uparrow}^1$ is bounded above by the uniform metric, it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\bigg(\sup_{0\le t\le 1} |M_n^{(j)}(t) - M_{n,m}^{(j)}(t)| > \epsilon \bigg) = 0,$$

for every $j = 1, \dots, d$, and further, as in the proof of Theorem 3.3, it is enough to show the last relation only for $j = 1$. Denote by $J_{n,m}$ the probability in the last relation (for $j = 1$). Now we treat separately the cases $\alpha \in (0, 1)$ and $\alpha \in [1, \infty)$.

Case $\alpha \in (0, 1)$. Recalling the definitions, we have

$$J_{n,m} \le \mathrm{P}\bigg(\bigvee_{i=1}^{n} \frac{|X_i^{(1)} - X_i^{m(1)}|}{a_n} > \epsilon \bigg) \le \mathrm{P}\bigg(\sum_{i=1}^{n} \frac{|X_i^{(1)} - X_i^{m(1)}|}{a_n} > \epsilon \bigg). \tag{36}$$

Similarly as in the univariate case treated in [9] we obtain

$$\begin{aligned} X_i^{(1)} - X_i^{m(1)} = \sum_{j=1}^{d} \bigg(& \sum_{k=m+1}^{\infty} C_{k;1,j} Z_{i-k}^{(j)} + (C_{m-1;1,j} - C_{1,j}^{(m,\vee)}) Z_{i-m+1}^{(j)} \\ & + (C_{m;1,j} - C_{1,j}^{(m,\wedge)}) Z_{i-m}^{(j)} \bigg), \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=1}^{n} |X_i^{(1)} - X_i^{m(1)}| \\ & \quad \le \sum_{j=1}^{d} \bigg[\sum_{i=-\infty}^{0} |Z_{i-m}^{(j)}| \sum_{s=1}^{n} \|C_{m-i+s}\| + \bigg(2 \sum_{l=m-1}^{\infty} \|C_l\| \bigg) \sum_{i=1}^{n+1} |Z_{i-m}^{(j)}| \bigg]. \end{aligned}$$

Therefore from (36) by applying condition (35) and the multivariate generalization of Lemma 3.2 in [8] (for the proof of this generalization see the arXiv preprint [6]) it follows that $\lim_{m\to\infty} \limsup_{n\to\infty} J_{n,m} = 0$, which means that $M_n \xrightarrow{d} M$ as $n \to \infty$ in $D_{\uparrow}^d$ with the weak M_1 topology.

Case $\alpha \in [1, \infty)$. Define

$$A_{k,j} = \begin{cases} C_{k;1,j} - C_{1,j}^{(m,\vee)}, & \text{if } k = m-1, \\ C_{k;1,j} - C_{1,j}^{(m,\wedge)}, & \text{if } k = m, \\ C_{k;1,j}, & \text{if } k \ge m+1, \end{cases}$$

for $k \geq m-1$ and $j \in \{1,\dots,d\}$. Then using the representation of $X_i^{(1)} - X_i^{m(1)}$ obtained in the previous case we get

$$|M_n^{(1)}(t) - M_{n,m}^{(1)}(t)| \leq \bigvee_{i=1}^{n} \frac{|X_i^{(1)} - X_i^{m(1)}|}{a_n} = \bigvee_{i=1}^{n} \sum_{j=1}^{d} \bigg| \sum_{k=m-1}^{\infty} A_{k,j} \frac{Z_{i-k}^{(j)}}{a_n} \bigg|$$

for every $t \in [0,1]$. This, (35) and Lemma 5.2 in the arXiv preprint [6] yield $\lim_{m\to\infty} \limsup_{n\to\infty} J_{n,m} = 0$. Thus in this case also $M_n \xrightarrow{d} M$ as $n \to \infty$ in $D_{\uparrow}^d$ with the weak M_1 topology. □

Remark 4.2. When the sequence of coefficients (C_i) is deterministic, the limiting process M in Theorem 4.1 has the representation

$$M(t) = \bigvee_{T_i \leq t} P_i S_i, \quad t \in [0,1],$$

where $S_i = (S_i^{(1)}, \dots, S_i^{(d)})$, with $S_i^{(k)} = \bigvee_{j=1}^{d} (D_+^{k,j} Q_i^{(j)+} \vee D_-^{k,j} Q_i^{(j)-})$ for $k = 1,\dots,d$. It is an extremal process with an exponent measure ρ, where for $x \in [0,\infty)^d$, $x \neq 0$,

$$\rho([[0,x]]^c) = \int_0^{\infty} \mathrm{P}\left(y \bigvee_{k=1}^{d} \frac{S_1^{(k)}}{x^{(k)}} > 1 \right) \alpha y^{-\alpha-1}\, \mathrm{d}y.$$

