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Integral representation with respect to fractional Brownian motion under a log-Hölder assumption

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Abstract We show that if a random variable is the final value of an adapted log-Hölder continuous process, then it can be represented as a stochastic integral with respect to a fractional Brownian motion with adapted integrand. In order to establish this representation result, we extend the definition of the fractional integral.

Keywords Fractional Brownian motion, integral representation, fractional integral, small deviation

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1 Introduction

This paper can be considered as a continuation of the research started in [5, 4, 7–9]. Namely, we are interested in representations of a random variable in the form of a stochastic integral

$$\xi = \int_0^1 \psi(s)\mathrm{d}B^H(s) \tag{1}$$

with respect to a fractional Brownian motion with Hurst parameter $H > 1/2$; the integrand ψ is assumed to be adapted to the natural filtration generated by B^H on the

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interval [0, 1]. The motivation comes from financial mathematics, where capitals of self-financing strategies are given by stochastic integrals with respect to asset pricing processes.

The main representation results reported in [5, 4, 7–9] involve the following assumption about ξ: it is the value at 1 of an adapted Hölder-continuous process. Moreover, it was shown in [4] (see Remark 2.5) that such an assumption is unavoidable in the methods used in the cited papers.

In this paper, we generalize the existing results by showing the existence of representation (1) under a weaker assumption that ξ is the value at 1 of an adapted *log-Hölder* continuous process. In order to establish such a representation, we extend the definition of the fractional integral introduced in [10].

The paper is organized as follows. Section 2 gives all necessary prerequisites about the fractional Brownian motion and the definition of the extended fractional integral. In Section 3, we prove the main representation result. The Appendix contains auxiliary results concerning the extended fractional integral.

2 Preliminaries

2.1 General conventions

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,1]}, P)$ be a standard stochastic basis. The adaptedness of processes will be understood with respect to the filtration $\mathbb{F}$.

Throughout the paper, the symbol C will mean a generic constant, the value of which may change from line to line. When the dependence on some parameter(s) is important, this will be indicated by superscripts. A finite random variable of no importance will be denoted by $C(\omega)$.

2.2 Fractional Brownian motion

Our main object of interest in this paper is a fractional Brownian motion (fBm) with Hurst index $H \in (1/2, 1)$ on $(\Omega, \mathcal{F}, \mathsf{P})$, that is, an $\mathbb{F}$-adapted centered Gaussian process $B^H = \{B^H(t)\}_{t\geq 0}$ with the covariance function

$$R_H(t,s) = \mathsf{E}\big[B^H(t)B^H(s)\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

By the Kolmogorov–Chentsov theorem, B^H has a continuous modification, so in what follows, we assume that B^H is continuous. Moreover, we will need the following statement on the uniform modulus of continuity of B^H (see, e.g., [3]).

Theorem 1. *For an fBm B^H, we have*

$$\sup_{t,s\in[0,1]} \frac{|B^H(t) - B^H(s)|}{|t-s|^H |\log(t-s)|^{1/2}} < \infty \textit{ almost surely.}$$

2.3 Small deviations of sum of squared increments of fractional Brownian motion

First, we state a small deviation estimate for sum of squares of Gaussian random variables (see, e.g., [2]).

Lemma 1. *Let $\{\xi_i\}_{i=1,\dots,n}$ be jointly Gaussian centered random variables. Then for all x such that $0 < x < \sum_{i=1}^n \mathsf{E}[\xi_i^2]$, we have*

$$\mathsf{P}\left\{\sum_{i=1}^n \xi_i^2 \le x\right\} \le \exp\left(-\frac{(x - \sum_{i=1}^n \mathsf{E}[\xi_i^2])^2}{\sum_{i,j=1}^n (\mathsf{E}[\xi_i \xi_j])^2}\right).$$

We will also need the following asymptotics of the covariance of a fractional Brownian motion. Its proof is given in the Appendix.

Lemma 2. *Let $H \in (1/2, 1)$. Set $\Delta B_i^H = B^H(i+1) - B^H(i)$ for $i = 0, \dots, n-1$. Then the following relation holds as $n \to \infty$:*

$$\sum_{i,j=0}^{n-1} \left(\mathsf{E}\left[\Delta B_i^H \Delta B_j^H\right]\right)^2 \sim C_H \begin{cases} n, & H \in (1/2, 3/4), \\ n \log n, & H = 3/4, \\ n^{4H-2}, & H \in (3/4, 1). \end{cases}$$

Lemmas 1 and 2 imply the following small deviation estimate for the sum of squares of fBm increments.

