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About stability of equilibria of one system of stochastic delay differential equations with exponential nonlinearity

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Abstract A system of two nonlinear delay differential equations under stochastic perturbations is considered. Nonlinearity of the exponential type in each equation of the system under consideration depends on the both variables of the system. The stability in probability of the zero and nonzero equilibria of the system is studied via the general method of Lyapunov functionals construction and the method of linear matrix inequalities (LMIs). The obtained results are illustrated via examples and figures with numerical simulations of solutions of a considered system of stochastic differential equations. The proposed way of investigation can be applied to nonlinear systems of higher dimension and with other types of nonlinearity, both for delay differential equations and for difference equations.

Keywords Nonlinear differential equation, stochastic perturbations, asymptotic mean square stability, stability in probability, linear matrix inequality (LMI), exponential nonlinearity
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1 Introduction

Systems of differential and difference equations with various types of exponential nonlinearities are very popular both in theoretical research and in different applications (see, for instance, [2, 4, 5, 7–9, 15, 18–22, 24, 26–29, 33–41] and references therein).

Here, similarly to [38], the stability of the positive equilibrium of a system with exponential nonlinearity is investigated under stochastic perturbations via the general method of Lyapunov functionals construction [16, 17, 31, 32] and the method of linear matrix inequalities (LMIs) [3, 6, 10–12, 14, 23, 25, 30]. However, unlike, for instance, [28, 37–39], where the exponential nonlinearity in each equation depends on only one variable, here each equation exponentially depends on all variables of the system under consideration. The obtained results are illustrated via examples and figures with numerical simulations of solutions of the considered system of stochastic differential equations. Numerical analysis of the considered LMI is carried out using MATLAB.

Consider the system of two nonlinear delay differential equations

$$\begin{aligned} \dot{x}_1(t) + a_1 x_1(t) + b_1 x_1(t-h) &= c_1 x_1(t) e^{-p_1 x_1(t) - q_1 x_2(t)}, \\ \dot{x}_2(t) + a_2 x_2(t) + b_2 x_2(t-h) &= c_2 x_2(t) e^{-p_2 x_1(t) - q_2 x_2(t)}, \\ x_i(s) = \phi_i(s), \quad s \in [-h, 0], \quad & i = 1, 2, \quad t \geq 0, \end{aligned} \tag{1.1}$$

with positive parameters and positive initial conditions.

It is clear that the system (1.1) has the zero equilibrium $E_0 = (0, 0)$, and the nonzero equilibrium $E^* = (x_1^*, x_2^*)$ of the system (1.1) is defined by the system of two algebraic equations

$$\begin{aligned} a_1 + b_1 &= c_1 e^{-p_1 x_1 - q_1 x_2}, \\ a_2 + b_2 &= c_2 e^{-p_2 x_1 - q_2 x_2}, \end{aligned} \tag{1.2}$$

that can be presented in the form of the system of two linear equations

$$\begin{aligned} p_1 x_1 + q_1 x_2 &= r_1, \\ p_2 x_1 + q_2 x_2 &= r_2, \\ r_i = \ln\big(c_i/(a_i + b_i)\big), & \quad i = 1, 2, \end{aligned} \tag{1.3}$$

with the solution

$$x_i^* = \frac{\Delta_i}{\Delta}, \quad i = 1, 2, \quad \Delta = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} r_1 & q_1 \\ r_2 & q_2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix}. \tag{1.4}$$

Note that via (1.3) for an equilibrium $E^* = (x_1^*, x_2^*)$ with both positive coordinates the condition $r_i > 0$, i.e. $c_i > a_i + b_i$, $i = 1, 2$, must be satisfied. It is clear that for $c_i < a_i + b_i$ we get a negative equilibrium, while for $c_i = a_i + b_i$ there is the zero equilibrium. However, the inequality $c_i > a_i + b_i$ is only a necessary condition for the solution of the system (1.3) to be positive, but not a sufficient one. Indeed, for example, the system $2x_1 + x_2 = 4$, $x_1 + x_2 = 1$ with $r_1 = 4$, $r_2 = 1$ has the solution $x_1 = 3$, $x_2 = -2$, the system $2x_1 + x_2 = 4$, $x_1 + x_2 = 2$ with $r_1 = 4$, $r_2 = 2$ has the solution $x_1 = 2$, $x_2 = 0$.

