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First-return time in fractional kinetics

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Abstract The first-return time is the time that it takes a random walker to go back to the initial position for the first time. In this paper, the first-return time is studied when random walkers perform fractional kinetics, specifically fractional diffusion, that is modelled within the framework of the continuous-time random walk on homogeneous space in the uncoupled formulation with Mittag-Leffler distributed waiting-times. Both the Markovian and non-Markovian settings are considered, as well as any kind of symmetric jump-size distributions, namely with finite or infinite variance. It is shown that the first-return time density is indeed independent of the jump-size distribution when it is symmetric, and therefore it is affected only by the waiting-time distribution that embodies the memory of the process. The analysis is performed in two cases: *first jump then wait* and *first wait then jump*, and several exact results are provided, including the relation between results in the Markovian and non-Markovian settings and the difference between the two cases.

Keywords First-return time, fractional kinetics, fractional diffusion, continuous-time random walk, first-passage time, Sparre Andersen theorem
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1 Introduction

Fractional kinetics [43, 23] concerns particles' motion whose random displacements follow a probability density function that evolves according to equations including

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fractional calculus operators. In a sloppy way, we can say that fractional kinetics concerns processes that follow a nonlocal generalisation of the Fokker–Planck equation. A noteworthy example is fractional diffusion [30].

The emerging evolution equation in fractional diffusion generalises the heat diffusion equation by replacing the first derivative in time and/or the second derivative in space by the corresponding fractional derivatives [17]. As a consequence, the underlying random walk model is not the classical random walk leading to the Brownian motion but its generalisation characterised by power-law distributions. Such generalised random walk is the continuous-time random walk (CTRW) [37] that extends classical random walk by replacing a fixed time-step with a random interval between consecutive jumps. These random intervals are called waiting-times and, in the basic formulation [37], are assumed to be i.i.d. random variables and independent of the jump-sizes, which also are assumed to be i.i.d. random variables. When the distribution of the waiting-times displays a power-law, then the evolution equation of the displacement density function contains time-fractional derivatives. On the other side, when the distribution of the jump-sizes displays a power-law, then such evolution equation contains space-fractional derivatives. In this second case, the corresponding random walk is called Lévy flight [5]. When the jump-size distribution displays power-law then the variance of the jumps and that of the walker's density function are infinite. When the waiting-time distribution displays a power-law and time-fractional derivatives appear, then the process is non-Markovian. The process is strictly Markovian when the waiting-time distribution is exponential [53, 33], however, this constraint can be relaxed by considering as Markovian all those processes with finite mean waiting-times, and as non-Markovian those with infinite mean waiting-times. Here, we study the first-return time when both waiting-time and jump-size distribution may display power-law.

The first-return time is the time after which a random walker comes back to the starting location for the first time. This kind of problem has a quite important application in the animal kingdom because it is linked to concepts as site fidelity, breeding, social associations, optimal foraging [18]. Mathematically, it is strongly related with the first-passage time problem [41]. The main result in first-passage time problems is the celebrated Sparre Andersen theorem [46], which concerns the survival probability on the half-line conditioned on the walker's initial position for symmetric Markovian random walks with fixed time-steps in the presence of an absorbing boundary. Its importance lays on the fact that it states that the conditional survival probability is independent of the jump-size distribution when the walker's trajectory starts from the location of the absorbing boundary, thus, it is universal. As a matter of fact, we show in this paper that this universality of first-passage time problems is reflected also in first-return time problems. Therefore, the results derived here are indeed independent of the tails of the walkers' density function and so the process can be governed by a Gaussian or by a Lévy stable density, as well. On the other side, the system is affected indeed by memory effects. Therefore, the results display different behaviour for the Markovian framework, namely with exponentially distributed waiting-times [53, 33], and for the non-Markovian one.

In particular, we report here that in discrete-time random walks, which are Markovian by construction, when a nearest-neighbour jump-law is taken into account then the probability mass function of the first-return time is linearly proportional to that of the first-passage time [41, 26, 19]. As it is discussed in the following, see formula (32), this linear proportionality holds also when the assumed jump-law is different of the nearest-neighbour rule. However, since the general framework of the CTRW is considered here, we study in addition the specific non-Markovian setting generated by Mittag-Leffler distributed waiting-times and we determine and investigate also the difference emerging between Markovian and non-Markovian formulations.

The formulation of the CTRW depends on the choice of the starting instant for measuring the elapsed time. In fact, the measurement of the duration of the first-return time can start synchronised with the first jump, and we label this case as the case *first jump then wait*, or it can start independently of the process so that the first jump occurs after the first random waiting-time, and we label this case as the case *first wait then jump*. We consider both cases, because, even if formulae relating Markovian and non-Markovian frameworks are equal, exact results in the Markovian setting differ for the two cases, and so in the non-Markovian setting. Moreover, we also quantify this difference.

We derive exact results in the Markovian setting and provide the relation between the Markovian and non-Markovian settings, so that non-Markovian results for Mittag-Leffler distributed waiting-times are also exact and fully determined. The derivation of the result is based on the Sparre Andersen theorem and it exploits the Laplace transform method. In particular, we provide some preliminary notions and results in Section 2 where the Sparre Andersen theorem is reminded and we also briefly remind the theory of the CTRW [37]. This allows us to state the Sparre Andersen theorem in the framework of the CTRW. In this section, we also derive integral (27), which is the main formula for the calculation of the remaining results. In Section 3, first we formulate the problem of first-return time and later we derive exact results for both cases *first jump then wait* and *first wait then jump*.