Lemma 3. *Let $B^H = \{B_t^H\}_{t \ge 0}$ be an fBm with Hurst index $H > 1/2$, $n \ge 2$, and let $\{\Delta B_k^H\}_{k=0,\dots,n-1}$ be as before. For all $\alpha \in (0, 1)$, we have*

$$\mathsf{P}\left\{\sum_{k=0}^{n-1} \left(\Delta B_k^H\right)^2 \le \alpha n\right\} \le \exp\left\{-C_H (1-\alpha)^2 r(n)\right\},$$

where

$$r(n) = \begin{cases} n, & H \in (1/2, 3/4), \\ n/\log n, & H = 3/4, \\ n^{4-4H}, & H \in (3/4, 1). \end{cases}$$

2.4 Extended fractional integral

To integrate with respect to a fractional Brownian motion, we use the fractional integral introduced in [10], but the definition is modified according to our purposes. For functions $f, g \colon [a, b] \to \mathbb{R}$ and $\alpha \in (0, 1)$, define the fractional Riemann–Liouville derivatives (in Weyl form)

$$\left(D_{a+}^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x-u)^{\alpha+1}} \mathrm{d}u\right),$$
$$\left(D_{b-}^{1-\alpha} g\right)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(u)}{(u-x)^{2-\alpha}} \mathrm{d}u\right).$$

Then the fractional integral $\int_a^b f(x) \mathrm{d}g(x)$ can be defined as

$$\int_a^b f(x) \mathrm{d}g(x) = e^{i\pi\alpha} \int_a^b \left(D_{a+}^\alpha f\right)(x) \left(D_{b-}^{1-\alpha} g_{b-}\right)(x) \mathrm{d}x, \tag{2}$$

provided that the last integral is finite. However, in order to have good properties of this integral (independence of α, additivity), we need to assume more than just the finiteness of the integral; see the Appendix for details.

Now we turn to the integration with respect to a fractional Brownian motion. We will restrict our exposition to the interval $[0, 1]$, which suffices for our purposes. Fix a number $\alpha \in (1-H, 1/2)$. Note that from Theorem 1 it is easy to derive the following estimate:

$$\big|D_{1-}^{1-\alpha} B_{1-}^{H}(x)\big| \le C(\omega)|1-x|^{H+\alpha-1}\big|\log(1-x)\big|^{1/2}.$$

For some $\mu > 1/2$, define the weight

$$\rho(x) = (1-x)^{H+\alpha-1}\big|\log(1-x)\big|^{\mu}.$$

Let a function $f\colon [0, 1] \to \mathbb{R}$ be such that $D_{0+}^{\alpha} f \in L^1([0, 1], \rho)$; this will be our class of admissible integrands. Then the *extended fractional integral*

$$\int_0^1 f(x)\mathrm{d}B^H(x) = e^{i\pi\alpha}\int_0^1 \big(D_{0+}^{\alpha} f\big)(x)\big(D_{1-}^{1-\alpha} B_{1-}^{H}\big)(x)\mathrm{d}x \tag{3}$$

is well defined (see the Appendix). In particular, it is possible to take f with $D_{0+}^{\alpha} f \in L^1[0, 1]$, and for such integrands, the definition agrees with the definition of the fractional integral given in [10]; see Remark on p. 340.

Furthermore, it is shown in the Appendix that if f satisfies the above assumption for a different value of α, the value of the extended fractional integral will be the same. The following estimate is obvious:

$$\left|\int_0^1 f(x)\mathrm{d}B^H(x)\right| \le C(\omega)\big\|D_{0+}^{\alpha} f\big\|_{L^1([0,1],\rho)}.$$

For each $t \in (0, 1)$, we will define the integral $\int_0^t f(x)\mathrm{d}B^H(x)$ by a similar formula understanding it in the sense of [10] since $D_{0+}^{\alpha} f \in L^1[0, t]$, $D_{t-}^{1-\alpha} B_{t-}^{H} \in L^{\infty}[0, t]$. Under the additional assumption that $D_{t+}^{\alpha} f \in L^1([t, 1], \rho)$, we can define the integral $\int_t^1 f(x)\mathrm{d}B^H(x)$ similarly to (3), and the additivity holds:

$$\int_0^1 f(x)\mathrm{d}B^H(x) = \int_0^t f(x)\mathrm{d}B^H(x) + \int_t^1 f(x)\mathrm{d}B^H(x).$$

Note that the additivity

$$\int_0^t f(x)\mathrm{d}B^H(x) = \int_0^s f(x)\mathrm{d}B^H(x) + \int_s^t f(x)\mathrm{d}B^H(x)$$

for $s < t$ follows from the results of [10].

Finally, it is worth to add that for $f \in C^{\gamma}[0, 1]$ with $\gamma > 1-H$, the extended fractional integral is well defined since the derivative $D_{0+}^{\alpha} f$ is bounded for any $\alpha < \gamma$, and thus we can take $\alpha \in (1-H, \gamma)$ in the definition. The value of the integral agrees with the so-called Young integral, which is given by a limit of integral sums. An important example is $f(x) = g(B^H(x))$ where g is Lipschitz continuous. In this case, the following change-of-variable formula holds:

$$\int_a^b g\big(B^H(x)\big)\mathrm{d}B^H(x) = G(b) - G(a),$$

where $G(x) = \int_0^x g(y)\mathrm{d}y$. The formula appears to be valid (with the integral defined in the sense of [10]) even for functions h of locally bounded variation; see [1]. However, Lipschitz continuity (even continuous differentiability) will suffice for our purposes.