Below, the system (1.1) is investigated under stochastic perturbations, the conditions of stability for both the zero and one positive equilibria are studied, the obtained results are illustrated by numerical simulations of solutions of the system under consideration.

2 Stochastic perturbations, centralization and linearization

Let (x_1^*, x_2^*) be one of the equilibria of the system (1.1). Let us suppose that the system (1.1) is exposed to stochastic perturbations of the white noise type, that are directly proportional to the deviation of the system (1.1) state $(x_1(t), x_2(t))$ from the equilibrium (x_1^*, x_2^*). As a result, we obtain the system of Ito's stochastic delay differential equations [13]

$$\begin{aligned} dx_1(t) &= \big(-a_1x_1(t) - b_1x_1(t-h) + c_1x_1(t)e^{-p_1x_1(t)-q_1x_2(t)}\big)dt \\ &\quad + \sigma_1\big(x_1(t) - x_1^*\big)dw_1(t), \\ dx_2(t) &= \big(-a_2x_2(t) - b_2x_2(t-h) + c_2x_2(t)e^{-p_2x_1(t)-q_2x_2(t)}\big)dt \\ &\quad + \sigma_2\big(x_2(t) - x_2^*\big)dw_2(t). \end{aligned} \tag{2.1}$$

Note that stochastic perturbations of the type of (2.1) were first used in [1] and later in many other works (see, for instance, [31, 32] and references therein). In this case, the equilibrium (x_1^*, x_2^*) of the deterministic system (1.1) is also the solution of the stochastic system (2.1).

2.1 Nonzero equilibrium

Let (x_1^*, x_2^*) be the nonzero equilibrium of the system (1.1). Using the new variables $y_i(t) = x_i(t) - x_i^*$, $i = 1, 2$, and (1.2), transform the first equation of the system (2.1) by the following way:

$$\begin{aligned} dy_1(t) &= \big[-a_1\big(y_1(t) + x_1^*\big) - b_1\big(y_1(t-h) + x_1^*\big) \\ &\quad + c_1\big(y_1(t) + x_1^*\big)e^{-p_1y_1(t)-q_1y_2(t)}e^{-p_1x_1^*-q_1x_2^*}\big]dt + \sigma_1y_1(t)dw_1(t) \\ &= \big[-a_1\big(y_1(t) + x_1^*\big) - b_1\big(y_1(t-h) + x_1^*\big) \\ &\quad + (a_1 + b_1)\big(y_1(t) + x_1^*\big)e^{-p_1y_1(t)-q_1y_2(t)}\big]dt + \sigma_1y_1(t)dw_1(t) \\ &= \big[\big(y_1(t) + x_1^*\big)\big((a_1 + b_1)e^{-p_1y_1(t)-q_1y_2(t)} - a_1\big) \\ &\quad - b_1\big(y_1(t-h) + x_1^*\big)\big]dt + \sigma_1y_1(t)dw_1(t). \end{aligned}$$

Similarly, for the second equation of the system (2.1) we have

$$\begin{aligned} dy_2(t) &= \big[-a_2\big(y_2(t) + x_2^*\big) - b_2\big(y_2(t-h) + x_2^*\big) \\ &\quad + c_2\big(y_2(t) + x_2^*\big)e^{-p_2y_1(t)-q_2y_2(t)}e^{-p_2x_1^*-q_2x_2^*}\big]dt + \sigma_2y_2(t)dw_2(t) \\ &= \big[-a_2\big(y_2(t) + x_2^*\big) - b_2\big(y_2(t-h) + x_2^*\big) \\ &\quad + (a_2 + b_2)\big(y_2(t) + x_2^*\big)e^{-p_2y_1(t)-q_2y_2(t)}\big]dt + \sigma_2y_2(t)dw_2(t) \\ &= \big[\big(y_2(t) + x_2^*\big)\big((a_2 + b_2)e^{-p_2y_1(t)-q_2y_2(t)} - a_2\big) \\ &\quad - b_2\big(y_2(t-h) + x_2^*\big)\big]dt + \sigma_2y_2(t)dw_2(t). \end{aligned}$$

