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Exponentially quasi-mixing limits for killed symmetric Lévy processes

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Abstract Quasi-mixing limits of the killed symmetric Lévy process are studied. It is proved that (intrinsic) ultracontractivity of the underlying process implies the existence of its (uniformly) exponentially quasi-mixing limits. As a by-product, this implication ensures that the process has (uniformly) exponential quasi-ergodicity and (uniformly) exponentially fractional quasi-ergodicity on L^p ($p \geq 1$). It is noteworthy that precise rates of convergence and precise limiting equalities are provided, which are determined by spectral gaps and eigenfunction ratios of the underlying process. Finally, three examples are provided to demonstrate the theoretical results.

Keywords Lévy process, Quasi-ergodicity, Fractional quasi-ergodicity, Quasi-mixing limit, spectral
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1 Introduction

We consider a d-dimensional ($d \geqslant 1$) symmetric Lévy process Y on $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\Omega = \{\omega : [0,\infty) \to \mathbb{R}^d \mid \omega \text{ is cádlág}\}$ is the collection of all cádlág-paths (right-

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continuous with left limits) from $[0, \infty)$ to $\mathbb{R}^d$ and $\mathcal{F}_t \triangleq \sigma\{Y_s, 0 \le s \le t\}$ is the σ-algebra generated by $\{Y_s, 0 \le s \le t\}$. Denote by $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$ the corresponding Markov family. Let $D \subset \mathbb{R}^d$ be a bounded connected nonempty open domain and τ_D (or τ for short) be the stopping time defined by $\inf\{t > 0 : Y_t \notin D\}$. We define

$$X_t = \begin{cases} Y_t, & \text{if } \tau > t, \\ \partial, & \text{if } \tau \le t, \end{cases}$$

and denote the Lévy measure of X_t by υ which is assumed to be nonzero. Here, ∂ is an extra point and $\inf \emptyset$ is defined as infinity by convention. Then we call $X = (X_t, \mathcal{F}_t, \mathbb{P}_x, \tau)$ the process on D obtained by killing Y upon exiting D, and its transition function is clearly given by

$$P_t(x, B) = P(t; x, B) = \mathbb{P}_x(X_t \in B; \tau > t), \quad t > 0,\ B \in \mathcal{B}(D),$$

where $\mathcal{B}(D)$ is the Borel σ-algebra of D. Under these circumstances, we are mainly interested in the long-term behavior of X.

As a typical model with wide application in the fields of finance, physics and signal processing, cf. [1, 7, 21, 22], Lévy process has a long research history. One of the most fundamental problems is to study its long-term behavior. This paper is devoted to investigating quasi-mixing limits of the Lévy process with killing. More specially, we intend to discuss the *quasi-stationary distribution* (*qsd*), *fractional quasi-stationary distribution* (*fqsd*), *quasi-ergodic distribution* (*qed*), *fractional quasi-ergodic distribution* (*fqed*). Recall that $\mu \in \mathcal{P}(D)$ is called a *qsd* of X if there exists a $\rho \in \mathcal{P}(D)$ such that

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_t \in A | \zeta > t) = \mu(A), \quad t > 0,\ A \in \mathcal{B}(D), \tag{1}$$

and $\nu \in \mathcal{P}(D)$ is called a θ-*fqsd* ($\theta \in (0, 1)$) of X if there is a $\rho \in \mathcal{P}(D)$ such that

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_{\theta t} \in A | \tau > t) = \nu(A), \quad t > 0,\ A \in \mathcal{B}(D), \tag{2}$$

where $\mathcal{P}(D)$ is the set of all probability measures on $(D, \mathcal{B}(D))$, $\mathbb{1}_A$ is the indicator function of A, $\mathbb{P}_\rho(\cdot) := \int_D \mathbb{P}_x(\cdot)\rho(dx)$ is the probability taken for X with an initial distribution ρ. Furthermore, μ and ν are said to be the *quasi-ergodic distribution* (*qed*) and *fractional quasi-ergodic distribution* (*fqed*) of X if (1) and (2) hold for all $\rho \in \mathcal{P}(D)$. In addition, we study quasi-mixing limits and double limit of X,

$$\lim_{t\to\infty} \mathbb{E}_\rho(f(X_{\theta t})g(X_{\eta t}) | \tau > t), \quad \theta, \eta \in (0, 1),$$
$$\lim_{t\to\infty} \lim_{T\to\infty} \mathbb{E}_\rho(f(X_t) | \tau > T),$$

being viewed as extensions of *qed* and *fqed*, where f and g are some suitable functions on D. For backgrounds and applications on *qsd*, *fqsd*, *qed* and *fqed*, we refer to Champagnat et al. [2–4], Chen et al. [5, 6], Guillin et al. [13], Kaleta et al. [17], Méléard and Villemonais [25] and Zhang et al. [29].

There are a large number of publications on the relationship between the conditional distributions of a Markov process converging (uniformly) exponentially to a unique *qsd* or *fqsd* and (intrinsic) ultracontractivity of its associated semigroup. For example, Knobloch and Partzsch [19] reveal that intrinsic ultracontractivity implies uniformly conditional ergodicity; Zhang et al. [29] develop that ultracontractivity ensure the existence and uniqueness of the *qed*, *fqed*, etc., for the general Markov processes in a finite measure space; Zhang et al. [28] indicate that a symmetric Markov semigroup having (intrinsic) ultracontractivity implies the (uniformly) exponential convergence of the conditional distributions to a unique *qsd*. It is worth mentioning that, to the best of our knowledge, there are almost no papers describing the relationship between quasi-mixing limits and ultracontractivity explicitly except [29]. In this paper, the property that quasi-mixing limits of X (uniformly) exponentially exist will be studied via (intrinsic) ultracontractivity of its associated semigroup. For more information on the long-term behavior of Markov process, the interested reader can consult [2–4, 13, 15, 16, 20, 23, 24, 27, 17].

Our paper is structured as follows. In Section 2, we give some basic settings and key lemmas. We exhibit the main results in Section 3. In Section 4, three examples are provided to demonstrate the theoretical results.

2 Preparations

Here is the basic setting of this paper. For each $p \in [1,\infty]$, p^* and $L^p(D)$ represent respectively its Hölder conjugate index and the usual Lebesgue space endowed with the norm $\|\cdot\|_p$. The scalar product and norm in $L^2(D)$ are written as $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. $\rho(fg)$ denotes the integral of fg w.r.t. the measure ρ on D if this integral exists. $a \wedge b$ denotes the minimum of $a, b \in \mathbb{R}$. The values of the constants c_1, c_2, ... may change from one appearance to another.

Definition 1. Let $\{T_t\}$ be a strongly continuous semigroup on $L^2(D)$. We say that T_t is ultracontractive if and only if $\{T_t\}$ has a kernel $k(t,x,y)$ satisfying $0 \le k(t,x,y) \le c_t < \infty$ a.e. for some constant c_t.

Definition 2. Provided that $\{T_t\}$, possessing a positive integral kernel $k(t,x,y)$, is a strongly continuous semigroup on $L^2(D)$, let $(-\mathcal{L}, D(\mathcal{L}))$ be its generator and φ_0 be the bottom eigenfunction (ground state) of $\mathcal{L}$. We say that T_t is intrinsically ultracontractive if

$$\sup_{x,y\in D} \frac{k(t,x,y)}{\varphi_0(x)\varphi_0(y)} < \infty, \quad t > 0.$$

For the reader's convenience, we recall the equivalent statements of Definition 2, see Davies and Simon [12, Theorem 2.1]. That is, for any $t > 0$ there exists a constant c_t such that

$$k(t,x,y) \ge c_t \sqrt{k(t,x,x)}\sqrt{k(t,y,y)}, \quad x, y \in D. \tag{3}$$

Hypothesis 1. The standing assumptions in this paper are the following:

H1 Y has a symmetric density $q(t,x,y) = q(t,y,x) = \tilde{q}(t,x-y)$ for any $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

H2 $q(t,x,y)$ is continuous and there exists a constant $c(\delta)$ such that $\tilde{q}(t,z) \le c(\delta)$ for $t > 0$, $|z| \ge \delta$.

