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Some continuity estimates for ruin probability and other ruin-related quantities

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Abstract In this paper we investigate continuity properties for ruin probability in the classical risk model. Properties of contractive integral operators are used to derive continuity estimates for the deficit at ruin. These results are also applied to obtain desired continuity inequalities in the setting of continuous time surplus process perturbed by diffusion. In this framework, the ruin probability can be expressed as the convolution of a compound geometric distribution K with a diffusion term. A continuity inequality for K is derived and an iterative approximation for this ruin-related quantity is proposed. The results are illustrated by numerical examples.

Keywords Classical risk model, ruin probability, deficit at ruin, contractive operators, diffusion
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1 Introduction

The concept of measuring distances between probability measures is a fundamental one with applications across several areas of mathematics. In ruin theory, the main objective is to model the surplus of an insurance business using a stochastic process and evaluate its ruin probability. This characteristic is a typical measure for the solvency of a portfolio. An extension of this, which accounts for the severity of ruin, is the distribution of the deficit at ruin (given that ruin occurs). In general, explicit expressions for the ruin probability and/or deficit at ruin are known only in some cases, for instance, when the claim sizes have exponential or phase-type distribution (see [1], pg 14-15). Therefore, several theoretical approaches have been proposed to approximate, bound, estimate and numerically compute the ruin probability.

In insurance mathematics, stochastic models are used to idealize input and output elements to approximate real insurance activities. The problem of stability stated as seeking an appropriate measure of closeness between the ideal and real input element in order to estimate the corresponding deviation in the output. Several theoretical approaches have been developed to analyze optimal choices of metrics between input elements. The problem of stability of the aggregate claim amount over a finite time horizon is formally analyzed in [3]. The estimation of the ruin probability in univariate risk models, using the strong stability method has been investigated in [18] and [6]. Then, the application of this approach has been extended in various directions. For instance, the stability of the ruin probability in a Markov modulated risk model with Lévy process with investments was studied in [24]. A two-dimensional classical risk model was considered in [4] using the strong stability method (see also, [27, 2] and [15]). Continuity properties of the surplus process in multidimensional renewal risk models were studied in [12]. Furthermore, a functional approach that was used to obtain approximations and bounds for some ruin-related quantities was developed in [20] and [19]. In the discrete-time setup, approximation techniques involving some integral operator were proposed in [7] and a general formula yielding approximations based on negative binomial mixtures were presented in [26].

Recently, the Banach contraction principle and fixed point results for contractive operators have been applied in the theory of risk models. An approximation of the ruin probability under a Markov modulated classical risk model based on the Banach contraction principle was presented in [8]. It was shown in [16] that the ultimate ruin probability can be expressed as the fixed point of a contraction mapping in terms of q-scale functions. Similar contractive approaches have been examined in several recent works (see for instance [9] and [25]). Gordienko and Vázquez-Ortega in [11] proposed continuity inequalities for ruin probability using properties of contractive mappings. Extending this approach, we derive new continuity bounds for the ruin probability and the deficit at ruin in terms of various choices of probability metrics and we also investigate properties of contractive operators and fixed point results in the context of a continuous time surplus process perturbed by diffusion. To the best of our knowledge, no existing work has employed probability metrics in the setting of a surplus process perturbed by diffusion.

The goal in this paper is to propose appropriate probability metrics that allow us to obtain suitable continuity estimates for ruin-related quantities, such as probability of ruin with and without diffusion and the deficit at ruin, in the classical of risk theory. The paper is organized as follows: in the next section, we introduce the main quantities of interest in the classical risk model of risk theory, and we present the concept of the continuity problem for the ruin probability. In Section 3, we derive a continuity inequality for ruin probabilities and provide an upper bound on the supremum distance between two deficits at ruin. In Section 4, we derive upper bounds for a ruin-related quantity in the classical risk process perturbed by diffusion and obtain an approximation-using the Banach fixed point theorem (BFPT) to this quantity. Proofs are given in Section 5, while Section 6 contains numerical examples that illustrate the validity of the results. The final section summarizes the paper.

2 Definitions and preliminaries

The paper concerns the compound Poisson model with risk process

$$U(t) = u + ct - \sum_{n=1}^{N(t)} X_n, \quad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial surplus, c is the premium rate and $N(t)$ denotes the number of claims up to time t. The X_n's represent the claim sizes and they are assumed to be independent identically distributed positive random variables (r.v.'s) with a common distribution function $F(x) = Pr(X \leq x)$, tail $\overline{F}(x) = 1 - F(x) = Pr(X > x)$ and mean $EX < +\infty$. Also, $N(t)$ is a Poisson process with rate λ, independent of X_n. We further assume $c = \lambda(1+\theta)EX$, where $\theta > 0$ is the relative security loading. Let

$$\psi(u) := Pr(\inf_{t\geq 0} U(t) < 0 | U(0) = u), \quad u \geq 0, \tag{2}$$

be the ruin probability in infinite time. In the classical risk model (1), it is well-known that $\psi(u)$ satisfies

$$\psi(u) = Pr(L > u) = \sum_{n=1}^{\infty} \frac{\theta}{1+\theta} \left(\frac{1}{1+\theta}\right)^n \overline{F_e^{*n}}(u), \quad u \geq 0, \tag{3}$$

where $\overline{F_e^{*n}}(u) = Pr(X_e^{(1)} + X_e^{(2)} + \cdots X_e^{(n)} > u)$ is the tail of the nth-fold convolution of $F_e(u) = \int_0^u \overline{F}(z)dz / \int_0^\infty \overline{F}(z)dz$ with itself. The variable L here is the maximal aggregate loss in the surplus process. The distribution F_e is known as the equilibrium distribution associated with F.

In general, the continuity problem is based on the following implication. Let $U(t, \alpha)$ be a risk process governed by a parameter $\alpha = (\lambda, c, F)$ with the ruin probability ψ_α. If $\mathcal{A}$ denotes the space of possible values of the parameter α then one can view the ruin probability as a mapping $\psi : \mathcal{A} \to \Psi$, where Ψ is the functional space of all possible functions ψ_α. Assume that $\mathcal{A}$ and Ψ are metric spaces with metrics δ and ν, respectively. In such terms, the problem of interest is reduced to investigation of (δ, ν)-continuity of this mapping: that is if the implication

$$\delta(\alpha, \widetilde{\alpha}) \to 0, \quad \text{then} \quad \nu(\psi_\alpha, \psi_{\widetilde{\alpha}}) \to 0, \tag{4}$$

for $\alpha, \widetilde{\alpha} \in \mathcal{A}$, holds at a fixed point α.

The metrics δ and ν should be computationally convenient and reflect the core of the problem. In the following, we consider various choices for these metrics. Other such metrics that may be suitable to obtain bounds of this form for ruin probabilities and ruin-related quantities can be found in [17]. If we can find an inequality

$$\nu(\psi_\alpha, \psi_{\widetilde{\alpha}}) \leq w(\delta(\alpha, \widetilde{\alpha})), \tag{5}$$

where $w(s) \geq 0, w(s) \overset{s\to 0}{\to} 0$ and $w(0) = 0$, then the inequality (5) is called a continuity inequality (estimate) and provides the possibility of bounding $\nu(\psi_\alpha, \psi_{\widetilde{\alpha}})$ in terms of the distance $\delta(\alpha, \widetilde{\alpha})$ (see e.g. [18]).

In applications, the vector parameter $\alpha = (\lambda, c, F)$ that governs the risk model is usually unknown. Therefore, the intensity of claim arrivals λ and the distribution of claim sizes F are approximated by some parameter $\widetilde{\lambda}$ and distribution $\widetilde{F}$, respectively, for which the ruin probability $\widetilde{\psi}$ can be found. This allows local estimations of the ruin probability alterations to be obtained with respect to disturbances of the parameter λ and distribution F. Throughout the paper, we make the assumption that $\mu := \int_0^\infty x dF(x) < \infty$ and $\widetilde{\mu} := \int_0^\infty x d\widetilde{F}(x) < \infty$ hold, as well as the net profit conditions:

$$\phi = \frac{\lambda\mu}{c} < 1, \quad \text{and} \quad \widetilde{\phi} = \frac{\widetilde{\lambda}\widetilde{\mu}}{c} < 1. \tag{6}$$

3 Continuity estimate for ruin probability and for deficit at ruin in the classical risk model

3.1 Ruin probability $\psi(u)$

In this section, we will study continuity conditions of the ruin probability between surplus processes as presented in (1). Firstly, we introduce the background needed to use properties of contractive operators in order to bound the ruin probability in terms of an appropriate distance.

