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Convergence of random walks in ℓ_p-spaces of growing dimension

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Abstract We prove a limit theorem for paths of random walks with n steps in $\mathbb{R}^d$ as n and d both go to infinity. For this, the paths are viewed as finite metric spaces equipped with the ℓ_p-metric for $p \in [1, \infty)$. Under the assumptions that all components of each step are uncorrelated, centered, have finite $2p$-th moments, and are identically distributed, we show that such random metric space converges in probability to a deterministic limit space with respect to the Gromov-Hausdorff distance. This result generalises earlier work by Kabluchko and Marynych [Ann. Inst. H. Poincaré Probab. Statist. 60(4): 2945–2974, 2024] for $p = 2$.

Keywords Random metric space, random walk, growing dimension
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1 Introduction

For each dimension $d \in \mathbb{N}$, consider a random walk $(S_i^{(d)})_{i=0,1,\dots}$ in $\mathbb{R}^d$ defined by

$$S_0^{(d)} = 0, \quad S_n^{(d)} = X_1^{(d)} + X_2^{(d)} + \cdots + X_n^{(d)}, \quad n \in \mathbb{N},$$

where $X_i^{(d)} = (X_{i,1}^{(d)}, \dots, X_{i,d}^{(d)}), i \geq 1$, are independent identically distributed random vectors in $\mathbb{R}^d$, and denote $S_i^{(d)} = (S_{i,1}^{(d)}, \dots, S_{i,d}^{(d)})$.

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The values of this random walk

$$\mathcal{Z}_n^{(d)} = \{S_0^{(d)}, \dots, S_n^{(d)}\}, \quad n \in \mathbb{N}.$$

are considered as a finite metric space which is embedded in $\mathbb{R}^d$ with the induced Euclidean metric.

In the regime where the dimension d and the distribution of $X_1^{(d)}$ are fixed, with $\mathbf{E}X_1^{(d)} = 0$ and an identity covariance matrix $\mathrm{Cov}(X_1^{(d)}) = I_d$. Donsker's invariance principle implies that, after rescaling by $n^{-1/2}$, the random set $\mathcal{Z}_n^{(d)}$ converges in distribution to the path of the d-dimensional standard Brownian motion on $[0, 1]$.

Let ℓ_2 be the space of square-summable real sequences with norm denoted by $\|\cdot\|_2$. For every $d \in \mathbb{N}$, we identify $\mathbb{R}^d$ with the d-dimensional coordinate subspace of ℓ_2, which leads to the embedding of $\mathbb{R}^d$ into ℓ_2. This identification allows us to view all random walks as subsets of a common ambient space.

When both n and d tend to infinity, under the square integrability and several further assumptions listed in [5], the random metric space $(n^{-1/2}\mathcal{Z}_n^{(d)}, \|\cdot\|_2)$ converges in probability to the Wiener spiral with respect to the Gromov–Hausdorff distance. The latter space is defined as the set of indicator functions $\mathbf{1}_{[0,t]}$, $t \in [0, 1]$, viewed as a subset of the Hilbert space $L^2([0, 1])$ and endowed with the metric induced by the L^2-norm. With respect to this metric, the space is isometric to the interval $[0, 1]$ equipped with the distance $r(t, s) = \sqrt{|t - s|}$. Later Jin [4] verified that the bridge variant of the random metric space also converges in probability to a deterministic limit in the Gromov–Hausdorff sense, that is $[0, 1]$ equipped with the pseudo-metric $\sqrt{|t - s|(1 - |t - s|)}$. Furthermore, the case of random walk with heavy-tailed increments is considered in [6].

The *Gromov–Hausdorff distance* between metric spaces $\mathbb{X} = (X, \rho_X)$ and $\mathbb{Y} = (Y, \rho_Y)$ is defined as

$$d_{GH}(\mathbb{X}, \mathbb{Y}) = \inf_{i:X\hookrightarrow Z, j:Y\hookrightarrow Z} d_H\big(i(X), j(Y)\big),$$

where the infimum is taken over all isometric embeddings i and j into all possible metric spaces (Z, d) which can embed X and Y. The Hausdorff distance between sets F and H in (Z, d) is defined as

$$d_H(F, H) = \inf\{\varepsilon > 0 : F \subset H^\varepsilon \text{ and } H \subset F^\varepsilon\},$$

where $F^\varepsilon = \{x : d(x, F) < \varepsilon\}$ is the ε-neighbourhood of F, see [1, Chapter 7].

