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Convergence of hitting times for jump-diffusion processes

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Abstract We investigate the convergence of hitting times for jump-diffusion processes. Specifically, we study a sequence of stochastic differential equations with jumps. Under reasonable assumptions, we establish the convergence of solutions to the equations and of the moments when the solutions hit certain sets.

Keywords Stochastic differential equation, Poisson measure, jump-diffusion process, stopping time, convergence
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1 Introduction

In this article, we consider a sequence of stochastic differential equations with jumps

$$\begin{aligned} X^n(t) &= X^n(0) + \int_0^t a^n\big(s, X^n(s)\big)ds + \int_0^t b^n\big(s, X^n(s)\big)dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}^m} c^n\big(s, X^n(s-), \theta\big)\widetilde{\nu}(d\theta, ds), \quad t \ge 0,\ n \ge 0. \end{aligned}$$

Here W is a standard Wiener process, $\widetilde{\nu}$ is a compensated Poisson random measure, and $X^n(0)$ is nonrandom (see Section 2 for precise assumptions). Assuming that $a^n \to a^0$, $b^n \to b^0$, $c^n \to c^0$, and $X^n(0) \to X^0(0)$ as $n \to \infty$ in an appropriate sense, we are interested in convergence of hitting times $\tau^n \to \tau^0$, $n \to \infty$, where

$$\tau^n = \inf\big\{t \ge 0 : \varphi^n\big(t, X^n(t)\big) \ge 0\big\}$$

is the first time when the process X^n hits the set $\mathcal{G}_t^n = \{x : \varphi^n(t, x) \ge 0\}$.

The study is motivated by the following observation. Jump-diffusion processes are commonly used to model prices of financial assets. When the parameters of a jump-diffusion process are estimated with the help of statistical methods, there is an estimation error. Thus, it is natural to investigate whether the optimal exercise strategies are close for two jump-diffusion processes with close parameters. Moreover, we should study particular hitting times since, in the Markovian setting, the optimal stopping time is the hitting time of the optimal stopping set.

There is a lot of literature devoted to jump-diffusion processes and their applications in finance. The book [1] gives an extensive list of references on the subject. The convergence of stopping times for diffusion and jump-diffusion processes was studied in [2, 3, 6]. All these papers are devoted to the one-dimensional case, and the techniques are different from ours. Here we generalize these results to the multidimensional case and also relax the assumptions on the convergence of coefficients. As an auxiliary result of independent interest, we prove the convergence of solutions under very mild assumptions on the convergence of coefficients.

2 Preliminaries and notation

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathsf{P})$ be a standard stochastic basis with filtration $\mathbf{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual assumptions. Let $\{W(t) = (W_1(t), \dots, W_k(t)), t \geq 0\}$ be a standard Wiener process in $\mathbb{R}^k$, and $\nu(d\theta, dt)$ be a Poisson random measure on $\mathbb{R}^m \times [0, \infty)$. We assume that W and ν are compatible with the filtration $\mathbf{F}$, that is, for any $t > s \geq 0$ and any $A \in \mathcal{B}(\mathbb{R}^m)$ and $B \in \mathcal{B}([s, t])$, the increment $W(t) - W(s)$ and the value $\nu(A \times B)$ are F_t-measurable and independent of $\mathcal{F}_s$.

Assume in addition that $\nu(d\theta, dt)$ is homogeneous, that is, for all $A \in \mathcal{B}(\mathbb{R}^m)$ and $B \in \mathcal{B}([0, \infty))$, $\mathsf{E}[\nu(A \times B)] = \mu(A)\lambda(B)$, where λ is the Lebesgue measure, μ is a σ-finite measure on $\mathbb{R}^m$ having no atom at zero. Denote by $\widetilde{\nu}$ the corresponding compensated measure, that is, $\widetilde{\nu}(A \times B) = \nu(A \times B) - \mu(A)\lambda(B)$ for all $A \in \mathcal{B}(\mathbb{R}^m)$, $B \in \mathcal{B}([0, \infty))$.

For each integer $n \geq 0$, consider a stochastic differential equation in $\mathbb{R}^d$

$$X_i^n(t) = X_i^n(0) + \int_0^t a_i^n\big(s, X^n(s)\big)ds + \sum_{j=1}^k \int_0^t b_{ij}^n\big(s, X^n(s)\big)dW_j(s) + \int_0^t \int_{\mathbb{R}^m} c_i^n\big(s, X^n(s-), \theta\big)\widetilde{\nu}(d\theta, ds), \quad t \geq 0,\ i = 1, \dots, d. \tag{1}$$

In this equation, the initial condition $X^n(0) \in \mathbb{R}^d$ is nonrandom, and the coefficients $a_i^n, b_{ij}^n : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, $c_i^n : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, d$, $j = 1, \dots, k$, are nonrandom and measurable.

In what follows, we abbreviate Eq. (1) as

$$X^n(t) = X^n(0) + \int_0^t a^n\big(s, X^n(s)\big)ds + \int_0^t b^n\big(s, X^n(s)\big)dW(s) + \int_0^t \int_{\mathbb{R}^m} c^n\big(s, X^n(s-), \theta\big)\widetilde{\nu}(d\theta, ds), \quad t \geq 0. \tag{2}$$

For the rest of the article, we adhere to the following notation. By $|\cdot|$ we denote the absolute value of a number, the norm of a vector, or the operator norm of a matrix, and by (x, y) the scalar product of vectors x and y; $B_k(r) = \{x \in \mathbb{R}^k : |x| \le r\}$. The symbol C means a generic constant whose value is not important and may change from line to line; a constant dependent on parameters $a, b, c, \dots$ will be denoted by $C_{a,b,c,\dots}$.

The following assumptions guarantee that Eq. (2) has a unique strong solution.

(A1) For all $n \ge 0$, $T > 0$, $t \in [0, T]$, $x \in \mathbb{R}^d$,

$$\big|a^n(t,x)\big|^2 + \big|b^n(t,x)\big|^2 + \int_{\mathbb{R}^m} \big|c^n(t,x,\theta)\big|^2 \mu(d\theta) \le C_T\big(1+|x|^2\big).$$

(A2) For all $n \ge 0$, $T \ge 0$, $t \in [0, T]$, $R > 0$, and $x, y \in B_d(R)$

$$\begin{aligned}&\big|a^n(t,x) - a^n(t,y)\big|^2 + \big|b^n(t,x) - b^n(t,y)\big|^2 \\ &\quad + \int_{\mathbb{R}^m} \big|c^n(t,x,\theta) - c^n(t,y,\theta)\big|^2 \mu(d\theta) \le C_{T,R}|x-y|^2.\end{aligned}$$

Moreover, under these assumptions, for any $T \ge 0$, we have the following estimate:

$$\mathsf{E}\Big[\sup_{t\in[0,T]} \big|X^n(t)\big|^2\Big] \le C_T\big(1 + \big|X^n(0)\big|^2\big) \tag{3}$$

(see, e.g., [5, Section 3.1]). From this estimate it is easy to see from Eq. (2) that for all $t, s \in [0, T]$,

$$\mathsf{E}\big[\big|X^n(t) - X^n(s)\big|^2\big] \le C_T\big(1 + \big|X^n(0)\big|^2\big)|t-s|. \tag{4}$$

Now we state the assumptions on the convergence of coefficients of (2).