Remark 4.3. A special case of multivariate linear processes studied in this paper is

$$X_i = \sum_{j=0}^{\infty} B_j Z_{i-j}, \quad i \in \mathbb{Z},$$

where $(B_i)_{i\geq 0}$ is a sequence of random variables independent of (Z_i). To obtain this linear process from the general one in (34) take

$$C_{i;k,j} = \begin{cases} B_i, & \text{if } k = j, \\ 0, & \text{if } k \neq j, \end{cases}$$

for $i \geq 0$ and $k, j \in \{1,\dots,d\}$. In this case, under the conditions from Theorem 4.1 the limiting process M reduces to

$$\begin{aligned} M(t) &= \left(\widetilde{D}_+^{k,k} M^{(k+)}(t) \vee \widetilde{D}_-^{k,k} M^{(k-)}(t) \right)_{k=1,\dots,d} \\ &= \left(\widetilde{B}_+ M^{(k+)}(t) \vee \widetilde{B}_- M^{(k-)}(t) \right)_{k=1,\dots,d} \end{aligned}$$

for $t \in [0,1]$, where $(\widetilde{B}_+, \widetilde{B}_-)$ is a two-dimensional random vector, independent of $(M^{(k+)}, M^{(k-)})_{k=1,\dots,d}^{*}$, such that $(\widetilde{B}_+, \widetilde{B}_-) \stackrel{d}{=} (\bigvee_{i\geq 0} B_i^+, \bigvee_{i\geq 0} B_i^-)$. By an application of Propositions 5.2 and 5.3 in [14] we can represent M in the form

$$M(t) = \widetilde{B}_+ M_+(t) \vee \widetilde{B}_- M_-(t), \quad t \in [0,1],$$

where $M_+(t) = (M^{(k+)})_{k=1,\dots,d}$ and $M_-(t) = (M^{(k-)})_{k=1,\dots,d}$ are extremal processes with exponent measures ν_+ and ν_- respectively, where for $x \in [0,\infty)^d$, $x \neq 0$,

$$\nu_+([[0,x]]^c) = \int_0^\infty \mathrm{P}\bigg(y \bigvee_{k=1}^{d} \frac{Q_1^{(k)+}}{x^{(k)}} > 1 \bigg) \alpha y^{-\alpha-1}\,\mathrm{d}y$$

and

$$\nu_-([[0,x]]^c) = \int_0^\infty \mathrm{P}\bigg(y \bigvee_{k=1}^{d} \frac{Q_1^{(k)-}}{x^{(k)}} > 1 \bigg) \alpha y^{-\alpha-1}\,\mathrm{d}y.$$

In the following example we show that the functional convergence in the weak M_1 topology in Theorems 3.3 and 4.1 in general cannot be replaced by convergence in the stronger standard M_1 topology.

Example 4.4. Let $(T_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. unit Fréchet random variables, i.e. $\mathrm{P}(T_i \leq x) = e^{-1/x}$ for $x > 0$. Take a sequence of positive real numbers (a_n) such that $n\,\mathrm{P}(T_1 > a_n) \to 1/2$ as $n \to \infty$, for instance, we can take $a_n = 2n$. Let

$$Z_i = (T_{2i-1}, T_{2i}), \quad i \in \mathbb{Z}.$$

Then it follows easily that $n\,\mathrm{P}(\|Z_1\| > a_n) \to 1$ as $n \to \infty$. It is straightforward to see that the random process $(Z_i)_{i\in\mathbb{Z}}$ satisfies all conditions of Theorem 3.3, and hence the partial maxima processes $M_n(\,\cdot\,)$ of the linear process

$$X_i = C_0 Z_i + C_1 Z_{i-1}, \quad i \in \mathbb{Z},$$

with

$$C_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

converges in distribution in $D_\uparrow^2$ with the weak M_1 topology.