We replace the ℓ_2-metric on the space of sequences with the ℓ_p-metric for a general $p \in [1, \infty)$. The study of random metric spaces relies on identifying the spaces up to isometries. The following remarks explain why the proof for $p \neq 2$ is not a routine modification of the ℓ_2 case. Contrary to the ℓ_2 setting, which admits a large group of rotations as isometries, the isometry group of ℓ_p for $p \neq 2$ is far more constrained, see [7] and [2, Theorem 7.4.1]. Furthermore, while we have the identity $\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\langle x, y\rangle$, no analogous simple expression exists for $\|x + y\|_p^p$ with $p \neq 2$. This complicates the analysis of path of the random walk.

It should be noted that Kabluchko and Marynych [5] established that for subsets of ℓ_2, convergence in the Gromov–Hausdorff sense is equivalent to convergence in the

Hausdorff distance up to isometries of ℓ_2, that is distance for the two subsets defined by taking the infimum over the Hausdorff distance between their images under all possible isometries of ℓ_2. However, this equivalence fails for compact subsets of ℓ_p when $p \neq 2$. For instance, consider the two-point metric spaces $F = \{(0,0,\ldots),(1,1,0,\ldots)\}$ and $H = \{(0,0,\ldots),(a^{1/p},b^{1/p},(2-a-b)^{1/p},0,\ldots)\}$ for any $a,b > 0$ such that $a+b<2$, $a+b\neq 1$ and $a,b\neq 1$. Equipped with the ℓ_p-metric, these spaces are isometric for any $p \in [1,\infty)$ and thus the Gromov–Hausdorff distance between them vanishes, while it is impossible to map F to H using an isometry of ℓ_p if $p \neq 2$.

Fix a $p \in [1,\infty)$. Impose a special structure on the increments of the random walk. Namely, we assume that

$$X_1^{(d)} = d^{-1/p}(\xi_1,\ldots,\xi_d), \quad \xi_1,\ldots,\xi_d \text{ share the same distribution,}$$
$$\mathbf{E}\xi_1 = 0, \quad \mathbf{E}|\xi_1|^{2p} < \infty, \quad \operatorname{Cov}(\xi_i,\xi_j) = 0 \quad \text{for all } 1 \le i \neq j \le d. \tag{1}$$

Denote $\mathbf{E}\xi^2 = \sigma^2$. Let M_p be the p-th absolute moment of the standard normal distribution.

Theorem 1. *Let $p \in [1,\infty)$ and $d = d(n)$ be an arbitrary sequence of positive integers such that $d(n) \to \infty$ as $n \to \infty$. Consider a random walk with increments given by (1). Then, as $n \to \infty$, the random metric space $(n^{-1/2}\mathcal{Z}_n^{(d)}, \|\cdot\|_p)$, converges in probability to $\left([0,1], \sqrt{|t-s|}\sigma M_p^{1/p}\right)$ under the Gromov-Hausdorff distance.*

Remark 1. In the case $p = 2$, we assume $\mathbf{E}\xi^4 < \infty$, while only the finite second moment of ξ is required in [5]. However, we do not see a possibility that the moment assumption can be weaker, since a different tool is applied in our proof.

The paper is organised as follows. In Section 2, we provide the univariate and bivariate moment convergence theorems, which serve as the main tools for proving the limit theorem. Section 3 contains some auxiliary theorems and the proof of the main result.

2 Moment convergence theorem

We need the following results.

Lemma 1 (Moment convergence theorem)**.** *Let $\eta,\eta_1,\eta_2,\ldots$ be independent identically distributed random variables with $\mathbf{E}\eta = \mu$ and $\operatorname{Var}\eta = \sigma^2$, and $S_n = \eta_1+\cdots+\eta_n$, $n \ge 1$. Then*

$$\mathbf{E}\left|\frac{S_n - n\mu}{\sigma\sqrt{n}}\right|^p \to M_p = 2^{p/2}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right),$$

if $p \in (0,2)$ and for $p \ge 2$ if $\mathbf{E}|\eta|^p < \infty$.