We recall two definitions from fixed point theory that will be used in the sequel.

Definition 3.1. A fixed point of a mapping $T : X \longrightarrow X$ of a set X is an $x^\star \in X$ that is mapped into itself, that is $T(x^\star) = x^\star$.

Definition 3.2. Let (X, ν) be a metric space. A mapping $T : X \longrightarrow X$ is a contractive operator on X if there exists a constant $\rho \in (0, 1)$ such that $\nu(Tx, Ty) \leq \rho\nu(x, y)$ for all $x, y \in X$.

We now state the Banach fixed point theorem (BFTP), which gives a unique fixed point of the mapping and provides a constructive method to find those fixed points.

Theorem 3.3 (Banach's fixed-point theorem)**.** *Let (X, ν) be a complete metric space and $T : X \longrightarrow X$ a contractive operator with $\rho \in (0, 1)$. Then T has a unique fixed point $x^\star \in X$ and $\lim_{n\to\infty} T^n x = x^\star$ for an arbitrary $x^\star \in X$, where $T^n x = T\left(T^{n-1}x\right) = T\left(T\left(T^{n-2}x\right)\right) = \cdots = T(T\cdots(Tx))$ for $n = 1, 2, \ldots$.*

A detailed proof can be found in [13].

Let $\mathcal{D}_\infty$ be the set of functions $h : [0, \infty] \longrightarrow \mathbb{R}$ which are right-continuous on $[0, \infty)$ and have left-hand limits on $[0, \infty]$ (cádlág functions). This space endowed with the supremum norm, $||h||_\infty = \sup_{t\in\mathbb{R}} |h(t)|$, is a nonseparable Banach space (see e.g. [21]). For $f \in \mathcal{D}_\infty$, we define a new function $h_\gamma : [0, \infty] \longrightarrow [0, \infty]$ such that

$$h_\gamma(x) = (1 + x)^\gamma |h(x)|, \quad x \geq 0.$$

Then, for $\gamma \geq 0$, the space of functions $\mathcal{D}_\gamma = \{h : [0, \infty) \longrightarrow \mathbb{R} : h_\gamma \in \mathcal{D}_\infty\}$ with the norm on $\mathcal{D}_\gamma$ defined by $||h||_\gamma = ||h_\gamma||_\infty$ is again a nonseparable Banach space (see e.g. [14]). For any $\gamma \geq 0$, let $\mathcal{D}_\gamma$ endowed with the metric ν_γ where, for $x, y \in \mathcal{D}_\gamma$,

$$\nu_\gamma(x, y) := \int_0^\infty (1 + t)^\gamma |x(t) - y(t)| dt, \quad \gamma \geq 0. \tag{7}$$

Then $(\mathcal{D}_\gamma, \nu_\gamma)$ is a complete metric space.

Suppose now that the claim size distribution has finite moments of order $\gamma + 1$ for $\gamma \geq 0$, so that $F \in \mathcal{D}_\gamma$. Then the following theorem holds.

Theorem 3.4. *Let X and $\widetilde{X}$ be two r.v.'s representing the individual claim amounts in two surplus processes as in (1) with distribution functions F and $\widetilde{F}$, respectively. We assume that the intensities of claim arrivals are λ and λ_X and that $EX^{(\gamma+1)} < +\infty$ and $E\widetilde{X}^{(\gamma+1)} < +\infty$ for some $\gamma \geq 0$. We denote by $\psi(u)$ and $\widetilde{\psi}(u)$ the ruin probabilities associated with the surplus processes $\{U(t) : t \geq 0\}$ and $\{\widetilde{U}(t) : t \geq 0\}$, respectively. Then,*

$$\int_0^\infty (1+t)^\gamma \left|\psi(t) - \widetilde{\psi}(t)\right| dt \leq \frac{c}{c - \lambda M_\gamma^X} \left(\frac{\nu_{\gamma+1}(F, \widetilde{F})}{\gamma + 1} + \nu_\gamma(F, \widetilde{F}) M_\gamma^L + \frac{|\lambda - \widetilde{\lambda}|}{c} M_{\gamma+1}^{\widetilde{X}} (1 + M_\gamma^L) \right), \tag{8}$$

where,

$$M_\gamma^X = \int_0^\infty (1+t)^\gamma \overline{F}_X(t) dt \quad \text{and provided that} \quad \lambda M_\gamma^X / c < 1.$$

It is noteworthy that the choice of the metric ν (see (4)) used to measure deviations between ruin probabilities is crucial for a successful stability analysis. Since the asymptotic behavior of the probability of ruin is of central interest in risk theory (see [1, 17] and [22]), we seek stability bounds in risk models that allow for a tail comparison of the ruin probabilities. Hence, weighted functions play an important role in the choice of an appropriate metric and may, under suitable conditions, induce a Banach space structure in which contractive operator properties can be established.

A function $\zeta(x)$ is said to be a weight function if it satisfies the following conditions, for $x \geq 0$ (see [18]):

1. $\zeta(x) \geq 1$,
2. $\zeta(x)$ is increasing (typically, to $+\infty$).

An important subclass of weight functions is the class $\mathcal{S}$ of submultiplicative functions on $\mathbb{R}$, that is, the class of positive, Borel measurable functions ζ_γ, satisfying

$$\zeta_\gamma(0) = 1, \quad \zeta_\gamma(x+y) \leq \zeta_\gamma(x)\zeta_\gamma(y) \quad \text{for all} \quad x, y \in \mathbb{R}.$$

It is well-known (see e.g. [23]) that, given any $\zeta_\gamma \in \mathcal{S}$, the collection $\mathcal{S}(\zeta_\gamma)$ of all complex measures ν defined on the σ-algebra of Borel sets on $\mathbb{R}$ endowed with the norm

$$||\nu||_{\zeta_\gamma} = \int_{\mathbb{R}} \zeta_\gamma(x)|\nu|(dx) < +\infty$$

(here $|\nu|$ denotes the weighted total variation measure associated with ν), is a Banach space.

The metric ν should be chosen in such a way as to reveal important features of the setup. For example, if the probability of ruin decays exponentially as in the Cramér case (light-tailed claim-size distributions, see [1]), then it is natural to consider a weight function of the form $\zeta_\gamma(x) = e^{\gamma x}$ for an appropriate constant $\gamma > 0$. If the probability of ruin has a power law decay (e.g. large claims with Pareto tails) then it is reasonable to consider the polynomial weight function $\zeta_\gamma(x) = (1+x)^\gamma$ for $\gamma \geq 0$ (see equation (7)). Another admissible choice of submultiplicative functions is $\zeta_\gamma(x) = e^{\tau x^\gamma}$ for $\tau > 0$ and $\gamma \in (0, 1)$.

Remark 1. For $\gamma = 0$, the inequality (8) reduces to the following inequality

$$\mathbb{K}(\psi, \widetilde{\psi}) \leq \frac{c}{c - \lambda\mu}\left(\nu_1(F, \widetilde{F}) + \frac{\mathbb{K}(F, \widetilde{F})EX^2}{2\theta\mu} + \frac{|\lambda - \widetilde{\lambda}|\widetilde{\mu}}{c}\left(1 + \frac{EX^2}{2\theta\mu}\right)\right),$$

where $M_\gamma^X = EX = \mu$, $M_\gamma^L = EL = EX^2/2\theta EX$ and the function $\mathbb{K} : \mathcal{F}\times\mathcal{F} \to [0, \infty]$ with $\mathbb{K}(F, \widetilde{F}) = \int_0^\infty \left|F(t) - \widetilde{F}(t)\right| dt$ is called the *Kantorovich metric* (see e.g. [17]) in $\mathcal{F}$ the set of distribution functions F of all positive r.v.'s.