3 Main result

In this section, for a given $\mathcal{F}_1$-measurable random variable ξ, we construct an $\mathbb{F}$-adapted process $\psi = \{\psi(t)\}_{t\in[0,1]}$ such that (1) holds almost surely under the following "log-Hölder" assumption on ξ.

Assumption 1. There exists an $\mathbb{F}$-adapted process $\{Z(t)\}_{t\in[0,1]}$ such that $Z(1) = \xi$ and, for some $a > 1$,

$$\big|Z(1) - Z(t)\big| \le C(\omega)\big|\log(1-t)\big|^{-a} \tag{4}$$

for all $t \in [0, 1)$.

Remark 1. Obviously, the process Z satisfies

$$\big|Z(1) - Z(t)\big| \le C(\omega)(1-t)^b \tag{5}$$

for any $b > 0$. So (4) is weaker than (5), which is the assumption made in [5].

In [5], the following example of a random variable not satisfying (5) was given. Assume that $\mathbb{F} = \{\mathcal{F}_t = \sigma(B_s^H, s \in [0,t])\}_{t\in[0,1]}$, and let $\xi = \int_{1/2}^1 g(t)dW_t$, where $g(t) = (1-t)^{-1/2}|\log(1-t)|^{-1}$, and W is a Wiener process such that its natural filtration coincides with $\mathbb{F}$. Using the same argument as in [5], it is possible to show that ξ does not satisfy (4) as well. However, if we take the same construction with $g(t) = (1-t)^{-1/2}|\log(1-t)|^{-d}$, $d > 1$, then the corresponding random variable satisfies (4), but not (5).

Next, we state a helpful lemma from [5], used in our construction of ψ.

Lemma 4. *There exists an $\mathbb{F}$-adapted process ϕ on $[0,1]$ such that for any $t > 0$, $D_{0+}^{\alpha}\phi \in L^1[0,t]$, so that the integral $\int_0^t \phi(s)\mathrm{d}B^H(s)$ is defined as the fractional integral, and*

$$\lim_{t\to 1-}\int_0^t \phi(s)\mathrm{d}B^H(s) = +\infty$$

almost surely.

We can now proceed with the main result.

Theorem 2. *Let ξ satisfy Assumption 1. Then there exists an $\mathbb{F}$-adapted process $\psi = \{\psi(t)\}_{t\in[0,1]}$ such that (1) holds with the integral defined in the extended fractional sense.*

Proof. The proof is divided into three parts.

Construction of ψ. Let $\kappa \in (2, 2^a)$. Put $t_n = 1 - e^{-\kappa^{n/a}}$, $n \geq 1$, and let $\Delta_n = t_{n+1} - t_n$. It is easy to see that

$$(1 - t_n) \leq C\Delta_n. \tag{6}$$

Denote for brevity $\xi_n = Z(t_n)$. Then, by Assumption 1, $|\xi_n - \xi| \leq C(\omega)\kappa^{-n}$, so that $|\xi_n - \xi| \leq 2^{-n}$ for all n large enough, say, for $n \geq N(\omega)$. In particular, we have

$$|\xi_n - \xi_{n-1}| \leq 2^{-n+2} \tag{7}$$

for all $n \geq N(\omega) + 1$.

The integrand ψ is constructed in an inductive way between the points $\{t_n, n \geq 1\}$. Set first $\psi(t) = 0$, $t \in [0, t_1]$. Assuming that $\psi(t)$ is defined on $[0, t_n)$, let $V(t) = \int_0^t \psi(s)\mathrm{d}B^H(s)$, $t \in [0, t_n]$. The construction of the integrand $[t_n, t_{n+1})$ depends on whether $V(t_n) = \xi_{n-1}$ or not.

Case I. $V(t_n) \neq \xi_{n-1}$. In this case, thanks to Lemma 4, there exists an adapted process $\{\phi(t), t \in [t_n, t_{n+1}]\}$ such that $\int_{t_n}^t \phi(s)\mathrm{d}B^H(s) \to +\infty$ as $t \to t_{n+1}-$. Define the stopping time

$$\tau_n = \inf\left\{t \geq t_n : \int_{t_n}^t \phi(s)\mathrm{d}B^H(s) \geq |\xi_n - V_{t_n}|\right\}$$

and the process

$$\psi(t) = \phi(t)\mathsf{sign}\big(\xi_n - V(t_n)\big)\mathbb{I}_{[t_n,\tau_n]}(t), \quad t \in [t_n, t_{n+1}).$$

It is obvious that $\int_{t_n}^{t_{n+1}} \psi(s)\mathrm{d}B^H(s) = \xi_n - V(t_n)$ and $V(t_{n+1}) = \xi_n$.