H3 There is a nonnegative and locally integrable Borel function F on $\mathbb{R}^d \setminus \{0\}$ such that $m(B) = \int_B F(x)\upsilon(dx)$ for $B \in \mathcal{B}(D)$, where m and υ respectively stand for the Lebesgue measure in $\mathbb{R}^d$ and the Lévy measure of X.

In what follows, let $X = (X_t, \mathcal{F}_t, \mathbb{P}_x, \tau)$ be the process on D obtained by killing Y upon exiting D, as discussed in the introduction. We infer from [14, Section 2] that under **H1** and **H2** the transition density $p(t, x, y)$ of X is exactly given by

$$p(t, x, y) = q(t, x, y) - \mathbb{E}_x[t > \tau; q(t - \tau, X_\tau, y)].$$

Now, we denote by $\{P_t\}$ the transition semigroup of X. It is then well-known, cf. [14, Section 2], that $\{P_t\}$ is a strongly continuous semigroup of contractions on $L^2(D)$. The following lemmas establish several analytical components for the semigroup $\{P_t\}$: quantitative bounds for its heat kernel; spectral estimates for eigenfunctions of its generator; and asymptotic characterization as $t \to \infty$.

Lemma 1. *Assume **H1** and **H2**. Let $(-\mathcal{A}, D(\mathcal{A}))$ be the generator of $\{P_t\}$ on $L^2(D)$.*

(i) *$\mathcal{A}$ has purely discrete spectrum consisting of eigenvalues $\{\lambda_i\}_{i=1}^\infty$ with $0 < \lambda_1 \le \lambda_2 \le \cdots \uparrow +\infty$, and there exists a complete orthonormal basic $\{\varphi_i\}_{i=1}^\infty$ of $L^2(D)$. Here, each λ_i is counted according to multiplicity, $\varphi_i \in D(\mathcal{A})$ is a continuous function on D such that $\mathcal{A}\varphi_i = \lambda_i\varphi_i$ for any $i \ge 1$, and φ_1 can be chosen to be strictly positive on D.*

(ii) *The transition density function $p(t, \cdot, \cdot)$ of X is symmetric, continuous, strictly positive and bounded on $D \times D$, $t > 0$. Additionally, $p(t, x, y)$ has the expansion*

$$p(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y), \quad t > 0,\ x, y \in D,$$

where the series is locally uniformly convergent on $(0, \infty) \times D \times D$.

(iii) *b_t is a continuous function in $L^2(D)$ and $|\varphi_i| \le e^{\lambda_i t} b_{2t} \in L^p(D)$ for any $t > 0$, $i \ge 1$ and $p \in [1, \infty]$, where $b_t(x) := \sqrt{p(t, x, x)}$ for $t > 0$ and $x \in D$.*

(iv) *For each $x \in D$, $b_t(x)$ and $e^{\lambda_1 t} b_{2t}(x)$ are analytic, logarithmically convex, monotonically decreasing functions of t.*

(v) *For any $t > s > 0$ and $x, y \in D$, we find*

$$|e^{\lambda_1 t} p(t, x, y) - \varphi_1(x)\varphi_1(y)| \le e^{\lambda_2 s} e^{-(\lambda_2 - \lambda_1)t} b_s(x) b_s(y).$$

Proof. To see (i)–(iv), we refer to [14, Section 2], [11, Theorem 7.2.3 and Theorem 7.2.5], [10, Lemma 2.1, Corollary 2.2, and Lemma 2.3] or [9, Theorem 2.1.4]. The item (v) follows specifically from (ii) and the Cauchy–Schwarz inequality,

$$\begin{aligned} &|e^{\lambda_1 (r+s)} p(r + s, x, y) - \varphi_1(x)\varphi_1(y)| \\ &= \left| \sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)(r+s)} \varphi_i(x)\varphi_i(y) \right| \end{aligned}$$

$$\le \left[\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(r+s)}\varphi_i^2(x)\right]^{\frac{1}{2}}\left[\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(r+s)}\varphi_i^2(y)\right]^{\frac{1}{2}}$$
$$\le \left[\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)r}e^{-(\lambda_i-\lambda_1)s}\varphi_i^2(x)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)r}e^{-(\lambda_i-\lambda_1)s}\varphi_i^2(y)\right]^{\frac{1}{2}}$$
$$\le e^{\lambda_1 s}e^{-(\lambda_2-\lambda_1)r}b_s(x)b_s(y), \quad r,s>0,\ x,y\in D.$$

Finally, we let $r = t - s$ in the above equation to derive (v). □

Lemma 2. *Assume **H1** and **H2**.*

(i) *For any $t > 0$, $x \in D$ and $f \in L^p(D)$ with $p \in [1,\infty]$, $P_t f(x)$ has the bounded continuous version*

$$P_t f(x) = \int_D p(t,x,y)f(y)dy = \sum_{i=1}^{\infty} e^{-\lambda_i t}(\varphi_i, f)\varphi_i(x),$$

where the series converges absolutely and uniformly in $(t,x) \in [\epsilon,\infty)\times D$ for any $\epsilon > 0$.

(ii) *For any $t > s > 0$, $\rho \in \mathcal{P}(D)$ and $f \in L^p(D)$ with $p \in [1,\infty]$, we find*

$$|\rho(e^{\lambda_1 t}P_t f) - \rho(\varphi_1)(\varphi_1, f)| \le e^{\lambda_2 s}e^{-(\lambda_2-\lambda_1)t}\rho(b_s)\|b_s\|_{p^*}\|f\|_p.$$

Proof. The item (ii) is an immediate consequence of Lemma 1(v). To confirm (i), owing to Lemma 1(i)(iii), all we must show is that the integration and summation in the second equation may be interchanged. But, this can be guaranteed by the Cauchy–Schwartz inequality and the dominated convergence. Indeed, if $p_n(t,x,y)$ is the nth partial sum of $p(t,x,y)$ for $x,y \in D$, then we arrive at

$$|p_n(t,x,\cdot)f(\cdot)| \le b_t(x)b_t(\cdot)|f(\cdot)| \in L^1(D), \quad n \ge 1,\ t > 0,\ f \in L^p(D),\ p \in [1,\infty].$$

□

We strengthen Lemma 2 to analyze uniformly quasi-ergodic behavior of X.

Lemma 3. *Assume **H1**, **H2** and **H3**. Put*

$$\alpha_t := \left[\sup_{x,y\in D}\frac{p(t,x,y)}{\varphi_1(x)\varphi_1(y)}\right]^{\frac{1}{2}}, \quad t > 0.$$

(i) *$\{P_t\}$ is intrinsically ultracontractive, or alternatively, $\alpha_t < \infty$.*

(ii) *α_t and $e^{\lambda_1 t}\alpha_{2t}$ are monotonically decreasing functions of t. What is more, we have $b_t(x) \le \alpha_t\varphi_1(x)$ and $|\varphi_i(x)| \le \alpha_{2t}e^{\lambda_i t}\varphi_1(x)$ for any $t > 0$, $x \in D$, $i \ge 1$.*

(iii) *For any $t > s > 0$ and $x, y \in D$,*

$$|e^{\lambda_1 t}p(t,x,y) - \varphi_1(x)\varphi_1(y)| \le \alpha_s^2 e^{\lambda_2 s}e^{-(\lambda_2-\lambda_1)t}\varphi_1(x)\varphi_1(y),$$

which in turn leads, for any $\rho \in \mathcal{P}(D)$ and $f \in L^p(D)$ with $p \in [1,\infty]$, to

$$|\rho(e^{\lambda_1 t}P_t f) - \rho(\varphi_1)(f,\varphi_1)| \le \alpha_s^2 e^{\lambda_2 s}e^{-(\lambda_2-\lambda_1)t}\rho(\varphi_1)(\varphi_1,|f|)$$
$$\le \alpha_s^2 e^{\lambda_2 s}e^{-(\lambda_2-\lambda_1)t}\|\varphi_1\|_\infty\|\varphi_1\|_{p^*}\|f\|_p.$$

Proof. The item (i) follows directly from [18, Theorem 3.11]. The item (iii) is just a result of (ii) and Lemma 2(ii). To build (ii), recalling Lemma 1(iii) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
e^{2\lambda_1(t+s)}\alpha_{2t+2s}^2 &= \sup_{x,y\in D}\frac{e^{2\lambda_1(t+s)}p(2t+2s,x,y)}{\varphi_1(x)\varphi_1(y)}\\
&\le \sup_{x,y\in D}\frac{e^{\lambda_1(t+s)}b_{2t+2s}(x)e^{\lambda_1(t+s)}b_{2t+2s}(y)}{\varphi_1(x)\varphi_1(y)}\\
&\le \sup_{x,y\in D}\frac{e^{\lambda_1 t}b_{2t}(x)e^{\lambda_1 t}b_{2t}(y)}{\varphi_1(x)\varphi_1(y)}\\
&\le e^{2\lambda_1 t}\alpha_{2t}^2, \quad t>0,\ s\ge 0,
\end{aligned}$$

proving the first part. The second is due to the definitions of α_t, b_t and Lemma 1(iii). □

3 Main results

In this section, we consider the existence of (uniformly) exponentially quasi-mixing limits of the killed symmetric Lévy process X given in the introduction. As a by-product, (uniformly) exponential quasi-ergodicity and (uniformly) exponentially fractional quasi-ergodicity on $L^p(D)$ $(p \ge 1)$ of X are established. It is worth emphasizing that our results are straightforward.