2 Preliminaries

2.1 The Sparre Andersen theorem

Let $\mathbb{R}$ be the set of real numbers, we denote by $\mathbb{R}^+$ and $\mathbb{R}^+_0$ the set of positive real numbers and the set of nonnegative real numbers, e.g., the half-line of spatial excursions excluded the origin $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ and the elapsed times $\mathbb{R}^+_0 = \{t \in \mathbb{R} | t \geq 0\}$. Analogously, we denote by $\mathbb{R}^-$ and $\mathbb{R}^-_0$ the set of negative real numbers and the set of nonpositive real numbers. Let furthermore $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $S_n = X_1 + \cdots + X_n$ be the sum of $n \in \mathbb{N}$ i.i.d. random variables $X_n \in \mathbb{R}$. Then the first ladder epoch $\{\mathcal{T} = n\} = \{S_1 \geq 0, \ldots, S_{n-1} \geq 0, S_n < 0\}$ is the epoch of the first entry of the walker into the negative semiaxis $\mathbb{R}^-$. By adding a constant $x_0 \in \mathbb{R}^+_0$ to all terms, we obtain a random walk with initial position at x_0. The probability for a walker started at $S_0 = x_0$ to remain in the initial half-axis after n steps, that is, the probability for $\mathcal{T}$ to be larger than n, is called survival probability conditioned on the initial position and the site $x = 0$ is termed as the location of an absorbing barrier. Thus, let the survival probability conditioned on the initial position be denoted by $\phi_n : \mathbb{R}^+_0 \to [0, 1]\,, \forall n \in \mathbb{N}$. Then for a walker starting at $S_0 = x_0$ we have $\phi_n(x_0) = \mathbb{P}(\mathcal{T} > n)$, and it yields

$$\int_0^{+\infty} k(z - x_0)\, \phi_n(z)\, dz = \phi_{n+1}(x_0)\,, \quad x_0 \in \mathbb{R}^+_0\,,\ n \in \mathbb{N}_0\,, \tag{1}$$

with initial condition $\phi_0(\xi) = 1\,, \forall\, \xi \in \mathbb{R}_0^+$, where $k : \mathbb{R} \to \mathbb{R}_0^+$ is the probability density function of the i.i.d. random variables X_n, $\forall\, n \in \mathbb{N}$. Within the framework of the CTRW, the density function $k(x)$ turns into the jump-size distribution.

Integral equation (1) is known as the Wiener–Hopf equation [12]. A first connection between random walks with an absorbing barrier and the Wiener–Hopf integrals was pointed out by D.V. Lindley in 1952 [28]. For a more general relations between the Wiener–Hopf integrals and probabilistic problems, see, e.g., reference [13, Sections XII.3a and XVIII.3]. However, as observed by W. Feller, the connections between first-passage problems and the Wiener–Hopf integrals *"are not as close as they are usually made to appear"* [13, Introduction to Chapter XII]. For more recent papers, see, e.g., references [21, 15, 35, 34, 3, 36, 7]. A general solution to the Wiener–Hopf equation (1) is known in the literature as the Pollaczek–Spitzer formula [3, formula (12)]. This name refers to a formula (see [49, theorem 5, formula (4.6)]) derived by F. Spitzer in 1957 [49, theorem 3, formula (3.1)] on the basis of an auxiliary formula obtained by F. Pollaczek in 1952 [40, formula (8)] but through a different method. It is possible to show that problem (1) is equivalent to a Sturm–Liouville problem with proper boundary conditions [8], and this is also an alternative and easier approach with respect to the Pollaczek–Spitzer formula for the calculation of the survival probability. In particular, one of the required boundary conditions for the Sturm–Liouville system emerged due to the Sparre Andersen theorem. Moreover, the Sparre Andersen theorem can be derived on the basis of formula (1) under the assumption that $\phi_{n+1}(\cdot)$ is of bounded variation [14]. Actually, the Sparre Andersen theorem [46] is a fundamental result in the study of the first-passage time problems.

Originally established in 1954 [46], the Sparre Andersen theorem states

$$\phi_n(0) = 2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{n\pi}}\,, \quad n \to \infty\,. \tag{2}$$

Thereby, the Sparre Andersen formula (2) is valid for arbitrary but symmetric jump-size distribution, i.e., $k(\xi) = k(-\xi)$ with $\xi \in \mathbb{R}$. In other words, the Sparre Andersen theorem [46] provides the exact survival probability conditioned on the initial position for symmetric Markovian random walkers starting in the same location of the absorbing barrier and states that it is independent of the jump-size distribution $k(x)$ (whenever it is assumed to be symmetric). See, for example, F. Spitzer [48] and W. Feller [13, Section XII.7]. This independency uncovers a universal nature of the Sparre Andersen theorem which is reflected also in other results, see, e.g., [4, 24, 25, 34, 10, 38, 27]. Thus, we refer to this independency as the *universality* of the Sparre Andersen theorem. Sparre Andersen formula (2) is indeed one of the many results emerging from a deep *corpus studiorum* on random walks based on combinatorial arguments by E. Sparre Andersen [44, 45, 47, 46] and F. Spitzer [48–50]. For more recent and general results on the basis of probabilistic arguments, see, e.g., [2]. The general formulation of the first-passage time problem for processes with discrete-time is available in [3, 36].

2.2 *From the discrete- to the continuous-time setting*

The survival probability problem can be formulated also in the continuous-time setting. In particular, we consider the CTRW on homogeneous and continuous space in the uncoupled formulation as originally introduced by Montroll and Weiss in the year

1965 [37]. This means that the i.i.d. random waiting-times and the i.i.d. random jump-sizes are, at any epoch, independent of each other and also of the current position and time. Therefore, in analogy with the discrete-time setting, the walker's position after n iterations, with $n \in \mathbb{N}$, is given by the sum of n i.i.d. random variables $X_n \in \mathbb{R}$ distributed according to the jump-size distribution $k(x)$, i.e., $S_n = x_0 + X_1 + \cdots + X_n$ where $x_0 \in \mathbb{R}_0^+$ is the initial position. The actual time $t \geq 0$ is given by the sum of n i.i.d. random waiting-times $\tau_j \in \mathbb{R}_0^+$ between two consecutive jumps, i.e., $t = \sum_{j=1}^n \tau_j$ with initial instant $t = 0$. Moreover, let the survival probability conditioned on the initial position in the continuous-time setting be denoted by $\Lambda : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to [0, 1]$. Thus, by using the same approach adopted to compute the walker's density function in the CTRW theory [37], the conditional survival probability Λ is given by the weighted superposition of all the possible discrete-time counterparts $\phi_n(\cdot)$, and it results in the series

$$\Lambda(x_0, t) = \sum_{n=0}^{\infty} \phi_n(x_0)\, \Psi_n(t)\,, \quad x_0, t \in \mathbb{R}_0^+\,, \tag{3}$$

where the index n counts the number of occurred jumps and $\Psi_n(\cdot)$ is the probability to have an elapsed time equals to $t \in \mathbb{R}_0^+$ after $n \in \mathbb{N}_0$ jumps such that

$$\Psi_n(t) = \int_0^t \Psi_{n-1}(t-\tau)\psi(\tau)\, d\tau\,, \quad \Psi_0(t) = \Psi(t) = 1 - \int_0^t \psi(\tau) d\tau\,, \tag{4}$$

and $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the distribution of the waiting-times.