Lemma 2 (Marcinkiewicz–Zygmund inequality. See Corollary 3.8.2 in [3])**.** *Let $p \ge 1$. Suppose that $X, X_1, \ldots, X_n$ are independent, identically distributed random variables with mean 0 and $\mathbf{E}|X|^p < \infty$. Set $S_n = \sum_{k=1}^n X_k$, $n \ge 1$. Then there exists a constant B_p depending only on p such that*

$$\mathbf{E}|n^{-1/2}S_n|^p \le \begin{cases} B_p n^{1-p/2}\mathbf{E}|X|^p, & \text{when } 1 \le p \le 2, \\ B_p \mathbf{E}|X|^p, & \text{when } p \ge 2. \end{cases}$$

Theorem 2 (Bivariate moment convergence theorem)**.** *Let* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *be independent copies of a centered* $2p$*-integrable random vector* (X, Y) *with the covariance matrix* Σ*. Denote* $S_n = X_1 + \cdots + X_n$ *and* $Z_n = Y_1 + \cdots + Y_n$*. Then*

$$\mathbf{E}\big|n^{-1/2}S_n\big|^p\big|n^{-1/2}Z_n\big|^p \to \mathbf{E}|\eta_1\eta_2|^p \quad \text{as } n \to \infty, \tag{2}$$

where $(\eta_1, \eta_2) \sim \mathcal{N}(0, \Sigma)$.

Proof. By the central limit theorem,

$$\big(n^{-1/2}S_n^{(d)}, n^{-1/2}Z_n\big) \xrightarrow{d} (\eta_1, \eta_2) \quad \text{as } n \to \infty.$$

Now we apply Skorokhod's representation theorem, which allows us to replace the distributional convergence above with a.s. convergence on a new probability space which accommodates the following objects.

- For every $d \in \mathbb{N}$, a distributional copy $(\overline{X}_k^{(d)}, \overline{Y}_k^{(d)})_{k\in\mathbb{N}}$ of the sequence $(X_k^{(d)}, Y_k^{(d)})$;
- Distributional copy $(\overline{\eta}_1, \overline{\eta}_2)$ of (η_1, η_2).

Denote $\overline{S}_n^{(d)} = \overline{X}_1^{(d)} + \cdots + \overline{X}_n^{(d)}$ and $\overline{Z}_n^{(d)} = \overline{Y}_1^{(d)} + \cdots + \overline{Y}_n^{(d)}$. Then

$$\big(n^{-1/2}\overline{S}_n^{(d)}, n^{-1/2}\overline{Z}_n^{(d)}\big) \xrightarrow{\text{a.s.}} (\eta_1, \eta_2) \quad \text{as } n \to \infty.$$

By Lemma 1, as $n \to \infty$,

$$\begin{aligned}\mathbf{E}\big(|n^{-1/2}\overline{S}_n^{(d)}|^{2p} + |n^{-1/2}\overline{Z}_n^{(d)}|^{2p}\big) &= \mathbf{E}\big(n^{-1/2}|S_n^{(d)}|^{2p} + |n^{-1/2}Z_n^{(d)}|^{2p}\big)\\ &\to \mathbf{E}(|\eta_1|^{2p} + |\eta_2|^{2p}) = \mathbf{E}(|\overline{\eta}_1|^{2p} + |\overline{\eta}_1|^{2p}).\end{aligned}$$

Furthermore, by Pratt's extension of the Lebesgue dominated convergence theorem, see [3, Theorem 5.5], and the inequality

$$|xy|^p = |x|^p|y|^p \le \frac{|x|^{2p} + |y|^{2p}}{2}, \quad x, y \in \mathbb{R},$$

we conclude that, as $n \to \infty$,

$$\mathbf{E}\big|n^{-1/2}S_n^{(d)}\big|\big|n^{-1/2}Z_n^{(d)}\big| = \mathbf{E}\big|n^{-1/2}\overline{S}_n^{(d)}\big|^p\big|n^{-1/2}\overline{Z}_n^{(d)}\big|^p \to \mathbf{E}|\overline{\eta}_1\overline{\eta}_2|^p = \mathbf{E}|\eta_1\eta_2|^p. \quad \square$$

3 Convergence of the ℓ_p-metric of random walks

The proof of Theorem 1 relies on the following theorems, while they follow the general scheme of [5], substantial adjustments are necessary to handle the ℓ_p-case with $p \neq 2$.

Theorem 3. *Let* $p \in [1, \infty)$*. Consider a random walk with increments given by* (1)*. Then*

$$n^{-p/2}\|S_{\lfloor nt\rfloor}^{(d)}\|_p^p \xrightarrow{p} t^{p/2}\sigma^p M_p \quad \text{as } n \to \infty$$

for all $t \in [0, 1]$.