3.2 The deficit at ruin

In this section, we apply the contraction mapping method to obtain a bound for the deficit at ruin in classical risk model. Contractive operator techniques to derive bounds for the deficit at ruin, though this approach requires the existence of the adjustment coefficient (light-tailed claims) were used in [9]. Gordienko and Vázquez-Ortega in [11] proposed a simple technique for continuity estimation for ruin probability in the compound Poisson risk model. In a similar manner, for $y > 0$ fixed, we study the comparison between two defective tails of the deficit at ruin $\overline{G}(u, y)$ and $\overline{\widetilde{G}}(u, y)$, respectively.

The deficit at ruin (given that ruin occurs), which was introduced in the paper of [10], represents the probability that, starting with a surplus u, the deficit at ruin does not exceed y, i.e. $G(u, y) = Pr\left(|U_T| \leq y, T < \infty | U(0) = u\right)$. If we set $\overline{G}(u, y) = \psi(u) - G(u, y)$ for $u \geq 0$ and $y \geq 0$, then $\overline{G}(u, y)$ satisfies the defective renewal equation (see [32])

$$\overline{G}(u, y) = \frac{\lambda}{c}\left[\int_0^u \overline{G}(u - t, y)\overline{F}(t)dt + \int_{u+y}^\infty \overline{F}(t)dt\right], \tag{9}$$

which is the associated tail of the defective distribution of $G(u, y)$ and we note that it satisfies $\lim_{y\to\infty} = \psi(u) < 1$.

Similarly to Section 3.1, we consider a surplus process as (1), $\widetilde{U}(t)$, but one that is governed by the parameters $\widetilde{\lambda}$ and $\widetilde{F}$ with associated defective tail of the deficit at ruin which satisfies the following integral equation

$$\overline{\widetilde{G}}(u, y) = \frac{\widetilde{\lambda}}{c}\left[\int_0^u \overline{\widetilde{G}}(u - t, y)\overline{\widetilde{F}}(t)dt + \int_{u+y}^\infty \overline{\widetilde{F}}(t)dt\right]. \tag{10}$$

In the following theorem we obtain a bound for the uniform metric between two deficits at ruin.

Theorem 3.5. *With the above notation, it holds that*

$$\sup_{u\geq 0}\left|\overline{G}(u,y)-\overline{\widetilde{G}}(u,y)\right| \leq \frac{1}{c-\lambda\mu}\left[\lambda Q_y(F,\widetilde{F})+|\lambda-\widetilde{\lambda}|\widetilde{\mu}\right], \tag{11}$$

where $Q_y(F,\widetilde{F})=\int_y^\infty|\overline{F}(t)-\overline{\widetilde{F}}(t)|dt.$

4 Ruin probability in a perturbed risk model

In the classical risk model the number of claims $N(t)$ in (1) follows a homogeneous Poisson process with intensity $\lambda>0$. An extension of this is to add a diffusion term to account for additional uncertainties in the aggregate claims or the premium income. Gerber (1970) introduced the classical model perturbed by diffusion, with surplus process

$$U(t)=u+ct-S(t)+\sigma B(t),\quad t\geq 0, \tag{12}$$

where the dispersion parameter $\sigma>0$ and $\{B(t):t\geq 0\}$ is a standard Wiener process that is independent of the compound Poisson process $\{S(t):t\geq 0\}$ and of the individual claim sizes.

Dufresne and Gerber in [5] studied three kinds of probabilities based on (12): $\psi_d(u)=Pr(T<\infty,U(T)=0|U(0)=u)$ is the probability for ruin that is caused by oscillation, $\psi_s(u)=Pr(T<\infty,U(T)<0|U(0=u)$ is the probability that ruin is caused by a claim and $\psi_t(u)=Pr(T<\infty|U(0)=u)$, the probability of ruin. We have that $\psi_t(u)=\psi_d(u)+\psi_s(u)$, with $\psi_d(0)=1$ and $\psi_s(0)=1$.

It was shown in [5] showed that $\psi_t(u)=Pr(L^*>u)$ is the tail probability of the maximal aggregate loss $L^*=max\{u-U(t):t\geq 0\}$. The r.v. L^* can be decomposed as

$$L^*=L_{o,0}+L_{c,1}+\cdots+L_{c,N}+L_{o,N}=\sum_{n=1}^{\infty}\left(L_{o,n-1}+L_{c,N}\right)+L_{o,N},$$

with $L^*=L_{o,0}$ if $N=0$, where L_o, N and $L_{c,N}$ are the amounts that result in the $(n+1)$-th and n-th record highs of the aggregate loss process $\{u-U(t)\}$ due to oscillation and a claim, respectively, and N it the number of record highs of the process $\{u-U(t)\}$ caused by a claim. In addition, the r.v's $L_{o,0},L_{o,1},L_{o,2}...$ are identically distributed (as L_o) with common distribution function $H_1(u)=1-e^{-(c/D)u}$, where $D=\sigma^2/2$ and $L_{c,1},L_{c,2},L_{c,3}...$ are identically distributed (as L_c) with common distribution $H_2(u)=F_e(u)$. Also, $N,L_{o,0},L_{c,1},L_{o,1},L_{c,2},L_{o,2}...$ are independent. For further details and a probabilistic viewpoint of ψ_t (see [5] and [29]).

Furthermore, Tsai in [28] showed that

$$\overline{K}(u)=Pr(L_K>u)=\frac{1}{1+\theta}\psi_d(u)+\psi_s(u)=\sum_{n=1}^{\infty}\frac{\theta}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^n\overline{A^{*n}}(u),\quad u\geq 0, \tag{13}$$

with $\overline{K}(0)=1/(1+\theta)$, where $\overline{A}(x)=1-H_1*H_2(x)=1-\int_0^x\overline{H_1}(x-t)dH_2(t)$ is the distribution function of L_o+L_c with density $a(x)$. Since $L^*=L_K+L_o$, we have that

$Pr(L^* > u)$, the ruin probability for surplus process (12), is a compound geometric convolution, given by

$$\psi_t(u) = \overline{K * H_1}(u) = \sum_{n=0}^{\infty} \frac{\theta}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^n \overline{A^{*n} * H_1}(u), \quad u \geq 0. \tag{14}$$

Similarly, the expression for $\overline{K}(u)$ in (13) can be considered as $\overline{K}(u) = Pr(L^K > u)$ where

$$L^K = L_{o,0} + L_{c,1} + \cdots + L_{o,N-1} = \sum_{n=1}^{N}\left(L_{o,n-1} + L_{c,n}\right),$$

with $L^K = 0$ if $N = 0$. When the diffusion term is removed (i.e., $\sigma = 0$), then all L_os disappear, implying model (12) reduce to the non-perturbed classical risk model and both $\psi_t(u)$ and $K(u)$ reduce to (3).

4.1 Continuity inequality for $K(u)$

As seen in equation (14), $\psi_t(u)$ is the tail of the convolution of distribution function $K(u)$ and the diffusion term $H_1(u)$. Therefore, if an analytical expression for $\overline{K}(u)$ in (13) is available, then $\psi_t(u)$ can also be obtained in explicit form. However, explicit expressions for $K(u)$ are generally unavailable, except in specific cases such as a combination of exponential claim distribution or a mixture of Erlangs (see [28]). Veraverbeke in [31] investigated the asymptotic behavior of $\psi_t(u)$ in a classical risk model perturbed by diffusion and showed that the tail of $\psi_t(u)$ is related to the tail decay of $K(u)$ and $H_1(u)$. Moreover, an effective method to construct an upper bound for $\overline{K}(u)$ was proposed in [30]. Hence, it is of interest to study the continuity properties of $K(u)$. In the following theorem, we derive a continuity inequality for the function in (13).