For the present purposes, we pass to the Laplace domain and obtain for formula (3) that

$$\mathcal{L}\{\Lambda(x_0, t); s\} = \widetilde{\Lambda}(x_0, s) = \widetilde{\Psi}(s) \sum_{n=0}^{\infty} \phi_n(x_0)\, \left[\widetilde{\psi}(s)\right]^n\,, \tag{5}$$

where $\mathcal{L}\{g(t); s\} := \widetilde{g}(s) = \int_0^{+\infty} \mathrm{e}^{-st}\, g(t)\, dt$ is the Laplace transform of a sufficiently well-behaving function $g(t)$. Hence, after splitting formula (5) in

$$\widetilde{\Lambda}(x_0, s) = \widetilde{\Psi}(s)\phi_0(x_0) + \widetilde{\Psi}(s) \sum_{n=1}^{\infty} \phi_n(x_0)\, \left[\widetilde{\psi}(s)\right]^n\,, \tag{6}$$

by recalling that $\phi_0(x_0) = 1\,, \forall\, x_0 \in \mathbb{R}_0^+$, and expressing $\phi_n(\cdot)$ through the Wiener–Hopf integral equation (1), from formula (6) we get

$$\widetilde{\Lambda}(x_0, s) = \widetilde{\Psi}(s) + \widetilde{\psi}(s) \int_0^{+\infty} k(\xi - x_0)\widetilde{\Lambda}(\xi, s)\, d\xi\,. \tag{7}$$

Finally, after the inverse Laplace transformation $g(t) = \mathcal{L}^{-1}\{\widetilde{g}(s); t\}$, the equation for the survival probability is

$$\Lambda(x_0, t) = \Psi(t) + \int_0^t \psi(t-\zeta) \int_0^{+\infty} k(\xi - x_0)\Lambda(\xi, \zeta)\, d\xi d\zeta\,, \quad x_0, t \in \mathbb{R}_0^+\,, \tag{8}$$

which is the analogue of formula (1) in continuous time, see also [7, 8, 42], and we recover $\Lambda(x_0, 0) = 1\,, \forall\, x_0 \in \mathbb{R}_0^+$, because $\Psi(0) = 1$ by definition (4).

2.3 The Sparre Andersen theorem in the framework of the CTRW

The Sparre Andersen theorem can also be put in the framework of the uncoupled CTRW, in particular, through formula (3). In fact, if we set $x_0 = 0$ in formula (5), and from the Sparre Anderson formula (2) for the discrete-time setting we take the expression of $\phi_n(0)$, then

$$\begin{aligned}\widetilde{\Lambda}(0,s) &= \widetilde{\Psi}(s) \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \left[\widetilde{\psi}(s)\right]^n \\ &= \frac{\widetilde{\Psi}(s)}{\sqrt{1-\widetilde{\psi}(s)}} = \frac{\sqrt{s\,\widetilde{\Psi}(s)}}{s} \\ &= \frac{1}{s}\sqrt{1-\widetilde{\psi}(s)}\,, \end{aligned} \tag{9}$$

where in the second line we used the series

$$\sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-z}}\,, \quad |z| < 1\,. \tag{10}$$

Formula (9) was already reported to some extent in [1, formula (7)] but not formally derived, yet. By applying the initial and final value theorems we have

$$\Lambda(0,0) = \lim_{s\to\infty} s\widetilde{\Lambda}(0,s) = 1\,, \tag{11a}$$

$$\Lambda(0,\infty) = \lim_{s\to 0} s\widetilde{\Lambda}(0,s) = 0\,, \tag{11b}$$

because $\widetilde{\psi}(0) = 1$ and $\widetilde{\psi}(\infty) = 0$. After the Laplace antitransformation of the first line of formula (9), the Sparre Andersen theorem reads

$$\Lambda(0,t) = \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \Psi_n(t)\,. \tag{12}$$

For Markovian random walks, we have that $\psi(t) = \psi_{\mathrm{M}}(t) = \mathrm{e}^{-t}$ [53, 33], such that

$$\widetilde{\psi}_{\mathrm{M}}(s) = \widetilde{\Psi}_{\mathrm{M}}(s) = \frac{1}{1+s}\,. \tag{13}$$

and then formula (9) reads

$$\widetilde{\Lambda}_{\mathrm{M}}(0,s) = \frac{1}{\sqrt{s(1+s)}}\,. \tag{14}$$

Consequently, formula (12) for the Markovian case gives in the original domain [42]

$$\Lambda_{\mathrm{M}}(0,t) = \mathrm{e}^{-t/2}\, I_0(t/2)\,, \tag{15}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order 0 defined by the series

$$I_0(\zeta) = \sum_{j=0}^{\infty} \frac{(\zeta^2/4)^j}{(j!)^2}\,, \quad \zeta \in \mathbb{R}_0^+\,, \tag{16}$$

with

$$I_0(0) = 1\,, \quad \left.\frac{dI_0(\zeta)}{d\zeta}\right|_{\zeta=0} = 0\,, \quad \left.\frac{d^2 I_0(\zeta)}{d\zeta^2}\right|_{\zeta=0} = \frac{1}{2}\,. \tag{17}$$

From formula (14) we have in the long-time limit, i.e., $s \to 0$, that $\widetilde{\Lambda}_{\mathrm{M}}(0,s) \sim 1/\sqrt{s}$ and after the Laplace inversion it gives $\Lambda_{\mathrm{M}}(0,t) \sim 1/\sqrt{t}$ for $t \to +\infty$. This last limit is the continuous-time counterpart of limit (2).