Case II. $V(t_n) = \xi_{n-1}$. We consider the uniform partition $s_{n,k} = t_n + k\delta_n$, $k = 1, \ldots, n$ of $[t_n, t_{n+1}]$ with mesh $\delta_n = \Delta_n n^{-1}$ and auxiliary function

$$\overline{\phi}(t) = a_n \sum_{k=0}^{n-1} \big(B^H(t) - B^H(s_{n,k})\big)\mathbb{I}_{[s_{n,k},s_{n,k+1})}(t),$$

with $a_n = 2^{-n+3}\delta_n^{-2H} n^{-1}$. Notice that by the change-of-variable formula,

$$\int_{t_n}^{t_{n+1}} \overline{\phi}(t)\mathrm{d}B^H(t) = a_n \sum_{k=0}^{n-1} \big(B^H(s_{n,k+1}) - B^H(s_{n,k})\big)^2. \tag{8}$$

Define the stopping time

$$\sigma_n = \inf\left\{t \geq t_n : \int_{t_n}^t \overline{\phi}(s)\mathrm{d}B^H(s) \geq |\xi_n - \xi_{n-1}|\right\} \wedge t_{n+1}$$

and set

$$\psi(t) = \mathsf{sign}(\xi_n - \xi_{n-1})\overline{\phi}(t)\mathbb{I}_{[t_n,\sigma_n]}(t), \quad t \in [t_n, t_{n+1}).$$

The construction approaches ξ. Our aim now is to prove that $V(t_n) = \xi_{n-1}$ for all n large enough. By construction it suffices to show that Case II happens for all n large enough. Equivalently, we need to show that $\sigma_n < t_{n+1}$ for all n large enough. In view of (8), the latter inequality holds if

$$a_n \sum_{k=0}^{n-1} \bigl(B^H(s_{n,k+1}) - B^H(s_{n,k})\bigr)^2 > |\xi_n - \xi_{n-1}|.$$

Thus, in view of the Borell–Cantelli lemma and inequality (7), it suffices to verify the convergence of the series

$$\sum_{n\ge 1} \mathsf{P}\left\{ a_n \sum_{k=1}^{n-1} \bigl(B^H(s_{n,k+1}) - B^H(s_{n,k})\bigr)^2 < 2^{-n+2} \right\}$$

or, equivalently, that

$$\sum_{n\ge 1} \mathsf{P}\left\{ \sum_{k=1}^{n-1} \bigl(\delta_n^{-H}\bigl(B^H_{s_{n,k+1}} - B^H_{s_{n,k}}\bigr)\bigr)^2 < n/2 \right\} < \infty.$$

The latter follows from Lemma 3 through the self-similarity and stationarity of increments of an fBm. Thus, for all n large enough, say, for $n \ge N_2(\omega)$, $V(t_n) = \xi_{n-1}$.

Integrability of ψ. It is easy to see that ψ is integrable w.r.t. B^H on any interval $[0, t_N]$. It remains to verify that the integral $\int_{t_N}^1 \psi(s)\mathrm{d}B^H(s)$ is well defined and vanishes as $N \to \infty$ (note that we did not establish the continuity of the integral as a function of the upper limit). For some $\mu > 1/2$ (which will be specified later), define

$$\rho(x) = (1-x)^{H+\alpha-1}\bigl|\log(1-x)\bigr|^{\mu}.$$

Clearly, it suffices to show that $\|D^{\alpha}_{t_N+}\psi\|_{L^1([t_N,1],\rho)} \to 0$, $N \to \infty$. Let $N \ge N_2(\omega)$. Write

$$\begin{aligned}\int_{t_N}^1 \bigl|\bigl(D^{\alpha}_{t_N+}\psi\bigr)(s)\bigr|\rho(s)\mathrm{d}s &= \sum_{n=N}^{\infty} \int_{t_n}^{t_{n+1}} \bigl|\bigl(D^{\alpha}_{t_N+}\psi\bigr)(s)\bigr|\rho(s)\mathrm{d}s \\ &\le C \sum_{n=N}^{\infty} \Delta_n^{H+\alpha-1}|\log \Delta_n|^{\mu} \int_{t_n}^{t_{n+1}} \bigl|\bigl(D^{\alpha}_{t_N+}\psi\bigr)(s)\bigr|\mathrm{d}s,\end{aligned}$$

where we have used (6) and the fact that ρ is decreasing in a left neighborhood of 1.

Now we estimate

$$\begin{aligned}\int_{t_n}^{t_{n+1}} \bigl|\bigl(D^{\alpha}_{t_N+}\psi\bigr)(s)\bigr|\mathrm{d}s &\le \int_{t_n}^{t_{n+1}} \left(\frac{|\psi(s)|}{(s-t_N)^{\alpha}} + \int_{t_N}^{s} \frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u \right)\mathrm{d}s \\ &\le C(\omega) a_n \Delta_n^{1-\alpha}\delta_n^H \bigl|\log(\delta_n)\bigr|^{1/2} + \int_{t_N}^{t_{n+1}} \int_{t_n}^{s} \frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s,\end{aligned}$$

where we have used Theorem 1 to estimate ψ.