Theorem 1. *Assume* ***H1*** *and* ***H2****. Let μ be the measure $\varphi_1 \cdot m/m(\varphi_1)$, $\mathbb{B}_p$ be the unit ball of $L^p(D)$, κ be the multiplicity of the second eigenvalue λ_2 of $\mathcal{A}$, and $\mathbb{1} = \mathbb{1}_D$.*

(i) *X admits the following exponential quasi-ergodicity on $L^p(D)$:*

$$\begin{aligned}
&\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)t}\sup_{f\in\mathbb{B}_p}|\mathbb{E}_\rho(f(X_t)|\tau>t)-\mu(f)|\\
&\quad=\frac{|\rho(\varphi_2)|}{\rho(\varphi_1)}\left\|\sum_{i=2}^{\kappa+1}\frac{m(\varphi_1)\varphi_i-m(\varphi_i)\varphi_1}{m(\varphi_1)^2}\right\|_{p^*}, \quad p\in[1,\infty],\ \rho\in\mathcal{P}(D).
\end{aligned}$$

In particular, μ is a qed of X by taking $p=\infty$ in the above limit.

(ii) *Assume in addition* ***H3****. X admits uniform and exponential quasi-ergodicity on $L^p(D)$:*

$$\begin{aligned}
&\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)t}\sup_{(\rho,f)\in\mathcal{P}(D)\times\mathbb{B}_p}|\mathbb{E}_\rho(f(X_t)|\tau>t)-\mu(f)|\\
&\quad=\sup_{\rho\in\mathcal{P}(D)}\frac{|\rho(\varphi_2)|}{\rho(\varphi_1)}\left\|\sum_{i=2}^{\kappa+1}\frac{m(\varphi_1)\varphi_i-m(\varphi_i)\varphi_1}{m(\varphi_1)^2}\right\|_{p^*}, \quad p\in[1,\infty].
\end{aligned}$$

Proof. (i) For any $\rho \in \mathcal{P}(D)$ and $f \in \mathbb{B}_p$ with $p \in [1,\infty]$. We calculate by using Lemma 2(i) that

$$\begin{aligned}\mathbb{E}_\rho(f(X_t)|\tau > t) - \mu(f) &= \frac{\rho(P_t f) - \rho(\mu(f)P_t \mathbb{1})}{\rho(P_t\mathbb{1})}\\ &= \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}[(f,\varphi_i)-(\varphi_i,\mathbb{1})\mu(f)]\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}, \quad t>0.\end{aligned} \tag{4}$$

We then multiply both sides of (4) by $e^{(\lambda_2-\lambda_1)t}$ to derive

$$\begin{aligned}&e^{(\lambda_2-\lambda_1)t}[\mathbb{E}_\rho(f(X_t)|\tau > t) - \mu(f)] - \frac{\rho\{\sum_{i=1}^{\kappa+1}[(f,\varphi_i)-(\varphi_i,\mathbb{1})\mu(f)]\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &= \frac{\rho\{\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}[(f,\varphi_i)-(\varphi_i,\mathbb{1})\mu(f)]\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.\end{aligned} \tag{5}$$

Observe that $\rho(e^{\lambda_1 t}P_t\mathbb{1}) \to \|\varphi_1\|_1\rho(\varphi_1)$ as $t \to \infty$ by Lemma 2(ii), and that Lemma 1(iii) yields, for any $0 < \epsilon < (\lambda_{\kappa+2}-\lambda_2)/\lambda_{\kappa+2}$,

$$\begin{aligned}&\rho\Big\{\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}[(f,\varphi_i)-(\varphi_i,\mathbb{1})\mu(f)]\varphi_i\Big\}\\ &\quad\le \rho\Big\{\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}\|f\|_p(\|\varphi_i\|_{p^*}+\|\varphi_1\|_{p^*}\|\varphi_i\|_1/\|\varphi_1\|_1)|\varphi_i|\Big\}\\ &\quad\le \sum_{i=\kappa+2}^\infty e^{-[(1-\epsilon)\lambda_i-\lambda_2]t}[\|b_{\epsilon t}\|_{p^*}+\|b_{\epsilon t}\|_1\|\varphi_1\|_{p^*}/\|\varphi_1\|_1]\|b_{\epsilon t}\|_\infty.\end{aligned}$$

Since b_t is decreasing by Lemma 1(iv), the previous inequality implies that the right-hand side of (5) converges to zero. We take successively the absolute value, the supremum w.r.t. $f \in \mathbb{B}_p$ and the limit as $t \to \infty$ in (5), and then use the continuous linear functional representation theorem to obtain the desired results.

(ii) For any given $0 < \epsilon < \min\{(\lambda_2-\lambda_1)/\lambda_2, (\lambda_{\kappa+2}-\lambda_2)/\lambda_{\kappa+2}\}$, when $t > 0$ is large enough such that $1-\alpha_{\epsilon t}^2 e^{-[(1-\epsilon)\lambda_2-\lambda_1]t} > 0$, Lemma 3 suggests that the right-hand side of (5) is bounded by

$$\begin{aligned}&\frac{\rho\big(\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}|(f,\varphi_i)-(\varphi_i,\mathbb{1})\mu(f)||\varphi_i|\big)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\quad\le \frac{\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}(\|\varphi_i\|_{p^*}\|\varphi_1\|_1\|f\|_p+\|\varphi_i\|_1\|\varphi_1\|_{p^*}\|f\|_p)\rho(|\varphi_i|)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})\|\varphi_1\|_1}\\ &\quad\le \frac{\sum_{i=\kappa+2}^\infty 2e^{-[(1-\epsilon)\lambda_i-\lambda_2]t}\alpha_{\epsilon t}^2\|\varphi_1\|_{p^*}\|\varphi_1\|_1\rho(\varphi_1)}{(1-\alpha_{\epsilon t}^2 e^{-[(1-\epsilon)\lambda_2-\lambda_1]t})\|\varphi_1\|_1^2\rho(\varphi_1)}\\ &\quad= \frac{\sum_{i=\kappa+2}^\infty 2e^{-[(1-\epsilon)\lambda_i-\lambda_2]t}\alpha_{\epsilon t}^2\|\varphi_1\|_{p^*}}{(1-\alpha_{\epsilon t}^2 e^{-[(1-\epsilon)\lambda_2-\lambda_1]t})\|\varphi_1\|_1}, \quad f\in\mathbb{B}_p,\ \rho\in\mathcal{P}(D).\end{aligned}$$

□

We next explore the (uniformly) exponentially fractional quasi-ergodicity of X.

Theorem 2. *Assume* ***H1*** *and* ***H2****, and let* ν *be the measure* $\varphi_1^2 \cdot m$.

