Modern Stochastics: Theory and Applications 2 (2015) 233–249
DOI: 10.15559/15-VMSTA36CNF

Pricing the European call option in the model with stochastic volatility driven by Ornstein–Uhlenbeck process. Exact formulas

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Received: 29 July 2015, Revised: 13 September 2015, Accepted: 14 September 2015,
Published online: 25 September 2015

Abstract We consider the Black–Scholes model of financial market modified to capture the stochastic nature of volatility observed at real financial markets. For volatility driven by the Ornstein–Uhlenbeck process, we establish the existence of equivalent martingale measure in the market model. The option is priced with respect to the minimal martingale measure for the case of uncorrelated processes of volatility and asset price, and an analytic expression for the price of European call option is derived. We use the inverse Fourier transform of a characteristic function and the Gaussian property of the Ornstein–Uhlenbeck process.

Keywords Financial markets, stochastic volatility, Ornstein–Uhlenbeck process, option pricing
2010 MSC 91B24, 91B25, 91G20

1 Introduction

One of the promising directions of enhancement of the classical Black–Scholes model is construction and research of diffusion models with volatility of risky asset governed by a stochastic process. Empirical studies [7, 11] evidence in favor of the fact that the classical model with constant volatility is unable to capture important features

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of volatility observed in real financial markets. This drawback of the Black–Scholes model has been widely investigated and to some extent eliminated by the extension of the theory in three directions: models with time-dependent deterministic volatility, models with state-dependent volatility, and models with stochastic volatility. The first and second of these categories may be viewed as intermediate between the classical model and third category, although equipping the market with certain constraints (the most essential is the limiting time period under consideration) allow less complex models to produce results of acceptable precision.

Despite recent popularity of the stochastic volatility modification of the Black–Scholes theory, the range of models under consideration is quite narrow. One of the first models of such a type is presented in [6], where the authors assume the volatility of the price of risky asset to be governed by the square root of the geometric Brownian motion. An expression for the price of European call option is derived under the following assumption: the volatility process is driven by a Brownian motion independent of the Brownian motion governing the price of risky asset. In [16], the authors choose the Ornstein-Uhlenbeck (OU) process to drive the volatility. The OU process is mean-reverting, and there is a strong evidence that the volatility in real financial markets has such a feature [4, 3]. Under this assumption, the authors of [16] describe the distribution of the price of risky asset and apply it to derive an estimate of the price of European call option. As an alternative, there is an option to choose the Cox–Ingersoll–Ross process to govern the volatility process [4, 3]. It is worth mentioning that although all cited works contain some significant results, they rely upon simplified models of real-world volatility process (e.g., ignoring the mean-reversion property). We remark that there is a vast amount of further investigations that consider more sophisticated and thus more realistic models. An extensive overview of these results is given in [14].

Nonnegativity is another desirable feature of the process modeling volatility. One of possible choices is to use the exponential function of the OU process (see [10, 13], and references therein).

Questions of existence of equivalent (local) martingale measures are investigated in different frameworks and different generality in [3, 5, 8, 17]. Often, after specifying the model, the authors state that a risk-neutral measure exists and continue investigation in the risk-neutral world without defining the measure.

A significant part of works (including aforementioned) use the Fourier transform to derive an analytical representation of the price of European call option. A great deal of information about developments in application of the Fourier transform to option pricing problems can be found in [12].

Our work investigates the market defined by a diffusion model with stochastic volatility being an arbitrary function governed by the Ornstein–Uhlenbeck process. Under general setting and quite mild assumptions, we prove that the market satisfies two distinct no-arbitrage properties for different classes of trading strategies. For the special case of uncorrelated Wiener processes, we derive an analytical expression for the price of European call option.

This paper is structured as follows: in Section 2, we define a general model. In Section 3, we present definitions and preliminary results necessary for further analysis. In Section 4, we investigate matters of existence of equivalent (local) martingale measures and arbitrage properties of the general model. In Section 5, we define a particular case of the general model and raise the problem of pricing European call option. Section 6 covers the derivation of an analytical expression for the option price.

2 Diffusion model with stochastic volatility governed by Ornstein–Uhlenbeck process

Let $\{\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t^{(B,W)}, t \geq 0\}, \mathbb{P}\}$ be a complete probability space with filtration generated by correlated Wiener processes $\{B_t, W_t, 0 \leq t \leq T\}$. We consider the model of the market where one risky asset is traded, its price evolves according to the geometric Brownian motion $\{S_t, \ 0 \leq t \leq T\}$, and its volatility is driven by a stochastic process. More precisely, the market is described by the pair of stochastic differential equations

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dB_t, \tag{1}$$

$$dY_t = -\alpha Y_t dt + k dW_t. \tag{2}$$

Denote by $S_0 = S$ and $Y_0 = Y$ the deterministic initial values of the processes specified by Eqs. (1)–(2).

In Sections 2–4, we impose the following assumptions:

(A1) The Wiener processes B and W are correlated with correlation coefficient $\rho \in [-1; 1]$, that is, $dB_t dW_t = \rho dt$;

(A2) the volatility function $\sigma : \mathbb{R} \to \mathbb{R}_+$ is measurable, bounded away from zero by a constant c, that is,

$$\sigma(x) \geq c > 0, \quad x \in \mathbb{R},$$

and satisfies the conditions $\int_0^T \sigma^2(Y_t)dt < \infty$ a.s.;

(A3) the coefficients α, μ, and k are positive.