As a result, we obtain the system of two nonlinear Ito's stochastic delay differential equations [13] with the zero solution:

$$\begin{aligned} dy_1(t) &= \big[\big(y_1(t)+x_1^*\big)\big((a_1+b_1)e^{-p_1y_1(t)-q_1y_2(t)}-a_1\big) \\ &\quad - b_1\big(y_1(t-h)+x_1^*\big)\big]dt+\sigma_1y_1(t)dw_1(t), \\ dy_2(t) &= \big[\big(y_2(t)+x_2^*\big)\big((a_2+b_2)e^{-p_2y_1(t)-q_2y_2(t)}-a_2\big) \\ &\quad - b_2\big(y_2(t-h)+x_2^*\big)\big]dt+\sigma_2y_2(t)dw_2(t). \end{aligned} \tag{2.2}$$

Using the representation $e^{-x}=1-x+o(x)$, where $\lim_{x\to 0}\frac{o(x)}{x}=0$, we obtain the linear part of the system (2.2). Really, neglecting the nonlinear terms in the square brackets of the first equation (2.2), we have

$$\begin{aligned} &\big(y_1(t)+x_1^*\big)\big((a_1+b_1)e^{-p_1y_1(t)-q_1y_2(t)}-a_1\big)-b_1\big(y_1(t-h)+x_1^*\big) \\ &\quad = \big(y_1(t)+x_1^*\big)\big((a_1+b_1)\big(1-p_1y_1(t)-q_1y_2(t)\big)-a_1\big)-b_1\big(y_1(t-h)+x_1^*\big) \\ &\quad = \big(y_1(t)+x_1^*\big)\big(b_1-(a_1+b_1)\big(p_1y_1(t)+q_1y_2(t)\big)\big)-b_1\big(y_1(t-h)+x_1^*\big) \\ &\quad = b_1\big(y_1(t)+x_1^*\big)-(a_1+b_1)x_1^*\big(p_1y_1(t)+q_1y_2(t)\big)-b_1\big(y_1(t-h)+x_1^*\big) \\ &\quad = \big(b_1-(a_1+b_1)x_1^*p_1\big)y_1(t)-(a_1+b_1)x_1^*q_1y_2(t)-b_1y_1(t-h). \end{aligned}$$

Similarly, for the second equation (2.2) we obtain

$$\begin{aligned} &\big(y_2(t)+x_2^*\big)\big((a_2+b_2)e^{-p_2y_1(t)-q_2y_2(t)}-a_2\big)-b_2\big(y_2(t-h)+x_2^*\big) \\ &\quad = \big(y_2(t)+x_2^*\big)\big((a_2+b_2)\big(1-p_2y_1(t)-q_2y_2(t)\big)-a_2\big)-b_2\big(y_2(t-h)+x_2^*\big) \\ &\quad = \big(y_2(t)+x_2^*\big)\big(b_2-(a_2+b_2)\big(p_2y_1(t)+q_2y_2(t)\big)\big)-b_2\big(y_2(t-h)+x_2^*\big) \\ &\quad = b_2\big(y_2(t)+x_2^*\big)-(a_2+b_2)x_2^*\big(p_2y_1(t)+q_2y_2(t)\big)-b_2\big(y_2(t-h)+x_2^*\big) \\ &\quad = -(a_2+b_2)x_2^*p_2y_1(t)+\big(b_2-(a_2+b_2)x_2^*q_2\big)y_2(t)-b_2y_2(t-h). \end{aligned}$$

As a result, we obtain the linear part of the system (2.2)

$$\begin{aligned} dz_1(t) &= \big[\big(b_1-(a_1+b_1)x_1^*p_1\big)z_1(t)-(a_1+b_1)x_1^*q_1z_2(t)-b_1z_1(t-h)\big]dt \\ &\quad +\sigma_1z_1(t)dw_1(t), \\ dz_2(t) &= \big[-(a_2+b_2)x_2^*p_2z_1(t)+\big(b_2-(a_2+b_2)x_2^*q_2\big)z_2(t)-b_2z_2(t-h)\big]dt \\ &\quad +\sigma_2z_2(t)dw_2(t), \end{aligned}$$

or in the matrix form

$$dz(t)=\big(Az(t)-Bz(t-h)\big)dt+\sum_{i=1}^{2}C_iz(t)dw_i(t), \tag{2.3}$$

where

$$\begin{aligned} z(t) &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad A=\begin{bmatrix} b_1-(a_1+b_1)x_1^*p_1 & -(a_1+b_1)x_1^*q_1 \\ -(a_2+b_2)x_2^*p_2 & b_2-(a_2+b_2)x_2^*q_2 \end{bmatrix}, \\ B &= \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad C_1=\begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2=\begin{bmatrix} 0 & 0 \\ 0 & \sigma_2 \end{bmatrix}. \end{aligned} \tag{2.4}$$