Proof. Without loss of generality, let $t = 1$. By the definition of convergence in probability, we need to verify that

$$\mathbf{P}\left\{\left|n^{-p/2}\sum_{i=1}^{d}|S_{n,i}^{(d)}|^p - \sigma^p M_p\right| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty. \tag{3}$$

Markov's inequality implies that (3) is bounded above by

$$\begin{aligned}
&\varepsilon^{-2}\mathbf{E}\left(n^{-p/2}\sum_{i=1}^{d}|S_{n,i}^{(d)}|^p - \sigma^p M_p\right)^2 \\
&\quad= \varepsilon^{-2}\left(\mathbf{E}\left(n^{-p/2}\sum_{i=1}^{d}|S_{n,i}^{(d)}|^p\right)^2 + \sigma^{2p}M_p^2 - 2\sigma^p M_p\mathbf{E}\left(n^{-p/2}\sum_{i=1}^{d}|S_{n,i}^{(d)}|^p\right)\right) \\
&\quad= \varepsilon^{-2}\left(n^{-p}d\mathbf{E}|S_{n,1}^{(d)}|^{2p} + n^{-p}\sum_{1\le i\ne j\le d}\mathbf{E}|S_{n,i}^{(d)}|^p|S_{n,j}^{(d)}|^p + \sigma^{2p}M_p^2\right. \\
&\qquad\qquad\left. - 2n^{-p/2}\sigma^p M_p d\mathbf{E}|S_{n,1}^{(d)}|^p\right) \\
&\quad= \varepsilon^{-2}(A_1 + A_2 + A_3),
\end{aligned}$$

where we have decomposed the terms as follows,

$$\begin{aligned}
A_1 &= n^{-p}d\mathbf{E}|S_{n,1}^{(d)}|^{2p}, \\
A_2 &= n^{-p}\sum_{1\le i\ne j\le d}\operatorname{Cov}\left(|S_{n,i}^{(d)}|^p, |S_{n,j}^{(d)}|^p\right),
\end{aligned}$$

and

$$A_3 = n^{-p}d(d-1)\left(\mathbf{E}|S_{n,1}^{(d)}|^p\right)^2 + \sigma^{2p}M_p^2 - 2n^{-p/2}\sigma^p M_p d\mathbf{E}|S_{n,1}^{(d)}|^p.$$

Let $(\xi_1^{(k)}, \dots, \xi_d^{(k)})$, $1 \le k \le n$, be independent copies of $(\xi_1, \dots, \xi_d)$. Denote

$$b_{nij,p} = \operatorname{Cov}\left(n^{-p/2}\left|\xi_i^{(1)} + \xi_i^{(2)} + \cdots + \xi_i^{(n)}\right|^p, n^{-p/2}\left|\xi_j^{(1)} + \xi_j^{(2)} + \cdots + \xi_j^{(n)}\right|^p\right)$$

for all $1 \le i \ne j \le d$. Thus, by Lemma 2,

$$A_1 = d^{-1}\mathbf{E}n^{-p}|\xi_1 + \xi_1^{(2)} + \cdots + \xi_1^{(n)}|^{2p} \le d^{-1}B_{2p}\mathbf{E}|\xi|^{2p},$$

where B_{2p} is a constant depending only on p, and the term A_1 converges to 0 as $d \to \infty$.

By Theorem 2, the term A_2 converges to 0 as $n \to \infty$ since $\lim_{n\to\infty} b_{ndd,p} = 0$.

Furthermore, the term A_3 is bounded above by

$$\begin{aligned}
&n^{-p}d^2\left(\mathbf{E}|S_{n,1}^{(d)}|^p\right)^2 - n^{-p/2}\sigma^p M_p d\mathbf{E}|S_{n,1}^{(d)}|^p + \sigma^{2p}M_p^2 - n^{-p/2}\sigma^p M_p d\mathbf{E}|S_{n,1}^{(d)}|^p \\
&\quad= n^{-p/2}d\mathbf{E}|S_{n,1}^{(d)}|^p\left(n^{-p/2}d\mathbf{E}|S_{n,1}^{(d)}|^p - \sigma^p M_p\right)
\end{aligned}$$

$$+ \sigma^p M_p \left(\sigma^p M_p - n^{-p/2} d\mathbf{E}|S_{n,1}^{(d)}|^p \right)$$

$$= \left(n^{-p/2} d\mathbf{E}|S_{n,1}^{(d)}|^p - \sigma^p M_p \right)^2,$$

which converges to 0 as $n \to \infty$ by the moment convergence theorem. □

Theorem 4 (Uniform convergence of the ℓ_p-norm of random walk)**.** *Let $p \in [1, \infty)$. Consider a random walk with increments given by* (1)*. Then*

$$\sup_{t \in [0,1]} \left| n^{-p/2} \|S_{\lfloor nt \rfloor}^{(d)}\|_p^p - t^{p/2} \sigma^p M_p \right| \xrightarrow{p} 0 \qquad \text{as } n \to \infty.$$