(i) *For* $\theta \in (0,1)$, *X admits exponentially* θ*-fractional quasi-ergodicity on* $L^p(D)$*:*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)[(t-\theta t)\wedge(\theta t)]} \sup_{f\in\mathbb{B}_p} |\mathbb{E}_\rho(f(X_{\theta t})|\tau > t) - \nu(f)| = \sup_{f\in\mathbb{B}_p} |\Phi(\rho,\kappa,\theta,f)|, \quad p \in [1,\infty],\ \rho \in \mathcal{P}(D),$$

where the function $\Phi(\rho,\kappa,\theta,f)$ *has the expression*

$$\begin{cases} \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)\rho(\varphi_i)}{\rho(\varphi_1)}, & \text{if } 0 < \theta < \frac{1}{2}, \\ \sum_{i=2}^{\kappa+1} \frac{m(\varphi_i)(\varphi_i, f\varphi_1)}{m(\varphi_1)}, & \text{if } \frac{1}{2} < \theta < 1, \\ \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)[\rho(\varphi_i)m(\varphi_1) + m(\varphi_i)\rho(\varphi_1)]}{\rho(\varphi_1)m(\varphi_1)}, & \text{if } \theta = \frac{1}{2}. \end{cases}$$

In particular, ν *is a* θ*-fqed of X by taking* $p = \infty$ *in the above limit.*

(ii) *Assume in addition* ***H3****. Then the limit in (i) is uniformly convergent w.r.t.* $(\rho, f) \in \mathcal{P}(D) \times \mathbb{B}_p$.

Proof. We invoke the properties of conditional expectation and the Markov property of X to discover that, for any $t \geq 0$, $0 < \theta < 1$, $x \in D$ and $f \in \mathbb{B}_p$,

$$\mathbb{E}_x[f(X_{\theta t})\mathbb{1}_{\{\tau>t\}}] = P_{\theta t}[f P_{t-\theta t}\mathbb{1}](x). \tag{6}$$

Thus we compute easily by using (6) and Lemma 2(i) that

$$\begin{aligned} \mathbb{E}_\rho(f(X_{\theta t})|\tau > t) - \nu(f) &= \frac{\rho\{P_{\theta t}(f P_{t-\theta t}\mathbb{1}) - \nu(f)P_{\theta t}\mathbb{1}\}}{\rho(P_t\mathbb{1})} \\ &= \frac{\rho\{e^{\lambda_1\theta t}(P_{\theta t}([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}))\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &= \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)\theta t}([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1},\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad + \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)(t-\theta t)}(\mathbb{1},\varphi_i)(f\varphi_1,\varphi_i)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}, \quad t > 0,\ \rho \in \mathcal{P}(D). \end{aligned} \tag{7}$$

To estimate the right-hand side of (7), we prepare some facts. Thanks to Lemma 1(iii) and the Cauchy–Schwarz inequality, for any $t > 0$, $p \in [1,\infty]$ and $f \in \mathbb{B}_p$,

$$\nu(|f|) = (|f|,\varphi_1^2) \leq \|f\|_p \|\varphi_1^2\|_{p^*} \leq \|\varphi_1\|_\infty^2 m(D)^{\frac{1}{p^*}} \triangleq c_1, \tag{8}$$

$$m(e^{\lambda_1 t}P_t|f|) \leq e^{\lambda_1 t}(b_t,|f|)m(b_t) \leq m(D)^{\frac{1+p^*}{p^*}} e^{\lambda_1 t}\|b_t\|_\infty^2 \triangleq h(t). \tag{9}$$

(i) For any $\epsilon > 0$, $t > 0$, $i \geq 1$ and $\theta \in (0, 1)$, we use the symmetry of $\{P_t\}$, Hölder's inequality, Lemma 1(iii) and (8)–(9) to estimate the right-hand side of (7):

$$\begin{aligned}|(fe^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}, \varphi_i)\varphi_i| &\leq h(t-\theta t)e^{\lambda_i\epsilon\theta t}\|b_{\epsilon\theta t}\|_\infty^2,\\ |(\nu(f)e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}, \varphi_i)\varphi_i| &\leq c_1 h(t-\theta t)e^{\lambda_i\epsilon\theta t}\|b_{\epsilon\theta t}\|_\infty^2,\\ |(\mathbb{1}, \varphi_i)(f\varphi_1, \varphi_i)\varphi_1| &\leq e^{\lambda_i\epsilon(t-\theta t)}m(D)^{\frac{1+p^*}{p^*}}\|b_{\epsilon(t-\theta t)}\|_\infty^2\|\varphi_1\|_\infty^2\\ &\triangleq c_2 e^{\lambda_i\epsilon(t-\theta t)}\|b_{\epsilon(t-\theta t)}\|_\infty^2.\end{aligned} \tag{10}$$

In order to obtain the expression of Φ, we divide θ into three cases and denote

$$\Phi_1(t, \rho, \kappa, \theta, f) := \frac{\rho\{\sum_{i=2}^{\kappa+1}([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}, \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})},$$

$$\Phi_2(t, \rho, \kappa, \theta, f) := \frac{\rho\{\sum_{i=2}^{\kappa+1}(\mathbb{1}, \varphi_i)(f\varphi_1, \varphi_i)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.$$

Case 1. If $\theta \in (0, \frac{1}{2})$, we multiply (7) by $e^{(\lambda_2-\lambda_1)\theta t}$ and use (10) to yield

$$\begin{aligned}&|e^{(\lambda_2-\lambda_1)\theta t}[\mathbb{E}_\rho(f(X_{\theta t})|\tau > t) - \nu(f)] - \Phi(\rho, \kappa, \theta, f)|\\ &\quad\leq \frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)\theta t}|([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}, \varphi_i)\varphi_i|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\qquad+ \frac{\rho\{\sum_{i=2}^{\infty} e^{(\lambda_2-\lambda_1)\theta t-(\lambda_i-\lambda_1)(t-\theta t)}|(\mathbb{1}, \varphi_i)(f\varphi_1, \varphi_i)\varphi_1|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\qquad+ |\Phi(\rho, \kappa, \theta, f) - \Phi_1(t, \rho, \kappa, \theta, f)|\\ &\quad\leq \frac{\sum_{i=\kappa+2}^{\infty}(1+c_1)e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)\theta t}h(t-\theta t)\|b_{\epsilon\theta t}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\qquad+ \frac{\sum_{i=2}^{\infty} c_2 e^{(\lambda_2-\lambda_1)\theta t-(\lambda_i-\lambda_1-\lambda_i\epsilon)(t-\theta t)}\|b_{\epsilon(t-\theta t)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\qquad+ |\Phi(\rho, \kappa, \theta, f) - \Phi_1(t, \rho, \kappa, \theta, f)|.\end{aligned} \tag{11}$$

We note that b_t and $h(t)$ are monotonically decreasing functions of t, and that $\rho(e^{\lambda_1 t}P_t\mathbb{1}) \to m(\varphi_1)\rho(\varphi_1)$ as $t \to \infty$. Accordingly, we conclude that the right-hand side of (11) converges to zero uniformly w.r.t. $f \in \mathbb{B}_p$ as $t \to \infty$ for any fixed $\epsilon \in (0, \epsilon_0)$, where

$$\epsilon_0 = \min\left\{\frac{\lambda_{\kappa+2}-\lambda_2}{\lambda_{\kappa+2}}, \frac{(1-2\theta)(\lambda_2-\lambda_1)}{(1-\theta)\lambda_2}\right\}.$$

Case 2. If $\theta \in (\frac{1}{2}, 1)$, multiplying (7) by $e^{(\lambda_2-\lambda_1)(t-\theta t)}$, we use (10) to obtain

$$\begin{aligned}&|e^{(\lambda_2-\lambda_1)(t-\theta t)}[\mathbb{E}_\rho(f(X_{\theta t})|\tau > t) - \nu(f)] - \Phi(\rho, \kappa, \theta, f)|\\ &\quad\leq \frac{\rho\{\sum_{i=2}^{\infty} e^{(\lambda_2-\lambda_1)(t-\theta t)-(\lambda_i-\lambda_1)\theta t}|([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1}, \varphi_i)\varphi_i|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &\qquad+ \frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)(t-\theta t)}|(\mathbb{1}, \varphi_i)(f\varphi_1, \varphi_i)\varphi_1|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\end{aligned}$$

$$
\begin{aligned}
&+|\Phi(\rho,\kappa,\theta,f)-\Phi_2(\rho,\kappa,\theta,f)| \qquad (12)\\
\le\ & \frac{\sum_{i=2}^{\infty}(1+c_1)h(t-\theta t)e^{(\lambda_2-\lambda_1)t-(\lambda_i+\lambda_2-2\lambda_1-\lambda_i\epsilon)\theta t}\|b_{\epsilon\theta t}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&+\frac{\sum_{i=\kappa+2}^{\infty}c_2e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)(t-\theta t)}\|b_{\epsilon(t-\theta t)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&+|\Phi(\rho,\kappa,\theta,f)-\Phi_2(\rho,\kappa,\theta,f)|.
\end{aligned}
$$