For example, the conditions mentioned in assumption (A2) are satisfied for a measurable function $\sigma(x)$ such that $c \leq \sigma^2(x) \leq C$ for $x \in \mathbb{R}$ and some constants $0 < c < C$. Moreover, given the square integrability of $\sigma(Y_s)$, the solution of the differential equation (1) is given by

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds + \int_0^t \sigma(Y_s)dB_s\right), \tag{3}$$

which yields that S_t is continuous. Hence, the product $\sigma(Y_s)S_t$ is square integrable: $\int_0^T \sigma^2(Y_t)S_t^2 dt < \infty$ a.s.

The unique solution of the Langevin equation (2) Y_t is the so-called Ornstein–Uhlenbeck (OU) process. Its properties make it a suitable tool for modeling the volatility in financial markets. One of the most important features is the mean-reversion property. The OU process is Gaussian with the following characteristics:

$$E[Y_t] = Y_0 \,\mathrm{e}^{-\alpha t}, \qquad \mathrm{Var}[Y_t] = \frac{k^2}{2\alpha}\left(1 - \mathrm{e}^{-2\alpha t}\right).$$

Moreover, the OU process is Markov and admits the explicit representation

$$Y_t = Y_0 \, \mathrm{e}^{-\alpha t} + k \int_0^t \mathrm{e}^{-\alpha(t-s)} \, dW_s.$$

We can represent the process W in the form

$$W_t = \rho B_t + \sqrt{1-\rho^2} Z_t,$$

where Z is a Wiener process independent of B. In what follows, we will use this representation. Notice that $\mathcal{F}^{(B,W)} = \mathcal{F}^{(B,Z)}$, where the filtration $\{\mathcal{F}_t^{(B,Z)}, 0 \le t \le T\}$ is generated by independent Wiener processes B and Z.

3 Definitions and preliminary results

Most of the information presented in this section can be found in more detail in [1, 15] (and other references below).

We consider the market with one risky asset and one risk-free asset. Evolutions of prices of both assets are given by a semimartingale process $(\hat{S}_t)_{t=0}^T$ and deterministic process $(B_t)_{t=0}^T = \mathrm{e}^{rt}$, respectively, where r is a constant risk-free rate of return. We introduce the discounted price process $(S_t)_{t=0}^T = \mathrm{e}^{-rt} \hat{S}_t$.

Agents acting in the market may buy or sell risky asset and make their decisions concerning the structure of their portfolios basing upon the information available at the moment of decision. This principle can be formalized by the following definition.

Definition 3.1. A trading strategy is a predictable process $(\pi_t)_{t=0}^T$. The value π_t of this process represents the amount of asset $\hat{S}$ in a portfolio at time t.

Certain amount of preliminary concepts is necessary in order to introduce the essential notion of admissible self-financing strategy. Let a semimartingale S admit the decomposition $S = S_0 + A + M$, where A is a bounded-variation process, and M is a local martingale. According, for example, to [15], p. 635, there is a nondecreasing adapted (to the filtration $(\mathcal{F}_t)_{t\ge 0}$) process $C = (C_t)_{t\ge 0}$, $C_0 = 0$, and adapted processes $c = (c_t)_{t\ge 0}$ and $\hat{c} = (\hat{c}_t)_{t\ge 0}$ such that

$$A_t = \int_0^t c_s dC_s, \quad t > 0,$$

and the quadratic variation equals

$$[M, M]_t = \int_0^t \hat{c}_s dC_s.$$

Definition 3.2. Let π be a predictable process. We shall say that:

- $\pi \in L_{var}(A)$ if for all $\omega \in \Omega$, we have $\int_0^t \pi_s c_s dC_s < \infty$, $t > 0$;
- $\pi \in L_{loc}^q(M)$, $q \ge 1$, if there exists a sequence of stopping times τ_n approaching ∞ as $n \to \infty$ such that

$$\mathbb{E}\left[\int_0^{\tau_n} \pi^2 \hat{c}_s dC_s\right]^{q/2} < \infty;$$

- $\pi \in L^q(S)$ if there exists a representation $S = S_0 + A + M$ such that $\pi \in L_{var}(A) \cap L^q_{loc}(M)$.

Definition 3.3. A trading strategy is called admissible (relative to the price process S) if $\pi \in L^1(S)$.

Definition 3.4. An admissible strategy is said to be self-financing (relative to the price process S) or, equivalently, $\pi \in SF(S)$ if its value $S^\pi_t = \pi_t S_t$ has a representation $S^\pi_t = S^\pi_0 + \int_0^t \pi_s dS_s$.

Further, we define two particular classes of trading strategies along with the corresponding classes of $\mathcal{F}_T$-measurable pay-off functions $\psi = \psi(\omega)$ that can be majorized by returns of strategies belonging to each class.

Definition 3.5. For each $a \geq 0$, define

$$\Pi_a(S) = \left\{\pi \in SF(S) : S^\pi_t \geq -a,\ t \in [0, T]\right\}$$

and

$$\Psi_+(S) = \left\{\psi \in L_\infty(\Omega, \mathcal{F}_T, \mathbb{P}) : \psi \leq \int_0^T (\pi_s, dS_s) \text{ for some } \pi \in \Pi_+(S)\right\},$$

where $\Pi_+(S) = \bigcup_{a\geq 0} \Pi_a(S)$.

Definition 3.6. Let $g(S_t) = g^0 + g^1 S_t$, $g_0 \geq 0$, $g_1 \geq 0$. Define

$$\Pi_g(S) = \left\{\pi \in SF(S) : S^\pi_t \geq -g(S_t),\ t \in [0, T]\right\}$$

and

$$\Psi_g(S) = \left\{\psi \in L_g(\Omega, \mathcal{F}_T, \mathbb{P}) : \psi \leq \int_0^T (\pi_s, dS_s) \text{ for some } \pi \in \Pi_g(S)\right\},$$

where $L_g(\Omega, \mathcal{F}_T, \mathbb{P})$ is the set of $\mathcal{F}_T$-measurable random variables ψ such that $|\psi| \leq g(S_T)$.