Proof. For all $p \geq 1$,

$$\|S_n^{(d)}\|_p^p = \sum_{i=1}^d |S_{n,i}^{(d)}|^p = T_n^{(d)} + Q_n^{(d)},$$

where

$$T_n^{(d)} = \sum_{i=1}^d |S_{n,i}^{(d)}|^p - p \sum_{i=1}^d \sum_{j=1}^n X_{j,i}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{p-2},$$

$$Q_n^{(d)} = p \sum_{i=1}^d \sum_{j=1}^n X_{j,i}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{p-2}. \tag{4}$$

For all $n \in \mathbb{N}$,

$$T_n^{(d)} - T_{n-1}^{(d)} = \sum_{i=1}^d \left(|S_{n,i}^{(d)}|^p - p \sum_{j=1}^n X_{j,i}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{p-2} - |S_{n-1,i}^{(d)}|^p \right.$$

$$\left. + p \sum_{j=1}^{n-1} X_{j,i}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{p-2} \right)$$

$$= \sum_{i=1}^d \left(|S_{n,i}^{(d)}|^p - |S_{n-1,i}^{(d)}|^p - p X_{n,i}^{(d)} S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2} \right)$$

$$= \sum_{i=1}^d \left(|S_{n,i}^{(d)}|^p - |S_{n-1,i}^{(d)}|^p - p(S_{n,i}^{(d)} - S_{n-1,i}^{(d)}) S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2} \right).$$

If $p > 1$, then the generic term of this sum is

$$|S_{n,i}^{(d)}|^p - |S_{n-1,i}^{(d)}|^p - p(S_{n,i}^{(d)} - S_{n-1,i}^{(d)}) S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2}$$

$$= |S_{n,i}^{(d)}|^p - |S_{n-1,i}^{(d)}|^p - p S_{n,i}^{(d)} S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2} + p |S_{n-1,i}^{(d)}|^p$$

$$= |S_{n,i}^{(d)}|^p + (p-1) |S_{n-1,i}^{(d)}|^p - p S_{n,i}^{(d)} S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2}$$

$$\geq |S_{n,i}^{(d)}|^p + (p-1) |S_{n-1,i}^{(d)}|^p - p \left(\frac{|S_{n,i}^{(d)}|^p}{p} + \frac{|S_{n-1,i}^{(d)}|^p}{p/(p-1)} \right) = 0,$$

where the last inequality follows from $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, for all $p, q > 1$ and $x, y > 0$.

If $p = 1$, then

$$T_n^{(d)} - T_{n-1}^{(d)} = \sum_{i=1}^{d} \left(|S_{n,i}^{(d)}| - |S_{n-1,i}^{(d)}| - X_{n,i}^{(d)} S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{-1} \right)$$
$$= \sum_{i=1}^{d} \begin{cases} |S_{n,i}^{(d)}| - S_{n,i}^{(d)}, & \text{if } S_{n-1,i}^{(d)} > 0, \\ |S_{n,i}^{(d)}| + S_{n,i}^{(d)}, & \text{if } S_{n-1,i}^{(d)} < 0, \end{cases} \ge 0.$$

Thus, the sequence $T_n^{(d)}$ is monotone increasing.

Next, $Q_n^{(d)}$ is a martingale, since

$$\mathbf{E}\left(Q_n^{(d)} - Q_{n-1}^{(d)} \,\middle|\, \mathcal{F}_{n-1}^{(d)} \right) = \mathbf{E}\left(\sum_{i=1}^{d} p X_{n,i}^{(d)} S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2} \,\middle|\, \mathcal{F}_{n-1}^{(d)} \right)$$
$$= p S_{n-1,i}^{(d)} |S_{n-1,i}^{(d)}|^{p-2} d^{1-1/p} \mathbf{E}\xi = 0,$$

where $\mathcal{F}_{n-1}^{(d)}$ is the σ-algebra generated by $X_1^{(d)}, \dots, X_{n-1}^{(d)}$. Then, by Doob's inequality,

$$\mathbf{P}\left\{ \sup_{t \in [0,1]} |Q_{\lfloor nt \rfloor}^{(d)}| \ge n^{p/2} \varepsilon \right\} \le n^{-p} \varepsilon^{-2} \mathbf{E} (Q_n^{(d)})^2. \tag{5}$$