Similar to **Case 1**, the right member of (12) converges to zero uniformly w.r.t. $f\in\mathbb{B}_p$ as $t\longrightarrow\infty$ for any fixed $\epsilon\in(0,\epsilon_1)$, where

$$\epsilon_1=\min\left\{\frac{\lambda_{\kappa+2}-\lambda_2}{\lambda_{\kappa+2}},\frac{(2\theta-1)(\lambda_2-\lambda_1)}{\theta\lambda_2}\right\}.$$

Case 3. If $\theta=1/2$, multiplying both sides of (7) by $e^{(\lambda_2-\lambda_1)\theta t}$ gives

$$
\begin{aligned}
&|e^{(\lambda_2-\lambda_1)\theta t}[\mathbb{E}_\rho(f(X_{\theta t})|\tau>t)-\nu(f)]-\Phi(\rho,\kappa,\theta,f)|\\
&\quad\le \frac{\rho\{\sum_{i=\kappa+2}^{\infty}e^{-(\lambda_i-\lambda_2)\theta t}|([f-\nu(f)]e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1},\varphi_i)\varphi_i|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\qquad+\frac{\rho\{\sum_{i=\kappa+2}^{\infty}e^{-(\lambda_i-\lambda_2)\theta t}|(\mathbb{1},\varphi_i)(f\varphi_1,\varphi_i)\varphi_1|\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\qquad+|\Phi(\rho,\kappa,\theta,f)-\Phi_1(\rho,\kappa,\theta,f)-\Phi_2(\rho,\kappa,\theta,f)| \qquad (13)\\
&\quad\le \frac{\sum_{i=\kappa+2}^{\infty}(1+c_1)h(t-\theta t)e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)\theta t}\|b_{\epsilon\theta t}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\qquad+\frac{\sum_{i=\kappa+2}^{\infty}c_2e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)\theta t}\|b_{\epsilon(t-\theta t)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\qquad+|\Phi(\rho,\kappa,\theta,f)-\Phi_1(\rho,\kappa,\theta,f)-\Phi_2(\rho,\kappa,\theta,f)|.
\end{aligned}
$$

Once again, the right member of (13) converges to zero uniformly w.r.t. $f\in\mathbb{B}_p$ as $t\longrightarrow\infty$ for any fixed $0<\epsilon<(\lambda_{\kappa+2}-\lambda_2)/\lambda_{\kappa+2}$.

(ii) In view of (9) and Lemma 3(ii), we discover, for any $\epsilon>0$, $t>0$ and $f\in\mathbb{B}_p$ with $p\in[1,\infty]$, that

$$
\begin{aligned}
&|(fe^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1},\varphi_i)\varphi_i|\le h(t-\theta t)\alpha_{\epsilon\theta t}^2e^{\lambda_i\epsilon\theta t}\|\varphi_1\|_\infty\varphi_1,\\
&|(\nu(f)e^{\lambda_1(t-\theta t)}P_{t-\theta t}\mathbb{1},\varphi_i)\varphi_i|\le c_1h(t-\theta t)\alpha_{\epsilon\theta t}^2e^{\lambda_i\epsilon\theta t}\|\varphi_1\|_\infty\varphi_1,\\
&|(\mathbb{1},\varphi_i)(f\varphi_1,\varphi_i)\varphi_1|\le\alpha_{\epsilon(t-\theta t)}^2e^{\lambda_i\epsilon(t-\theta t)}\|\varphi_1\|_1\|\varphi_1\|_\infty\|\varphi_1\|_{p^*}\varphi_1.
\end{aligned}
$$

Besides, for any fixed $0<\epsilon<(\lambda_2-\lambda_1)/\lambda_2$ and $t>0$ large enough such that $1-\alpha_{\epsilon t}^2e^{-[(1-\epsilon)\lambda_2-\lambda_1]t}>0$, we apply Lemma 3(iii) to get the estimate

$$\rho(e^{\lambda_1 t}P_t\mathbb{1})\ge(1-\alpha_{\epsilon t}^2e^{-[(1-\epsilon)\lambda_2-\lambda_1]t})\rho(\varphi_1)m(\varphi_1).$$

Consequently, (ii) is obtained by performing similar tricks and steps as in (i). □

We exhibit the quasi-mixing limit theorem and the double limit theorem for X.

Theorem 3. *Assume* ***H1*** *and* ***H2****.*

(i) *For any $\rho \in \mathcal{P}(D)$ and $a \in (0,1)$, there exists a finite function Ψ such that*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)\alpha t} \sup_{(f,g)\in\mathbb{B}_p\times\mathbb{B}_q} |\mathbb{E}_\rho[f(X_{at})g(X_t)|\tau > t] - \nu(f)\mu(g)|$$
$$= \sup_{(f,g)\in\mathbb{B}_p\times\mathbb{B}_q} |\Psi(\rho,\kappa,\alpha,f,g)|, \quad \alpha := a \wedge (1-a),\ p,q \in [1,\infty],$$

where the function $\Psi(\rho,\kappa,\alpha,f,g)$ has the expression

$$\begin{cases} \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)\nu(g)\rho(\varphi_i)}{\rho(\varphi_1)}, & \text{if } a < 1/2,\\ \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)(g,\varphi_i)}{m(\varphi_1)}, & \text{if } a > 1/2,\\ \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)[\nu(g)\rho(\varphi_i)m(\varphi_1) + (g,\varphi_i)\rho(\varphi_1)]}{m(\varphi_1)\rho(\varphi_1)}, & \text{if } a = 1/2. \end{cases}$$

(ii) *For any $\rho \in \mathcal{P}(D)$ and $0 < a < b < 1$, there exists a finite function Γ such that*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)\beta t} \sup_{(f,g)\in\mathbb{B}_p\times\mathbb{B}_q} |\mathbb{E}_\rho[f(X_{at})g(X_{bt})|\tau > t] - \nu(f)\nu(g)|$$
$$= \sup_{(f,g)\in\mathbb{B}_p\times\mathbb{B}_q} |\Gamma(\rho,\kappa,\beta,f,g)|, \quad \beta := a \wedge (b-a) \wedge (1-b),\ p,q \in [1,\infty],$$

where the function $\Gamma(\rho,\kappa,\beta,f,g)$ has the expression

$$\begin{cases} \sum_{i=2}^{\kappa+1} \frac{(f\varphi_1,\varphi_i)\nu(g)\rho(\varphi_i)}{\rho(\varphi_1)} \triangleq \Gamma_1(\rho,\kappa,f,g), & \text{if } a < [(b-a)\wedge(1-b)],\\ \sum_{i=2}^{\kappa+1} (f\varphi_1,\varphi_i)(g\varphi_1,\varphi_i) \triangleq \Gamma_2(\kappa,f,g), & \text{if } (b-a) < [a\wedge(1-b)],\\ \sum_{i=2}^{\kappa+1} \frac{(g\varphi_1,\varphi_i)\nu(f)m(\varphi_i)}{m(\varphi_1)} \triangleq \Gamma_3(\kappa,f,g), & \text{if } (1-b) < [a\wedge(b-a)],\\ \Gamma_1(\rho,\kappa,f,g) + \Gamma_2(\kappa,f,g), & \text{if } a = b-a < 1-b,\\ \Gamma_2(\rho,\kappa,f,g) + \Gamma_3(\kappa,f,g), & \text{if } b-a = 1-b < a,\\ \Gamma_1(\kappa,f,g) + \Gamma_3(\kappa,f,g), & \text{if } a = 1-b < b-a,\\ \Gamma_1(\rho,\kappa,f,g) + \Gamma_2(\kappa,f,g) + \Gamma_3(\kappa,f,g), & \text{if } a = 1-b = b-a. \end{cases}$$