Consider the second term. It equals

$$\sum_{k=1}^{n}\int_{s_{n,k-1}}^{s_{n,k}}\left(\int_{t_N}^{t_n}+\int_{t_n}^{s_{n,k-1}}+\int_{s_{n,k-1}}^{s}\right)\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s=:I_1+I_2+I_3.$$

Start with I_1, observing that ψ vanishes on $(\sigma_n, t_{n+1}]$:

$$\begin{aligned}
I_1&\le\int_{t_n}^{t_{n+1}}\sum_{j=N}^{n}\int_{t_{j-1}}^{t_j}\frac{|\psi(s)|+|\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s\\
&\le C(\omega)a_n\delta_n^H|\log\delta_n|^{1/2}\int_{t_n}^{t_{n+1}}(s-t_n)^{-\alpha}\mathrm{d}s\\
&\quad+\sum_{j=N}^{n-1}a_j\delta_j^H|\log\delta_j|^{1/2}\int_{t_n}^{t_{n+1}}(s-t_{j+1})^{-\alpha}\mathrm{d}s\\
&\le C(\omega)\left(a_n\Delta_n^{1-\alpha}\delta_n^H|\log\delta_n|^{1/2}+\sum_{j=N}^{n-1}a_j\delta_j^H|\log\delta_j|^{1/2}\Delta_n^{1-\alpha}\right).
\end{aligned}$$

Proceed with the second term:

$$\begin{aligned}
I_2&\le C(\omega)a_n\delta_n^H|\log\delta_n|^{1/2}\sum_{k=1}^{n}\int_{s_{n,k-1}}^{s_{n,k}}\int_{t_n}^{s_{n,k-1}}|s-u|^{-1-\alpha}\mathrm{d}u\,\mathrm{d}s\\
&\le C(\omega)a_n\delta_n^H|\log\delta_n|^{1/2}\sum_{k=1}^{n}\int_{s_{n,k-1}}^{s_{n,k}}(s-s_{n,k-1})^{-\alpha}\mathrm{d}s\\
&\le C(\omega)a_n n\delta_n^{H+1-\alpha}|\log\delta_n|^{1/2}=C(\omega)a_n\Delta_n\delta_n^{H-\alpha}|\log\delta_n|^{1/2}.
\end{aligned}$$

Finally, assuming that $\sigma_n\in[s_{n,l-1},s_{n,l})$, we have

$$\begin{aligned}
I_3&\le C(\omega)\sum_{k=1}^{l-1}\int_{s_{n,k-1}}^{s_{n,k}}\int_{s_{n,k-1}}^{s}a_n\frac{(s-u)^H|\log(s-u)|^{1/2}}{(s-u)^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s\\
&\quad+\int_{s_{n,l-1}}^{\sigma_n}\int_{s_{n,l-1}}^{s}\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s+\int_{\sigma_n}^{s_{n,l}}\int_{s_{n,l-1}}^{\sigma_n}\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s\\
&\le C(\omega)a_n\sum_{k=1}^{n}\int_{s_{n,k-1}}^{s_{n,k}}(s-s_{n,k-1})^{H-\alpha}\big|\log(s-s_{n,k-1})\big|^{1/2}\mathrm{d}s\\
&\quad+C(\omega)a_n\delta_n^H|\log\delta_n|^{1/2}\int_{\sigma_n}^{s_{n,l}}\int_{s_{n,l-1}}^{\sigma_n}\frac{1}{|s-u|^{1+\alpha}}\mathrm{d}u\,\mathrm{d}s\\
&\le C(\omega)a_n n\delta_n^{H+1-\alpha}|\log\delta_n|^{1/2}n\delta_n^{1-\alpha}=C(\omega)a_n\Delta_n\delta_n^{H-\alpha}|\log\delta_n|^{1/2}.
\end{aligned}$$

Gathering all estimates, we get

$$\int_{t_N}^{1}\big|D_{t_N+}^{\alpha}(\psi)(s)\big|\rho(s)\mathrm{d}s\le C(\omega)\sum_{n=N}^{\infty}\Big(a_n\Delta_n^H\delta_n^H|\log\delta_n|^{1/2}|\log\Delta_n|^{\mu}$$

$$+ a_n \Delta_n^{H+\alpha} \delta_n^{H-\alpha} |\log \delta_n|^{1/2} |\log \Delta_n|^{\mu} + \Delta_n^H |\log \Delta_n|^{\mu} \sum_{j=N}^{n-1} a_j \delta_j^H |\log \delta_j|^{1/2} \Bigg).$$

Consider the second sum. After changing the order of summation, we get

$$\sum_{n=N}^{\infty} \sum_{j=N}^{n-1} a_j \delta_j^H |\log \delta_j|^{1/2} \Delta_n^H |\log \Delta_n|^{\mu} = \sum_{j=N}^{\infty} a_j \delta_j^H \sum_{n=j+1}^{\infty} \Delta_n^H |\log \Delta_n|^{\mu}$$
$$\le C \sum_{j=N}^{\infty} a_j \delta_j^H \Delta_j^H |\log \Delta_j|^{\mu} \sum_{n=j+1}^{\infty} e^{\kappa^{j/a} - \kappa^{n/a}} \kappa^{\mu(n-j)/a}$$
$$\le C \sum_{j=N}^{\infty} a_j \delta_j^H \Delta_j^H |\log \Delta_j|^{\mu}$$
$$\le C \sum_{j=N}^{\infty} a_j \delta_j^H \Delta_j^H |\log \Delta_j|^{\mu} \sum_{n=j+1}^{\infty} e^{-\kappa^{(n-j)n/a}} \kappa^{\mu(n-j)/a}$$
$$\le C \sum_{j=N}^{\infty} a_j \delta_j^H \Delta_j^H |\log \Delta_j|^{\mu}.$$