For non-Markovian random walks, we consider the specific model [20]

$$\psi(t) = t^{\beta-1}E_{\beta,\beta}(-t^\beta)\,, \quad \Psi(t) = E_{\beta,1}(-t^\beta) = E_\beta(-t^\beta)\,, \tag{18}$$

where $E_{\beta,\alpha}(z)$ is the Mittag-Leffler function [29, Appendix E]

$$E_{\beta,\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j + \alpha)}\,, \quad \mathrm{Re}\{\beta\} > 0\,,\ \alpha \in \mathbb{C}\,,\ z \in \mathbb{C}\,, \tag{19}$$

which in the Laplace domain correspond to

$$\widetilde{\psi}(s) = \frac{1}{1+s^\beta}\,, \quad \widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}\,. \tag{20}$$

Hence, formula (9) reads

$$\widetilde{\Lambda}(0,s) = \frac{s^{\beta/2-1}}{\sqrt{1+s^\beta}}\,. \tag{21}$$

We observe that

$$\widetilde{\Lambda}(0,s) = s^{\beta-1}\widetilde{\Lambda}_{\mathrm{M}}(0,s^\beta)\,, \tag{22}$$

and by applying the Efros formula [11, 51], i.e.,

$$\mathcal{L}^{-1}\Big\{v(s)\widetilde{\mathcal{W}}[q(s)]\Big\} = \int_0^\infty \mathcal{W}(\zeta)\,\mathcal{L}^{-1}\Big\{v(s)\mathrm{e}^{-\zeta q(s)}\Big\}\,d\zeta\,, \tag{23}$$

with $v(s) = s^{\beta-1}$ and $q(s) = s^\beta$, in the original domain we have

$$\Lambda(0,t) = t^{-\beta}\int_0^\infty \mathrm{e}^{-\zeta/2} I_0(\zeta/2) M_\beta(\zeta/t^\beta)\,d\zeta\,, \tag{24}$$

where $M_\beta(\cdot)$ is the Mainardi/Wright function defined by the series [29, Appendix F]

$$M_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\frac{z^j}{\Gamma[-\nu j + (1-\nu)]}\,, \quad 0 < \nu < 1\,,\ z \in \mathbb{C}\,, \tag{25}$$

whose Laplace transform is known in the literature [30, (4.26)]. From formula (21) we have that $\widetilde{\Lambda}(0,s) \sim s^{\beta/2-1}$, when $s \to 0$, and after the Laplace inversion $\Lambda(0,t) \sim 1/t^{\beta/2}$ for $t \to +\infty$.

Moreover, after reminding that the unconditional survival probability is determined by the integral $\int_0^\infty \rho(\xi)\Lambda(\xi,t)\,d\xi$, where $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the distribution of the initial position, we can state the following theorem.

Theorem 1. *The unconditional survival probability of a symmetric CTRW is independent of the jump-size distribution if this distribution is equal to the distribution of the initial position.*

Proof. From the third line of formula (9) we have $\widetilde{\Psi}(s) = s\widetilde{\Lambda}^2(0,s)$ and, after summing and subtracting $\widetilde{\Lambda}(0,s)$, we can derive

$$\widetilde{\Lambda}(0,s) = \widetilde{\Psi}(s) + \widetilde{\Lambda}(0,s)\left[1 - s\widetilde{\Lambda}(0,s)\right], \tag{26}$$

which, after the comparison against formula (7) with $x_0 = 0$, gives

$$\begin{aligned}\int_0^\infty k(\xi)\widetilde{\Lambda}(\xi,s)\,d\xi &= \frac{\widetilde{\Lambda}(0,s)}{\widetilde{\psi}(s)}\left[1 - s\widetilde{\Lambda}(0,s)\right] \\ &= \frac{\widetilde{\Psi}(s)}{\widetilde{\psi}(s)}\left[\frac{1}{\sqrt{1-\widetilde{\psi}(s)}} - 1\right] \\ &= \frac{\sqrt{1-\widetilde{\psi}(s)} - 1 + \widetilde{\psi}(s)}{s\,\widetilde{\psi}(s)},\end{aligned} \tag{27}$$

and this is independent of the jump-size distribution $k(x)$ as a consequence of the universality of the Sparre Andersen theorem (2) when plugged into (3). □

3 First-return time for CTRW

3.1 Problem formulation and definitions

The duration of excursions of a random walker until its first comeback to the starting position is called first-return time (FRT). This observable can be understood by an example from the animal kingdom in terms of the return of an animal to a previously occupied area, such as the duration of the flight of birds when they leave and then return to their nests. This is a wide-spread behaviour associated with a number of ecological processes as site fidelity, breeding, social associations, optimal foraging [18] and is successfully modelled also by fractional kinetics [16, 52].

The FRT for the uncoupled CTRW reported at the beginning of Subsection 2.2 can be exactly calculated by using integral formula (27). In particular, we can exactly calculate the FRT for the fractional kinetics emerging from a CTRW with Mittag-Leffler distributed waiting-times (18). In this section, we consider a random walk starting at the origin $x = 0$ and, provided that the first jump away is of length, say, ξ, we are interested in how much time it takes to the random walker to pass through the origin for the first time after the departure. As a matter of fact, the problem is equivalent to a first-passage time problem when the random walker starts at the initial instant τ_0 from the random first-landing position ξ and the associated absorbing barrier is located at the origin $x = 0$. Actually, the first-landing position ξ is distributed according to the jump-size distribution $k(\xi)$ and τ_0 is the delay elapsed between the two locations $x = 0$ and $x = \xi$. The CTRW formalism allows for two different formulations of the problem: *first jump then wait* (jw), such that $\tau_0 = 0$, and *first wait then jump* (wj), such that τ_0 is a random variable distributed according to $\psi(t)$. Thus, by following the literature [9, 6, 22], we define the FRT as follows:

$$\text{FRT} \equiv \begin{cases} \mathcal{T}_\ell(\xi) + \tau_0 \,, & \xi \in \mathbb{R}^- \,, \\ \mathcal{T}_r(\xi) + \tau_0 \,, & \xi \in \mathbb{R}^+ \,, \end{cases} \tag{28}$$

where $\mathcal{T}$ is the first-passage time and, in particular, $\mathcal{T}_\ell$ is the first-passage time when the first jump away from the origin is towards the left into the position $\xi < 0$ and $\mathcal{T}_r$ when it is towards the right into the position $\xi > 0$.