(iii) *Assume in addition* ***H3****. Then the limits in (i) and (ii) are uniformly convergent w.r.t. $(\rho,f,g) \in \mathcal{P}(D)\times\mathbb{B}_p\times\mathbb{B}_q$.*

Proof. Using the properties of conditional expectation and the Markov property of X, we have, for any $t \geq 0$, $x \in D$, $0 \leq a \leq b \leq 1$ and $(f,g) \in \mathbb{B}_p \times \mathbb{B}_q$,

$$\mathbb{E}_x[f(X_{at})g(X_{bt})\mathbb{1}_{\{t<\tau\}}] = P_{at}\{fP_{(b-a)t}[g(P_{t-bt}\mathbb{1})]\}(x). \tag{14}$$

For any $t > 0$, $\rho \in \mathcal{P}(D)$ and $a \in (0,1)$, we employ (14) and Lemma 2(i) to get

$$\begin{aligned}
&\mathbb{E}_\rho[f(X_{at})g(X_t)|\tau > t] - \nu(f)\mu(g)\\
&= \frac{\rho\{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)at}(fe^{\lambda_1(t-at)}P_{t-at}g,\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\quad - \frac{\rho\{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)at}(\nu(f)\mu(g)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1},\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\quad + \frac{\rho\{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(t-at)}(g,\varphi_i)(f\varphi_i,\varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.
\end{aligned} \tag{15}$$

(i) Similar to (8)–(10) and utilizing the symmetry of $\{P_t\}$, we have the following estimates for the right-hand side of (15), for any $a \in (0,1)$, $t > 0$, $f \in \mathbb{B}_p$ and $g \in \mathbb{B}_q$ with $p, q \in [1,\infty]$:

$$\begin{aligned}
\mu(|g|) = (|g|,\varphi_1)/\|\varphi_1\|_1 &\le \|g\|_q\|\varphi_1\|_{q^*}/\|\varphi_1\|_1 \le \|\varphi_1\|_{q^*}/\|\varphi_1\|_1 \triangleq c_3,\\
|(fe^{\lambda_1(t-at)}P_{t-at}g,\varphi_i)\varphi_i| &\le h(t/2-at/2)^2 e^{\lambda_i\epsilon at}\|b_{\epsilon at}\|_\infty^2,\\
|(\nu(f)\mu(g)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1},\varphi_i)\varphi_i| &\le c_1c_3h(t-at)e^{\lambda_i\epsilon at}\|b_{\epsilon at}\|_\infty^2,\\
|(g,\varphi_i)(\varphi_i f,\varphi_1)\varphi_1| &\le m(D)^{\frac{1}{q^*}+\frac{1}{p^*}} e^{\lambda_i\epsilon(t-at)}\|b_{\epsilon(t-at)}\|_\infty^2\|\varphi_1\|_\infty^2\\
&\triangleq c_4 e^{\lambda_i\epsilon(t-at)}\|b_{\epsilon(t-at)}\|_\infty^2.
\end{aligned} \tag{16}$$

In order to derive the expression of Ψ, we now divide a into three cases and denote

$$\begin{aligned}
&\Psi_1(t,\rho,\kappa,a,f,g)\\
&\quad := \frac{\rho\{\sum_{i=2}^{\kappa+1}([fe^{\lambda_1(t-at)}P_{t-at}g - \nu(f)\mu(g)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}],\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})},\\
&\Psi_2(t,\rho,\kappa,a,f,g) := \frac{\rho\{\sum_{i=2}^{\kappa+1}(g,\varphi_i)(f\varphi_i,\varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.
\end{aligned}$$

Case 1. If $a \in (0,\frac{1}{2})$, we multiply (15) by $e^{(\lambda_2-\lambda_1)at}$ and use (16) to deduce

$$\begin{aligned}
&|e^{(\lambda_2-\lambda_1)at}\mathbb{E}_\rho[f(X_{at})g(X_t)\mathbb{1}_{\{t<\tau\}}] - \Psi(\rho,\kappa,a,f,g)|\\
&\le \frac{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)at}h(t/2-at/2)^2\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\quad + \frac{\sum_{i=\kappa+2}^{\infty} c_1c_3e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)at}h(t-at)\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\quad + \frac{\sum_{i=2}^{\infty} c_4e^{(\lambda_2-\lambda_1)at-(\lambda_i-\lambda_1-\lambda_i\epsilon)(t-at)}\|b_{\epsilon(t-at)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\
&\quad + |\Psi(\rho,\kappa,a,f,g) - \Psi_1(t,\rho,\kappa,a,f,g)|.
\end{aligned} \tag{17}$$

We note that b_t and $h(t)$ are monotonically decreasing functions of t and $\rho(e^{\lambda_1 t}P_t\mathbb{1}) \to m(\varphi_1)\rho(\varphi_1)$ as $t \to \infty$. Thus, the right member of (17) converges to zero uniformly w.r.t. $\rho \in \mathcal{P}(D)$ and $(f,g) \in \mathbb{B}_p \times \mathbb{B}_q$ as $t \to \infty$ for any fixed $\epsilon \in (0,\epsilon_2)$, where

$$\epsilon_2 = \min\left\{\frac{\lambda_{\kappa+2}-\lambda_2}{\lambda_{\kappa+2}}, \frac{(1-2a)(\lambda_2-\lambda_1)}{(1-a)\lambda_2}\right\}.$$

Case 2. If $a \in (\frac{1}{2}, 1)$, we multiply (15) by $e^{(\lambda_2-\lambda_1)(t-at)}$ and use (16) to get

$$\begin{aligned} &|e^{(\lambda_2-\lambda_1)(t-at)}\mathbb{E}_\rho[f(X_{at})g(X_t)\mathbb{1}_{\{t<\tau\}}] - \Psi(\rho,\kappa,a,f,g)| \\ &\quad\le \frac{\sum_{i=2}^\infty e^{(\lambda_2-\lambda_1)(t-at)-(\lambda_i-\lambda_1-\lambda_i\epsilon)at} h(t/2-at/2)^2\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\sum_{i=2}^\infty c_1c_3 e^{(\lambda_2-\lambda_1)(t-at)-(\lambda_i-\lambda_1-\lambda_i\epsilon)at} h(t-at)\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\sum_{i=\kappa+2}^\infty c_4 e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)(t-at)}\|b_{\epsilon(t-at)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ |\Psi(\rho,\kappa,a,f,g) - \Psi_2(t,\rho,\kappa,a,f,g)|. \end{aligned} \tag{18}$$

Similarly, the right member of (18) converges to zero uniformly w.r.t. $\rho \in \mathcal{P}(D)$ and $(f,g) \in \mathbb{B}_p \times \mathbb{B}_q$ as $t \to \infty$ for any fixed $\epsilon \in (0, \epsilon_3)$, where

$$\epsilon_3 = \min\left\{\frac{(2a-1)(\lambda_2-\lambda_1)}{a\lambda_2}, \frac{\lambda_{\kappa+2}-\lambda_2}{\lambda_{\kappa+2}}\right\}.$$

Case 3. If $a = 1/2$, we multiply (15) by $e^{(\lambda_2-\lambda_1)at}$ and employ (16) to derive

$$\begin{aligned} &|e^{(\lambda_2-\lambda_1)at}\mathbb{E}_\rho[f(X_{at})g(X_t)\mathbb{1}_{\{t<\tau\}}] - \Psi(\rho,\kappa,a,f,g)| \\ &\quad\le \frac{\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)at} h(t/2-at/2)^2\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\sum_{i=\kappa+2}^\infty c_1c_3 e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)at} h(t-at)\|b_{\epsilon at}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\sum_{i=\kappa+2}^\infty c_4 e^{-(\lambda_i-\lambda_2-\lambda_i\epsilon)at}\|b_{\epsilon(t-at)}\|_\infty^2}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ |\Psi(\rho,\kappa,a,f,g) - \Psi_1(t,\rho,\kappa,a,f,g) - \Psi_2(t,\rho,\kappa,a,f,g)|. \end{aligned} \tag{19}$$

Analogously, the right member of (19) converges to zero uniformly w.r.t. $\rho \in \mathcal{P}(D)$ and $(f,g) \in \mathbb{B}_p \times \mathbb{B}_q$ as $t \to \infty$ for any fixed $0 < \epsilon < (\lambda_{\kappa+2} - \lambda_2)/\lambda_{\kappa+2}$.