Theorem 4.1. *Let $\overline{K}, \overline{\widetilde{K}} \in \mathcal{K}_d$ be the tails of (13) with $D \geq \widetilde{D}$ and $\mu \geq \widetilde{\mu}$. We also assume the net profit conditions $\phi, \widetilde{\phi} < 1$. Then, we derive*

$$\sup_{u\geq 0}\left|\overline{K}(u) - \overline{\widetilde{K}}(u)\right| \leq \frac{1}{c-\lambda\mu}\left[\lambda\mu\left(\frac{c}{D}\mathbb{K}(H_1,\widetilde{H}_1) + \frac{|\widetilde{D}-D|}{D} + \frac{\mathbb{K}(F,\widetilde{F})}{\mu} + \frac{|\widetilde{\mu}-\mu|}{\mu}\right) + |\lambda\mu - \widetilde{\lambda}\widetilde{\mu}|\right].$$

4.2 Estimation for K(u)

An approximation method for the ruin probability, $\psi(u)$, based on the contractive properties and the BFPT was proposed in [25]. Following similar arguments, we use a suitable contracting operator T_d on a certain Banach space $(\mathcal{K}_d, \nu_d)$, so that it can be applied to obtain an approximation of $\overline{K}$.

By the BFPT for contraction mappings, there exists a unique function (fixed point) $\overline{K} \in \mathcal{K}_d$, such that $T_d\overline{K} = \overline{K}$ and $\overline{K}$ is the limit of the functions

$$\overline{K}_n := T_d\overline{K}_{n-1} = T_d^n\overline{K}_0, \quad n \geq 1, \tag{15}$$

where

$$T_d x(u) = \frac{\lambda\mu}{c}\left(\overline{A}(u) + \int_0^u x(u-t)dA(t)\right), \quad u \geq 0, \tag{16}$$

for some arbitrary function $\overline{K}_0$ of $\mathcal{K}_d$. Next, we consider the iterative sequence $\{\overline{K}_n\}_{n\geq 0}$ associated with $\overline{K}$ as defined in (15),

$$\overline{K}_n(u) := \frac{1}{1+\theta}\left(\overline{A}(u) + \int_0^u \overline{K}_{n-1}(u-x)dA(x)\right), \tag{17}$$

where $\overline{K}_0 \in \mathcal{K}_d$.

Since T_d is a contractive operator in Banach space $(\mathcal{K}_d, \nu_d)$, the BFPT (see Theorem 3.3) ensures that iterative sequence defined in (15) converges to the unique fixed point $\overline{K}$, that is

$$\nu_d(\overline{K}_n, \overline{K}) \to 0.$$

The following result provides an explicit formula to compute the elements of the sequence $\{\overline{K}_n\}_{n\geq 0}$.

Lemma 1. *Let $\overline{K}_n \in \mathcal{K}_d$, if $\overline{K}_0(u) = k \in (0,1)$ and for $u \geq 0$, then*

$$\overline{K}_n(u) = \begin{cases} \phi - (1-k)\phi A(u), & n = 1, \\ \phi - (1-k)\phi^n A^{*(n)}(u) - (1-\phi)\sum_{i=1}^{n-1}\phi^i A^{*(i)}(u), & n \geq 2, \end{cases}$$

where $\phi = 1/(1+\theta) < 1$.

Remark 2. It was proven in [28] that $\overline{K}(u)$ satisfies the defective renewal equation:

$$\overline{K}(u) = \frac{\lambda\mu}{c}\left(\overline{A}(u) + \int_0^u \overline{K}(u-t)dA(t)\right), \quad u \geq 0. \tag{18}$$

As $n \to \infty$, we reduce to equation (18) as an immediate consequence of BFPT and Lemma 1. In particular, if $\overline{K}_0(u) = \phi$ and under the net profit conditions (see equation (6)) for classical model with $\overline{K}(u)$, then it follows that

$$\overline{K}(u) = \lim_{n\to\infty} \overline{K}_n(u) = (1-\phi)\sum_{i=1}^{\infty} \phi^i \overline{A^{*i}}(u), \quad u \geq 0,$$

with $\overline{K}(0) = \phi$.

5 Proofs

In this section, we provide proofs of all results stated in Sections 3 and 4.

Proof of Theorem 3.4. The following integral equation for ruin probabilities is commonly known (see e.g. [22]),

$$\psi(u) = \frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u \psi(u-t)\overline{F}(t)dt\right). \tag{19}$$

It is easy to check that for the operators

$$Tx(u) = \frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u x(u-t)\overline{F}(t)dt\right), \quad u \geq 0, \tag{20}$$

and

$$\widetilde{T}x(u) = \frac{\widetilde{\lambda}}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt\right), \quad u \geq 0, \tag{21}$$

we have $T\mathcal{D}_\gamma \subset \mathcal{D}_\gamma$ and $\widetilde{T}\mathcal{D}_\gamma \subset \mathcal{D}_\gamma$.

By (20) and (21) for every $x, y \in \mathcal{D}_\gamma$,

$$\begin{aligned}
\nu_\gamma(Tx, Ty) &= \frac{\lambda}{c}\int_0^\infty (1+u)^\gamma \left|\int_0^u x(u-t)\overline{F}(t)dt - \int_0^u y(u-t)\overline{F}(t)dt\right| du \\
&\leq \frac{\lambda}{c}\int_0^\infty \int_0^u (1+u)^\gamma \overline{F}(t)\,|x(u-t)-y(u-t)|\,dtdu \\
&= \frac{\lambda}{c}\int_0^\infty \int_t^\infty (1+u)^\gamma \overline{F}(t)\,|x(u-t)-y(u-t)|\,dudt,
\end{aligned}$$

where the last equality was obtained by changing the order of integration. Hence, setting $u = t + z$ and using the inequality $(1+z+t)^\gamma \leq (1+z)^\gamma(1+t)^\gamma$, for all $z \geq 0$, $t \geq 0$ and $\gamma \geq 0$, it follows that

$$\begin{aligned}
\nu_\gamma(Tx, Ty) &\leq \frac{\lambda}{c}\int_0^\infty \int_t^\infty (1+u)^\gamma \overline{F}(t)\,|x(u-t)-y(u-t)|\,dudt \\
&= \frac{\lambda}{c}\int_0^\infty \int_0^\infty (1+z+t)^\gamma \overline{F}(t)\,|x(z)-y(z)|\,dzdt \\
&\leq \frac{\lambda}{c}\int_0^\infty (1+t)^\gamma \overline{F}(t)\int_0^\infty (1+z)^\gamma\,|x(z)-y(z)|\,dzdt \\
&= \nu_\gamma(x,y)\frac{\lambda}{c}\int_0^\infty (1+t)^\gamma\overline{F}(t)dt = \nu_\gamma(x,y)\frac{\lambda}{c}M_\gamma^X.
\end{aligned}$$

Therefore, the operators (20) and (21) are contractive on $\mathcal{D}_\gamma$ with modules $\lambda\mu/c$ and $\widetilde{\lambda}\widetilde{\mu}/c$, respectively, since net profit conditions hold (see equation (6)) Now, integrating by parts and noting that $\gamma \geq 0$, we get

$$\begin{aligned}
M_\gamma^X = \int_0^\infty (1+t)^\gamma \overline{F}(t)dt &= \left[\overline{F}(t)\frac{(1+t)^{\gamma+1}}{\gamma+1}\right]_0^\infty + \int_0^\infty f(t)\frac{(1+t)^{\gamma+1}}{\gamma+1}dt \\
&= \frac{E(X+1)^{\gamma+1}-1}{\gamma+1}.
\end{aligned}$$

Also, it is well known that $E(|X|^p) < \infty \Leftrightarrow E(|X-a|^p) < \infty$, $\forall a \in \mathbb{R}$ and $0 < p < \infty$. Therefore, M_γ^X exists if the moments $EX^{(\gamma+1)}$ of the claim-size distribution are finite for $\gamma \geq 0$.