2.2 Zero equilibrium

Let us linearize the system (2.1) with the zero equilibrium. Neglecting the nonlinear terms, transform the expression in brackets of the first equation (2.1) by the following way:

$$\begin{aligned}
&- a_1x_1(t) - b_1x_1(t-h) + c_1x_1(t)e^{-p_1x_1(t)-q_1x_2(t)} \\
&\quad = -a_1x_1(t) - b_1x_1(t-h) + c_1x_1(t)\big(1 - p_1x_1(t) - q_1x_2(t)\big) \\
&\quad = (c_1 - a_1)x_1(t) - b_1x_1(t-h).
\end{aligned}$$

Similarly, for the second equation (2.1) we have

$$\begin{aligned}
&- a_2x_2(t) - b_2x_2(t-h) + c_2x_2(t)e^{-p_2x_1(t)-q_2x_2(t)} \\
&\quad = -a_2x_2(t) - b_2x_2(t-h) + c_2x_2(t)\big(1 - p_2x_1(t) - q_2x_2(t)\big) \\
&\quad = (c_2 - a_2)x_2(t) - b_2x_2(t-h).
\end{aligned}$$

As a result, we obtain that the linearized system decomposes into two unrelated equations of the same type

$$\begin{aligned}
dz_1(t) &= \big[(c_1 - a_1)z_1(t) - b_1z_1(t-h)\big]dt + \sigma_1z_1(t)dw_1(t), \\
dz_2(t) &= \big[(c_2 - a_2)z_2(t) - b_2z_2(t-h)\big]dt + \sigma_2z_2(t)dw_2(t).
\end{aligned} \tag{2.5}$$

3 Stability

3.1 Some necessary definitions and statements

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space, $\{\mathfrak{F}_t\}_{t\geq 0}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e. $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the braces, $\mathbf{E}$ be the mathematical expectation, H_2 be the space of $\mathfrak{F}_0$-adapted stochastic processes $\varphi(s) = (\varphi_1(s), \varphi_2(s))$, $s \in [-h, 0]$, $\|\varphi\|_0 = \sup_{s\in[-h,0]} |\varphi(s)|$, $\|\varphi\|_1^2 = \sup_{s\in[-h,0]} \mathbf{E}|\varphi(s)|^2$, $|\varphi(s)| = \sqrt{\varphi_1^2(s) + \varphi_2^2(s)}$, $|\varphi(s)|^2 = \varphi_1^2(s) + \varphi_2^2(s)$.

Definition 3.1. The zero solution of the system (2.2) with the initial condition $y(s) = (y_1(s), y_2(s)) = \phi(s)$, $s \in [-h, 0]$, is called stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y(t, \phi)$ of the system (2.2) satisfies the condition $\mathbf{P}\{\sup_{t\geq 0} |y(t, \phi)| > \varepsilon_1\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\|\phi\|_0 < \delta\} = 1$.

Definition 3.2. The zero solution of the equation (2.3) with the initial condition $z(s) = \phi(s)$, $s \in [-h, 0]$, is called:

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z(t, \phi)|^2 < \varepsilon$, $t \geq 0$, provided that $\|\phi\|_1^2 < \delta$;
- asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} \mathbf{E}|z(t, \phi)|^2 = 0$ for each initial function ϕ.

Remark 3.1. Note that the level of nonlinearity of the system (2.2) is more than one. It is known [32] that in this case sufficient conditions for asymptotic mean square stability of the zero solution of the linear part of this system, i.e. of the equation (2.3), are also sufficient conditions for stability in probability of the zero solution of the nonlinear system (2.2) and therefore are sufficient conditions for stability in probability of the equilibrium (x_1^*, x_2^*) of the system (2.1).

Let $z(t)$ be a value of the solution of the equation (2.3) in the time moment t and $z_t = z(t+s)$, $s \in [-h, 0]$, be a trajectory of the solution of the equation (2.3) until the time moment t.