Next we show that $M_n(\,\cdot\,)$ do not converge in distribution under the standard M_1 topology on $D_\uparrow^2$. This shows that the weak M_1 topology in Theorems 3.3 and 4.1 in general cannot be replaced by the standard M_1 topology. Let

$$V_n(t) = M_n^{(1)}(t) - M_n^{(2)}(t), \quad t \in [0,1],$$

where

$$M_n^{(1)}(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{Z_i^{(1)} + Z_i^{(2)}}{a_n} = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{T_{2i-1} + T_{2i}}{a_n}$$

and

$$M_n^{(2)}(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{Z_{i-1}^{(1)} + Z_{i-1}^{(2)}}{a_n} = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{T_{2i-3} + T_{2i-2}}{a_n}.$$

The first step is to show that $V_n(\,\cdot\,)$ does not converge in distribution in D^1 endowed with the standard M_1 topology. According to [15] (see also Proposition 2 in [1], where the term "weak M_1 convergence" is used for convergence in distribution in the standard M_1 topology) it suffices to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}(\omega_\delta(V_n) > \epsilon) > 0 \tag{37}$$

for some $\epsilon > 0$, where

$$\omega_\delta(x) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} M(x(t_1), x(t), x(t_2))$$

($x \in D^1, \delta > 0$) and

$$M(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\ \min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise.} \end{cases}$$

Denote by $i' = i'(n)$ the index at which $\max_{1 \le i \le n-1} T_i$ is obtained. Fix $\epsilon > 0$ and let $A_{n,\epsilon} = \{T_{i'} > \epsilon a_n\}$ and

$$B_{n,\epsilon} = \{T_{i'} > \epsilon a_n \text{ and } \exists\, k \in \{-i' - 1, \ldots, 3\} \setminus \{0\} \text{ such that } T_{i'+k} > \epsilon a_n/8\}.$$

The regular variation property of T_1 yields $n\,\mathrm{P}(T_1 > c a_n) \to (2c)^{-1}$ as $n \to \infty$ for $c > 0$, and this, together with the fact that (T_i) is a sequence of i.i.d. variables, yield

$$\lim_{n \to \infty} \mathrm{P}(A_{n,\epsilon}) = 1 - \lim_{n \to \infty} \left(1 - \frac{n\,\mathrm{P}(T_1 > \epsilon a_n)}{n}\right)^{n-1} = 1 - e^{-(2\epsilon)^{-1}} \tag{38}$$

and

$$\begin{aligned} \limsup_{n \to \infty} \mathrm{P}(B_{n,\epsilon}) &\le \limsup_{n \to \infty} \sum_{i=1}^{n-1} \sum_{\substack{k = -n \\ k \neq 0}}^{3} \mathrm{P}(T_i > \epsilon a_n, T_{i+k} > \epsilon a_n/8) \\ &\le \limsup_{n \to \infty} (n-1)(n+3)\,\mathrm{P}(T_1 > \epsilon a_n)\,\mathrm{P}(T_1 > \epsilon a_n/8) = 2\epsilon^{-2}. \end{aligned} \tag{39}$$

Note that on the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$ it holds that $T_{i'} > \epsilon a_n$ and $T_{i'+k} \le \epsilon a_n/8$ for every $k \in \{-i' - 1, \ldots, 3\} \setminus \{0\}$. Now we distinguish two cases.

(i) i' is an even number. Then $i' = 2i^*$ for some integer i^*. Observe that on the set $A_{n,\epsilon} \setminus B_{n,\epsilon}$ we have

$$M_n^{(1)}\Big(\frac{i^*}{n}\Big) = \frac{T_{i'-1} + T_{i'}}{a_n} > \epsilon \quad \text{and} \quad M_n^{(2)}\Big(\frac{i^*}{n}\Big) = \bigvee_{i=1}^{i^*} \frac{T_{2i-3} + T_{2i-2}}{a_n} \le \frac{\epsilon}{4},$$

and similarly

$$M_n^{(1)}\Big(\frac{i^* - 1}{n}\Big) \le \frac{\epsilon}{4} \quad \text{and} \quad M_n^{(2)}\Big(\frac{i^* - 1}{n}\Big) \le \frac{\epsilon}{4}.$$