(ii) We find, for any $t > 0$, $\rho \in \mathcal{P}(D)$ and $0 < a < b < 1$, that

$$\begin{aligned} &\mathbb{E}_\rho[f(X_{at})g(X_{bt})|\tau > t] - \nu(f)\nu(g) \\ &\quad= \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)at}(fe^{(b-a)\lambda_1 t}P_{bt-at}(ge^{(t-bt)\lambda_1}P_{t-bt}\mathbb{1}),\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad- \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)at}(\nu(f)\nu(g)e^{(t-at)\lambda_1}P_{t-at}\mathbb{1},\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)(b-a)t}((ge^{(t-bt)\lambda_1}P_{t-bt}\mathbb{1},\varphi_i)f\varphi_i,\varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &\quad+ \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)(t-bt)}(((\mathbb{1},\varphi_i)g\varphi_i,\varphi_1)f\varphi_1,\varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}. \end{aligned}$$

The desired conclusion is completed by performing calculations similar to (i).

(iii) The proof is almost precisely like that of Theorem 2(ii). □

Theorem 4. *Assume **H1** and **H2**. For any $p \in [1,\infty]$ and $\rho \in \mathcal{P}(D)$, we have*

$$\lim_{s\to\infty}\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)s}\sup_{f\in\mathbb{B}_p}|\mathbb{E}_\rho(f(X_s)|\tau>t)-\nu(f)| = \sup_{f\in\mathbb{B}_p}\left|\sum_{i=2}^{\kappa+1}\frac{(f\varphi_1,\varphi_i)\rho(\varphi_i)}{\rho(\varphi_1)}\right|.$$

*Assume in addition **H3**. Then the limit above is uniformly convergent w.r.t. $(\rho, f) \in \mathcal{P}(D)\times\mathbb{B}_p$.*

Proof. For any $t > s \geq 0$, $\rho \in \mathcal{P}(D)$ and $f \in \mathbb{B}_p$ with $p \in [1,\infty]$, the Markov property and Lemma 2(i) yield

$$\mathbb{E}_\rho(f(X_s)|\tau>t)-\nu(f) = \frac{\rho(e^{\lambda_1 s}P_s(fe^{\lambda_1(t-s)}P_{t-s}\mathbb{1}))-\rho(e^{\lambda_1 s}P_s(f\varphi_1))m(\varphi_1)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}$$
$$+\frac{\rho(e^{\lambda_1 s}P_s(f\varphi_1))m(\varphi_1)-\nu(f)\rho(\varphi_1)m(\varphi_1)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}$$
$$+\frac{\nu(f)\rho(\varphi_1)m(\varphi_1)-\nu(f)\rho(e^{\lambda_1 t}P_t\mathbb{1})}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.$$

We now claim that, for any fixed $s > 0$,

$$\lim_{t\to\infty}\sup_{f\in\mathbb{B}_p}|\mathbb{E}_\rho(f(X_s)|\tau>t)-\nu(f)| = \sup_{f\in\mathbb{B}_p}\left|\sum_{i=2}^{\infty}\frac{e^{-(\lambda_i-\lambda_1)s}(f\varphi_1,\varphi_i)\rho(\varphi_i)}{\rho(\varphi_1)}\right|. \quad (20)$$

Indeed, the Cauchy–Schwarz inequality gives rise to

$$\begin{aligned}&|\rho\{e^{\lambda_1 s}P_s[f(e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}-\|\varphi_1\|_1\varphi_1)]\}|\\ &\quad\leq e^{\lambda_1 s}\|b_s\|_\infty(b_s,|f(e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}-\|\varphi_1\|_1\varphi_1)|)\\ &\quad\leq e^{\lambda_1 s}\|b_s\|_\infty\|b_s\|_{p^*}\|f(e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}-\|\varphi_1\|_1\varphi_1)\|_p\\ &\quad\leq e^{\lambda_1 s}\|b_s\|_\infty\|b_s\|_{p^*}\|f\|_p\|e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}-\|\varphi_1\|_1\varphi_1\|_\infty,\end{aligned}$$

proving the claim. Finally, we multiply both sides of (20) by $e^{(\lambda_2-\lambda_1)s}$ and then perform simple calculations similar to the proof in Theorem 1. □

4 Examples

In this section, we present three typical processes – a Lévy process with a Gaussian component, killed Brownian motion, and a symmetric α-stable process with no Gaussian component – to demonstrate how Theorems 1–4 in Section 3 apply to these cases.

The first example is the usual Lévy process whose long-term behavior has been analyzed in [29], but it seems that our results are more straightforward.

Example 1. Let $Y = \{Y_t : t \geq 0\}$ be a symmetric Lévy process on $\mathbb{R}^d$ with a Gaussian component. Y has a smooth transition density $q(t,x,y)$ w.r.t. the Lebesgue measure m. Its regular Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is given by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d}(\nabla u(x), A\nabla v(x))dx + \int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-v(y))^2\upsilon(dy-x)dx,$$

and $\mathcal{D}(\mathcal{E}) = \overline{C_0(\mathbb{R}^d)}^{\mathcal{E}_1}$, where A is a symmetric positive definite $d \times d$ matrix, υ is the Lévy measure of Y and $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)$. Let D be a bounded connected open subset of $\mathbb{R}^d$ and $X = \{X_t : t \geq 0\}$ be the process on D obtained by killing Y upon exiting D.

It is known, cf. [29, Example 4.2], that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the Nash inequality,

$$\|u\|_2^{2+4/d} \leq c_1 \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \cdot \|u\|_1^{4/d} \leq c_2 \mathcal{E}(u,u) \|u\|_1^{4/d}, \quad u \in \mathcal{D}(\mathcal{E}).$$

Thus, X has a strictly positive continuous density $p(t,x,y)$ and there exists a $c > 0$ such that

$$p(t,x,y) \leq q(t,x,y) \leq ct^{-d/2}, \quad t > 0,\ x,y \in \mathbb{R}^d;$$

consult, e.g., [9, Theorem 2.4.6] or [26, Theorem 4.1.1]. This also suggests that Lemma 2 holds for the transition semigroup of X. If furthermore the Lebesgue measure m and Lévy measure υ meet **H3**, then Lemma 3 holds for the transition semigroup of X.

Next we discuss the one-dimensional killed Brownian motion on a finite interval.

Example 2. Let $D = (a,b)$ be a finite open interval. Consider the operator $\mathcal{A} = -\frac{1}{2}\frac{d^2}{dx^2}$ on $L^2(D)$ and define a bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as

$$\mathcal{E}(u,v) = (\mathcal{A}u, v) = \frac{1}{2}(u', v'), \quad \forall u,v \in \mathcal{D}(\mathcal{E}),$$

where $D(\mathcal{E})$ stands for the closure of $C_c^\infty(D)$ w.r.t. the norm $\|\cdot\|_{\mathcal{E}_1}$, $C_c^\infty(D)$ denotes the space of infinitely differentiable functions with compact supports in D and $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)$.

(i) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet form on $L^2(D)$, $(-\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and $\{P_t := e^{-\mathcal{A}t}\}$ are its generator and semigroup, where $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{E}) \cap \{u : \mathcal{A}u \in L^2(D)\}$.

(ii) $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ has discrete spectrum $\lambda_i = i^2\pi^2/[2(b-a)^2]$ with the eigenfunction

$$\varphi_i(x) = \sqrt{2/(b-a)} \sin\left(i\pi \frac{x-a}{b-a}\right), \quad x \in D,\ i \geq 1.$$

(iii) We have $|\varphi_i(x)| \leq i\varphi_1(x)$ for any $i \geq 1$, and the heat kernel of P_t admits the expansion

$$p(t,x,y) = \sum_{i=1}^{\infty} e^{\lambda_i t} \varphi_i(x)\varphi_i(y) \leq \sum_{i=1}^{\infty} e^{-\lambda_i t} i^2 \varphi_1(x)\varphi_1(y) < \infty,\ t > 0,\ x,y \in D.$$

By (iii) and the definition of intrinsic ultracontractivity, $\{P_t\}$ is intrinsically ultracontractive. Therefore, Theorems 1–4 are satisfied by X.