According to (19), ψ and $\widetilde{\psi}$ are the unique fixed points of T and $\widetilde{T}$ that is $\psi = T\psi$ and $\widetilde{\psi} = \widetilde{T}\widetilde{\psi}$. Now,

$$\begin{aligned}
\nu_\gamma(\psi, \widetilde{\psi}) &= \nu_\gamma(T\psi, \widetilde{T}\widetilde{\psi}) \\
&\leq \nu_\gamma(T\psi, T\widetilde{\psi}) + \nu_\gamma(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}) \\
&\leq \frac{\lambda}{c}M_\gamma^X \nu_\gamma(\psi, \widetilde{\psi}) + \nu_\gamma(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}),
\end{aligned}$$

or,

$$\nu_\gamma(\psi, \widetilde{\psi}) \leq \frac{c}{c - \lambda M_\gamma^X} \nu_\gamma(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}), \tag{22}$$

where $M_\gamma^X = \int_0^\infty (1+t)^\gamma \overline{F}(t)dt = \frac{1}{\gamma+1}\left(E(X+1)^{\gamma+1} - 1\right)$. In view of (20) and (21), for each $\psi \in \mathcal{D}_\gamma$, we have

$$\begin{aligned}
\nu_\gamma(T\psi, \widetilde{T}\psi) \leq & \int_0^\infty (1+u)^\gamma \left| \frac{\lambda}{c} \left(\int_u^\infty \overline{F}(t)dt + \int_0^u \psi(u-t)\overline{F}(t)dt \right) \right. \\
& \left. - \frac{\lambda}{c} \left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u \psi(u-t)\overline{\widetilde{F}}(t)dt \right) \right| du \\
& + \int_0^\infty (1+u)^\gamma \left| \left(\frac{\lambda - \widetilde{\lambda}}{c} \right) \left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u \psi(u-t)\overline{\widetilde{F}}(t)dt \right) \right| du \\
& = I_1 + I_2.
\end{aligned}$$

For the first term I_1 on the last inequality, we have

$$\begin{aligned}
I_1 &\leq \frac{\lambda}{c} \int_0^\infty (1+u)^\gamma \left(\int_u^\infty \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dt + \int_0^u \psi(u-t) \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dt \right) du \\
&= \frac{\lambda}{c} \left[\int_0^\infty \int_0^t (1+u)^\gamma \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dudt \right. \\
&\left. + \int_0^\infty \int_t^\infty (1+u)^\gamma \psi(u-t) \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dudt \right],
\end{aligned}$$

where for the first term inside the square bracket it follows that

$$\begin{aligned}
\int_0^\infty \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| \int_0^t (1+u)^\gamma dudt &= \int_0^\infty \frac{(1+t)^{\gamma+1} - 1}{\gamma+1} \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dt \\
&\leq \frac{1}{\gamma+1} \nu_{\gamma+1}(F, \widetilde{F}),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\int_0^\infty \int_t^\infty (1+u)^\gamma \psi(u-t) \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dudt &= \int_0^\infty \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| \\
&\times \int_0^\infty (1+z+t)^\gamma \psi(z) dzdt \\
&\leq \int_0^\infty (1+t)^\gamma \left| \overline{F}(t) - \overline{\widetilde{F}}(t) \right| dt \\
&\times \int_0^\infty (1+z)^\gamma \psi(z) dz \\
&\leq \nu_\gamma(F, \widetilde{F}) M_\gamma^L.
\end{aligned} \tag{24}$$

Therefore, by (23) and (24) it follows that

$$I_1 \leq \frac{1}{\gamma+1} \nu_{\gamma+1}(F, \widetilde{F}) + \nu_\gamma(F, \widetilde{F}) M_\gamma^L. \tag{25}$$

Similarly, for the term I_2, we have

$$\begin{aligned}
I_2 &= \frac{|\lambda - \widetilde{\lambda}|}{c}\left(\int_0^\infty \overline{\overline{F}}(t)\int_0^t (1+u)^\gamma du dt + \int_0^\infty \overline{\overline{F}}(t)\int_t^\infty (1+u)^\gamma \psi(u-t) du dt\right)\\
&= \frac{|\lambda - \widetilde{\lambda}|}{c}\left(\int_0^\infty \overline{\overline{F}}(t)\frac{(1+t)^{\gamma+1}-1}{\gamma+1}dt + \int_0^\infty \overline{\overline{F}}(t)\int_0^\infty (1+z+t)^\gamma \psi(z) dz dt\right)\\
&\le \frac{|\lambda - \widetilde{\lambda}|}{c}\left(\int_0^\infty \overline{\overline{F}}(t)\frac{(1+t)^{\gamma+1}}{\gamma+1}dt + M_\gamma^L \int_0^\infty \overline{\overline{F}}(t)(1+t)^\gamma dt\right)\\
&\le \frac{|\lambda - \widetilde{\lambda}|}{c}\left(\int_0^\infty \overline{\overline{F}}(t)(1+t)^{\gamma+1}dt + M_\gamma^L \int_0^\infty \overline{\overline{F}}(t)(1+t)^{\gamma+1} dt\right)
\end{aligned}$$

Hence,

$$I_2 \le \frac{|\lambda - \widetilde{\lambda}|}{c} M_{\gamma+1}^{\widetilde{X}}\left(1 + M_\gamma^L\right). \tag{26}$$

By (22), (25) and (26) the result follows. □

Proof of Theorem 3.5. Let $\mathcal{X}$ be the space of all functions $x : [0,\infty)\times[0,\infty) \to [0,1]$ endowed with the *uniform metric* $\nu_d(x,y) := \sup_{t\ge 0}|x(t) - y(t)|$. Also, let the variable z as a constant so that $\overline{G}(x,z) = \overline{\Gamma}_z(x)$ is a function of x. Then $(\mathcal{X}, \nu_d)$ is easy to see that it is a complete metric space.

We have $T^z\mathcal{X} \subset \mathcal{X}$ and $\widetilde{T}^z\mathcal{X} \subset \mathcal{X}$ for the following operators

$$T^z x(u) = \frac{\lambda}{c}\left[\int_{z+u}^\infty \overline{F}(t)dt + \int_0^u x(u-t)\overline{F}(t)dt\right], \quad u \ge 0, \tag{27}$$

and

$$\widetilde{T}^z x(u) = \frac{\widetilde{\lambda}}{c}\left[\int_{z+u}^\infty \overline{\overline{F}}(t)dt + \int_0^u x(u-t)\overline{\overline{F}}(t)dt\right], \quad u \ge 0. \tag{28}$$

Therefore, by (27), for every $x, y, \in \mathcal{X}$, we have

$$\begin{aligned}
\nu_d(T^z x, T^z y) &= \frac{\lambda}{c}\sup_{u\ge 0}\left|\int_0^u x(u-t)\overline{F}(t)dt - \int_0^u y(u-t)\overline{F}(t)dt\right|\\
&\le \frac{\lambda}{c}\sup_{u\ge 0}\int_0^u \overline{F}(t)\sup_{s\in[0,u]}|x(s)-y(s)|dt\\
&\le \frac{\lambda}{c}\nu_d(x,y)\int_0^\infty \overline{F}(t)dt = \frac{\lambda\mu}{c}\nu_d(x,y).
\end{aligned}$$

According to (9) and (10), $\Gamma_z(x)$ and $\widetilde{\Gamma}_z(x)$ are the unique fixed points of T^z and $\widetilde{T}^z$ that is $\Gamma_z = T^z\Gamma_z$ and $\widetilde{\Gamma}_z = \widetilde{T}^z\widetilde{\Gamma}_z$. Thus, it follows that

$$\begin{aligned}
\nu_d(\Gamma_z, \widetilde{\Gamma}_z) &= \nu_d(T^z\Gamma_z, \widetilde{T}^z\widetilde{\Gamma}_z)\\
&\le \nu_d(T^z\Gamma_z, T^z\widetilde{\Gamma}_z) + \nu_d(T^z\widetilde{\Gamma}_z, \widetilde{T}^z\widetilde{\Gamma}_z)\\
&\le \frac{\lambda\mu}{c}\nu_d(\Gamma_z, \widetilde{\Gamma}_z) + \nu_d(T^z\widetilde{\Gamma}_z, \widetilde{T}^z\widetilde{\Gamma}_z),
\end{aligned}$$

or equivalently,

$$\nu_d(\Gamma_z, \widetilde{\Gamma}_z) \leq \frac{c}{c - \lambda\mu} \nu_d(T^z \widetilde{\Gamma}_z, \widetilde{T}^z \widetilde{\Gamma}_z). \tag{29}$$