We denote the closures of the sets $\Psi_+(S)$ and $\Psi_g(S)$ with respect to norms $\|\cdot\|_\infty$ and $\|\cdot\|_g$ (see [15], p. 648, for definitions of these norms) by $\overline{\Psi}_+(S)$ and $\overline{\Psi}_g(S)$, respectively.

Now following the notation presented in [15], we proceed to the main definitions of absence of arbitrage.

Definition 3.7. We say that the property $\overline{NA}_+$ (or equivalently that the market is $\overline{NA}_+$) holds if

$$\overline{\Psi}_+(S) \cap L^+_\infty(\Omega, \mathcal{F}_T, \mathbb{P}) = \{0\},$$

where $L^+_\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ is the subset of nonnegative random variables in $L_\infty(\Omega, \mathcal{F}_T, \mathbb{P})$.

Definition 3.8. We say that the property $\overline{NA}_g$ holds (or equivalently that the market is $\overline{NA}_g$) if

$$\overline{\Psi}_g(S) \cap L^+_\infty(\Omega, \mathcal{F}_T, \mathbb{P}) = \{0\}.$$

There are two theorems that establish necessary and sufficient conditions for the absence of arbitrage in the market in terms of equivalent (local) martingale measures. An important condition that will be addressed further is the local boundedness of the price process.

Definition 3.9. A probability measure $\mathbb{Q}$, which is equivalent to the objective measure $\mathbb{P}$, is called an equivalent (local) martingale measure if the discounted price process is a (local) martingale under the measure $\mathbb{Q}$.

Definition 3.10. A stochastic process S is called locally bounded if there exists a sequence $(\tau_n)_{n=1}^{\infty}$ of stopping times increasing a.s. to $+\infty$ and such that the stopped processes $S_t^{\tau_n} = S_{t\wedge\tau_n}$ are uniformly bounded for each $n \in \mathbb{N}$.

Theorem 3.1. ([15]) *Let a semimartingale S be locally bounded. Then the market is $\overline{NA}_+$ if and only if there exists an equivalent local martingale measure* (*ELMM*)*.*

Theorem 3.2. ([15]) *Let a semimartingale S be locally bounded. Then the market is $\overline{NA}_g$ if and only if there exists an equivalent martingale measure* (*EMM*)*.*

The following theorem is a corollary of Proposition 6.1 from [5] and defines the construction of ELMM in the model (1)–(2).

Theorem 3.3. *A probability measure $\mathbb{Q}$, which is equivalent to the objective measure $\mathbb{P}$ on $\mathcal{F}_T$, is an ELMM for the process S defined by the model (1)–(2) on $\mathcal{F}_T$ if and only if there exists a progressively measurable process $\nu = (\nu_t)_{0\le t\le T}$, $\int_0^T \nu_s^2 ds < \infty$ $\mathbb{P}$-a.s., such that the local martingale $(L_t)_{0\le t\le T}$ defined by*

$$
\begin{aligned}
L_t = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = \exp\biggl(&\int_0^t (r-\mu)/\sigma(Y_s)dB_s + \int_0^t \nu_s dZ_s \\
&- \frac{1}{2}\int_0^t \bigl((r-\mu)^2/\sigma^2(Y_s) + \nu_s^2\bigr)ds\biggr)
\end{aligned} \tag{4}
$$

satisfies $\mathbb{E}L_T = 1$.

Denote by $\mathcal{LM}^S(\mathbb{P})$ and $\mathcal{M}^S(\mathbb{P})$ the sets of ELMM and EMM in the market modeled by (1)–(2). It is obvious that $\mathcal{M}^S(\mathbb{P}) \subset \mathcal{LM}^S(\mathbb{P})$.

Recall that there is a decomposition of a $\mathbb{P}$-semimartingale S into the sum of a local $\mathbb{P}$-martingale M and an adapted finite-variation process A: $S = S_0 + M + A$.

Definition 3.11. A probability measure $\mathbb{Q}$, which is equivalent to the objective measure $\mathbb{P}$, is called a minimal martingale measure (MMM) if $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_0$ and any square-integrable $\mathbb{P}$-martingale strictly orthogonal to the process M is a local $\mathbb{Q}$-martingale.

A minimal martingale measure is unique (see [2]).

4 Absence of arbitrage in the general model

In this section, we investigate the absence of arbitrage in the model (1)–(2). Notice that we further deal with an undiscounted process S defined in Section 2.

Theorem 4.1. *The market defined by the model (1)–(2) with assumptions (A1)–(A3):*

(i) *satisfies $\overline{NA}_+$ property;*

(ii) *satisfies $\overline{NA}_g$ property, provided that for some ELMM $\mathbb{Q}$,*

$$\mathbb{E}^{\mathbb{Q}} \int_0^T \sigma^2(Y_s) X_s^2 ds < \infty. \tag{5}$$

Proof. (i) Since S is locally bounded due to its continuity, Theorem 3.1 yields that in order to prove the first part of the theorem, it suffices to show that $\mathcal{LM}^S(\mathbb{P}) \neq \emptyset$.

Consider the process L_t defined by (4) with $\nu = (\nu_t)_{0 \le t \le T} = 0$. Let $L_T = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_T}$. In view of Theorem 3.3, it suffices to show that, under such a choice of ν, we have

$$\mathbb{E} L_T = 1. \tag{6}$$

It suffices to verify the Novikov condition

$$\mathbb{E} \exp\left(\frac{1}{2} \int_0^T \left((r-\mu)^2/\sigma^2(Y_s) + \nu_s^2 \right) ds \right) < \infty. \tag{7}$$

It follows from the boundedness away from zero of the function σ (assumption (A2)) and our choice of ν that inequality (7) holds. Hence, $\mathbb{Q} \subset \mathcal{LM}^S(\mathbb{P})$, which proves part (i) of the theorem.