Consider a functional $V(t,\varphi) : [0,\infty) \times H_2 \to \mathbf{R}_+$ that can be represented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s \in [-h, 0)$, and for $\varphi = z_t$ put

$$\begin{gathered} V_\varphi(t,z) = V(t,\varphi) = V(t,z_t) = V\big(t,z,z(t+s)\big), \\ z = \varphi(0) = z(t), \quad s \in [-h,0). \end{gathered} \tag{3.1}$$

Let D be the set of the functionals, for which the function $V_\varphi(t,z)$ defined by (3.1) has a continuous derivative with respect to t and two continuous derivatives with respect to z. The generator L of the equation (2.3) is defined on the functionals from D and has the form [13, 32] (here and everywhere below $'$ is the sign of transposition)

$$\begin{aligned} LV(t,z_t) = {} & \frac{\partial}{\partial t} V_\varphi\big(t,z(t)\big) + \nabla V_\varphi'\big(t,z(t)\big)\big(Az(t) - Bx(t-h)\big) \\ & + \frac{1}{2}\sum_{i=1}^{2} z'(t) C_i' \nabla^2 V_\varphi\big(t,z(t)\big) C_i z(t). \end{aligned} \tag{3.2}$$

Theorem 3.1 ([32]). *Let there exist a functional $V(t,\varphi) \in D$, positive constants c_1, c_2, c_3, such that the following conditions hold:*

$$\mathbf{E}V(t,x_t) \geq c_1 \mathbf{E}|x(t)|^2, \quad \mathbf{E}V(0,\phi) \leq c_2 \|\phi\|^2, \quad \mathbf{E}LV(t,x_t) \leq -c_3 \mathbf{E}|x(t)|^2.$$

Then the zero solution of the equation (2.3) is asymptotically mean square stable.

3.2 Nonzero equilibrium

Theorem 3.2. *Let there exist positive definite matrices P and R, such that the linear matrix inequality (LMI)*

$$\Phi = \begin{bmatrix} PA + A'P + S_0 + R & -PB \\ -B'P & -R \end{bmatrix} < 0 \tag{3.3}$$

holds, where via (2.4)

$$S_0 = \sum_{i=1}^{2} C_i' P C_i = \begin{bmatrix} \sigma_1^2 p_{11} & 0 \\ 0 & \sigma_2^2 p_{22} \end{bmatrix}$$

and p_{11}, p_{22} are diagonal elements of the matrix P. Then the equilibrium (x_1^*, x_2^*) of the system (2.1) is stable in probability.

Proof. Following Remark 3.1, it is enough to prove the asymptotic mean square stability of the zero solution of the linear equation (2.3). Using the general method of Lyapunov functionals construction [16, 17, 31, 32], consider the functional $V = V_1 + V_2$, where $V_1(z(t)) = z'(t)Pz(t)$, $P > 0$, the additional functional V_2 will be chosen below. Via (2.3), (3.2) and $\nabla V_1(z(t) = 2Pz(t)$, $\nabla^2 V_1(z(t)) = 2P$, we have

$$\begin{aligned} LV_1\big(z(t)\big) &= 2z'(t)P\big(Az(t) - Bx(t-h)\big) + z'(t)S_0z(t) \\ &= z'(t)\big(PA + A'P + S_0\big)z(t) - 2z'(t)PBz(t-h). \end{aligned}$$

Putting $V_2(z_t) = \int_{t-h}^{t} z'(s)Rz(s)ds$, $R > 0$, with $LV_2(z_t) = z'(t)Rz(t) - z'(t-h)Rz(t-h)$, as a result for the functional $V = V_1 + V_2$, we obtain

$$\begin{aligned} LV\big(z(t)\big) &= z'(t)\big(PA + A'P + S_0 + R\big)z(t) \\ &\quad - 2z'(t)PBz(t-h) - z'(t-h)Rz(t-h) \\ &= \eta'(t)\Phi\eta(t), \end{aligned}$$

where $\eta(t) = (z(t), z(t-h))'$ and the matrix Φ is defined in (3.3). From here and the LMI (3.3) it follows that $LV(z(t)) \le -c|\eta(t)|^2 \le -c|z(t)|^2$ for some $c > 0$. Via Theorem 3.1 it means that the zero solution of the linear equation (2.3) is asymptotically mean square stable.