(C1) For all $t \ge 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}&a^n(t,x) \to a^0(t,x), \qquad b^n(t,x) \to b^0(t,x), \\ &\int_{\mathbb{R}^m} \big|c^n(t,x,\theta) - c^0(t,x,\theta)\big|^2 \mu(d\theta) \to 0, \quad n \to \infty.\end{aligned}$$

(C2) $X^n(0) \to X^0(0)$, $n \to \infty$.

3 Convergence of solutions to stochastic differential equations with jumps

First, we establish a result on convergence of solutions to stochastic differential equations.

Theorem 3.1. *Let the coefficients of Eq. (2) satisfy assumptions (A1), (A2), (C1), and (C2). Then, for any $T > 0$, we have the convergence in probability*

$$\sup_{t\in[0,T]} \big|X^n(t) - X^0(t)\big| \xrightarrow{\mathsf{P}} 0, \quad n \to \infty.$$

If additionally the constant in assumption (A2) *is independent of R, then for any* $T > 0$,

$$\mathsf{E}\Big[\sup_{t\in[0,T]}\big|X^n(t) - X^0(t)\big|^2\Big] \to 0, \quad n \to \infty.$$

Proof. Denote $\Delta^n(t) = \sup_{s\in[0,t]} |X^n(t) - X^0(t)|$, $a_s^{n,m} = a^n(s, X^m(s))$, $b_s^{n,m} = b^n(s, X^m(s))$, $c_s^{n,m}(\theta) = c^n(s, X^m(s-), \theta)$,

$$I_a^n(t) = \int_0^t a_s^{n,n} ds, \qquad I_b^n(t) = \int_0^t b_s^{n,n} dW(s),$$
$$I_c^n(t) = \int_0^t \int_{\mathbb{R}^m} c_s^{n,n}(\theta)\tilde{\nu}(d\theta, ds).$$

It is easy to see that I_b^n and I_c^n are martingales.

Write

$$\Delta^n(t)^2 \le C\Big(\big|X^n(0) - X^0(0)\big|^2 + \sup_{s\in[0,t]}\big|I_a^n(s) - I_a^0(s)\big|^2$$
$$+ \sup_{s\in[0,t]}\big|I_b^n(s) - I_b^0(s)\big|^2 + \sup_{s\in[0,t]}\big|I_c^n(s) - I_c^0(s)\big|^2\Big).$$

For $N \ge 1$, define

$$\sigma_N^n = \inf\big\{t \ge 0 : \big|X^0(t)\big| \vee \big|X^n(t)\big| \ge N\big\}$$

and denote $1_t = \mathbf{1}_{t\le\sigma_N^n}$. Then

$$\mathsf{E}\big[\Delta^n(t)^2 1_t\big] \le \mathsf{E}\big[\Delta^n\big(t \wedge \sigma_N^n\big)^2\big]$$
$$\le C\bigg(\big|X^n(0) - X^0(0)\big|^2 + \sum_{x\in\{a,b,c\}} \mathsf{E}\Big[\sup_{s\in[0,t\wedge\sigma_N^n]}\big|I_x^n(s) - I_x^0(s)\big|^2\Big]\bigg).$$

We estimate

$$\mathsf{E}\Big[\sup_{s\in[0,t\wedge\sigma_N^n]}\big|I_a^n(s) - I_a^0(s)\big|^2\Big] \le \mathsf{E}\bigg[\sup_{s\in[0,t]}\bigg(\int_0^s \big|a_u^{n,n} - a_u^{0,0}\big| 1_u du\bigg)^2\bigg]$$
$$\le \mathsf{E}\bigg[\bigg(\int_0^t \big|a_u^{n,n} - a_u^{0,0}\big| 1_u du\bigg)^2\bigg] \le t\int_0^t \mathsf{E}\big[\big|a_u^{n,n} - a_u^{0,0}\big|^2 1_u\big]du$$
$$\le C_t \int_0^t \big(\mathsf{E}\big[\big|a_u^{n,n} - a_u^{n,0}\big|^2 1_u\big] + \mathsf{E}\big[\big|a_u^{n,0} - a_u^{0,0}\big|^2 1_u\big]\big)du. \tag{5}$$

In turn,

$$\int_0^t \mathsf{E}\big[\big|a_u^{n,n} - a_u^{n,0}\big|^2 1_u\big]du = \int_0^t \mathsf{E}\big[\big|a^n\big(u, X^n(u)\big) - a^n\big(u, X^0(u)\big)\big|^2 1_u\big]du$$
$$\le C_{N,t}\int_0^t \mathsf{E}\big[\big|X^n(u) - X^n(0)\big|^2 1_u\big]du \le C_{N,t}\int_0^t \mathsf{E}\big[\Delta^n(u)^2 1_u\big]du.$$

By the Doob inequality and Itô isometry we obtain

$$\mathsf{E}\Big[\sup_{s\in[0,t\wedge\sigma_N^n]}\big|I_b^n(s)-I_b^0(s)\big|^2\Big]\le C\mathsf{E}\big[\big|I_b^n\big(t\wedge\sigma_N^n\big)-I_b^0\big(t\wedge\sigma_N^n\big)\big|^2\big]$$
$$=C\int_0^t\mathsf{E}\big[\big|b_s^{n,n}-b_s^{0,0}\big|^2 1_s\big]ds.$$

Estimating as in (5), we arrive at

$$\int_0^t\mathsf{E}\big[\big|b_s^{n,n}-b_s^{0,0}\big|^2 1_s\big]ds$$
$$\le C_{N,t}\int_0^t\mathsf{E}\big[\Delta^n(s)^2 1_s\big]ds+C\int_0^t\mathsf{E}\big[\big|b_s^{n,0}-b_s^{0,0}\big|^2 1_s\big]ds.$$