(ii) Now let us show that the measure $\mathbb{Q}$ from (i) is an EMM. Denote $\alpha(s) := (r-\mu)/\sigma(Y_s)$. Knowing that the measure $\mathbb{Q}$ is equivalent to the measure $\mathbb{P}$ and is defined by the Radon–Nikodym derivative $L_T = \frac{d\mathbb{Q}}{d\mathbb{P}}|_T$, we may apply the Girsanov theorem to derive that the processes $B_t^{\mathbb{Q}} := B_t - \int_0^t \alpha(s) ds$, $Z_t^{\mathbb{Q}} := Z_t$, $0 \le t \le T$, are Wiener processes w.r.t. $\mathbb{Q}$. The asset price process under the measure $\mathbb{Q}$ is a solution of the stochastic differential equation

$$dS_t = rS_t dt + \sigma(Y_t) S_t dB_t^{\mathbb{Q}},$$

which yields the following representation for the discounted price process $X_t := \mathrm{e}^{-rt} S_t$:

$$X_t = S + \int_0^t \sigma(Y_s) X_s dB_s^{\mathbb{Q}}. \tag{8}$$

Hence, provided that assumption (A2), as mentioned before, yields the square integrability of $\sigma(Y_s)X_s$ on $[0, T]$, X_t is a martingale. Therefore, $\mathbb{Q} \subset \mathcal{M}^S(\mathbb{P})$, and by Theorem 3.2 we deduce that the market (S, Y) is $\overline{NA}_g$. □

In order to prove $\overline{NA}_g$ and $\overline{NA}_+$ properties of the market, we have proved the existence of one EMM. However, we actually have a family of equivalent martingale measures in the market. The discounted price process is of the form (8) and thus is always a martingale given the square integrability of $\sigma(Y_s)X_s$ and any *admissible* choice of process ν.

Lemma 4.1. *Let the market defined by (1)–(2) with assumptions (A1)–(A3) and additional condition (5), and let the measure* $\mathbb{Q}$ *be such that*

$$\mathbb{E}^{\mathbb{Q}} \exp\left(\frac{1}{2}\int_0^T \nu_s^2 ds\right) < \infty.$$

Then $\mathbb{Q} \subset \mathcal{M}^S(\mathbb{P})$.

Since we have more than one equivalent martingale measure in the market, it is straightforward that the market is incomplete. Each EMM in the market is defined by the process $\nu(s) = \nu(s, Y_s, S_s)$ associated with it. In the financial literature, the process ν_s is called the market price of volatility risk.

Under the EMM $\mathbb{Q}^\nu$, the pair of processes (S_t, Y_t) has the following representation:

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t)S_t dB_t^{\mathbb{Q}}, \\ dY_t &= \left(-\alpha Y_t - k\left(\rho\frac{\mu - r}{\sigma(Y_t)} + \nu(t)\sqrt{1-\rho^2}\right)\right)dt + kdW_t^{\mathbb{Q}}, \end{aligned} \tag{9}$$

where the processes

$$\begin{aligned} B_t^{\mathbb{Q}} &= B_t + \int_0^t \frac{\mu - r}{\sigma(Y_s)} ds, \\ W_t^{\mathbb{Q}} &= \rho B_t^{\mathbb{Q}} + \sqrt{1-\rho^2} Z_t^{\mathbb{Q}}, \quad \text{and} \\ Z_t^{\mathbb{Q}} &= Z_t + \int_0^t \nu(s) ds \end{aligned}$$

are Wiener processes w.r.t. $\mathbb{Q}$ according to the Girsanov theorem, and $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are independent.

In the risk-neutral model (9), the volatility process is not the Ornstein–Uhlenbeck process anymore. So, generally speaking, there is no analytic solution to the corresponding differential equation. Therefore, we further consider a particular case of the general model, which is defined by a set of assumptions concerning the form and behavior of certain parameters of the model.

5 Case of uncorrelated processes

Let us define a modified set of assumptions:

(B1) The Wiener processes B and W are independent, that is, $\rho = 0$;

(B2) = (A2);

(B3) = (A3).

Assumption (B1) simplifies the risk-neutral model to the following form:

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t)S_t dB_t^{\mathbb{Q}}, \\ dY_t &= \big(-\alpha Y_t - k\nu(t)\big)dt + kdZ_t^{\mathbb{Q}}, \end{aligned} \tag{10}$$

where

$$B_t^{\mathbb{Q}} = B_t + \int_0^t \frac{\mu - r}{\sigma(Y_s)} ds \quad \text{and}$$
$$Z_t^{\mathbb{Q}} = Z_t + \int_0^t \nu(s) ds$$

are independent Wiener processes w.r.t. $\mathbb{Q}$.

Our purpose is to price a European call option in the model (10). We limit further investigation to the valuation w.r.t. the minimal martingale measure.