Consequently, noting that $\Delta_n^H \delta_n^H \le \Delta_n^{H+\alpha} \delta_n^{H-\alpha}$, we have

$$\int_{t_N}^{1} \big|D_{t_N+}^{\alpha}(\psi)(s)\big| \rho(s) \mathrm{d}s \le C(\omega) \sum_{n=N}^{\infty} a_n \Delta_n^{H+\alpha} \delta_n^{H-\alpha} |\log \delta_n|^{1/2} |\log \Delta_n|^{\mu}$$
$$\le \sum_{n=N}^{\infty} 2^{-n} n^{-1} \Delta_n^{H+\alpha} \delta_n^{-H-\alpha} |\log \delta_n|^{\mu+1/2}$$
$$\le \sum_{n=N}^{\infty} 2^{-n} n^{H+\alpha-1} \big(|\log \Delta_n|^{\mu+1/2} + n^{\mu+1/2}\big)$$
$$\le \sum_{n=N}^{\infty} 2^{-n} n^{H+\alpha-1} \big(C + \kappa^{(\mu+1/2)n/a} + n^{\mu+1/2}\big).$$

Now, in order for the last series to converge, it is sufficient that $(\mu + 1/2)/a < \log 2/\log \kappa$ or, equivalently, $\mu < a \log 2/\log \kappa - 1/2$. Since $a \log 2/\log \kappa > 1$ by our choice, it is possible to take some $\mu > 1/2$ satisfying this requirement, thus finishing the proof. □

A Appendix. Properties of the extended fractional integral

In this section, we establish important properties of the extended fractional integral defined by (2). We will restrict ourselves to the case $[a, b] = [0, 1]$. Assume that the function $g: [0, 1] \to \mathbb{R}$ satisfies the following regularity requirement: there exist $\lambda \in (0, 1)$ and $\nu > 0$ such that

$$\big|g(x) - g(y)\big| \le C|x - y|^{\lambda} \big|\log |x - y|\big|^{\nu} \tag{9}$$

for all $x, y \in [0, 1]$. We first state some properties of g that follow from this regularity.

Lemma 5. *Let* $g: [0,1] \to \mathbb{R}$ *satisfy* (9). *Then for any* $\beta \in (0, \lambda)$,

$$\left|\left(D_{1-}^{\beta} g_{1-}\right)(x) - \left(D_{1-}^{\beta} g_{1-}\right)(y)\right| \le C_\beta |x-y|^{\lambda-\beta} \left|\log|x-y|\right|^{\nu}$$

for all $x, y \in [0,1]$.

The proof is similar to that of Lemma 13.1 in [6] and thus omitted.

Now, for some $\mu > \nu$, define

$$\rho(x) = |1-x|^{\lambda-\beta} \left|\log(1-x)\right|^{\mu}, \quad x \in [0,1]. \tag{10}$$

Lemma 6. *Let* $g: [0,1] \to \mathbb{R}$ *satisfy* (9), *and* $\psi \in C^\infty[-1,0] \to \mathbb{R}$ *be a positive function with* $\int_{-1}^{0} \psi(x)\mathrm{d}x = 1$. *Define* $\psi_N(x) = N\psi(Nx)$,

$$g_N(x) = (g_{1-} * \psi_N)(x) = \int_x^{(x+1/N)\wedge 1} g_{1-}(y)\psi_N(x-y)\mathrm{d}y, \quad N \ge 1.$$

Then for any $\mu > \nu$, $\beta \in (0, \lambda)$, $x \in [0,1]$, *and all* $N \ge 1$ *large enough, we have*

$$\left|\left(D_{1-}^{\beta} g_N\right)(x) - \left(D_{1-}^{\beta} g_{1-}\right)(x)\right| \le C_\beta (\log N)^{\nu-\mu} \rho(x). \tag{11}$$

Proof. Note that $D_{1-}^{\beta} g_N = \mathbf{1}_{[0,1]}(D_{1-}^{\beta} g_{1-}) * \psi_N$. For brevity, set $h(x) = (D_{1-}^{\beta} g_{1-})(x)$, $x \in [0,1]$. First, suppose that $x \in [0, 1-1/N]$. Thanks to Lemma 5,

$$\begin{aligned}
\left|\left(D_{1-}^{\beta} g_N\right)(x) - \left(D_{1-}^{\beta} g_{1-}\right)(x)\right| &\le C \int_x^{x+1/N} \left|h(x) - h(y)\right| N\psi\left(N(x-y)\right)\mathrm{d}y \\
&\le C \int_{-1}^{0} \left|h(x-y/N) - h(y)\right| \psi(y)\mathrm{d}y \\
&\le C_\beta N^{\beta-\lambda} \int_{-1}^{0} |y|^{\lambda-\beta}\left(N - \log|y|\right)^{\nu} \psi(y)\mathrm{d}y \\
&\le C N^{\beta-\lambda}(\log N)^{\nu} \le C(\log N)^{\nu-\mu}\rho(x),
\end{aligned}$$

where the last inequality holds for all N large enough since $\rho(x)$ is decreasing for x close to 1.