We shall now estimate the second moment of $Q_n^{(d)}$ for $p > 1$. Firstly, by Lemma 1, for all $\delta > 0$, there exists an integer $N(\delta)$ such that for all $j \ge N(\delta)$,

$$M_{2p-2} - \delta \le \mathbf{E} \left| \frac{\sum_{l=1}^{j-1} X_{l,1}^{(d)}}{\sqrt{j-1} \sqrt{\mathbf{E}(X_{1,1}^{(d)})^2}} \right|^{2p-2} \le M_{2p-2} + \delta.$$

Using this inequality we infer

$$\mathbf{E}(Q_n^{(d)})^2 = p^2 \sum_{i=1}^{d} \sum_{j=1}^{n} \sum_{i'=1}^{d} \sum_{j'=1}^{n} \mathbf{E}\left(X_{j,i}^{(d)} X_{j',i'}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{p-2} S_{j'-1,i'}^{(d)} |S_{j'-1,i'}^{(d)}|^{p-2} \right)$$
$$= p^2 \sum_{i=1}^{d} \sum_{j=1}^{n} \mathbf{E}(X_{j,i}^{(d)})^2 \mathbf{E} |S_{j-1,i}^{(d)}|^{2p-2}$$
$$= p^2 d \mathbf{E}(X_{1,1}^{(d)})^2 \sum_{j=1}^{n} (\sqrt{j-1})^{2p-2} \left(\mathbf{E}(X_{1,1}^{(d)})^2 \right)^{p-1} \mathbf{E} \left| \frac{\sum_{l=1}^{j-1} X_{l,1}^{(d)}}{\sqrt{j-1} \sqrt{\mathbf{E}(X_{1,1}^{(d)})^2}} \right|^{2p-2}$$
$$\le p^2 d \mathbf{E}(X_{1,1}^{(d)})^2 \sum_{j=1}^{N(\delta)} (\sqrt{j-1})^{2p-2} \left(\mathbf{E}(X_{1,1}^{(d)})^2 \right)^{p-1} \mathbf{E} \left| \frac{\sum_{l=1}^{j-1} X_{l,1}^{(d)}}{\sqrt{j-1} \sqrt{\mathbf{E}(X_{1,1}^{(d)})^2}} \right|^{2p-2}$$
$$+ p^2 d \mathbf{E}(X_{1,1}^{(d)})^2 \sum_{j=1}^{n} (\sqrt{j-1})^{2p-2} \left(\mathbf{E}(X_{1,1}^{(d)})^2 \right)^{p-1} (M_{2p-2} + \delta).$$

Hence,

$$\begin{aligned}\mathbf{E}(Q_n^{(d)})^2 &\le p^2 d\mathbf{E}(X_{1,1}^{(d)})^2 \sum_{j=1}^{N(\delta)} \mathbf{E}\Big|\sum_{l=1}^{j-1} X_{l,1}^{(d)}\Big|^{2p-2} + p^2(M_{2p-2}+\delta)n^p d\left(\mathbf{E}(X_{1,1}^{(d)})^2\right)^p \\ &\le p^2 d\mathbf{E}(X_{1,1}^{(d)})^2 N(\delta) B_{2p-2} \max(N(\delta)^{p-1}, N(\delta))\mathbf{E}|X_{1,1}^{(d)}|^{2p-2} \\ &\qquad + p^2(M_{2p-2}+\delta)n^p d\left(\mathbf{E}(X_{1,1}^{(d)})^2\right)^p,\end{aligned}$$

where Lemma 2 is used to bound the first term and B_{2p-2} is a constant which depends on p. The right-hand side is bounded above by

$$p^2 N(\delta) B_{2p-2} \max(N(\delta)^{p-1}, N(\delta)) d^{-1}\mathbf{E}\xi^2 \mathbf{E}|\xi|^{2p-2} + p^2 c n^p d^{-1}\left(\mathbf{E}\xi^2\right)^p.$$