Theorem 5.1. *The EMM $\mathbb{Q}$ in the market defined by the model (10) is minimal iff the process ν corresponding to $\mathbb{Q}$ is identically zero.*

Proof. Suppose $\nu(t) = 0$, $t \in [0, T]$. Let B and $B^{\mathbb{Q}}$ be $\mathcal{F}_t$-adapted Wiener processes w.r.t. measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. If N_t is a square-integrable $\mathbb{P}$-martingale, then we can apply the Kunita–Watanabe decomposition to derive

$$N_t = N_0 + \int_0^t l_u dB_u + \int_0^t l_u^{\mathbb{Q}} dB_u^{\mathbb{Q}} + Z_t,$$

where

$$\langle B, Z \rangle = \langle B^{\mathbb{Q}}, Z \rangle = 0.$$

Let N be strictly orthogonal to $\int_0^t \sigma(u) dB_u$. Then

$$0 = \left\langle N, \int_0^{\cdot} \sigma_u dB_u \right\rangle_t = \int_0^t l_u \sigma_u du,$$

where $l_t = 0$, $t \in [0, T]$, a.s.. Then for $L_t = d\mathbb{Q}/d\mathbb{P}|_t$ and $\gamma_t := (r - \mu)/\sigma_t$,

$$\begin{aligned} d(N_t L_t) &= N_t dL_t + L_t dN_t + d\langle N, L \rangle_t \\ &= N_t dL_t + L_t dN_t + \gamma_t l_t dt \\ &= N_t dL_t + L_t dN_t. \end{aligned}$$

The process $N_t L_t$ is a local $\mathbb{P}$-martingale; hence, N_t is a local $\mathbb{Q}$-martingale. By definition $\mathbb{Q}$ is MMM.

The converse statement of the theorem comes straightforward from the uniqueness of MMM. $\square$

The solution of the differential equation defining the evolution of the price of asset has the following representation:

$$S_t = S \exp\left\{ rt + \int_0^t \sigma(Y_s) dB_s^{\mathbb{Q}} - \frac{1}{2} \int_0^t \sigma^2(Y_s) ds \right\}, \quad 0 \le t \le T. \tag{11}$$

For a fixed trajectory of Y_s, the argument of the exponential function in the right-hand side of (11) is a Gaussian process, and S_t, $1 \le t \le T$, has the log-normal distribution with $\ln S_t \sim N(\ln S + (r - \frac{1}{2}\bar{\sigma}_t^2)t, \bar{\sigma}_t^2 t)$, where $\hat{\sigma}_t^2 = \hat{\sigma}_t^2(Y_s) := \frac{1}{t} \int_0^t \sigma^2(Y_s) ds$,

$0 \le t \le T$. This fact is crucial for the derivation of expression for the value of European call option.

The value of European call option at time 0 w.r.t. the MMM is defined by the general formula

$$V_0 = \mathrm{e}^{-rT} E^{\mathbb{Q}}\big(S_T^{\mathbb{Q}} - K\big)^+.$$

We apply the telescopic property of mathematical expectation to transform the previous expression as follows:

$$V_0 = \mathrm{e}^{-rT} \mathbb{E}^{\mathbb{Q}}\big\{\mathbb{E}^{\mathbb{Q}}\big\{\big(S_T^{\mathbb{Q}} - K\big)^+ \big| Y_s, 0 \le s \le T\big\}\big\}. \tag{12}$$

The inner expectation is conditional on the path of Y_s, $0 \le s \le T$, and therefore, it is actually the Black–Scholes price for a model with deterministic time-dependent volatility. According to Lemma 2.1 in [9], the inner expectation in (12) has the following representation:

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\big\{\big(S_T^{\mathbb{Q}} - K\big)^+ | Y_s, 0 \le s \le T\big\} \\ &\quad = \mathrm{e}^{\ln S + rT} \varPhi\left(\frac{\ln S + (r + \frac{1}{2}\bar{\sigma}_0^2)T - \ln K}{\bar{\sigma}_0\sqrt{T}}\right) \\ &\qquad - K\varPhi\left(\frac{\ln S + (r - \frac{1}{2}\bar{\sigma}_0^2)T - \ln K}{\bar{\sigma}_0\sqrt{T}}\right), \end{aligned} \tag{13}$$

where $\bar{\sigma}_t := \sqrt{\frac{1}{T}\int_t^T \sigma^2(Y_s)ds} \ge 0$, and $\varPhi$ is the standard normal distribution function. Notice that $\bar{\sigma}_0^2(Y_s) = \hat{\sigma}_T^2(Y_s)$. The former notation may be viewed as the volatility averaged from the current moment to maturity, whereas the latter is the volatility averaged from the initial moment to the current one.

Notice that the inner conditional expectation is an increasing function of $\bar{\sigma}_0^2$ (see Lemma 3.1 in [9]), which is the type of behavior one may expect to be exhibited by the Black–Scholes price of European call option.

Taking into account the form of inner integral, in order to derive an analytic expression for the price of an option V_0, it is necessary to deal with expectation of $\varPhi$. Instead of trying to evaluate the integral analytically, it is possible to use the Monte Carlo method.

6 Derivation of analytic expression for the option price

From Eqs. (12)–(13) we can see that in order to derive the formula for the option price, it is necessary to present the exact formula for the expectation of $\varPhi$. In this section, we apply the inverse Fourier transform after rearranging of the right-hand side of (13).

We introduce the following deterministic functions $\sigma_i = \sigma_i(s)$, $i = \overline{1, 4}$:

$$\sigma_{1,2}(s) = \frac{s}{\sqrt{T}} \mp \frac{\sqrt{s^2 T - 2T(\ln(S/K) + rT)}}{T}, \tag{14}$$

$$\sigma_{3,4}(s) = \frac{-s}{\sqrt{T}} \mp \frac{\sqrt{s^2 T + 2T(\ln(S/K) + rT)}}{T}. \tag{15}$$

We define the domains of each of these functions to guarantee the nonnegativity of the expressions under square root, that is, $s^2 T \geq 2T(\ln(S/K) + rT$ for σ_1, σ_2, and $s^2 T \geq -2T(\ln(S/K) + rT$ for σ_3, σ_4.