Let $\lambda(\xi, t) = -\dfrac{\partial \Lambda}{\partial t}$ be the first-passage time density conditioned to the starting position ξ and satisfying the normalisation condition $\int_0^\infty \lambda(\xi, t)\, dt = 1$. Then the density function of the first-passage time weighted over all the possibile first-jump landing positions is

$$\begin{aligned} f(t) &= \int_{-\infty}^{+\infty} k(\xi)\lambda(\xi, t)\, d\xi \\ &= 2\int_0^{+\infty} k(\xi)\lambda(\xi, t)\, d\xi \,, \end{aligned} \tag{29}$$

where the symmetry property of the jump-size distribution, $k(\xi) = k(-\xi)$, is used.

Thus, since $\mathcal{T}$ and τ_0 are statistically independent, we have that the density function of their sum, namely the FRT (28), is

$$\mathcal{P}^{\mathrm{jw}}(t) = f(t) \,, \quad \text{when } \tau_0 = 0 \,, \tag{30a}$$

$$\mathcal{P}^{\mathrm{wj}}(t) = \int_0^t \psi(t - \zeta)\, \mathcal{P}^{\mathrm{jw}}(\zeta)\, d\zeta \,, \quad \text{when } \tau_0 \sim \psi(t) \,. \tag{30b}$$

An alternative formulation is possible by starting from the discrete-time setting with the extension to the continuous-time setting in the fashion of formula (3) [1]. In particular, by defining the discrete-time first-passage time density $\lambda_n(\xi) = \phi_{n-1}(\xi) - \phi_n(\xi)$, with $n \geq 1$, and exploiting the Sparre Andersen theorem (2) at $\xi = 0$, we have [41]

$$\begin{aligned} \lambda_n(0) &= 2^{-2(n-1)}\binom{2(n-1)}{n-1} - 2^{-2n}\binom{2n}{n} \\ &= 2^{-2(n-1)}\binom{2n}{n}\left\{\frac{n}{4n-2} - \frac{1}{4}\right\} \\ &= 2^{-2(n-1)}\binom{2n}{n}\frac{1}{2(4n-2)} \\ &= \frac{2^{-2n}}{2n-1}\binom{2n}{n} \,, \end{aligned} \tag{31}$$

which correctly gives $\sum_{n=1}^\infty \lambda_n(0) = 1$. Moreover, by definition (29), in the discrete-time setting we have

$$f_n = 2\int_0^\infty k(\xi)\lambda_n(\xi)\, d\xi$$

$$
\begin{aligned}
&= 2 \int_0^{\infty} k(\xi)\, [\phi_{n-1}(\xi) - \phi_n(\xi)]\; d\xi \\
&= 2\, [\phi_n(0) - \phi_{n+1}(0)] = 2\lambda_{n+1}(0)\,, \quad n \geq 1\,,
\end{aligned} \tag{32}
$$

where in the third line we used (1), and it holds that $f_0 = 0$.

3.2 The (jw)-case

We remind that in this case $\mathcal{P}^{\mathrm{jw}}(t) = f(t)$ (see (30a)), thus in the following we study the density function $f(t)$.

In particular, thanks to the relation in the Laplace domain between the first-passage time density and the corresponding survival probability, i.e., $\widetilde{\lambda}(\xi, s) = 1 - s\,\widetilde{\Lambda}(\xi, s)$, from (29) we have

$$
\begin{aligned}
\widetilde{f}(s) &= 2 \int_0^{+\infty} k(\xi) \widetilde{\lambda}(\xi, s)\, d\xi \\
&= 1 - 2s \int_0^{\infty} k(\xi) \widetilde{\Lambda}(\xi, s)\, d\xi \\
&= 1 - 2\, \frac{\sqrt{1 - \widetilde{\psi}(s)} - 1 + \widetilde{\psi}(s)}{\widetilde{\psi}(s)} \\
&= \frac{2 - \widetilde{\psi}(s) - 2\sqrt{1 - \widetilde{\psi}(s)}}{\widetilde{\psi}(s)}\,,
\end{aligned} \tag{33}
$$

where formula (27) has been used in the third line. It can be checked that the normalisation condition $\widetilde{f}(0) = 1$ is met because $\widetilde{\psi}(0) = 1$. Since formula (27) from Theorem 1 has been used for deriving (33), we underline the following remark.

Remark 1. The density function of the FRT of a symmetric CTRW in the (jw)-case is independent of the jump-size distribution.

Moreover, if we consider the non-Markovian setting (20), from formula (33) we finally obtain that the Laplace transform of the density function of the FRT for a CTRW of (jw)-type is

$$
\widetilde{f}(s) = 1 + 2s^{\beta} - 2\sqrt{s^{\beta}}\sqrt{s^{\beta} + 1}\,. \tag{34}
$$

When $\beta = 1$, the system is Markovian. Therefore, if $f_{\mathrm{M}}(t)$ denotes the corresponding density of the FRT, the following equality holds in the Laplace domain:

$$
\widetilde{f}(s) = \widetilde{f}_{\mathrm{M}}(s^{\beta})\,. \tag{35}
$$

Hence, we can state the next theorem.

Theorem 2. *Let $f(t)$ and $f_{\mathrm{M}}(t)$ be the density functions of the FRT in the (jw)-case for a symmetric CTRW with Mittag-Leffler (18) and exponentially distributed waiting-times, respectively, then*

$$
f(t) = \int_0^{\infty} \zeta^{-1/\beta} \ell_{\beta}(t/\zeta^{1/\beta}) f_{\mathrm{M}}(\zeta)\, d\zeta\,, \quad 0 < \beta < 1\,, \tag{36}
$$

where $\ell_{\beta}(t)$ is the one-sided Lévy stable density characterised by the Laplace transform $\mathrm{e}^{-s^{\beta}}$.

Proof. By reminding the Efros theorem (23), formula (35) can be inverted setting $v(s) = 1$ and $q(s) = s^\beta$, so that we have the integral relation (36). □

We observe that from (36) the Markovian case is recovered when $\beta = 1$ because $\ell_1(t) = \delta(t-1)$.