Finally, the Doob inequality yields

$$\mathsf{E}\Big[\sup_{s\in[0,t\wedge\sigma_N^n]}\big|I_c^n(s)-I_c^0(s)\big|^2\Big]\le C\mathsf{E}\big[\big|I_c^n\big(t\wedge\sigma_N^n\big)-I_c^0\big(t\wedge\sigma_N^n\big)\big|^2\big]$$
$$=C\int_0^t\int_{\mathbb{R}^m}\mathsf{E}\big[\big|c_s^{n,n}(\theta)-c_s^{0,0}(\theta)\big|^2 1_s\big]\mu(d\theta)ds$$
$$\le C\int_0^t\int_{\mathbb{R}^m}\big(\mathsf{E}\big[\big|c_s^{n,n}(\theta)-c_s^{n,0}(\theta)\big|^2 1_s\big]+\mathsf{E}\big[\big|c_s^{n,0}(\theta)-c_s^{0,0}(\theta)\big|^2 1_s\big]\big)\mu(d\theta)ds.$$

By (A2) we have

$$C\int_0^t\int_{\mathbb{R}^m}\mathsf{E}\big[\big|c_s^{n,n}(\theta)-c_s^{n,0}(\theta)\big|^2 1_s\big]\mu(d\theta)ds$$
$$\le C_{N,t}\int_0^t\mathsf{E}\big[\big|X^n(s)-X^0(s)\big|^2 1_s\big]ds\le C_{N,t}\int_0^t\mathsf{E}\big[\Delta^n(s)^2 1_s\big]ds.$$

Collecting all estimates, we arrive at the estimate

$$\mathsf{E}\big[\Delta^n(t)^2 1_t\big]\le C\big|X^n(0)-X^0(0)\big|^2+C_{N,t}\int_0^t\mathsf{E}\big[\Delta^n(s)1_s\big]ds$$
$$+C_t\int_0^t\mathsf{E}\big[\big|\tilde a_s^{n,0}-\tilde a_s^{0,0}\big|^2 1_s\big]ds+C\int_0^t\mathsf{E}\big[\big|b_s^{n,0}-b_s^{0,0}\big|^2 1_s\big]ds$$
$$+C\int_0^t\int_{\mathbb{R}^m}\mathsf{E}\big[\big|c_s^{n,0}(\theta)-c_s^{0,0}(\theta)\big|^2 1_s\big]\mu(d\theta)ds,$$

where we can assume without loss of generality that the constants are nondecreasing in t. The application of the Gronwall lemma leads to

$$\begin{aligned}
&\mathsf{E}\big[\Delta^n(T)^2 1_T\big]\\
&\quad\le C_{N,T}\bigg(\big|X^n(0)-X^0(0)\big|^2+\int_0^T \mathsf{E}\big[\big|\tilde a_s^{n,0}-\tilde a_s^{0,0}\big|^2 1_s\big]ds\\
&\qquad+\int_0^T \mathsf{E}\big[\big|b_s^{n,0}-b_s^{0,0}\big|^2 1_s\big]ds+\int_0^T\int_{\mathbb{R}^m}\mathsf{E}\big[\big|c_s^{n,0}(\theta)-c_s^{0,0}(\theta)\big|^2 1_s\big]\mu(d\theta)ds\bigg).
\end{aligned}$$

We claim that the right-hand side of the latter inequality vanishes as $n\to\infty$. Indeed, the integrands are bounded by $C_T(1+|X(s)|^2)$ due to (A1) and vanish pointwise due to (C1). Hence, the convergence of integrals follows from the dominated convergence theorem. The first term vanishes due to (C2); thus,

$$\mathsf{E}\big[\Delta^n(T)^2 1_T\big]\to 0,\quad n\to\infty.$$

Now to prove the first statement, for any $\varepsilon>0$, write

$$\begin{aligned}
\mathsf{P}\big(\Delta^n(T)>\varepsilon\big)&\le\frac{1}{\varepsilon^2}\mathsf{E}\big[\Delta^n(T)^2 1_T\big]+\mathsf{P}\big(\sigma_N^n<T\big)\\
&\le\frac{1}{\varepsilon^2}\mathsf{E}\big[\Delta^n(T)^2 1_T\big]+\mathsf{P}\Big(\sup_{t\in[0,T]}\big|X^n(0)\big|\ge N\Big)\\
&\quad+\mathsf{P}\Big(\sup_{t\in[0,T]}\big|X^0(0)\big|\ge N\Big).
\end{aligned}$$

This implies

$$\varlimsup_{n\to\infty}\mathsf{P}\big(\Delta^n(T)>\varepsilon\big)\le 2\sup_{n\ge 0}\mathsf{P}\Big(\sup_{t\in[0,T]}\big|X^n(0)\big|\ge N\Big).$$

By the Chebyshev inequality we have

$$\varlimsup_{n\to\infty}\mathsf{P}\big(\Delta^n(T)>\varepsilon\big)\le\frac{2}{N^2}\sup_{n\ge 0}\mathsf{E}\Big[\sup_{t\in[0,T]}\big|X^n(0)\big|^2\Big].$$

Therefore, using (3) and letting $N\to\infty$, we get

$$\varlimsup_{n\to\infty}P\big(\Delta^n(T)>\varepsilon\big)=0,$$

as desired.

In order to prove the second statement, we repeat the previous arguments with $\sigma_N^n\equiv T$, getting the estimate

$$\begin{aligned}
\mathsf{E}\big[\Delta^n(T)^2\big]\le C_T\bigg(&\big|X^n(0)-X^0(0)\big|^2+\int_0^T\mathsf{E}\big[\big|\tilde a_s^{n,0}-\tilde a_s^{0,0}\big|^2\big]ds\\
&+\int_0^T\mathsf{E}\big[\big|b_s^{n,0}-b_s^{0,0}\big|^2\big]ds\\
&+\int_0^T\int_{\mathbb{R}^m}\mathsf{E}\big[\big|c_s^{n,0}(\theta)-c_s^{0,0}(\theta)\big|^2\big]\mu(d\theta)ds\bigg).
\end{aligned}$$

Hence, we get the required convergence as before, using the dominated convergence theorem. □

4 Convergence of hitting times

For each $n \geq 0$, define the stopping time

$$\tau^n = \inf\bigl\{t \geq 0 : \varphi^n\bigl(t, X^n(t)\bigr) \geq 0\bigr\} \tag{6}$$

with the convention $\inf \varnothing = +\infty$; φ^n is a function satisfying certain assumptions to be specified later. In this section, we study the convergence $\tau^n \to \tau^0$ as $n \to \infty$.

The motivation to study stopping times of the form (6) comes from the financial modeling. Specifically, let a financial market model be driven by the process X^n solving Eq. (2), and $q > 0$ be a constant discount factor. Consider the problem of optimal exercise of an American-type contingent claim with payoff function f and maturity T, that is, the maximization problem

$$\mathsf{E}\bigl[e^{-q\tau} f\bigl(X^n(\tau)\bigr)\bigr] \to \max,$$

where τ is a stopping time taking values in $[0, T]$. Define the value function

$$v^n(t, x) = \sup_{\tau \in [t,T]} \mathsf{E}\bigl[e^{-q(\tau-t)} f\bigl(X^n(\tau)\bigr) \mid X^n(t) = x\bigr]$$

as the maximal expected discounted payoff provided that the price process X^n starts from x at the moment t; the supremum is taken over all stopping times with values in $[t, T]$.