The last example is the symmetric α-stable process with no Gaussian component.

Example 3. Let $Y = \{Y_t : t \geq 0\}$ be a symmetric α-stable process on $\mathbb{R}^d$ with $d \geq 2$ and $0 < \alpha < 2$. Its regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by

$$\mathcal{E}(u,v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} dxdy, \quad u,v \in \mathcal{D}(\mathcal{E}),$$

$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy < \infty \right\}.$$

For any bounded connected open subset $D \subset \mathbb{R}^d$, we use $\tau = \inf\{t > 0 : Y_t \notin D\}$ to denote the first exit time of the symmetric α-stable process Y from D. Adjoin an extra point ∂ to D and set

$$X_t = \begin{cases} Y_t, & \text{if } \tau > t, \\ \partial, & \text{if } \tau \le t. \end{cases}$$

The process $X = \{X_t : t \ge 0\}$ is called the symmetric α-stable process killed upon leaving D, or simply the killed symmetric α-stable process on D.

Denote by $\{P_t\}$ the transition semigroup of X, then [8, Theorem 2.5] asserts that $\{P_t\}$ is ultracontractive. Therefore, Lemma 2 holds for the transition semigroup $\{P_t\}$.

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References

[1] Berrhazi, B.E., Bouggar, D., Mrhardy, F.N.: Reflected backward doubly stochastic differential equations driven by Teugels martingales associated to a Lévy process with discontinuous barrier. Markov Process. Relat. **29**, 329–346 (2023). MR4683711

[2] Benaïm, M., Champagnat, N., Oçafrain, W., Villemonais, D.: Transcritical bifurcation for the conditional distribution of a diffusion process. J. Theor. Probab. **36**, 1555–1571 (2022). MR4621075. https://doi.org/10.1007/s10959-022-01216-7

[3] Champagnat, N., Villemonais, D.: General criteria for the study of quasi-stationarity. Electron. J. Probab. **28**, 1–84 (2023). MR4546021. https://doi.org/10.1214/22-ejp880

[4] Champagnat, N., Villemonais, D.: Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes. Stoch. Process. Appl. **135**, 51–74 (2021). MR4222402. https://doi.org/10.1016/j.spa.2020.12.005

[5] Chen, J.W., Li, H.T., Jian, S.Q.: Some limit theorems for absorbing Markov processes. J. Phys. A, Math. Theor. **45** (2012). MR2960634. https://doi.org/10.1088/1751-8113/45/34/345003

[6] Chen, J.W., Jian, S.Q.: A remark on quasi-ergodicity of ultracontractive Markov processes. Stat. Probab. Lett. **87**, 184–190 (2014). MR3168953. https://doi.org/10.1016/j.spl.2014.01.006

[7] Chen, Z.Q., Zhang, X.C., Zhao, G.H.: Supercritical SDEs driven by multiplicative stable-like Lévy processes. Trans. Am. Math. Soc. **374**, 7621–7655 (2021). MR4328678. https://doi.org/10.1090/tran/8343

[8] Chen, Z.Q., Song, R.M.: Intrinsic Ultracontractivity and Conditional Gauge for Symmetric Stable Processes. J. Funct. Anal. **150**, 204–239 (1997). MR1473631. https://doi.org/10.1006/jfan.1997.3104

[9] Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge University Press, (1989). MR0990239. https://doi.org/10.1017/CBO9780511566158

[10] Davies, E.B., Simon, B.: Ultracontractive semigroups and some problems in analysis. In: Aspects of Mathematics and Its Applications. North-Holland, Amsterdam (1984). MR0849557. https://doi.org/10.1016/S0924-6509(09)70260-0

[11] Davies, E.B.: Linear operators and their spectra. Cambridge University Press, (2007). MR2359869. https://doi.org/10.1017/CBO9780511618864

[12] Davies, E.B., Simon, B.: Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. J. Funct. Anal. **59**, 335–395 (1984). MR0766493. https://doi.org/10.1016/0022-1236(84)90076-4

[13] Guillin, A., Nectoux, B., Wu, L.M.: Quasi-stationary distribution for Hamiltonian dynamics with singular potentials. Probab. Theory Relat. Fields **185**, 921–959 (2023). MR4556285. https://doi.org/10.1007/s00440-022-01154-9

[14] Grzywny, T.: Intrinsic ultracontractivity for Lévy processes. Probab. Math. Stat-Pol. **28**, 91–106 (2008). MR2445505

[15] He, G.M., Zhang, H.J.: Exponential convergence to a quasi-stationary distribution for birth-death processes with an entrance boundary at infinity. J. Appl. Probab. **59**, 983–991 (2022). MR4507677. https://doi.org/10.1017/jpr.2021.100

[16] Journel, L., Monmarché, P.: Convergence of a particle approximation for the quasi-stationary distribution of a diffusion process: Uniform estimates in a compact soft case. ESAIM, Probab. Stat. **26**, 1–25 (2022). MR4363453. https://doi.org/10.1051/ps/2021017

[17] Kaleta, K., Lőrinczi, J.: Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes. Ann. Probab. **43**, 1350–1398 (2015). MR3342665. https://doi.org/10.1214/13-AOP897

[18] Kim, P., Song, R.M.: Intrinsic ultracontractivity for non-symmetric Lévy processes. Forum Math. **21**, 43–66 (2009). MR2494884. https://doi.org/10.1515/FORUM.2009.056

[19] Knobloch, R., Partzsch, L.: Uniform Conditional Ergodicity and Intrinsic Ultracontractivity. Potential Anal. **33**, 107–136 (2010). MR2658978. https://doi.org/10.1007/s11118-009-9161-5

[20] Li, H.S., Zhang, H.J., Liao, S.X.: On quasi-stationaries for symmetric Markov processes. J. Math. Anal. Appl. **528** (2023). MR4604336. https://doi.org/10.1016/j.jmaa.2023.127498

[21] Li, Y.X., Cohen, D., Kottos, T.: Enforcing Lévy relaxation for multi-mode fibers with correlated disorder. New J. Phys. **24** (2022). MR4455508. https://doi.org/10.1088/1367-2630/ac6318

[22] Li, Z., Feng, T.Q., Xu, L.P.: Non-confluence of fractional stochastic differential equations driven by Lévy process. Fract. Calc. Appl. Anal. **27**, 1414–1427 (2024). MR4746851. https://doi.org/10.1007/s13540-024-00278-0

[23] Lee, J.: Analytic form of the quasi-stationary distribution of a simple birth-death process. J. Korean Phys. Soc. **77**, 457–462 (2020). https://doi.org/10.3938/jkps.77.457

[24] Leliévre, T., Ramil, M., Reygner, J.: Quasi-stationary distribution for the Langevin process in cylindrical domains, Part I: Existence, uniqueness and long-time convergence. Stoch. Process. Appl. **144**, 173–201 (2022). MR4347490. https://doi.org/10.1016/j.spa.2021.11.005

[25] Méléard, S., Villemonais, D.: Quasi-stationary distributions and population processes. Probab. Surv. **9**, 340–410 (2012). MR2994898. https://doi.org/10.1214/11-PS191

[26] Saloff-Coste, L.: Aspects of Sobolev-Type Inequalities. Cambridge University Press, (2001). MR1872526

[27] Zhang, X.L., Zhang, X.C.: Ergodicity of supercritical SDEs driven by α-stable processes and heavy-tailed sampling. Bernoulli **29**, 1933–1958 (2023). MR4580902. https://doi.org/10.3150/22-bej1526

[28] Zhang, H.J., Li, H.S., Liao, S.X.: Intrinsic ultracontractivity and uniform convergenceto the Q-process for symmetric Markov processes. Electron. Commun. Probab. **28**, 1–12 (2023). MR4684057. https://doi.org/10.1214/23-ecp550

[29] Zhang, J.F., Li, S.M., Song, R.M.: Quasi-stationarity and quasi-ergodicity of general Markov processes. Sci. China, Math. **57**, 2013–2024 (2014). MR3247530. https://doi.org/10.1007/s11425-014-4835-x