In the Markovian case with $\beta = 1$, from (34) we have

$$\begin{aligned}\widetilde{f_{\mathrm{M}}}(s) &= (1+s)\,\frac{1+2s-2\sqrt{s}\sqrt{s+1}}{1+s} \\ &= \frac{1}{1+s} + 2s\left[\frac{1}{1+s} - \frac{1}{\sqrt{s(1+s)}}\right] \\ &\quad + s\left\{\frac{1}{1+s} + 2s\left[\frac{1}{1+s} - \frac{1}{\sqrt{s(1+s)}}\right]\right\}. \end{aligned} \tag{37}$$

Thus, by remembering the properties of the Laplace transform, and also the Laplace transform pair (14) and (15), we can calculate the exact result

$$\begin{aligned} f_{\mathrm{M}}(t) &= \mathrm{e}^{-t/2}\left[I_0(\zeta) - \frac{dI_0(\zeta)}{d\zeta}\right]_{\zeta=t/2} - \mathrm{e}^{-t} \\ &\quad + \frac{d}{dt}\left\{\mathrm{e}^{-t/2}\left[I_0(\zeta) - \frac{dI_0(\zeta)}{d\zeta}\right]_{\zeta=t/2} - \mathrm{e}^{-t}\right\} \\ &= \frac{1}{2}\mathrm{e}^{-t/2}\left[I_0(\zeta) - \frac{d^2I_0(\zeta)}{d\zeta^2}\right]_{\zeta=t/2}, \end{aligned} \tag{38}$$

and then the integral formula (36) is fully determined.

From (38), by using the values given in (17), we have that

$$f_{\mathrm{M}}(0) = \frac{1}{4}, \tag{39}$$

and, when plugged in (29), it gives the following universal result independent of $k(\xi)$

$$f_{\mathrm{M}}(0) = \int_{-\infty}^{+\infty} k(\xi)\lambda_{\mathrm{M}}(\xi, 0)\, d\xi = \frac{1}{4}, \tag{40}$$

in opposition to the first-passage time density which is indeed dependent on $k(\xi)$ also at $t = 0$ according to $\lambda_{\mathrm{M}}(\xi, 0) = \int_\xi^\infty k(y)dy$. In the non-Markovian case

$$f(0) = +\infty \tag{41}$$

in analogy with the first-passage time density $\lambda(\xi, 0)$. In fact, after the change of variable $\zeta = (t/\chi)^\beta$ in (36), we have

$$\begin{aligned} f(t) &= \beta t^{\beta-1}\int_0^\infty \ell_\beta(\chi) f_{\mathrm{M}}(t^\beta/\chi^\beta)\,\frac{d\chi}{\chi^\beta} \\ &\sim C\, t^{\beta-1}, \quad t \to 0, \quad \text{with } C = \frac{\beta}{4}\int_0^\infty \ell_\beta(\chi)\,\frac{d\chi}{\chi^\beta}. \end{aligned} \tag{42}$$

From (34) we can derive the asymptotic behaviour for large elapsed times. Actually, we have $\widetilde{f}(s) \sim 1-2s^{\beta/2}$ when $s \to 0$ which is consistent with the behaviour of $\mathrm{e}^{-2s^{\beta/2}}$ for small s and gives

$$f(t) \sim \frac{1}{2^{2/\beta}} \ell_{\beta/2}(t/2^{2/\beta}) \sim \frac{2}{t^{\beta/2+1}}, \quad t \to +\infty . \tag{43}$$

From (43) it follows that the mean FRT in the (jw)-case is infinite both in the Markovian and non-Markovian setting but the right tail of the density function decreases slower in non-Markovian systems. Because of formula (29), we have that scaling (43) is also the scaling of the corresponding first-passage time density $\lambda(\xi, t)$ [22], notwithstanding the exact functions may differ.

Tools of fractional calculus can be used for studying the integral formula (36). In fact, let J^μ, with $\mu > 0$, be the Riemann–Liouville fractional integral defined by the Laplace symbol $s^{-\mu}$ [17], then in the Laplace domain

$$\mathcal{L}\{J^{1-\beta} f(t); s\} = \frac{\widetilde{f}(s)}{s^{1-\beta}} = \int_0^\infty \frac{\mathrm{e}^{-\zeta s^\beta}}{s^{1-\beta}} f_{\mathrm{M}}(\zeta)\, d\zeta , \tag{44}$$

which, by using the formula $\mathcal{L}\{t^{-\beta} M_\beta(\zeta t^{-\beta}); s\} = s^{\beta-1}\mathrm{e}^{-\zeta s^\beta}$, with $\mathrm{Re}\{s\} > 0$, [30, (4.26)], gives

$$J^{1-\beta} f(t) = \int_0^\infty \frac{1}{t^\beta} M_\beta(\zeta/t^\beta) f_{\mathrm{M}}(\zeta)\, d\zeta . \tag{45}$$

Since from the study of time-fractional diffusion equations [30] we have that $\lim_{t\to 0} t^{-\beta} M_\beta(\zeta/t^\beta) = \delta(\zeta)$, then

$$J^{1-\beta} f(t)\big|_{t=0} = f_{\mathrm{M}}(0) = \frac{1}{4} . \tag{46}$$

Furthermore, we can also have the Laplace transform of the cumulative density function $F^{\mathrm{jw}}(t) = \int_0^t f(\zeta) d\zeta$ which, by using (35), leads to the equality

$$\widetilde{F}^{\mathrm{jw}}(s) = \frac{\widetilde{f}(s)}{s} = s^{\beta-1} \frac{\widetilde{f}_{\mathrm{M}}(s^\beta)}{s^\beta} = s^{\beta-1} \widetilde{F}^{\mathrm{jw}}_{\mathrm{M}}(s^\beta) . \tag{47}$$

Therefore, we can state the following theorem.

Theorem 3. *Let $F^{\mathrm{jw}}(t)$ and $F^{\mathrm{jw}}_{\mathrm{M}}(t)$ be the cumulative density functions of the FRT in the (jw)-case for a symmetric CTRW with Mittag-Leffler* (18) *and exponentially distributed waiting-times, respectively. Then*

$$F^{\mathrm{jw}}(t) = t^{-\beta} \int_0^\infty M_\beta(\zeta/t^\beta) F^{\mathrm{jw}}_{\mathrm{M}}(\zeta)\, d\zeta , \quad 0 < \beta < 1 , \tag{48}$$

where $M_\beta(\zeta)$ is the Mainardi/Wright function defined in (25).