Now consider the case $x \in [1-1/N, 1]$. Using Lemma (5), we have

$$\begin{aligned}
&\left|\left(D_{1-}^{\beta} g_N\right)(x) - \left(D_{1-}^{\beta} g_{1-}\right)(x)\right| \le \left|\left(D_{1-}^{\beta} g_N\right)(x)\right| + \left|\left(D_{1-}^{\beta} g_{1-}\right)(x)\right| \\
&\le \int_x^1 N\psi\left(N(x-y)\right)\left|h(1) - h(y)\right|\mathrm{d}y + \left|h(1) - h(x)\right| \\
&\le C|1-x|^{\lambda-\beta}\left|\log(1-x)\right|^{\nu} \le C\rho(x)\left|\log(1-x)\right|^{\nu-\mu} \\
&\le C\rho(x)(\log N)^{\nu-\mu}
\end{aligned}$$

since $x > 1 - 1/N$ and $\nu < \mu$. This finishes the proof. □

Theorem 3. *Let $g\colon [0,1] \to \mathbb{R}$ satisfy* (9)*, $\alpha \in (1-\lambda, 1)$, and let ρ be given by* (10) *with $\beta = 1-\alpha$. Assume further that $f\colon [0,1] \to \mathbb{R}$ is such that $D^{\alpha}_{0+} f \in L^1([0,1], \rho)$. Then the extended fractional integral*

$$\int_0^1 f(x)\mathrm{d}g(x) = e^{i\pi\alpha} \int_0^1 \bigl(D^{\alpha}_{1-} f\bigr)(x)\bigl(D^{1-\alpha}_{1-} g_{1-}\bigr)(x)\mathrm{d}x$$

is well defined. Moreover, if f and g satisfy the above assumptions for different α, the value of the integral is preserved. Additionally, if $D^{\alpha}_{t+} f \in L^1([t,1], \rho)$ for some $t \in (0,1)$, then

$$\int_0^1 f(x)\mathrm{d}g(x) = \int_0^t f(x)\mathrm{d}g(x) + \int_t^1 f(x)\mathrm{d}g(x),$$

where the first integral is understood in the sense of [10] *($D^{\alpha}_{0+} f \in L^1[0,t]$, $D^{1-\alpha}_{t-} g_{t-} \in L^{\infty}[0,t]$), and the second in the above sense.*

Proof. From Lemma 5 we have

$$\bigl|\bigl(D^{1-\alpha}_{1-} g\bigr)(x)\bigr| \le C(1-x)^{\lambda+\alpha-1}\bigl|\log(1-x)\bigr|^{\nu} \le C\rho(x),$$

whence the finiteness of the integral follows. Defining g_N, $N \ge 1$, as in Lemma 6, we have

$$\begin{aligned} &e^{i\pi\alpha} \int_0^1 \bigl(D^{\alpha}_{1-} f\bigr)(x)\bigl(D^{1-\alpha}_{1-} g_{1-}\bigr)(x)\mathrm{d}x \\ &\quad = e^{i\pi\alpha} \lim_{N\to\infty} \int_0^1 \bigl(D^{\alpha}_{1-} f\bigr)(x)\bigl(D^{1-\alpha}_{1-} g_N\bigr)(x)\mathrm{d}x \end{aligned} \tag{12}$$

in view of the dominated convergence theorem.

Since $g_N \in C^{\infty}[0,1] \to \mathbb{R}$, it follows from the properties of fractional integrals and derivatives (see [6, 10]) that $(D^{1-\alpha}_{1-} g_N)(x) = (I^{\alpha}_{1-} g'_N)(x)$, where

$$\bigl(I^{\alpha}_{1-} h\bigr)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \int_x^1 (y-x)^{\alpha-1} h(y)\mathrm{d}y$$

is the right-sided fractional Riemann–Liouville integral of order α. Note that

$$\begin{aligned} &\int_0^1 \int_x^1 \bigl|\bigl(D^{\alpha}_{1-} f\bigr)(x)\bigr|(y-x)^{\alpha-1}\bigl|g'_N(y)\bigr|\mathrm{d}y\,\mathrm{d}x \\ &\quad \le C_N \int_0^1 \bigl|\bigl(D^{\alpha}_{1-} f\bigr)(x)\bigr|(1-x)^{\alpha}\mathrm{d}x \le C_N \int_0^1 \bigl|\bigl(D^{\alpha}_{1-} f\bigr)(x)\bigr|\rho(x)\mathrm{d}x < \infty \end{aligned}$$

thanks to our assumptions.