For $p \ge 2$, a direct application of Lemma 2 yields the alternative bound

$$\begin{aligned}\mathbf{E}(Q_n^{(d)})^2 &= p^2 \sum_{i=1}^{d}\sum_{j=1}^{n} \mathbf{E}(X_{j,i}^{(d)})^2 \mathbf{E}|S_{j-1,i}^{(d)}|^{2p-2} \\ &\le p^2 dn\mathbf{E}(X_{1,1}^{(d)})^2 B_{2p-2} n^{p-1}\mathbf{E}|X_{1,1}^{(d)}|^{2p-2} = p^2 n^p B_{2p-2} d^{-1}\mathbf{E}\xi^2\mathbf{E}|\xi|^{2p-2}.\end{aligned}$$

Secondly, for $p = 1$,

$$\begin{aligned}\mathbf{E}(Q_n^{(d)})^2 &= \mathbf{E}\Big(\sum_{i=1}^{d}\sum_{j=1}^{n} X_{j,i}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{-1}\Big)^2 \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n}\sum_{i'=1}^{d}\sum_{j'=1}^{n} \mathbf{E}\Big(X_{j,i}^{(d)} X_{j',i'}^{(d)} S_{j-1,i}^{(d)} |S_{j-1,i}^{(d)}|^{-1} S_{j'-1,i'}^{(d)} |S_{j'-1,i'}^{(d)}|^{-1}\Big) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n} \mathbf{E}(X_{j,i}^{(d)})^2 = nd\mathbf{E}(X_{1,1}^{(d)})^2 = nd^{-1}\mathbf{E}\xi^2.\end{aligned}$$

Then we conclude that

$$n^{-p}\varepsilon^{-2}\mathbf{E}(Q_n^{(d)})^2 \to 0 \qquad \text{as } n \to \infty,$$

and

$$n^{-p/2}\sup_{t\in[0,1]} |Q_{\lfloor nt\rfloor}^{(d)}| \xrightarrow{p} 0 \qquad \text{as } n \to \infty.$$

By Theorem 3,

$$n^{-p/2} T_{\lfloor nt\rfloor}^{(d)} = n^{-p/2}\left(\|S_{\lfloor nt\rfloor}^{(d)}\|_p^p - Q_{\lfloor nt\rfloor}^{(d)}\right) \xrightarrow{p} t^{p/2}\sigma^p M_p \qquad \text{as } n \to \infty,$$

for all $t \in [0,1]$. By monotonicity of the function $t \mapsto T_{\lfloor nt\rfloor}^{(d)}$, Dini's theorem yields that

$$\sup_{t\in[0,1]} \left|n^{-p/2} T_{\lfloor nt\rfloor}^{(d)} - t^{p/2}\sigma^p M_p\right| \xrightarrow{p} 0 \qquad \text{as } n \to \infty.$$

Therefore,

$$\sup_{t\in[0,1]}\left|n^{-p/2}\|S^{(d)}_{\lfloor nt\rfloor}\|_p^p - t^{p/2}\sigma^p M_p\right| \le \sup_{t\in[0,1]}\left|n^{-p/2}T^{(d)}_{\lfloor nt\rfloor} - t^{p/2}\sigma^p M_p\right| + n^{-p/2}\sup_{t\in[0,1]}|Q^{(d)}_{\lfloor nt\rfloor}| \xrightarrow{p} 0,$$

which completes the proof. □

Theorem 5 (Uniform convergence of the ℓ_p-metric of differences)**.** *Let $p \in [1,\infty)$. Consider a random walk with increments given by* (1)*. Then*

$$\sup_{0\le s\le t\le 1}\left|n^{-1/2}\|S^{(d)}_{\lfloor nt\rfloor} - S^{(d)}_{\lfloor ns\rfloor}\|_p - \sqrt{t-s}\sigma M_p^{1/p}\right| \xrightarrow{p} 0 \qquad \text{as } n\to\infty.$$

Proof. Take some $m \in \mathbb{N}$. By Theorem 3, for every $i = 0,\dots,m-1$,

$$n^{-1/2}\|S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p \xrightarrow{p} \sqrt{i/m}\sigma M_p^{1/p}.$$

Moreover, for every integer $0 \le i \le j \le m$, by stationarity of the increments of random walks, we obtain that

$$n^{-1/2}\|S^{(d)}_{\lfloor n(j/m)\rfloor} - S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p \xrightarrow{p} \sqrt{(j-i)/m}\sigma M_p^{1/p}.$$