Lemma 6.1. *Suppose that the market is defined by the model (10) with assumptions (B1)–(B3), $\mathbb{Q}$ is MMM, and V_0 is the price of European call option at time 0. Then we have the following representations:*

1) *for* $\ln(S/K) + rT \geq 0$ *and* $k = \sqrt{2(\ln(S/K) + rT)}$,

$$\begin{aligned} V_0 = S\,\mathrm{e}^{rT}\bigg(\varPhi(k) + \frac{1}{\sqrt{2\pi}}\int_k^\infty \big(\mathbb{Q}\big(\bar{\sigma}_0 < \sigma_1(s)\big) + \mathbb{Q}\big(\bar{\sigma}_0 > \sigma_2(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg) \\ - K\bigg(\varPhi(0) + \frac{1}{\sqrt{2\pi}}\bigg(\int_0^\infty \mathbb{Q}\big(\bar{\sigma}_0 < \sigma_4(s)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ - \int_{-\infty}^0 \mathbb{Q}\big(\bar{\sigma}_0 > \sigma_4(s)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg)\bigg); \end{aligned} \tag{16}$$

2) *for* $\ln(S/K) + rT < 0$ *and* $l = \sqrt{-2(\ln(S/K) + rT)}$,

$$\begin{aligned} V_0 = S\,\mathrm{e}^{rT}\bigg(\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\bigg(\int_0^\infty \mathbb{Q}\big(\bar{\sigma}_0 > \sigma_2(s)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ - \int_{-\infty}^0 \mathbb{Q}\big(\bar{\sigma}_0 < \sigma_2(s)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg)\bigg) \\ - K\bigg(\varPhi(-l) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-l} \big(\mathbb{Q}\big(\bar{\sigma}_0 < \sigma_3(s)\big) \\ + \mathbb{Q}\big(\bar{\sigma}_0 > \sigma_4(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg). \end{aligned} \tag{17}$$

Proof. From (12) and (13) we have:

$$V_0 = S\,\mathrm{e}^{rT}\,\mathbb{E}^{\mathbb{Q}}\big(\varPhi(d_1)\big) - K\mathbb{E}^{\mathbb{Q}}\big(\varPhi(d_2)\big),$$

where d_1 and d_2 are defined as follows:

$$d_1 = \frac{\ln S + (r + \frac{1}{2}\bar{\sigma}_0^2)T - \ln K}{\bar{\sigma}_0\sqrt{T}}, \qquad d_2 = d_1 - \bar{\sigma}_0\sqrt{T}. \tag{18}$$

Since

$$\begin{aligned} \varPhi(d_1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \mathrm{e}^{-s^2/2}\,ds \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,\mathrm{I}_{\{d_1>0\}}\int_0^{d_1} \mathrm{e}^{-s^2/2}\,ds - \frac{1}{\sqrt{2\pi}}\,\mathrm{I}_{\{d_1<0\}}\int_{d_1}^0 \mathrm{e}^{-s^2/2}\,ds, \end{aligned} \tag{19}$$

we have

$$\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_1)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \mathbb{Q}(s < d_1)\, \mathrm{e}^{-s^2/2}\, ds - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathbb{Q}(s > d_1)\, \mathrm{e}^{-s^2/2}\, ds.$$

The probabilities in the integrands may be represented as follows:

$$\mathbb{Q}(s < d_1) = \mathbb{Q}\bigg(\frac{1}{2}\bar{\sigma}_0^2 T - s\bar{\sigma}_0\sqrt{T} + \ln(S/K) + rT > 0\bigg),$$
$$\mathbb{Q}(s > d_1) = \mathbb{Q}\bigg(\frac{1}{2}\bar{\sigma}_0^2 T - s\bar{\sigma}_0\sqrt{T} + \ln(S/K) + rT < 0\bigg).$$

Similarly, for $\Phi(d_2)$, we have

$$\begin{aligned}\Phi(d_2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \mathrm{e}^{-s^2/2}\, ds \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\, \mathrm{I}_{\{d_2>0\}} \int_0^{d_2} \mathrm{e}^{-s^2/2}\, ds - \frac{1}{\sqrt{2\pi}}\, \mathrm{I}_{\{d_2<0\}} \int_{d_2}^{0} \mathrm{e}^{-s^2/2}\, ds.\end{aligned}$$

Hence,

$$\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_2)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \mathbb{Q}(s < d_2)\, \mathrm{e}^{-s^2/2}\, ds - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathbb{Q}(s > d_2)\, \mathrm{e}^{-s^2/2}\, ds.$$

The probabilities from the integrands may be represented as follows:

$$\mathbb{Q}(s < d_2) = \mathbb{Q}\bigg(\frac{1}{2}\bar{\sigma}_0^2 T + s\bar{\sigma}_0\sqrt{T} - \ln(S/K) - rT < 0\bigg),$$
$$\mathbb{Q}(s > d_2) = \mathbb{Q}\bigg(\frac{1}{2}\bar{\sigma}_0^2 T + s\bar{\sigma}_0\sqrt{T} - \ln(S/K) - rT > 0\bigg).$$

Solutions of the quadratic equations, which correspond to the above quadratic inequalities, do not necessarily exist; therefore, we consider different cases:

1) The discriminant $D_{12} := s^2T - 2T(\ln(S/K) + rT)$ is a quadratic form w.r.t. s. There are two possibilities:

   1.1) $\ln(S/K) + rT > 0$. Then for $k = \sqrt{2(\ln(S/K) + rT)}$: $D_{12} < 0$, $s \in (-k; k)$; $D_{12} > 0$, $s \in (-\infty; -k) \cup (k; \infty)$; so

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_1)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\bigg(\int_0^{k} \mathrm{e}^{-s^2/2}\, ds + \int_k^{\infty} \big(\mathbb{Q}\big(\bar{\sigma}_0 < \sigma_1(s)\big) \\ + \mathbb{Q}\big(\bar{\sigma}_0 > \sigma_2(s)\big)\big)\, \mathrm{e}^{-s^2/2}\, ds\bigg) \\ - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-k} \big(\mathbb{Q}\big(\bar{\sigma}_0 < \sigma_2(s)\big) - \mathbb{Q}\big(\bar{\sigma}_0 < \sigma_1(s)\big)\big)\, \mathrm{e}^{-s^2/2}\, ds.\end{aligned}$$