In view of (20) and (21) for each $x \in \mathcal{X}$ we have

$$\begin{aligned}
\nu_d(T^z x, \widetilde{T}^z x) &\leq \sup_{u \geq 0} \bigg| \int_{u+z}^{\infty} \overline{F}(t)dt + \int_0^u x(u-t)\overline{F}(t)dt \\
&\quad - \frac{\lambda}{c} \int_{u+z}^{\infty} \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt \bigg| \\
&\quad + \sup_{u \geq 0} \left| \frac{\lambda - \widetilde{\lambda}}{c} \left(\int_{u+z}^{\infty} \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt \right) \right| \\
&\leq \frac{\lambda}{c} \int_z^{\infty} |\overline{F}(t) - \overline{\widetilde{F}}(t)|dt + \frac{|\lambda - \widetilde{\lambda}|}{c} \int_0^{\infty} \overline{\widetilde{F}}(t)dt.
\end{aligned}$$

Combining the last inequality with (29), we obtain

$$\nu_d(\overline{G}(u, y), \overline{\widetilde{G}}(u, y)) \leq \frac{1}{c - \lambda\mu} \left[\lambda \int_y^{\infty} |\overline{F}(t) - \overline{\widetilde{F}}(t)|dt + |\lambda - \widetilde{\lambda}|\widetilde{\mu} \right],$$

which completes the proof. □

Proof of Theorem 4.1. Let $\mathcal{K}_d$ be the space of all functions $x : [0, \infty) \to [0, 1]$. The space $\mathcal{K}_d$ is a Banach space with the uniform metric $\nu_d(x, y) := \sup_{u \geq 0} |x(u) - y(u)|$, i.e. $(\mathcal{K}_d, \nu_d)$ is a complete metric space.

For each $x \in \mathcal{K}_d$, we consider the operators T_d given in (16) and $\widetilde{T}_d : \mathcal{K}_d \to \mathcal{K}_d$, defined by

$$\widetilde{T}_d x(u) = \frac{\widetilde{\lambda}\widetilde{\mu}}{c} \left(\overline{\widetilde{A}}(u) + \int_0^u x(u-t)d\widetilde{A}(t) \right), \quad u \geq 0.$$

Given $u \geq 0$ and for all $x \in \mathcal{K}_d$ we have $T_d \mathcal{K}_d \subset \mathcal{K}_d$, since it follows immediately

$$\begin{aligned}
T_d x(u) &= \frac{\lambda\mu}{c} \left(\overline{A}(u) + \int_0^u x(u-t)a(t)dt \right) \\
&\leq \frac{\lambda\mu}{c} \left(\overline{A}(u) + \int_0^u a(t)dt \right) \leq \frac{\lambda\mu}{c} = \phi < 1.
\end{aligned}$$

Furthermore, these operators are contractive on $\mathcal{K}_d$ with modules ϕ and $\widetilde{\phi}$, respectively, since for all $u \geq 0$

$$\begin{aligned}
\nu_d(T_d x, T_d y) &= \frac{\lambda\mu}{c} \sup_{u \geq 0} \left| \int_0^u x(u-t)dA(t) - \int_0^u y(u-t)dA(t) \right| \\
&\leq \frac{\lambda\mu}{c} \sup_{u \geq 0} \int_0^u |x(u-t) - y(u-t)|\, a(t)dt \\
&\leq \frac{\lambda\mu}{c} \sup_{u \geq 0} \int_0^u \sup_{s \in [0,u]} |x(s) - y(s)|\, a(t)dt
\end{aligned}$$

$$\le \frac{\lambda\mu}{c}\nu_d(x,y)\int_0^\infty a(t)dt = \frac{\lambda\mu}{c}\nu_d(x,y).$$

Therefore, it follows that $\nu_d(T_d\overline{K}, T_d\overline{\overline{K}}) \le \phi\nu_d(\overline{K}, \overline{\overline{K}})$, where $\phi \le 1$. Similarly, $\widetilde{T}_d$ is also a contractive operator on $\mathcal{K}_d$.

According to (18), it is easy to show that $\overline{K}$ and $\overline{\overline{K}}$ are the unique fixed points of $\overline{K} = T_d\overline{K}$ and $\overline{\overline{K}} = \widetilde{T}_d\overline{\overline{K}}$. Hence, by triangle inequality it follows that

$$\begin{aligned}\nu_d(\overline{K}, \overline{\overline{K}}) &= \nu_d(T_d\overline{K}, \widetilde{T}_d\overline{\overline{K}})\\ &\le \nu_d(T_d\overline{K}, T_d\overline{\overline{K}}) + \nu_d(T_d\overline{\overline{K}}, \widetilde{T}_d\overline{\overline{K}})\\ &\le \frac{\lambda\mu}{c}\nu_d(\overline{K}, \overline{\overline{K}}) + \nu_d(T_d\overline{\overline{K}}, \widetilde{T}_d\overline{\overline{K}}),\end{aligned}$$

or equivalently,

$$\nu_d(\overline{K}, \overline{\overline{K}}) \le \frac{c}{c-\lambda\mu}\nu_d(T_d\overline{\overline{K}}, \widetilde{T}_d\overline{\overline{K}}). \tag{30}$$

Furthermore,

$$\begin{aligned}\nu_d(T_dx, \widetilde{T}_dx) \le& \sup_{u\ge 0}\left|\frac{\lambda\mu}{c}\left(\overline{A}(u) + \int_0^u x(u-t)a(t)dt\right)\right.\\ &\left.-\frac{\lambda\mu}{c}\left(\overline{\overline{A}}(u) + \int_0^u x(u-t)\widetilde{a}(t)dt\right)\right|\\ &+\sup_{u\ge 0}\left|\frac{\lambda\mu - \widetilde{\lambda}\widetilde{\mu}}{c}\left(\overline{\overline{A}}(u) + \int_0^u x(u-t)\widetilde{a}(t)dt\right)\right|\\ \le& \frac{\lambda\mu}{c}\int_0^\infty |a(t)-\widetilde{a}(t)|dt + \frac{|\lambda\mu - \widetilde{\lambda}\widetilde{\mu}|}{c}\int_0^\infty \widetilde{a}(t)dt,\end{aligned} \tag{31}$$

where

$$a(t) = \int_0^t h_1(t-z)dH_2(z) = \int_0^t \frac{c}{D}e^{-(c/D)(t-z)}\frac{\overline{F}(z)}{\mu}dz.$$

It follows that

$$\begin{aligned}\int_0^\infty |a(t)-\widetilde{a}(t)|dt \le& \int_0^\infty \left|\int_0^t \frac{c}{D}e^{-(c/D)(t-z)}\frac{\overline{F}(z)}{\mu}dz - \int_0^t \frac{c}{\widetilde{D}}e^{-(c/\widetilde{D})(t-z)}\frac{\overline{F}(z)}{\mu}dz\right|\\ &+\left|\int_0^t \frac{c}{\widetilde{D}}e^{-(c/\widetilde{D})(t-z)}\frac{\overline{F}(z)}{\mu}dz - \int_0^t \frac{c}{\widetilde{D}}e^{-(c/\widetilde{D})(t-z)}\frac{\overline{\overline{F}}(z)}{\widetilde{\mu}}dz\right|dt\\ \le& \int_0^\infty \frac{\overline{F}(z)}{\mu}\int_z^\infty \left|\frac{c}{D}e^{-(c/D)(t-z)} - \frac{c}{\widetilde{D}}e^{-(c/\widetilde{D})(t-z)}\right|dtdz\\ &+\int_0^\infty \left|\frac{\overline{F}(z)}{\mu} - \frac{\overline{\overline{F}}(z)}{\widetilde{\mu}}\right|\int_z^\infty \frac{c}{\widetilde{D}}e^{-(c/\widetilde{D})(t-z)}dtdz.\end{aligned}$$

Changing the integration variable by setting $t = y + z$, we obtain

$$\int_0^\infty |a(t)-\widetilde{a}(t)|dt \le \frac{c}{D}\mathbb{K}(H_1, \widetilde{H}_1) + \frac{|\widetilde{D}-D|}{D} + \frac{1}{\mu}\mathbb{K}(F, \widetilde{F}) + \frac{|\widetilde{\mu}-\mu|}{\mu}. \tag{32}$$

By inequalities (30), (31) and (32) we conclude the proof and the desired statement holds. □