Hence, by the Fubini theorem,

$$e^{i\pi\alpha} \int_0^1 \bigl(D^{\alpha}_{1-} f\bigr)(x)\bigl(D^{1-\alpha}_{1-} g_N\bigr)(x)\mathrm{d}x$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^y (y-x)^{1-\alpha} \big(D_{1-}^{\alpha} f\big)(x) \mathrm{d}x \, g_N'(y) \mathrm{d}y$$
$$= \int_0^1 \big(I_{0+}^{\alpha} D_{0+}^{\alpha} f\big)(y) g_N'(y) \mathrm{d}y,$$

where

$$\big(I_{0+}^{\alpha} h\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} h(y) \mathrm{d}y$$

is the left-sided fractional Riemann–Liouville integral of order α. Note that $D_{0+}^{\alpha} f \in L^1[0,t]$ for any $t \in (0,1)$, so by [6, Theorem 2.4], for almost all $y \in [0,t]$, we have $(I_{0+}^{\alpha} D_{0+}^{\alpha} f)(y) = f(y)$. Therefore, this equality holds for almost all $y \in [0,1]$, and

$$e^{i\pi\alpha} \int_0^1 \big(D_{1-}^{\alpha} f\big)(x) \big(D_{1-}^{1-\alpha} g_N\big)(x) \mathrm{d}x = \int_0^1 f(y) g_N'(y) \mathrm{d}y.$$

Taking into account (12), we get that, indeed, the extended fractional integral $\int_0^1 f(s)$ does not depend on α.

Concerning additivity, under the assumption $D_{t+}^{\alpha} f \in L^1([t,1],\rho)$, repeating the previous argument, we get

$$\int_t^1 f(x) \mathrm{d}g(x) = \lim_{N\to\infty} \int_t^1 f(y) g_N'(y) \mathrm{d}y.$$

On the other hand, from the proof of [10, Theorem 2.5] we have

$$\int_0^t f(x) \mathrm{d}g(x) = \lim_{N\to\infty} \int_0^t f(y) g_N'(y) \mathrm{d}y.$$

Therefore,

$$\int_0^t f(x) \mathrm{d}g(x) + \int_t^1 f(x) \mathrm{d}g(x) = \lim_{N\to\infty} \int_0^1 f(y) g_N'(y) \mathrm{d}y = \int_0^1 f(x) \mathrm{d}g(x),$$

as required. □

B Appendix. Proof of Lemma 2

Proof. We have $\mathsf{E}[\Delta B_i^H \Delta B_j^H] = \rho_H(|i-j|)$ with

$$\rho_H(m) = \frac{1}{2}\big((m+1)^{2H} + |m-1|^{2H} - 2m^{2H}\big), \quad m \ge 0.$$

It is easy to see that

$$\rho_H(m) \sim H(2H-1) m^{2H-2}, \quad m \to \infty. \tag{13}$$

The double sum in the question can be transformed as follows:

$$S_n^H := \sum_{i,j=0}^{n-1} \big(\mathsf{E}\big[\Delta B_i^H \Delta B_j^H\big]\big)^2 = \sum_{i,j=0}^{n-1} \rho_H\big(|i-j|\big)^2 = |m = i - j|$$

$$= \sum_{m=1-n}^{n-1} (n - |m|)\rho_H(|m|)^2.$$

Let $H \in (1/2, 3/4)$. Then

$$\frac{S_n^H}{n} = \sum_{m=1-n}^{n-1} \left(1 - \frac{|m|}{n}\right)\rho_H(|m|)^2.$$

In view of (13), the series $\sum_{m=-\infty}^{+\infty} \rho_H(|m|)^2$ converges, so by the dominated convergence theorem,

$$\frac{S_n^H}{n} \to \sum_{m=-\infty}^{+\infty} \rho_H(|m|)^2, \quad n \to \infty.$$

Now let $H \in [3/4, 1)$. Define $a_n = n \log n$ if $H = 3/4$, $a_n = n^{4H-2}$ if $H \in (3/4, 1)$. Applying the Stolz–Cèsaro theorem, we have

$$\lim_{n\to\infty} \frac{S_n^H}{a_n} = \lim_{n\to\infty} \frac{S_{n+1}^H - S_n^H}{a_{n+1} - a_n} = \lim_{n\to\infty} \frac{1}{a_{n+1} - a_n} \sum_{m=-n}^{n} \rho_H(|m|)^2,$$

provided that the latter limit exists. Using the Stolz–Cèsaro theorem once more, we have

$$\lim_{n\to\infty} \frac{1}{a_{n+1} - a_n} \sum_{m=-n}^{n} \rho_H(|m|)^2 = 2 \lim_{n\to\infty} \frac{\rho_H(n)^2}{a_{n+1} + a_{n-1} - 2a_n},$$

provided that the latter limit exists. It is easy to see that

$$a_{n+1} + a_{n-1} - 2a_n \sim \begin{cases} n^{-1}, & H = 3/4, \\ (4H-2)(4H-3)n^{4H-4}, & H \in (3/4, 1), \end{cases}$$

as $n \to \infty$. Thus, we get the required statement thanks to (13). □

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