Via Remark 3.1 the sufficient conditions for asymptotic mean square stability of the zero solution of the linear equation (2.3) are also sufficient conditions for stability in probability of the zero solution of the nonlinear system (2.2) and therefore are sufficient conditions for stability in probability of the equilibrium (x_1^*, x_2^*) of the system (2.1). The proof is completed. □

Example 3.1. Consider the system (2.1) with the following values of the parameters:

$$\begin{aligned} &a_1 = 1, \quad b_1 = 0.02, \quad c_1 = 4, \quad p_1 = 2, \quad q_1 = 0.2, \quad \sigma_1 = 1, \quad h = 1, \\ &\quad a_2 = 2, \quad b_2 = 0.03, \quad c_2 = 5, \quad p_2 = 0.2, \quad q_2 = 2, \quad \sigma_2 = 1. \end{aligned} \tag{3.4}$$

By that from (1.4) and (2.4) we obtain

$$\begin{gathered} x_1^* = 0.6446, \quad x_2^* = 0.3862, \\ A = \begin{bmatrix} -1.2950 & -0.1315 \\ -0.1568 & -1.5381 \end{bmatrix}, \quad B = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{gathered} \tag{3.5}$$

Via MATLAB it was shown that for the parameters (3.4), (3.5) there exist the positive definite matrices

$$P = \begin{bmatrix} 301.6380 & -23.3803 \\ -23.3803 & 252.7107 \end{bmatrix}, \quad R = \begin{bmatrix} 268.6902 & 4.3515 \\ 4.3515 & 281.0097 \end{bmatrix},$$

for which the LMI (3.3) holds. It means that the equilibrium $(x_1^*, x_2^*) = (0.6446, 0.3862)$ of the system (2.1) is stable in probability. In Fig. 1 100 trajectories of the solution of the system (2.1) are shown, with the initial conditions $x_1(s) = 0.77$, $x_2(s) = 0.27$, $s \in [-h, 0]$. One can see that all trajectories converge to the equilibrium.

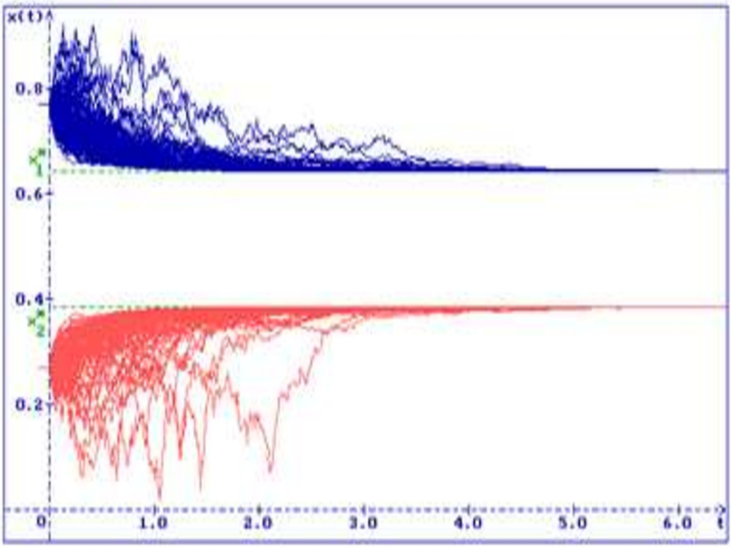

Fig. 1. 100 trajectories of the solution of the system (2.1): $x_1(t)$ (blue) and $x_2(t)$ (red)

3.3 Zero equilibrium

Note that both equations of the system (2.5) have the form

$$\dot{x}(t) = ax(t) + bx(t-h) + \sigma x(t)\dot{w}(t), \tag{3.6}$$

where $a = c_i - a_i$, $b = -b_i$, σ, $h \geq 0$ are known constants.

Lemma 3.1 ([32]). *The necessary and sufficient condition for asymptotic mean square stability of the zero solution of the equation* (3.6) *is*

$$a + b < 0, \quad G^{-1} > \frac{1}{2}\sigma^2, \tag{3.7}$$

where

$$G = \begin{cases} \frac{bq^{-1}\sin(qh)-1}{a+b\cos(qh)}, & b+|a|<0, \quad q=\sqrt{b^2-a^2}, \\ \frac{1+|a|h}{2|a|}, & b=a<0, \\ \frac{bq^{-1}\sinh(qh)-1}{a+b\cosh(qh)}, & a+|b|<0, \quad q=\sqrt{a^2-b^2}. \end{cases} \tag{3.8}$$

If, in particular, $b = 0$, *then this necessary and sufficient stability condition takes the form* $2a + \sigma^2 < 0$.