1.2) $\ln(S/K) + rT \le 0$. Then for any $s \in (-\infty;\infty)$, $D_{12} > 0$, So

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_1)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty &\big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_1(s)\big) \\ &+ \mathbb{Q}\big(\bar\sigma_0 > \sigma_2(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ &- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_2(s)\big) - \mathbb{Q}\big(\bar\sigma_0 < \sigma_1(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds.\end{aligned}$$

2) The discriminant $D_{34} := s^2T + 2T(\ln(S/K) + rT)$ is a quadratic form w.r.t. s. There are two possibilities:

2.1) $\ln(S/K) + rT < 0$. Then for $l = \sqrt{-2(\ln(S/K) + rT)}$, $D_{34} < 0$, $s \in (-l;l)$; $D_{34} > 0$, $s \in (-\infty;-l) \cup (l;\infty)$. So

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_2)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_l^\infty &\big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_4(s)\big) \\ &- \mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds - \frac{1}{\sqrt{2\pi}}\bigg(\int_{-l}^0 \mathrm{e}^{-s^2/2}\,ds \\ &+ \int_{-\infty}^{-l} \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big) + \mathbb{Q}\big(\bar\sigma_0 > \sigma_4(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg).\end{aligned}$$

2.2) $\ln(S/K) + rT \ge 0$. Then for any $s \in (-\infty;\infty)$, $D_{34} > 0$. So

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(\Phi(d_2)\big) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty &\big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_4(s)\big) \\ &- \mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ &- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big) + \mathbb{Q}\big(\bar\sigma_0 > \sigma_4(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds.\end{aligned}$$

Combining these cases, we get the following expressions for the option price:

1) for $\ln(S/K) + rT \ge 0$,

$$\begin{aligned}V_0 = S\,\mathrm{e}^{rT}\bigg(\Phi(k) + \frac{1}{\sqrt{2\pi}}\bigg(&\int_k^\infty \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_1(s)\big) \\ &+ \mathbb{Q}\big(\bar\sigma_0 > \sigma_2(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ &- \int_{-\infty}^{-k} \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_2(s)\big) - \mathbb{Q}\big(\bar\sigma_0 < \sigma_1(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg)\bigg) \\ &- K\bigg(\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\bigg(\int_0^\infty \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_4(s)\big) - \mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds \\ &- \int_{-\infty}^0 \big(\mathbb{Q}\big(\bar\sigma_0 < \sigma_3(s)\big) + \mathbb{Q}\big(\bar\sigma_0 > \sigma_4(s)\big)\big)\,\mathrm{e}^{-s^2/2}\,ds\bigg)\bigg); \qquad (20)\end{aligned}$$

2) for $\ln(S/K) + rT < 0$,

$$
\begin{aligned}
V_0 = S\,\mathrm{e}^{rT}\biggl(\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\biggl(\int_0^{\infty}\bigl(\mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_1(s)\bigr) \\
+ \mathbb{Q}\bigl(\bar{\sigma}_0 > \sigma_2(s)\bigr)\bigr)\,\mathrm{e}^{-s^2/2}\,ds \\
- \int_{-\infty}^{0}\bigl(\mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_2(s)\bigr) - \mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_1(s)\bigr)\bigr)\,\mathrm{e}^{-s^2/2}\,ds\biggr)\biggr) \\
- K\biggl(\Phi(-l) + \frac{1}{\sqrt{2\pi}}\biggl(\int_l^{\infty}\bigl(\mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_4(s)\bigr) \\
- \mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_3(s)\bigr)\bigr)\,\mathrm{e}^{-s^2/2}\,ds \\
- \int_{-\infty}^{-l}\bigl(\mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_3(s)\bigr) + \mathbb{Q}\bigl(\bar{\sigma}_0 > \sigma_4(s)\bigr)\bigr)\,\mathrm{e}^{-s^2/2}\,ds\biggr)\biggr). \qquad (21)
\end{aligned}
$$

Recalling that $\bar{\sigma}_0 \geq 0$ and noticing that some of the probabilities presented are identically zero, we simplify (20) and (21) to the forms (16) and (17), respectively. □

Let $S_i \subset \mathbb{R}$ be the domains of positivity of functions $\sigma_i(s)$, $i = \overline{1,4}$. It is easy to check that the functions appearing in the integrals (16)–(17) are positive on the integration domains.

Assume that the probability density function of $\bar{\sigma}_0^2$ is piecewise continuous on $\mathbb{R}$. Then due to the Fourier inversion theorem, for almost all $s \in S_i$, the probabilities in the integrands in (16)–(17) have the following representation:

$$
\begin{aligned}
\mathbb{Q}\bigl(\bar{\sigma}_0 < \sigma_i(s)\bigr) &= \mathbb{Q}\bigl(\bar{\sigma}_0^2 < \sigma_i^2(s)\bigr) \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\sigma_i^2(s)}\biggl(\int_{-\infty}^{\infty}\exp\biggl(iyu - \frac{\varepsilon^2 u^2}{2}\biggr)\phi(u)du\biggr)dy, \qquad (22)
\end{aligned}
$$

where $\phi(u) = \mathbb{E}^{\mathbb{Q}}(\mathrm{e}^{iu\bar{\sigma}_0^2})$ is the characteristic function of $\bar{\sigma}_0^2$.

We are now in a position to state the main result of this section.