Proof. By applying Efros formula (23) in analogy with the pair (22) and (24), we have (48). □

To conclude, we remind the formula [32, (6.3)]

$$t^{-\beta} M_\beta(\zeta/t^\beta) = t^{-\nu} \int_0^\infty M_\eta(\zeta/y^\eta) M_\nu(y/t^\nu)\, \frac{dy}{y^\eta}\,, \quad \beta = \nu\eta\,, \tag{49}$$

and then from Theorem 3 we have the following corollary.

Corollary 1. *The relation between the cumulative density function with the anomalous parameter β and that of parameter $\eta > \beta$ is*

$$F^{\mathrm{jw}}(t;\beta) = t^{-\nu} \int_0^\infty M_\nu(\zeta/t^\nu) F^{\mathrm{jw}}(\zeta;\eta)\, d\zeta\,, \quad \beta = \nu\eta\,. \tag{50}$$

Proof. By using (49) in (48) we have (50). □

Remark 2. An interesting special case of formula (50) is obtained when $\nu = 1/2$, that is,

$$F^{\mathrm{jw}}(t;\beta/2) = 2\int_0^\infty \frac{\mathrm{e}^{-\zeta^2/(4t)}}{\sqrt{4\pi t}} F^{\mathrm{jw}}(\zeta;\beta)\, d\zeta\,, \tag{51}$$

where the identity $M_{1/2}(\zeta) = \mathrm{e}^{-\zeta^2/4}/\sqrt{\pi}$ has been used.

A large number of other formulae can be obtained using the existing results for the Mainardi/Wright function, e.g., [30, 29, 31, 39].

3.3 The (wj)-case

The FRT density function in the (wj)-case (30b) can be studied in the Laplace domain. In particular, by using (29) and (33), from (30b) we have

$$\begin{aligned} \widetilde{\mathcal{P}}^{\mathrm{wj}}(s) &= \widetilde{\psi}(s)\widetilde{f}(s) \\ &= 2 - \widetilde{\psi}(s) - 2\sqrt{1-\widetilde{\psi}(s)}\,. \end{aligned} \tag{52}$$

It can be checked that the normalisation condition $\widetilde{\mathcal{P}}^{\mathrm{wj}}(0) = 1$ is met and by applying the initial and the final value theorems one has

$$\mathcal{P}^{\mathrm{wj}}(0) = \lim_{s\to+\infty} s\widetilde{\mathcal{P}}^{\mathrm{wj}}(s) = 0\,, \tag{53a}$$

$$\mathcal{P}^{\mathrm{wj}}(+\infty) = \lim_{s\to 0} s\widetilde{\mathcal{P}}^{\mathrm{wj}}(s) = 0\,, \tag{53b}$$

because $\widetilde{\psi}(0) = 1$ and $\widetilde{\psi}(+\infty) = 0$. In analogy with Remark 1, we underline also in this case that, since Theorem 1 lays behind the derivation of (52), we have the following remark.

Remark 3. The density function of the FRT of a symmetric CTRW in the (wj)-case is independent of the jump-size distribution.

Formulae (53a) and (53b) hold both in the Markovian and non-Markovian case. More explicitly, from (20) we have that the density function of the FRT for a CTRW of (wj)-type is

$$\widetilde{\mathcal{P}}^{\mathrm{wj}}(s) = \frac{1 + 2s^\beta - 2\sqrt{s^\beta}\sqrt{s^\beta+1}}{s^\beta+1}\,. \tag{54}$$

When $\beta = 1$, the system is Markovian and therefore, denoting by $\mathcal{P}_{\mathrm{M}}^{\mathrm{wj}}(t)$ the corresponding density of the FRT, in analogy with (35), in the Laplace domain one has

$$\widetilde{\mathcal{P}}^{\mathrm{wj}}(s) = \widetilde{\mathcal{P}}_{\mathrm{M}}^{\mathrm{wj}}(s^\beta) \,. \tag{55}$$

Formula (55) can be inverted by applying the Efros formula (23), and we have the following theorem.

Theorem 4. *Let $P^{\mathrm{wj}}(t)$ and $P_{\mathrm{M}}^{\mathrm{wj}}(t)$ be the density functions of the FRT in the (wj)-case for a symmetric CTRW with Mittag-Leffler* (18) *and exponentially distributed waiting-times, respectively. Then*

$$\mathcal{P}^{\mathrm{wj}}(t) = \int_0^\infty \zeta^{-1/\beta} \ell_\beta(t/\zeta^{1/\beta})\, \mathcal{P}_{\mathrm{M}}^{\mathrm{wj}}(\zeta)\, d\zeta \,, \quad 0 < \beta < 1 \,, \tag{56}$$

which recasts formula (36) *for the (jw)-case.*

In the Markovian case with $\beta = 1$, from (54) we have

$$\widetilde{\mathcal{P}}_{\mathrm{M}}^{\mathrm{wj}}(s) = \frac{1}{1+s} + 2s \left[\frac{1}{1+s} - \frac{1}{\sqrt{s(1+s)}} \right] , \tag{57}$$

and, by remembering the properties of the Laplace transform together with the Laplace transform pair (14) and (15), we can calculate the exact result

$$\mathcal{P}_{\mathrm{M}}^{\mathrm{wj}}(t) = \mathrm{e}^{-t/2} \left[I_0(\zeta) - \frac{dI_0(\zeta)}{d\zeta} \right]_{\zeta=t/2} - \mathrm{e}^{-t} \,, \quad \mathcal{P}_{\mathrm{M}}^{\mathrm{wj}}(0) = 0 \,, \tag{58}$$

so that the integral formula (56) is fully determined.

We can quantify the difference between the two cases (jw) and (wj). In fact, first by comparing (38) and (58) we have the proposition.