Proof of Lemma 1. The result can be demonstrated by mathematical induction. For $n = 1$ applying (17) with $\overline{K}_0 = k$, we have

$$\overline{K}_1(u) = TK_0(u) = Tk = \phi\left(\overline{A}(u) + \int_0^u k \cdot a(x)dx\right) = \phi - (1-k)\phi A(u). \quad (33)$$

Now, suppose that the result holds for $n \geq 2$ with $u \geq 0$ fixed and for the next iteration we have

$$\begin{aligned}\overline{K}_{n+1}(u) &= T\overline{K}_n(u)\\ &= \phi\left(\overline{A}(u) + \int_0^u [\phi - (1-k)p^n A^{*n}(u-x)\right.\\ &\left.-(1-\phi)\sum_{i=1}^{n-1}\phi^i A^{*i}(u-x)\right] a(x)dx\Bigg)\\ &= \phi\Bigg(1 - A(u) + \phi A(u) - (1-k)\phi^n A^{*(n+1)}(u)\\ &-(1-\phi)\sum_{i}^{n-1}\phi^i A^{*(i+1)}(u)\Bigg)\\ &= \phi - (1-k)\phi^{n+1}A^{*(n+1)}(u) - (1-\phi)\sum_{j=1}^{n}\phi^j A^{*(j+1)}(u).\end{aligned}$$

This completes the induction. □

6 Numerical examples

In this section, we illustrate the applicability of theorems formulated in Sections 3 and 4. The numerical examples illustrate applications of the properties of contractive operators and of the approximation method presented in Section 4. We first present an example related to Theorem 3.4.

Example 6.1. We consider the classical risk model in (1), where the r.v. X follows a mixture of two exponential distributions with tail

$$\overline{F}_X(t) = \frac{1}{2}e^{-5t/4} + \frac{1}{2}e^{-5t/6}, \quad t > 0,$$

and $\widetilde{X} \sim Exp(1)$ such that $EX = E\widetilde{X} = 1$. For convenience, we refer to the continuity bound in (8) as $DK1$. In Table 1 we compare the results of $DK1$ with the exact values of the distance $\nu_\gamma(\overline{F}, \overline{\widetilde{F}})$. For fixed parameters, the values corresponding to $\gamma = 1$ are larger than those for $\gamma = 0$, reflecting the increased sensitivity of the weighted norm ν_γ. The numerical results further show that higher values of the claim arrival rate λ are associated with larger values of the bound $DK1$. This observation is consistent with the stability analysis discussed in [2], which indicates that increasing discrepancies in the claim arrival rates lead to larger deviations between the corresponding ruin probabilities ψ and $\widetilde{\psi}$. Moreover, a decrease in both exact differences and bound $DK1$ is observed as the premium rate increases.

Table 1. Comparison between the bound $DK1$ in (8) and the exact values of $\nu_\gamma(F, \widetilde{F})$ for a mixture of two exponential distributions versus an exponential distribution

(a) $\gamma = 0,\ \lambda = \frac{5}{6}$

	$c = 3$	$c = 5$	$c = 7$
$\\|\psi - \widetilde{\psi}\\|_{1,\gamma}$	0.0154	0.0080	0.0054
$DK1$	0.1211	0.0999	0.0929

(b) $\gamma = 0,\ \lambda = \frac{10}{11}$

	$c = 3$	$c = 5$	$c = 7$
$\\|\psi - \widetilde{\psi}\\|_{1,\gamma}$	0.0174	0.0089	0.0059
$DK1$	0.1271	0.1024	0.0944

(c) $\gamma = 1,\ \lambda = \frac{5}{6}$

	$c = 3$	$c = 5$	$c = 7$
$\\|\psi - \widetilde{\psi}\\|_{1,\gamma}$	0.0736	0.0353	0.0231
$DK1$	0.6340	0.3548	0.2922

(d) $\gamma = 1,\ \lambda = \frac{10}{11}$

	$c = 3$	$c = 5$	$c = 7$
$\\|\psi - \widetilde{\psi}\\|_{1,\gamma}$	0.0850	0.0396	0.0257
$DK1$	0.7512	0.3795	0.3047

Example 6.2. Let us consider the classical risk model in (1) with three non-negative r.v.'s $X^{(1)} \sim Erlang(3,3)$, $X^{(2)} \sim Exp(1)$ and $X^{(3)}$ with tail $\overline{F}_{X^{(3)}}(t) = \frac{1}{2}e^{-5t/4} + \frac{1}{2}e^{-5t/6}$. We also assume the defective distribution function of the deficit at ruin $G^{(1)}(u,y)$, $G^{(2)}(u,y)$ and $G^{(3)}(u,y)$ in risk models with individual claim sizes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$, respectively.

In Table 2 we present the exact values of the distance $\left|\overline{G}^{(i)}(u,y) - \overline{G}^{(j)}(u,y)\right|$, $i, j = 1,2,3, i \neq j$ versus the bound $DK2$ derived from (11) for various values of u and y. For each case, we use the same λ (see [33] and [32]). Specifically, the deviations observed between $DK2$ and the exact value are natural since the bound is derived to hold uniformly with respect to u. Moreover, the numerical results show that the bound $DK2$ decreases as y increases, indicating a monotone behavior with respect to y. Finally, it appears to behave more stably for smaller values of y.

To illustrate the results of Theorem 4.1 we obtain the following example.

Example 6.3. For the surplus process (12), let X and $\widetilde{X}$ have the survival functions $\overline{F}_X(t) = e^{-3t}$ and $\overline{F}_{\widetilde{X}}(t) = (1/2)e^{-2t} + (1/2)e^{-6t}$, respectively, with $EX = E\widetilde{X} = 1/3$, $\theta = \theta = 1$, and $c = \widetilde{c} = 1$. For various choices of D and $\widetilde{D}$, we can easily compute the real values of the distance $\nu_d(\overline{K}, \overline{\widetilde{K}})$, respectively (see Example 2 in [29]). Table 3 presents a comparison between the exact values of these distances and the results obtained from the bound $DK3$ in the Theorem 4.1, with the last column showing the deviation of the ratio (Bound/Exact) from 1.

Example 6.4. Suppose that the individual claim amounts follow an exponential distribution with parameter β in the surplus process in (12). We notice that X satisfies the net profit condition if $\beta \in (\lambda/c, \infty)$. In this case the explicit solution for $\overline{K}(u)$ is given by

$$K(u) = \theta\left(D_1 e^{-s_1 u} + D_2 e^{-s_2 u}\right), \tag{34}$$

where s_1 and s_2 are the roots of the equation $s^2 - (b_0 + \beta)s + [\theta/(1+\theta)b_0\beta = 0$ with

Table 2. Comparison between $\left|\overline{G}^{(i)}(u,y) - \overline{G}^{(j)}(u,y)\right|$, $i, j = 1, 2, 3$, $i \neq j$ and $DK2$

(a) $\theta = 1$, $X^{(1)}$ **vs** $X^{(2)}$

y	u	$\left\lvert\overline{G}^{(1)} - \overline{G}^{(2)}\right\rvert$	$DK2$
0.10	0.10	0.0080	0.2938
	0.25	0.0184	
	0.50	0.0367	
	1.00	0.0640	
	2.00	0.0754	
0.25	0.10	0.0227	0.2726
	0.25	0.0365	
	0.50	0.0550	
	1.00	0.0742	
	2.00	0.0736	
0.50	0.10	0.0495	0.2220
	0.25	0.0625	
	0.50	0.0758	
	1.00	0.0821	
	2.00	0.0098	
1.00	0.10	0.0773	0.1547
	0.25	0.0818	
	0.50	0.0832	
	1.00	0.0748	
	2.00	0.0521	
2.00	0.10	0.0521	0.1081
	0.25	0.0506	
	0.50	0.0463	
	1.00	0.0372	
	2.00	0.0233	