Then it is well known that the minimal optimal stopping time is given as

$$\tau^{*,n} = \inf\bigl\{t \geq 0 : v^n\bigl(t, X^n(t)\bigr) = f\bigl(X^n(t)\bigr)\bigr\},$$

that is, it is the first time when the process X^n hits the so-called optimal stopping set

$$\mathcal{G}^n = \bigl\{(t, x) \in [0, T] \times \mathbb{R}^d : v^n(t, x) = f(x)\bigr\}.$$

Note that $\tau^{*,n} \leq T$ since $v(T, x) = g(x)$. Since, obviously, $v^n(t, x) \geq f(x)$, we may represent $\tau^{*,n}$ in the form (6) with $\varphi^n = f(x) - v^n(t, x)$.

5 Convergence of hitting times for finite horizon

Let $T > 0$ be a fixed number playing the role of finite maturity of an American contingent claim. Let also the stopping times τ^n, $n \geq 0$, be given by (6) with $\varphi^n : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following assumptions.

(G1) $\varphi^0 \in C^1([0, T) \times \mathbb{R}^d)$, and the derivative $D_x\varphi^0$ is locally Lipschitz continuous in x, that is, for all $t \in [0, T)$, $R > 0$, $s \in [0, t]$, and $x, y \in B_d(R)$,

$$\bigl|D_x\varphi^0(s, x) - D_x\varphi^0(s, y)\bigr| \leq C_{t,R}|x - y|.$$

(G2) For all $n \geq 0$ and $x \in \mathbb{R}^d$, $\varphi^n(T, x) = 0$.

(G3) For all $t \in [0, T)$ and $x \in \mathbb{R}^d$,

$$\bigl|b^0(t, x)^\top D_x\varphi^0(t, x)\bigr| > 0. \tag{7}$$

Here by $b^0(t,x)^\top D_x\varphi^0(t,x)$ we denote the vector in $\mathbb{R}^k$ with jth coordinate equal to

$$\sum_{i=1}^{d} b^0_{ij}(t,x)\partial_{x_i}\varphi^0(t,x), \quad j=1,\dots,k.$$

Remark 5.1. Assumption (7) means that the diffusion is acting strongly enough toward the border of the set $\mathcal{G}^0_t := \{x \in \mathbb{R}^d : \varphi^0(t,x) \le 0\}$. In which situations does this assumption hold, will be studied elsewhere. Here we just want to remark that it is more delicate than it might seem. For example, consider the optimal stopping problem described in the beginning of this section with $n=0$ in (2). Then, under suitable assumptions (see, e.g., [4, 7]), we have the smooth fit principle: $\partial_x v^0(t,x) = \partial_x f(x)$ on the boundary of the optimal stopping set. This means that we cannot set $\varphi^0(t,x) = f(x) - v^0(t,x)$ in order for (7) to hold, contrary to what was proposed in the beginning of the section.

We will also assume the locally uniform convergence $\varphi^n \to \varphi^0$.

(G4) For all $t \in [0,T)$ and $R > 0$,

$$\sup_{(s,x)\in[0,t]\times B_d(R)} \left|\varphi^n(s,x) - \varphi^0(s,x)\right| \to 0, \quad n \to \infty.$$

Remark 5.2. The convergence of value functions in optimal stopping problems usually holds under fairly mild assumptions on the convergence of coefficients and payoffs. However, as we explained in Remark 5.1, we cannot use the value function for φ^n. This means that we should find a function φ^n defining $\mathcal{G}$ *different* from $v^n(t,x) - f(x)$, but it still should satisfy the convergence assumption (G4).

The question in which cases such functions exist and the convergence assumption (G4) takes places will be a subject of our future research.

In the case where ν has infinite activity, that is, $\mu(\mathbb{R}^m) = \infty$, we will also need some additional assumptions on the components of Eq. (2).

(A3) For each $r > 0$, $\mu(\mathbb{R}^m \setminus B_m(r)) < \infty$.

(A4) For all $t \ge 0$, $x \in \mathbb{R}^d$, and $\theta \in \mathbb{R}^m$,

$$\left|c^0(t,x,\theta)\right| \le h(t,x)g(\theta),$$

where the functions g, h are locally bounded, $g(0)=0$, and $g(\theta) \to 0, \theta \to 0$.

Remark 5.3. Assumption (A3) means that only small jumps of μ can accumulate on a finite interval; assumption (A4) means that small jumps of μ are translated by Eq. (2) to small jumps of X^n. An important and natural example of a situation where these assumptions are satisfied is an equation

$$X^0(t) = X^0(0) + \int_0^t a^0\big(s, X^0(s)\big)ds + \int_0^t b^0\big(s, X^0(s)\big)dW(s) + \int_0^t h^0\big(s, X^0(s-)\big)dZ(s), \quad t \ge 0,$$

driven by a Lévy process $Z(t) = \int_0^t \int_{\mathbb{R}^m} \theta\, \widetilde{\nu}(d\theta, ds)$.

Now we are in a position to state the main result of this section.

Theorem 5.1. *Assume (A1)–(A4), (C1), (C2), (G1)–(G4). Then we have the following convergence in probability:*

$$\tau^n \xrightarrow{\mathsf{P}} \tau^0, \quad n \to \infty.$$

Proof. Let ε, δ be small positive numbers. We are to show that for all n large enough,

$$\mathsf{P}\big(\big|\tau^n - \tau^0\big| > \varepsilon\big) < \delta. \tag{8}$$

Using estimate (3) and the Chebyshev inequality, we obtain that for some $R > 0$,

$$\mathsf{P}\Big(\sup_{t\in[0,T]}\big|X^0(t)\big| \ge R\Big) < \frac{\delta}{4}.$$

Denote $\mathcal{K} = [0, T - \varepsilon/2] \times B_d(R+2)$,

$$M = 1 + R + C_{T,R+2} + C_T + C_{T-\varepsilon/2,R+2} + \sup_{(t,x)\in\mathcal{K}}\big(\big|a(t,x)\big| + \big|b(t,x)\big| + \big|\partial_t\varphi^0(t,x)\big| + \big|D_x\varphi^0(t,x)\big| + \big|b^0(t,x)^\top D_x\varphi^0(t,x)\big|^{-1}\big),$$

where, with some abuse of notation, $C_{T,R+2}$ is the constant from (A2) corresponding to T and $R+2$, C_T is the sum of constants from (A1) and (4), and $C_{T-\varepsilon/2,R+2}$ is the constant from (G1) corresponding to $T - \varepsilon/2$ and $R+2$.