Theorem 6.1. *Suppose that the market is defined by the model (10) with assumptions (B1)–(B3), $\mathbb{Q}$ is the MMM, and V_0 is the price at time 0 of European call option. Let the probability density function of $\bar{\sigma}_0^2$ be piecewise continuous on $\mathbb{R}$. Then we have the following representations:*

1) for $\ln(S/K) + rT \geq 0$ *and* $k = \sqrt{2(\ln(S/K) + rT)}$,

$$
\begin{aligned}
V_0 = \lim_{\varepsilon\to 0}\biggl(S\,\mathrm{e}^{rT}\biggl(\Phi(k) + \frac{1}{(2\pi)^{3/2}} \\
\times\biggl(\int_k^{\infty}\biggl(\int_{-\infty}^{\sigma_1^2(s)}\int_{-\infty}^{\infty}\exp\biggl(iyu - \frac{\varepsilon^2 u^2}{2}\biggr)\phi(u)du dy \\
+ \int_{\sigma_2^2(s)}^{\infty}\int_{-\infty}^{\infty}\exp\biggl(iyu - \frac{\varepsilon^2 u^2}{2}\biggr)\phi(u)du dy\biggr)\,\mathrm{e}^{-s^2/2}\,ds\biggr)\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&- K\Bigg(\frac{1}{2} + \frac{1}{(2\pi)^{3/2}} \\
&\times \bigg(\int_0^\infty \int_{-\infty}^{\sigma_4^2(s)} \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy\, \mathrm{e}^{-s^2/2}\, ds \\
&+ \int_{-\infty}^0 \int_{\sigma_4^2(s)}^\infty \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy\, \mathrm{e}^{-s^2/2}\, ds\bigg)\Bigg)\Bigg);
\end{aligned}
$$

2) for $\ln(S/K) + rT < 0$ *and* $l = \sqrt{-2(\ln(S/K) + rT)}$,

$$
\begin{aligned}
V_0 = \lim_{\varepsilon \to 0} \Bigg(& S\, \mathrm{e}^{rT}\Bigg(\frac{1}{2} + \frac{1}{(2\pi)^{3/2}} \\
&\times \bigg(\int_0^\infty \int_{\sigma_2^2(s)}^\infty \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy\, \mathrm{e}^{-s^2/2}\, ds \\
&- \int_{-\infty}^0 \int_{-\infty}^{\sigma_2^2(s)} \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy\, \mathrm{e}^{-s^2/2}\, ds\bigg)\Bigg) \\
&- K\Bigg(\Phi(-l) - \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{-l} \bigg(\int_{-\infty}^{\sigma_3^2(s)} \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy \\
&+ \int_{\sigma_4^2(s)}^\infty \int_{-\infty}^\infty \exp\bigg(iyu - \frac{\varepsilon^2 u^2}{2}\bigg)\phi(u)du dy\bigg)\, \mathrm{e}^{-s^2/2}\, ds\Bigg)\Bigg),
\end{aligned}
$$

where $\phi(u) = \mathbb{E}^{\mathbb{Q}}(\mathrm{e}^{iu\bar{\sigma}_0^2})$ *is the characteristic function of the random variable* $\bar{\sigma}_0^2$, *and* $\sigma_i = \sigma_i(s)$, $i = \overline{1,4}$, *are of the form (14)–(15).*

Remark 6.1. If $\bar{\sigma}_0 \in L^2(\mathbb{R})$, then the limit in Theorem 6.1 may be moved inside the integrals. Thus, ϵ may be equated to zero, and the expression for the option price is simplified.

Remark 6.2. Under the assumption that σ is bounded, we can rewrite the analytical expression in terms of moments of $\bar{\sigma}_0^2$.

Indeed, in this case, $\bar{\sigma}_0^2$ is bounded as well, so the characteristic function $\phi(u)$ admits the Taylor series expansion around zero:

$$
\phi(u) = 1 + \sum_{j=1}^{\infty} \frac{i^j u^j}{j!} m_j, \tag{23}
$$

where m_n is the nth moment of the random variable $\bar{\sigma}_0^2$, and $i = \sqrt{-1}$.

The moments of the random variable $\bar{\sigma}_0^2$ can be represented by applying the fact that the finite-dimensional distributions of the Ornstein–Uhlenbeck process are Gaussian vectors. Bearing in mind that the covariance matrix of the process Y_s is nondegenerate and consists of the elements of the form

$$
\big(\Sigma^{i,l}\big)_{i,l=1}^{j} = \frac{k^2}{2\alpha} \exp\big(-\alpha(t_i + t_l)\big)\big(\exp\big(2\alpha \min(t_i, t_l)\big) - 1\big), \tag{24}
$$

we get the following representation for the moments of the random variable $\bar{\sigma}_0^2$:

$$m_j = \frac{1}{T^j} \int_0^T \cdots \int_0^T \int_{\mathbb{R}^j} \frac{\sigma^2(y_1) \dots \sigma^2(y_j)}{(2\pi)^{j/2} |\Sigma|^{1/2}} \, \mathrm{e}^{-\frac{1}{2}(\mathbf{y}-\mu)^\top \Sigma^{-1} (\mathbf{y}-\mu)} \, \mathbf{dy} dt_1 \dots dt_j, \tag{25}$$

where

$$\mathbf{y} = (y_1, \dots, y_j), \qquad \mathbf{dy} = dy_1 \times \cdots \times dy_j, \qquad \mu = (Y_0 \, \mathrm{e}^{-\alpha y_1}, \dots, Y_0 \, \mathrm{e}^{-\alpha y_j}).$$

We have demonstrated that there is an analytic solution to the problem of pricing of European call option in the model. However, the resulting formula is complicated and cumbersome. Therefore, our further investigation will be aimed at comparison of numeric results produced by it with approximate calculations and possible simplifications.

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