Proposition 1. *The relationship between the density functions of the FRT with exponentially distributed waiting-times in the (wj)- and (jw)-case is*

$$\mathcal{P}_{\mathrm{M}}^{\mathrm{wj}}(t) = 2\mathcal{P}_{\mathrm{M}}^{\mathrm{jw}}(t) - \mathrm{e}^{-t/2} \left[\frac{dI_0}{d\zeta} - \frac{d^2 I_0}{d\zeta^2} \right]_{\zeta=t/2} - \mathrm{e}^{-t} \,. \tag{59}$$

Later, by using (36) and (56), we find

$$\begin{aligned} \mathcal{P}^{\mathrm{wj}}(t) = 2\mathcal{P}^{\mathrm{jw}}(t) &- \int_0^\infty \zeta^{-1/\beta} \ell_\beta(t/\zeta^{1/\beta})\, \mathrm{e}^{-\zeta}\, d\zeta \\ &- \int_0^\infty \zeta^{-1/\beta} \ell_\beta(t/\zeta^{1/\beta})\, \mathrm{e}^{-\zeta/2} \left[\frac{dI_0}{d\chi} - \frac{d^2 I_0}{d\chi^2} \right]_{\chi=\zeta/2} d\zeta \,. \end{aligned} \tag{60}$$

The second term in the r.h.s. can be solved and put in a more clear form. In fact, by using some formula concerning Lévy density function [30, (4.26)], Mainardi/Wright function [30, (4.32)] and Mittag-Leffler function [29, (1.45)], we have the equalities

$$\int_0^\infty \zeta^{-1/\beta} \ell_\beta(t/\zeta^{1/\beta})\, \mathrm{e}^{-\zeta s^\beta}\, d\zeta = \frac{\beta}{t} \int_0^\infty \frac{\zeta}{t^\beta} M_\beta(\zeta/t^\beta)\, \mathrm{e}^{-\zeta s^\beta}\, d\zeta$$

$$= -\frac{s^{1-\beta}}{t}\frac{\partial}{\partial s}\int_0^\infty M_\beta(y)\,\mathrm{e}^{-t^\beta s^\beta y}\,dy$$
$$= -\frac{s^{1-\beta}}{t}\frac{\partial}{\partial s}E_\beta(-t^\beta s^\beta)$$
$$= t^{\beta-1}E_{\beta,\beta}(-t^\beta s^\beta)\,. \tag{61}$$

Thus, by remembering (18) and setting $s = 1$, one has

$$\int_0^\infty \zeta^{-1/\beta}\ell_\beta(t/\zeta^{1/\beta})\,\mathrm{e}^{-\zeta}\,d\zeta = \psi(t)\,. \tag{62}$$

And so we have the following proposition.

Proposition 2. *The relationship between the density functions of the FRT with the Mittag-Leffler distributed waiting-times* (18) *in the* (wj)*- and* (jw)*-case is*

$$\mathcal{P}^{\mathrm{wj}}(t) = 2\mathcal{P}^{\mathrm{jw}}(t) - \psi(t) - \int_0^\infty \zeta^{-1/\beta}\ell_\beta(t/\zeta^{1/\beta})\,\mathrm{e}^{-\zeta/2}\left[\frac{dI_0}{d\chi} - \frac{d^2I_0}{d\chi^2}\right]_{\chi=\zeta/2} d\zeta\,. \tag{63}$$

From (54) we can derive the asymptotic behaviour for large elapsed times and, in analogy with (43), it is

$$\mathcal{P}^{\mathrm{wj}}(t) \sim \frac{1}{2^{2/\beta}}\,\ell_{\beta/2}(t/2^{2/\beta}) \sim \frac{2}{t^{\beta/2+1}}\,, \quad t \to +\infty\,, \tag{64}$$

from which it follows that the mean FRT also in the (wj)-case is infinite both in the Markovian and non-Markovian settings, and the right tail of the density function decreases slower in non-Markovian systems.

The analogies between the (jw)- and the (wj)-cases also include formulae (45) and (46) that now read as

$$J^{1-\beta}\mathcal{P}^{\mathrm{wj}}(t) = \int_0^\infty \frac{1}{t^\beta}M_\beta(\zeta/t^\beta)\mathcal{P}^{\mathrm{wj}}_{\mathrm{M}}(\zeta)\,d\zeta\,, \tag{65}$$
$$J^{1-\beta}\mathcal{P}^{\mathrm{wj}}(t)\big|_{t=0} = \mathcal{P}^{\mathrm{wj}}_{\mathrm{M}}(0) = 0\,, \tag{66}$$

as well as the analogue of Theorem 3, given below.

Theorem 5. *Let* $F^{\mathrm{wj}}(t)$ *and* $F^{\mathrm{wj}}_{\mathrm{M}}(t)$ *be the cumulative density functions of the FRT in the* (jw)*-case for a symmetric CTRW with Mittag-Leffler* (18) *and exponentially distributed waiting-times, respectively. Then*

$$F^{\mathrm{wj}}(t) = t^{-\beta}\int_0^\infty M_\beta(\zeta/t^\beta)F^{\mathrm{wj}}_{\mathrm{M}}(\zeta)\,d\zeta\,, \quad 0 < \beta < 1\,, \tag{67}$$

where $M_\beta(\zeta)$ *is the Mainardi/Wright function defined in* (25).

Proof. The proof is the same as of Theorem 3. □

Moreover, from Theorem 5 and by using (49), we have the following corollary.

Corollary 2. *The relation between the cumulative density function with the anomalous parameter β and that of parameter $\eta > \beta$ is*

$$F^{\mathrm{wj}}(t;\beta) = t^{-\nu} \int_0^\infty M_\nu(\zeta/t^\nu) F^{\mathrm{wj}}(\zeta;\eta)\, d\zeta \,, \quad \beta = \nu\eta \,,$$

which is the analogue of formula (50).

Proof. The proof is the same as of Corollary 1. □

Remark 4. An interesting special case of Corollary 2 is obtained when $\nu = 1/2$, that is,

$$F^{\mathrm{wj}}(t;\beta/2) = 2\int_0^\infty \frac{\mathrm{e}^{-\zeta^2/(4t)}}{\sqrt{4\pi t}} F^{\mathrm{wj}}(\zeta;\beta)\, d\zeta \,,$$

which is the analogue of formula (51).

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