Let $\kappa \in (0, M]$ be a number, which we will specify later. Now we claim that there exists a function $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^d)$ such that

$$\sup_{(t,x)\in\mathcal{K}}\big|\varphi(t,x) - \varphi^0(t,x)\big| < \varkappa/2$$

and, moreover,

$$\begin{aligned}
&\sup_{\substack{t\in[0,T-\varepsilon/2]\\ x\in B_d(R+1)}}\big(\big|\partial_t\varphi(t,x)\big| + \big|D_x\varphi(t,x)\big| + \big|D^2_{xx}\varphi(t,x)\big| + \big|b^0(t,x)^\top D_x\varphi(t,x)\big|^{-1}\big)\\
&\quad\le C_{T-\varepsilon/2,R+2} + \sup_{(t,x)\in\mathcal{K}}\big(\big|\partial_t\varphi^0(t,x)\big| + \big|D_x\varphi^0(t,x)\big|\\
&\qquad + \big|b^0(t,x)^\top D_x\varphi^0(t,x)\big|^{-1}\big)\\
&\quad\le M.
\end{aligned}$$

Indeed, we can take the convolution $\varphi(t,x) = (\varphi^0(t,\cdot) \star \psi)(x)$ with a delta-like smooth function ψ, supported on a ball of radius less than 1.

Further, by (G4) there exists $n_1 \ge 1$ such that for all $n \ge n_1$,

$$\sup_{(t,x)\in\mathcal{K}}\big|\varphi^n(t,x) - \varphi^0(t,x)\big| < \varkappa/2. \tag{9}$$

On the other hand, by Theorem 3.1 there exists $n_2 \ge 1$ such that for all $n \ge n_2$,

$$\mathsf{P}\Big(\sup_{t\in[0,T]}\big|X^n(t) - X^0(t)\big| \ge \frac{\varkappa}{M}\Big) < \frac{\delta}{4}. \tag{10}$$

In what follows, we consider $n \ge n_1 \vee n_2$.

Define the stopping time

$$\sigma^n = \inf\left\{t \geq 0 : \left|X^n(t) - X^0(t)\right| \geq \frac{\varkappa}{M} \text{ or } \left|X^0(t)\right| \geq R\right\} \wedge T.$$

Write

$$\begin{aligned}\mathsf{P}\big(\left|\tau^n - \tau^0\right| > \varepsilon\big) &\leq \mathsf{P}\big(\left|\tau^n - \tau^0\right| > \varepsilon, \sigma^n > T - \varepsilon/2\big)\\ &\quad + \mathsf{P}\Big(\sup_{t\in[0,T]}\left|X^0(t)\right| \geq R\Big) + \mathsf{P}\Big(\sup_{t\in[0,T]}\left|X^n(t) - X^0(t)\right| \geq \frac{\varkappa}{M}\Big)\\ &< \mathsf{P}\big(\left|\tau^n - \tau^0\right| > \varepsilon, \sigma^n > T - \varepsilon/2\big) + \frac{\delta}{2}. \end{aligned} \tag{11}$$

For any $t \leq \sigma^n$,

$$\left|X^n(t)\right| \leq \left|X^0(t)\right| + \frac{\varkappa}{M} < R + 1,$$

and hence,

$$\begin{aligned}&\left|\varphi^n\big(t, X^n(t)\big) - \varphi\big(t, X^0(t)\big)\right|\\ &\quad\leq \left|\varphi^n\big(t, X^n(t)\big) - \varphi\big(t, X^n(t)\big)\right| + \left|\varphi\big(t, X^n(t)\big) - \varphi\big(t, X^0(t)\big)\right|\\ &\quad\leq \varkappa + M\left|X^n(t) - X^0(t)\right| \leq 2\varkappa.\end{aligned}$$

Now take some $\eta \in (0, \varepsilon/2]$ whose exact value will be specified later and write the obvious inequality

$$\begin{aligned}&\mathsf{P}\big(\tau_T^{*,0} + \varepsilon < \tau_T^{*,n}, \sigma^n > T - \varepsilon/2\big)\\ &\quad\leq \mathsf{P}\big(\tau^0 < T - \varepsilon, \tau^0 + \eta < \tau^n, \sigma^n > T - \varepsilon/2\big).\end{aligned} \tag{12}$$

Assume that $\tau^0 < T - \varepsilon$, $\tau^0 + \eta < \tau^n$, $\sigma^n > T - \varepsilon/2$. Then, for all $t \in [\tau^0, \tau^0 + \eta] =: \mathcal{I}_\eta$,

$$\left|\varphi^n\big(s, X^0(s)\big) - \varphi\big(s, X^0(s)\big)\right| \leq 2\varkappa, \qquad \varphi^n\big(t, X^n(t)\big) < 0.$$

Therefore, in view of the inequality $\varphi(\tau^0, X^0(\tau^0)) \geq 0$, we obtain

$$\inf_{t\in\mathcal{I}_\eta} \varphi\big(t, X^0(t)\big) \geq \varphi\big(\tau^0, X^0\big(\tau^0\big)\big) - 2\varkappa. \tag{13}$$

Further, we will work with the expression $\varphi(t, X^0(t)) - \varphi(\tau^0, X^0(\tau^0))$ for $t \in \mathcal{I}_\eta$. For convenience, we will abbreviate $f_s = f(s, X^0(s))$; for example, $\varphi_s = \varphi(s, X^0(s))$.

Let $r > 0$ be a positive number, which we will specify later, and assume that ν does not have jumps on $\mathcal{I}_\eta$ greater than r, that is, $\nu((\mathbb{R}^m \setminus B_m(r)) \times \mathcal{I}_\eta) = 0$. Write, using the Itô formula,

$$\begin{aligned}&\varphi\big(t, X^0(t)\big) - \varphi\big(\tau^0, X^0\big(\tau^0\big)\big)\\ &\quad= \int_{\tau^0}^t L_s\varphi_s ds + \int_{\tau^0}^t \big(D_x\varphi_s, b_s^0 dW(s)\big) + \int_{\tau_0}^t\int_{B_m(r)} \varDelta_s(\theta)\,\widetilde{\nu}(d\theta, ds)\\ &\quad=: I_1(t) + I_2(t) + I_3(t),\end{aligned}$$

where

$$
\begin{aligned}
L_t\varphi_t &= \partial_t\varphi_t + \big(D_x\varphi_t, a_t^0\big) + \frac{1}{2}\operatorname{tr}\big(b_t^0\big(b_t^0\big)^\top D_{xx}^2\varphi_t\big) \\
&\quad + \int_{B_m(r)} \big(\Delta_s(\theta) - \big(D_x\varphi_s, c^0\big(s, X^0(s-), \theta\big)\big)\big)\mu(d\theta), \\
\Delta_s(\theta) &= \varphi\big(s, X^0(s-) + c\big(s, X^0(s-), \theta\big)\big) - \varphi\big(s, X^0(s-)\big).
\end{aligned}
$$