By the union bound, it follows that, for every fixed $m \in \mathbb{N}$,

$$\max_{0\le i\le j\le m}\left|n^{-1/2}\|S^{(d)}_{\lfloor n(j/m)\rfloor} - S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p - \sqrt{(j-i)/m}\sigma M_p^{1/p}\right| \xrightarrow{p} 0.$$

If $0 \le s \le t \le 1$ are such that $s \in [i/m,(i+1)/m)$ and $t \in [j/m,(j+1)/m)$, then by the triangle inequality,

$$\begin{aligned}&\left|n^{-1/2}\|S^{(d)}_{\lfloor nt\rfloor} - S^{(d)}_{\lfloor ns\rfloor}\|_p - n^{-1/2}\|S^{(d)}_{\lfloor n(j/m)\rfloor} - S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p\right| \\ &\quad\le n^{-1/2}\sup_{z\in[\frac{i}{m},\frac{i+1}{m}]}\|S^{(d)}_{\lfloor nz\rfloor} - S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p + n^{-1/2}\sup_{z\in[\frac{j}{m},\frac{j+1}{m}]}\|S^{(d)}_{\lfloor nz\rfloor} - S^{(d)}_{\lfloor n(j/m)\rfloor}\|_p.\end{aligned}$$

Consider the random variable

$$\varepsilon_{m,d} = n^{-1/2}\max_{i\in\{0,\dots,m-1\}}\sup_{z\in[\frac{i}{m},\frac{i+1}{m}]}\|S^{(d)}_{\lfloor nz\rfloor} - S^{(d)}_{\lfloor n(i/m)\rfloor}\|_p.$$

To complete the proof, it suffices to show that, for every $\varepsilon > 0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbf{P}\{\varepsilon_{m,d} \ge \varepsilon\} = 0.$$

By the union bound, it follows that, for every fixed $m \in \mathbb{N}$,

$$\mathbf{P}\{\varepsilon_{m,d} \ge \varepsilon\} \le m\mathbf{P}\left\{\sup_{t\in[0,\frac{1}{m}]}\|S^{(d)}_{\lfloor nt\rfloor}\|_p^p \ge n^{p/2}\varepsilon^p\right\}.$$

Since $\|S^{(d)}_{\lfloor nt\rfloor}\|_p^p = T^{(d)}_{\lfloor nt\rfloor} + Q^{(d)}_{\lfloor nt\rfloor}$ with $T^{(d)}_{\lfloor nt\rfloor}$ and $Q^{(d)}_{\lfloor nt\rfloor}$ defined by (4), it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} m\mathbf{P}\{T^{(d)}_{\lfloor n/m\rfloor} \geq n^{p/2}\varepsilon^p/2\} = 0, \tag{6}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} m\mathbf{P}\{\sup_{t\in[0,\frac{1}{m}]} Q^{(d)}_{\lfloor nt\rfloor} \geq n^{p/2}\varepsilon^p/2\} = 0. \tag{7}$$

For fixed $m \in \mathbb{N}$,

$$n^{-p/2}T^{(d)}_{\lfloor n/m\rfloor} \xrightarrow{p} m^{-p/2}\sigma^p M_p \quad \text{as } n \to \infty.$$

Hence, (6) holds for every $m > 2^{2/p}\sigma^2(M_p)^{2/p}/\varepsilon^2$. By Doob's inequality,

$$m\mathbf{P}\{\sup_{t\in[0,\frac{1}{m}]} Q^{(d)}_{\lfloor nt\rfloor} \geq n^{p/2}\varepsilon^p/2\} \leq mn^{-p}(\varepsilon^p/2)^{-2}\mathbf{E}(Q^{(d)}_{\lfloor n/m\rfloor})^2 \to 0,$$

where the last step is implied by (5), hence (7) holds. □

Proof of Theorem 1. By Corollary 7.3.28 of [1], the Gromov-Hausdorff distance between $(n^{-1/2}\mathcal{Z}_n, \|\cdot\|_p)$ and $\left([0,1], \sqrt{|t-s|}\sigma M_p^{1/p}\right)$ is bounded by

$$2\sup_{0\leq s\leq t\leq 1}\left|n^{-1/2}\|S^{(d)}_{\lfloor nt\rfloor} - S^{(d)}_{\lfloor ns\rfloor}\|_p - \sqrt{t-s}\sigma M_p^{1/p}\right|.$$

The proof is completed by referring to Theorem 5. □

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