Via Remark 3.1 and Lemma 3.1, we obtain

Theorem 3.3. *Let the conditions*

$$c_i < a_i + b_i, \quad G_i^{-1} > \frac{1}{2}\sigma_i^2, \quad i = 1, 2, \tag{3.9}$$

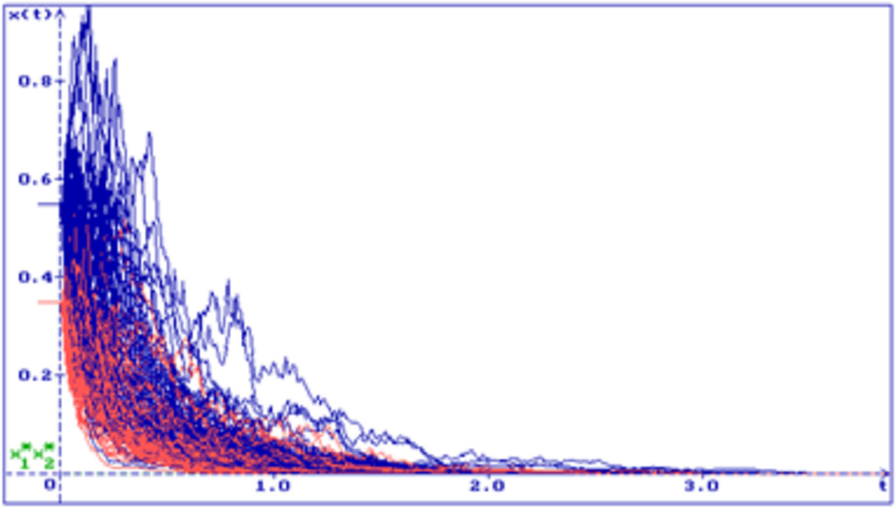

Fig. 2. 100 trajectories of the solution of the system (2.1): $x_1(t)$ (blue) and $x_2(t)$ (red)

hold, where

$$G_i = \begin{cases} \frac{1+b_i q_i^{-1}\sin(q_i h)}{a_i-c_i+b_i\cos(q_i h)}, & |a_i-c_i|<b_i, \quad q_i=\sqrt{b_i^2-(a_i-c_i)^2}, \\ \frac{1+(a_i-c_i)h}{2(a_i-c_i)}, & b_i=a_i-c_i>0, \\ \frac{1+b_i q_i^{-1}\sinh(q_i h)}{a_i-c_i+b_i\cosh(q_i h)}, & b_i<a_i-c_i, \quad q_i=\sqrt{(a_i-c_i)^2-b_i^2}. \end{cases} \tag{3.10}$$

Then the zero equilibrium $(x_1^*, x_2^*) = (0,0)$ *of the system* (2.1) *is stable in probability.*

Proof. Via the equation (3.6), for the proof it is enough to use Lemma 3.1, putting into (3.7) and (3.8)

$$a = c_i - a_i, \quad b = -b_i, \quad G = G_i, \quad \sigma = \sigma_i.$$

By that, (3.7) and (3.8) are transformed into (3.9) and (3.10). The proof is completed. □

Example 3.2. Consider the system (2.1) with the following values of the parameters:

$$\begin{gathered} a_1 = 2, \quad b_1 = 1, \quad c_1 = 1, \quad p_1 = 2, \quad q_1 = 0.2, \quad \sigma_1 = 0.9, \quad h = 1, \\ a_2 = 3, \quad b_2 = 1, \quad c_2 = 2, \quad p_2 = 0.2, \quad q_2 = 2, \quad \sigma_2 = 0.9. \end{gathered} \tag{3.11}$$

It is easy to see, that for the values of the parameters (3.11) the conditions (3.9) hold. Therefore, the zero equilibrium of the system (2.1) is stable in probability. In Fig. 2 100 trajectories of the solution of the system (2.1) are shown, with the initial conditions $x_1(s) = 0.55$, $x_2(s) = 0.35$, $s \in [-h, 0]$. One can see that all trajectories converge to the zero.

4 Conclusions

A system of two differential equations with exponential nonlinearity is considered. The nonlinearity in each equation depends on both variables of the system. It is supposed that this system is exposed to stochastic perturbations of the white noise type, which are directly proportional to the deviation of the system state from one of the system's equilibria (the zero or nonzero). Some sufficient conditions for stability in probability of the both system equilibria are obtained by virtue of the general method of Lyapunov functionals construction and the method of linear matrix inequalities (LMIs). The obtained results are illustrated by numerical simulations of solutions of the considered system. The method of stability investigation described here can be applied to systems of higher dimension and with other types of nonlinearity.

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