Start with estimating $I_1(t)$. Since $t \le \sigma^n \wedge (T - \varepsilon/2)$ for any $t \in \mathcal{I}_\eta$, by the definition of M and σ^n we have

$$
\left|\partial_t\varphi_t + \big(D_x\varphi_t, a_t^0\big) + \frac{1}{2}\operatorname{tr}\big(b_t^0\big(b_t^0\big)^\top D_{xx}^2\varphi_t\big)\right| \le M + M^2 + M^3 \le 3M^3.
$$

Further, by (A4), for $t \in \mathcal{I}_\eta$ and $\theta \in B_r$, $|c(t, X(t-), \theta)| \le h(t, X(t-))g(\theta) \le K_1 m_r$, where $K_1 = \sup_{t\in[0,T],|x|\le R} h(t,x)$ and $m_r = \sup_{\theta\in B_m(r)} g(\theta)$. Since $m_r \to 0$, $r \to 0$, we can assume that r is such that $m_r \le 1/K_1$. Then, for $t \in \mathcal{I}_\eta$, by the Taylor formula

$$
\begin{aligned}
&\left|\int_{B_m(r)} \big(\Delta_t(\theta) - \big(D_x\varphi_t, c^0\big(t, X^0(t-), \theta\big)\big)\big)\mu(d\theta)\right| \\
&\quad \le \frac{1}{2} \sup_{(u,x)\in[0,T]\times B_d(R+1)} \big|D_{xx}^2\varphi(u,x)\big| \int_{B_m(r)} \big|c\big(t, X^0(t-), \theta\big)\big|^2 \mu(d\theta) \\
&\quad \le \frac{1}{2}M^2\big(1 + \big|X(t)\big|^2\big) \le \frac{1}{2}M^2\big(1 + R^2\big) \le M^4.
\end{aligned}
$$

Summing up the estimates, we get

$$
\big|I_1(t)\big| \le \big(3M^3 + M^4\big)\eta \le 4M^4\eta. \tag{14}
$$

Now proceed to $I_3(t)$. By the Doob inequality, for any $a > 0$,

$$
\begin{aligned}
&\mathsf{P}\Big(\sup_{t\in\mathcal{I}_\eta}\big|I_3(t)\big| \ge a, \sigma_n > T - \varepsilon/2\Big) \le \mathsf{P}\Big(\sup_{t\in[\tau_0,(\tau_0+\eta)\wedge\sigma_n]}\big|I_3(t)\big| \ge a\Big) \\
&\quad \le Ca^{-2}\mathsf{E}\bigg[\bigg(\int_0^T \int_{B_m(r)} \Delta_s(\theta) 1_{[\tau_0,(\tau_0+\eta)\wedge\sigma_n]}(s)\widetilde{\nu}(d\theta, ds)\bigg)^2\bigg] \\
&\quad = Ca^{-2}\int_0^T \int_{B_m(r)} \mathsf{E}\big[\Delta_s(\theta)^2 1_{[\tau_0,(\tau_0+\eta)\wedge\sigma_n]}(s)\big]\mu(d\theta)ds \\
&\quad \le Ca^{-2}M^2\int_0^T \int_{B_m(r)} \mathsf{E}\big[\big|c\big(s, X^0(s-), \theta\big)\big|^2 1_{[\tau_0,(\tau_0+\eta)\wedge\sigma_n]}(s)\big]\mu(d\theta)ds \\
&\quad \le Ca^{-2}M^2\int_0^T \int_{B_m(r)} \mathsf{E}\big[K_1^2 m_r^2 1_{[\tau_0,(\tau_0+\eta)\wedge\sigma_n]}(s)\big]\mu(d\theta)ds \le K_2 a^{-2} m_r^2 \eta
\end{aligned}
$$

with some constant K_2. Further, we fix $a = \delta^2\eta^{1/2}$ and some $r > 0$ such that $m_r^2 \le \delta^5/(16K_2)$ and $m_r \le 1/K_1$. Then

$$
\mathsf{P}\Big(\sup_{t\in\mathcal{I}_\eta}\big|I_3(t)\big| \ge \delta^2\eta^{1/2}, \sigma_n > T - \varepsilon/2\Big) \le \frac{\delta}{16}.
$$

Hence, in view of (12)–(14), we obtain

$$
\begin{aligned}
&\mathsf{P}\big(\tau^0+\varepsilon<\tau^n, \sigma^n>T-\varepsilon/2\big)\\
&\quad\le \mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta} I_2(t)\ge -2\varkappa-4M^4\eta-\delta^2\eta^{1/2}, \sigma^n>T-\varepsilon/2\Big)\\
&\qquad+\mathsf{P}\Big(\sup_{t\in\mathcal{I}_\eta}\big|I_3(t)\big|\ge\delta^2\eta^{1/2}, \sigma_n>T-\varepsilon/2\Big)+\mathsf{P}\big(\nu\big(\big(\mathbb{R}^m\setminus B_m(r)\big)\times\mathcal{I}_\eta\big)>0\big)\\
&\quad\le \mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta} I_2(t)\ge -2\varkappa-4M^4\eta-\delta^2\eta^{1/2}, \sigma^n>T-\varepsilon/2\Big)\\
&\qquad+\eta\,\mu\big(\mathbb{R}^m\setminus B_m(r)\big)+\frac{\delta}{16}.
\end{aligned}
$$

Assume further that $\eta\le\eta_1:=\delta\,\mu(\mathbb{R}^m)/16$ (not yet fixing its exact value). Setting $\varkappa=(\eta M^4)\wedge M$, we get

$$
\begin{aligned}
&\mathsf{P}\big(\tau^0+\varepsilon<\tau^n, \sigma^n>T-\varepsilon/2\big)\\
&\quad\le \mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta} I_2(t)\ge -5\eta M^4-\delta^2\eta^{1/2}, \sigma^n>T-\varepsilon/2\Big)+\frac{\delta}{8}.
\end{aligned}
\tag{15}
$$

Write $I_2(t)=J_1(t)+J_2(t)+J_3(t)$, where

$$
\begin{aligned}
J_1(t)&=\int_{\tau^0}^t\big(D_x\varphi_s-D_x\varphi_{\tau^0}, b_s^0\,dW(s)\big),\\
J_2(t)&=\int_{\tau^0}^t\big(D_x\varphi_{\tau^0}, \big(b_s^0-b_{\tau^0}^0\big)dW(s)\big),\\
J_3(t)&=\big(D_x\varphi_{\tau^0}, b_{\tau^0}^0\big(W(t)-W\big(\tau^0\big)\big)\big)=\big(u_{\tau^0}, W(t)-W\big(\tau^0\big)\big);\\
u_s&=b^0\big(s, X^0(s)\big)^\top D_x\varphi\big(s, X^0(s)\big).
\end{aligned}
$$