(b) $\theta = 4$, $X^{(1)}$ **vs** $X^{(2)}$

y	u	$\left\lvert\overline{G}^{(1)} - \overline{G}^{(2)}\right\rvert$	$DK2$
0.10	0.10	0.0034	0.0735
	0.25	0.0084	
	0.50	0.0176	
	1.00	0.0290	
	2.00	0.0241	
0.25	0.10	0.0091	0.0681
	0.25	0.0100	
	0.50	0.0233	
	1.00	0.0304	
	2.00	0.0228	
0.50	0.10	0.0194	0.0555
	0.25	0.0243	
	0.50	0.0294	
	1.00	0.0303	
	2.00	0.0194	
1.00	0.10	0.0300	0.0387
	0.25	0.0309	
	0.50	0.0301	
	1.00	0.0247	
	2.00	0.0132	
2.00	0.10	0.0205	0.0270
	0.25	0.0189	
	0.50	0.0161	
	1.00	0.0114	
	2.00	0.0054	

(c) $\theta = 4$, $X^{(1)}$ **vs** $X^{(3)}$

y	u	$\left\lvert\overline{G}^{(1)} - \overline{G}^{(3)}\right\rvert$	$DK2$
0.10	0.10	0.0035	0.0779
	0.25	0.0087	
	0.50	0.0183	
	1.00	0.0306	
	2.00	0.0272	
0.25	0.10	0.0094	0.0723
	0.25	0.0156	
	0.50	0.0244	
	1.00	0.0322	
	2.00	0.0252	
0.50	0.10	0.0202	0.0592
	0.25	0.0254	
	0.50	0.0309	
	1.00	0.0324	
	2.00	0.0217	
1.00	0.10	0.0317	0.0413
	0.25	0.0328	
	0.50	0.0322	
	1.00	0.0270	
	2.00	0.0152	
2.00	0.10	0.0227	0.0297
	0.25	0.0211	
	0.50	0.0183	
	1.00	0.0134	
	2.00	0.0068	

(d) $\theta = 4$, $X^{(3)}$ **vs** $X^{(2)}$

y	u	$\left\lvert\overline{G}^{(3)} - \overline{G}^{(2)}\right\rvert$	$DK2$
0.10	0.10	0.0001	0.0054
	0.25	0.0003	
	0.50	0.0007	
	1.00	0.0016	
	2.00	0.0023	
0.25	0.10	0.0003	0.0052
	0.25	0.0006	
	0.50	0.0011	
	1.00	0.0018	
	2.00	0.0024	
0.50	0.10	0.0008	0.0046
	0.25	0.0011	
	0.50	0.0015	
	1.00	0.0021	
	2.00	0.0023	
1.00	0.10	0.0017	0.0035
	0.25	0.0019	
	0.50	0.0021	
	1.00	0.0023	
	2.00	0.0020	
2.00	0.10	0.0022	0.0027
	0.25	0.0022	
	0.50	0.0022	
	1.00	0.0020	
	2.00	0.0014	

Table 3. A mixture of two exponentials vs an exponential perturbed by diffusion

D	$\widetilde{D}$	$\sup_{u\geq 0} \lvert\overline{K}(u) - \overline{\widetilde{K}}(u)\rvert$	Bound $DK3$	$\lvert 1 - \text{Ratio}\rvert$
1	1/10	0.0854	0.4837	4.66
1/2	1/10	0.0559	0.4337	6.75
1/2	1/3	0.0271	0.2004	6.39
2	1	0.0496	0.2837	4.71
2	1/10	0.1148	0.5087	3.43
3	1/10	0.1305	0.5171	2.96
3	1/20	0.1334	0.5254	2.94

$b_0 = c/D$. The corresponding constants are given by

$$D_1 = \frac{s_2}{\theta(1+\theta)\sqrt{(b_0-\beta)^2 + 4b_0\beta/(1+\theta)}} > 0,$$

and

$$D_2 = -\frac{s_1}{\theta(1+\theta)\sqrt{(b_0-\beta)^2 + 4b_0\beta/(1+\theta)}} < 0$$

(see Example 2 in Tsai 2006).

Therefore, we apply BFPT and consider the corresponding iterative sequence of $\overline{K}(u)$, as defined in (17), in order to obtain an approximation of the function $\overline{K}(u)$. In the special case $\beta = c/D$, where $A(u)$ follows an $Erlang(2, \beta)$, the following expression of $\overline{K}_n$ is obtained after some straightforward algebra and using mathematical induction,

$$\overline{K}_n(u) = \begin{cases} \phi k + \phi(1-k)e^{-\beta u}(1+\beta u), & n = 1, \\ \phi e^{-\beta u}\mathcal{S}_1(\beta u) + \sum_{m=2}^{n-1} \phi^m e^{-\beta u}\left[\mathcal{S}_{2m-1}(\beta u) - \mathcal{S}_{2m-3}(\beta u)\right] \\ +\phi^n k \left(1 - e^{-\beta u}\mathcal{S}_{2n-3}(\beta u)\right) \\ +\phi^n(1-k)e^{-\beta u}\left[\mathcal{S}_{2n-1}(\beta u) - \mathcal{S}_{2n-3}(\beta u)\right], & n \geq 2, \end{cases} \tag{35}$$

where $\mathcal{S}_m(z) = \sum_{r=0}^{m} \frac{z^r}{r!}$ (where $\mathcal{S}_{-1}(z) = 0$ by convention).

Tables 4 and 5 present the numerical approximations of $\overline{K}(u)$ obtained from the first five iterations of expression (35), using the initial values $\overline{K}_0 := k = 0.0, 0.1, 0.2, \ldots, 1.0$. Table 4 corresponds to the parameter set $\beta = 2$, $\lambda = c = 1/2$, $D = 1/4$, and $u = 1$, for which the exact value is $\overline{K}(1) = 0.3325717$. Table 5 refers to the case $\beta = 3/2$, $\lambda = 3/4$, $c = 2/3$, and $D = 4/9$, where the exact value equals $\overline{K}(1) = 0.6573777$.

Table 4. Approximations of the tail $\overline{K}(u)$ for the first five iterations when $X_i \sim Exp(2)$, $D = 1$, $c = \lambda = 1$ and $u = 1$

n	$k = 0.0$	$k = 0.2$	$k = 0.4$	$k = 0.6$	$k = 0.8$	$k = 1.0$
1	0.2030029	0.2624023	0.3218018	0.3812012	0.4406006	0.5000000
2	0.3157823	0.3229262	0.3300700	0.3372138	0.3443576	0.3515015
3	0.3315714	0.3319855	0.3323996	0.3328137	0.3332278	0.3336419
4	0.3325381	0.3325518	0.3325655	0.3325793	0.3325930	0.3326067
5	0.3325709	0.3325712	0.3325717	0.3325720	0.3325721	0.3325724

Table 5. Approximations of the tail $\overline{K}(u)$ for the first five iterations when $X_i \sim Exp(3/2)$, $c = 2/3$, $D = 4/9$, $\lambda = 3/4$ and $u = 1$

n	$k = 0.0$	$k = 0.2$	$k = 0.4$	$k = 0.6$	$k = 0.8$	$k = 1.0$
1	0.4183691	0.4846952	0.5510214	0.6173476	0.6836738	0.7500000
2	0.6301684	0.6375532	0.6449379	0.6523227	0.6597075	0.6670923
3	0.6559814	0.6563574	0.6567334	0.6571093	0.65745853	0.6578613
4	0.6573377	0.6573484	0.6573591	0.6573699	0.6573806	0.6573913
5	0.6573769	0.6573771	0.6573773	0.6573775	0.6573777	0.6573779

7 Conclusions

In this work, continuity inequalities were established for ruin-related quantities within the classical risk model. Specifically, Theorem 3.4 provides a continuity inequality for the ruin probability under claim distributions with finite $(\gamma + 1)$-moments, while Theorem 3.5 presents a continuity estimate for the deficit at ruin. Both results were obtained using properties of contractive integral operators.

In Section 4, the analysis was extended to the classical risk model perturbed by diffusion, where a continuity inequality for the supremum distance of the function $K(u)$ was derived. In this framework, although the ruin probability $\psi_t(u) = \overline{K * H_1}(u)$, can be decomposed using the geometric r.v. N to denote the number of recorded highs, the $\psi_t(u)$ itself is not a compound geometric distribution. This observation motivates us to study the behavior of the compound geometric distribution $K(u)$. Consequently, we derived a continuity estimate for $K(u)$ and proposed an iterative approximation based on the Banach fixed point theorem.

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