This implies

$$V_n\Big(\frac{i^*}{n}\Big) = M_n^{(1)}\Big(\frac{i^*}{n}\Big) - M_n^{(2)}\Big(\frac{i^*}{n}\Big) > \frac{3\epsilon}{4},$$

and
$$V_n\Big(\frac{i^*-1}{n}\Big) = M_n^{(1)}\Big(\frac{i^*-1}{n}\Big) - M_n^{(2)}\Big(\frac{i^*-1}{n}\Big) \in \Big[-\frac{\epsilon}{4}, \frac{\epsilon}{4}\Big].$$
Further, on the set $A_{n,\epsilon} \setminus B_{n,\epsilon}$ it holds that
$$M_n^{(1)}\Big(\frac{i^*+1}{n}\Big) = \frac{T_{i'-1}+T_{i'}}{a_n} \quad \text{and} \quad M_n^{(2)}\Big(\frac{i^*+1}{n}\Big) = \frac{T_{i'-1}+T_{i'}}{a_n},$$
which yields
$$V_n\Big(\frac{i^*+1}{n}\Big) = 0.$$

(ii) i' is an odd number. Then $i' = 2i^*-1$ for some integer i^*. Similarly as in the case (i) on the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$ one obtains
$$V_n\Big(\frac{i^*}{n}\Big) > \frac{3\epsilon}{4}, \quad V_n\Big(\frac{i^*-1}{n}\Big) \in \Big[-\frac{\epsilon}{4}, \frac{\epsilon}{4}\Big] \quad \text{and} \quad V_n\Big(\frac{i^*+1}{n}\Big) = 0.$$
Hence from (i) and (ii) we conclude that on the set $A_{n,\epsilon} \setminus B_{n,\epsilon}$ it holds that
$$\Big|V_n\Big(\frac{i^*}{n}\Big) - V_n\Big(\frac{i^*-1}{n}\Big)\Big| > \frac{3\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{2} \tag{40}$$
and
$$\Big|V_n\Big(\frac{i^*+1}{n}\Big) - V_n\Big(\frac{i^*}{n}\Big)\Big| > \frac{3\epsilon}{4}. \tag{41}$$
Note that on the set $A_{n,\epsilon} \setminus B_{n,\epsilon}$ one also has
$$V_n\Big(\frac{i^*}{n}\Big) \notin \Big[V_n\Big(\frac{i^*-1}{n}\Big), V_n\Big(\frac{i^*+1}{n}\Big)\Big],$$
and therefore taking into account (40) and (41) we obtain
$$\omega_{2/n}(V_n) \geq M\Big(V_n\Big(\frac{i^*-1}{n}\Big), V_n\Big(\frac{i^*}{n}\Big), V_n\Big(\frac{i^*+1}{n}\Big)\Big) > \frac{\epsilon}{2}$$
on the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$. Therefore, since $\omega_\delta(\,\cdot\,)$ is nondecreasing in δ, it holds that
$$\begin{aligned}\liminf_{n\to\infty} \mathrm{P}(A_{n,\epsilon} \setminus B_{n,\epsilon}) &\leq \liminf_{n\to\infty} \mathrm{P}(\omega_{2/n}(V_n) > \epsilon/2) \\ &\leq \lim_{\delta\to 0} \limsup_{n\to\infty} \mathrm{P}(\omega_\delta(V_n) > \epsilon/2).\end{aligned} \tag{42}$$

Since $x^2(1-e^{-(2x)^{-1}})$ tends to infinity as $x \to \infty$, we can find $\epsilon > 0$ such that $\epsilon^2(1-e^{-(2\epsilon)^{-1}}) > 2$, that is $1-e^{-(2\epsilon)^{-1}} > 2\epsilon^{-2}$. For this ϵ, by relations (38) and (39), we have
$$\lim_{n\to\infty} \mathrm{P}(A_{n,\epsilon}) > \limsup_{n\to\infty} \mathrm{P}(B_{n,\epsilon}),$$
i.e.
$$\liminf_{n\to\infty} \mathrm{P}(A_{n,\epsilon} \setminus B_{n,\epsilon}) \geq \lim_{n\to\infty} \mathrm{P}(A_{n,\epsilon}) - \limsup_{n\to\infty} \mathrm{P}(B_{n,\epsilon}) > 0.$$

This and (42) imply (37), and hence $V_n(\,\cdot\,)$ does not converge in distribution in D^1 with the standard M_1 topology.

To finish, if $M_n(\,\cdot\,)$ would converge in distribution in the standard M_1 topology on $D^2_{\uparrow}$, and then also on D^2, using the fact that linear combinations of the coordinates are continuous in the same topology (see Theorems 12.7.1 and 12.7.2 in [17]) and the continuous mapping theorem, we would obtain that $V_n(\,\cdot\,) = M_n^{(1)}(\,\cdot\,) - M_n^{(2)}(\,\cdot\,)$ converges in D^1 with the standard M_1 topology, which is impossible, as is shown above.

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