Taking into account that $(s, X^0(s))\in\mathcal{K}$ for $s\le\sigma^n$, we estimate with the help of Doob's inequality

$$
\begin{aligned}
\mathsf{E}\Big[\sup_{t\in\mathcal{I}_\eta} J_1(t)^2\mathbf{1}_{\sigma^n>T-\epsilon/2}\Big]&\le\mathsf{E}\Big[\sup_{t\in[\tau^0,(\tau^0+\eta)\wedge\sigma^n]} J_1(t)^2\Big]\\
&\le C\mathsf{E}\bigg[\bigg(\int_{\tau^0}^{(\tau^0+\eta)\wedge\sigma^n}\big(D_x\varphi_s-D_x\varphi_{\tau^0}, b_s^0\,dW(s)\big)\bigg)^2\bigg]\\
&\le C\mathsf{E}\bigg[\int_{\tau^0}^{(\tau^0+\eta)\wedge\sigma^n}|D_x\varphi_s-D_x\varphi_{\tau^0}|^2\big|b_s^0\big|^2ds\bigg]\\
&\le CM^3\mathsf{E}\bigg[\int_{\tau^0}^{\tau^0+\eta}\big|X^0(s)-X^0\big(\tau^0\big)\big|^2ds\bigg]\\
&\le CM^4\big(1+\big|X^0(0)\big|^2\big)\eta^2\le CM^4\big(1+R^2\big)\eta^2\le CM^6\eta^2.
\end{aligned}
$$

Similarly, using (A2), we get

$$
\mathsf{E}\Big[\sup_{t\in\mathcal{I}_\eta} J_2(t)^2\mathbf{1}_{\sigma^n>T-\epsilon/2}\Big]\le CM^6\eta^2.
$$

The Chebyshev inequality yields

$$\mathsf{P}\Big(\sup_{t\in\mathcal{I}_\eta}\big(|J_1(t)|+|J_2(t)|\big)\geq\eta^{2/3},\sigma^n>T-\varepsilon/2\Big)\leq K_3M^6\eta^{2/3}$$

with certain constant K_3. Assume further that

$$\eta\leq\eta_2:=\left(\frac{\delta}{16K_3M^6}\right)^{3/2},$$

in which case the right-hand side of the last inequality does not exceed $\delta/16$, and that

$$\eta\leq\eta_3:=\frac{1}{125M^9},$$

so that $\eta^{2/3}\geq 5\eta M^3$. Hence, in view of (15), we obtain

$$\begin{aligned}&\mathsf{P}\big(\tau^0+\varepsilon<\tau^n,\sigma^n>T-\varepsilon/2\big)\\&\quad\leq\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta}J_3(t)\geq-5\eta M^3-\eta^{2/3}-\delta^2\eta^{1/2},\sigma^n>T-\varepsilon\Big)+\frac{3\delta}{16}\\&\quad\leq\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta}J_3(t)\geq-2\eta^{2/3}-\delta^2\eta^{1/2},\big(\tau^0,X^0\big(\tau^0\big)\big)\in\mathcal{K}\Big)+\frac{3\delta}{16}.\end{aligned}\tag{16}$$

Further, due to the strong Markov property of W,

$$\begin{aligned}&\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta}J_3(t)\geq-2\eta^{2/3}-\delta^2\eta^{1/2},\big(\tau^0,X^0\big(\tau^0\big)\big)\in\mathcal{K}\Big)\\&\quad=\mathsf{E}\Big[\mathbf{1}_\mathcal{K}\big(\tau^0,X^0\big(\tau^0\big)\big)\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta}J_3(t)\geq-2\eta^{2/3}-\delta^2\eta^{1/2}\mid F_{\tau^0}\Big)\Big]\\&\quad=\mathsf{E}\Big[\mathbf{1}_\mathcal{K}\big(\tau^0,X^0\big(\tau^0\big)\big)\\&\qquad\times\mathsf{P}\Big(\inf_{z\in[0,\eta]}\big(u(s,x),W(s+z)-W(s)\big)\geq-2\eta^{2/3}-\delta^2\eta^{1/2}\Big)\,|_{(s,x)=(\tau^0,X^0(\tau^0))}\Big],\end{aligned}$$

where $u(s,x)=b^0(s,x)^\top D_x\varphi(s,x)$. Observe now that $\{(u(s,x),W(z+s)-W(s)),\ z\geq 0\}$ is a standard Wiener process multiplied by $|u(s,x)|$. Therefore,

$$\begin{aligned}&\mathsf{P}\Big(\inf_{z\in[0,\eta]}\big(u(s,x),W(s+z)-W(s)\big)\geq-2\eta^{2/3}-\delta^2\eta^{1/2}\Big)\\&\quad=1-2\mathsf{P}\big(\big(u(s,x),W(s+\eta)-W(s)\big)<-2\eta^{2/3}-\delta^2\eta^{1/2}\big)\\&\quad=1-2\Phi\left(-\frac{2\eta^{2/3}+\Delta^2\eta^{1/2}}{|u(s,x)|\eta^{1/2}}\right)=1-2\Phi\left(-\frac{2\eta^{1/6}+\delta^2}{|u(s,x)|}\right),\end{aligned}$$

where Φ is the standard normal distribution function. Thus,

$$\begin{aligned}
&\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta} J_3(t) \ge -2\eta^{2/3} - \delta^2\eta^{1/2}, \big(\tau^0, X^0(\tau^0)\big) \in \mathcal{K}\Big)\\
&\quad \le \mathsf{E}\bigg[\mathbf{1}_{\mathcal{K}}\big(\tau^0, X^0(\tau^0)\big)\bigg(1 - 2\Phi\bigg(-\frac{2\eta^{1/6}+\delta^2}{|u(\tau^0, X^0(\tau^0))|}\bigg)\bigg)\bigg]\\
&\quad \le 1 - 2\Phi\big(-M\big(2\eta^{1/6}+\delta^2\big)\big) \le \frac{M\sqrt{2}}{\sqrt{\pi}}\big(2\eta^{1/6}+\delta^2\big).
\end{aligned}$$

Note that the definition of M does not depend on δ. Thus, we can assume without loss of generality that $\delta \le \sqrt{\pi}/(32M\sqrt{2})$. Finally, if

$$\eta \le \eta_4 := \left(\frac{\delta\sqrt{\pi}}{64\sqrt{2}}\right)^6,$$

then

$$\mathsf{P}\Big(\inf_{t\in\mathcal{I}_\eta} J_3(t) \ge -2\eta^{2/3} - \Delta^2\eta^{-1/2}, \big(\tau^0, X^0(\tau^0)\big) \in \mathcal{K}\Big) \le \frac{\delta}{16}. \tag{17}$$

Now we can fix $\eta = \min\{\varepsilon/2, \eta_1, \eta_2, \eta_3, \eta_4\}$, making all previous estimates to hold. Combining (16) with (17), we arrive at

$$\mathsf{P}\big(\tau^0 + \varepsilon < \tau^n, \sigma^n > T - \varepsilon/2\big) \le \frac{\delta}{4}.$$

Similarly,

$$\mathsf{P}\big(\tau^n + \varepsilon < \tau^0, \sigma^n > T - \varepsilon/2\big) \le \frac{\delta}{4},$$

and hence

$$\mathsf{P}\big(\big|\tau^n - \tau^0\big| > \varepsilon, \sigma^n > T - \varepsilon/2\big) \le \frac{\delta}{2}.$$

Plugging this estimate into (11), we arrive at the desired inequality (8). □

Remark 5.4. It is easy to modify the proof for the case where (7) holds for all $(t,x) \in \mathcal{G}^0 := \{(t,x) \in [0,T) \times \mathbb{R}^d : \varphi(t,x) = 0\}$. Indeed, the continuity would imply that (7) holds in some neighborhood of $\mathcal{G}^0$, which is sufficient for the argument.

Remark 5.5. As we have already mentioned, assumptions (A3) and (A4) are not needed in the case $\mu(\mathbb{R}^m) < \infty$. Indeed, we can set $r = 0$ in the previous argument and skip the estimation of $I_3(t)$. Nevertheless, these assumptions does not seem very restrictive, as we pointed out in Remark 5.3.

5.1 Convergence of hitting times for infinite horizon

Here we extend the results of the previous subsection to the case of infinite time horizon. Let, as before, the stopping times τ^n, $n \ge 0$, be given by (6). We impose the following assumptions.

(H1) $\varphi^0 \in C^1([0,\infty) \times \mathbb{R}^d)$, and $D_x\varphi^0$ is locally Lipschitz continuous in x, that is, for all $T > 0$, $R > 0$, $t \in [0,T]$, and $x, y \in B_d(R)$,

$$\big|D_x\varphi^0(t,x) - D_x\varphi^0(t,y)\big| \le C_{T,R}|x-y|.$$

(H2) $\tau^0 < \infty$ a.s.

(H3) For all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\left| D_x \varphi^0(t,x) b^0(t,x)^\top \right| > 0.$$

(H4) For all $t \geq 0$ and $R > 0$,

$$\sup_{(s,x)\in[0,t]\times B_d(R)} \left| \varphi^n(t,x) - \varphi^0(t,x) \right| \to 0, \quad n \to \infty.$$

Theorem 5.2. *Assume (A1), (A2), (C1), (C2), (H1)–(H4). Then we have the following convergence in probability:*

$$\tau^n \xrightarrow{\mathsf{P}} \tau^0, \quad n \to \infty.$$

Proof. Fix arbitrary $\varepsilon \in (0,1)$ and $\delta > 0$. Since $\tau^0 < \infty$ a.s., $\mathsf{P}(\tau^0 > T-1) \leq \delta$ for some $T > 1$. For $n \geq 0$, $t \in [0,T]$, and $x \in \mathbb{R}^d$, define $\tilde{\varphi}^n(t,x) = \varphi^n(t,x)\mathbf{1}_{[0,T)}(t)$, $\tau_T^n = \tau^n \wedge T$. Then the functions $\tilde{\varphi}^n$, $n \geq 0$, satisfy (G1)–(G3) and $\tau_T^n = \inf\{t \geq 0 : \tilde{\varphi}^n(t, X^n(t)) \geq 0\}$. Therefore, in view of Theorem 5.1,

$$\mathsf{P}\left(\left|\tau_T^n - \tau_T^{*,0}\right| > \varepsilon\right) \to 0, \quad n \to \infty.$$

We estimate

$$\begin{aligned} \mathsf{P}\left(\left|\tau^n - \tau^0\right| > \varepsilon\right) &\leq \mathsf{P}\left(\left|\tau_T^n - \tau_T^0\right| > \varepsilon\right) + \mathsf{P}\left(\tau^0 > T-1\right) \\ &\leq \mathsf{P}\left(\left|\tau_T^n - \tau_T^0\right| > \varepsilon\right) + \delta. \end{aligned}$$

Hence,

$$\varlimsup_{n\to\infty} \mathsf{P}\left(\left|\tau^n - \tau^0\right| > \varepsilon\right) \leq \delta.$$

Letting $\delta \to 0$, we arrive at the desired convergence. □

Example 5.1. Let $d = k = m = 1$ and for all $t \geq 0$, $x, \theta \in \mathbb{R}$, $a^n(t,x) = a^n$, $b^n(t,x) = b^n$, $c^n(t,x,\theta) = c^n\theta$, where $a^n, b^n, c^n \in \mathbb{R}$. Then we have a sequence of Lévy processes

$$X^n(t) = X^n(0) + a^n t + b^n W(t) + c^n \int_0^t \int_{\mathbb{R}} \theta \, \widetilde{\nu}(ds, d\theta).$$

Consider the following times:

$$\tau^n = \inf\left\{t \geq 0 : X^n(t) \geq h^n(t)\right\} \wedge T, \quad n \geq 0,$$

of crossing some curve $h \in C^1([0,T))$.

Assume that $a^n \to a^0$, $b^n \to b^0 \neq 0$, $c^n \to c^0$, and $X^n(0) \to X^0(0)$ as $n \to \infty$ and, for any $t \in [0,T)$, $\sup_{s\in[0,t]} |h^n(t) - h^0(t)| \to 0$ as $n \to \infty$. Then $\tau^n \xrightarrow{\mathsf{P}} \tau^0$, $n \to \infty$. Indeed, setting $\varphi^n(t,x) = (h^n(t) - x)\mathbf{1}_{[0,T)}(t)$, we can check that all assumptions of Theorem 5.1 are in force.

Example 5.2. Let $d = k = m = 1$. Suppose that the coefficients a^n, b^n, c^n satisfy (A1), (A2) and that the convergence (C1)–(C3) takes place. Assume that $b^0(t,x) > 0$ for all $t \ge 0$ and $x \in \mathbb{R}$. Define

$$\tau^n = \inf\{t \ge 0 : X^n(t) \notin (l^n, r^n)\}, \quad n \ge 0.$$

It is not hard to check that, due to the nondegeneracy of b^0, $\tau^0 < \infty$ a.s. Assume that $l^n \to l^0$, $r^n \to r^0$, $n \to \infty$. Then, setting $\varphi^n(t,x) = (x - l^n)(r^n - x)$ and using Theorem 5.2, we get the convergence $\tau^n \xrightarrow{\mathsf{P}} \tau^0$, $n \to \infty$.

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