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Fractional Lévy-driven SPDEs with discontinuous coefficients

Salem Lamine[a], **Mounir Zili**[b], **Eya Zougar**[c,d,*]

[a]*Research Laboratory LR18ES17-University of Monastir & ESPRIT School of Engineering, 1-2 Rue André Ampère, Pôle Technologique El Ghazala, 2083, Tunisia*
[b]*Research Laboratory LR18ES17-University of Monastir & Tunisian Military Academy, Department of Mathematics, Faculty of Sciences, Monastir, 5000, Tunisia*
[c]*Univ. Polytechnique Hauts-de-France, CERAMATHS - Laboratoire de Matériaux Céramiques et de Mathématiques, Le Mont Houy, Valenciennes, 59313, France*
[d]*Univ. Clermont Auvergne, Laboratoire de Mathématiques Blaise Pascal (LMBP, UMR 6620 CNRS), F-63000 Clermont-Ferrand, France*

Salem.Lamine@hotmail.fr (S. Lamine), Mounir.Zili@fsm.rnu.tn (M. Zili),
Eya.Zougar@uca.fr (E. Zougar)

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Abstract This paper introduces a novel stochastic partial differential equation (SPDE) driven by space-only fractional Lévy noise $\dot{L}_x^d$ with $d \in (0, 1/2)$, governed by a second-order elliptic operator with piecewise constant coefficients. We establish a new framework for analyzing diffusion in heterogeneous media, proving the existence, uniqueness, and equivalence of mild and weak solutions for the resulting non-Gaussian random field. A key original contribution is the explicit characterization of the covariance kernel in terms of transmission-reflection paths induced by the interface. We provide sharp estimates for the sample path regularity, proving that the solution is almost surely Hölder continuous of order $\gamma < d$ in space and $\beta < \frac{2d+1}{4}$ in time. Our results demonstrate that the fractional order of the noise robustly dictates path regularity, which remains preserved despite the singular transmission conditions and the jump structure of the driving noise.

Keywords Stochastic Partial Differential Equations, fractional Lévy noise, discontinuous coefficients, non-Gaussian processes, Hölder continuity, skew-diffusion

*Corresponding author.

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1 Introduction

In recent decades, there has been a surge of interest in Stochastic Partial Differential Equations (SPDEs) due to their pivotal role in modeling various applied fields, ranging from geophysics and ecology to mathematical finance. For a comprehensive overview of the foundational theory and its broad applications, we refer the reader to [5, 11, 22, 26] and the references therein. A significant portion of the existing literature has focused on SPDEs driven by space-time Gaussian noises across various operators, including standard Laplacians as well as more complex operators with variable or discontinuous coefficients (see, e.g., [9, 14, 15, 13, 23, 2, 25, 37, 31, 32, 35]). Alongside the Gaussian framework, numerous works have explored SPDEs driven by space-time Lévy noises, offering a broader framework for modeling phenomena characterized by jumps and heavy tails [16, 21, 27].

A particularly interesting class of operators involves those with discontinuous coefficients, which represent the mathematical counterpart to heat propagation and mass diffusion in heterogeneous media. We focus on the second-order elliptic operator in divergence form:

$$\mathcal{L} = \frac{1}{2\rho(x)} \frac{d}{dx} \left(\rho(x) a(x) \frac{d}{dx} \right), \tag{1}$$

where the diffusion coefficient $a(x)$ and the density function $\rho(x)$ are piecewise constant:

$$a(x) = a_1 \mathbb{1}_{\{x \leqslant 0\}} + a_2 \mathbb{1}_{\{0 < x\}} \quad \text{and} \quad \rho(x) = \rho_1 \mathbb{1}_{\{x \leqslant 0\}} + \rho_2 \mathbb{1}_{\{0 < x\}}. \tag{2}$$

The non-smoothness of these coefficients at the interface $x = 0$ introduces significant analytical challenges, as the classical regularity of the Laplacian is replaced by flux continuity conditions and transmission-reflection phenomena. While the deterministic theory for such operators is well-established [24, 29, 30], their extension to the stochastic regime has primarily been limited to space-time white noise [33] or Gaussian fractional regimes [34–36].

However, increasing attention has been directed toward SPDEs driven by space-only noise, a framework particularly relevant for systems where the primary source of randomness is a fixed, highly irregular spatial environment [1, 3, 4, 10]. While Salem and Zili [18] recently investigated the heat equation driven by space-only fractional Lévy noise, their study was restricted to the standard Laplacian, leaving the case of operators with discontinuous coefficients, which necessitate complex transmission conditions, unexplored.

The current work bridges this gap by introducing a novel framework that couples the singular operator $\mathcal{L}$ with space-only fractional Lévy noise $\dot{L}_x^d$. Fractional Lévy processes generalize fractional Brownian motion by replacing the underlying Gaussian process with a general Lévy process; this preserves the second-order dependence structure inherent to fractional Brownian motion (fBm) while inherently introducing non-Gaussian features, such as heavy tails and jumps, that broaden the modeling scope beyond the Gaussian framework. Although Mishura, Ralchenko, and Zili [14] established a framework for $\mathcal{L}$ under Gaussian space-time white noise, our approach

uniquely accounts for both the non-homogeneous medium and the non-Gaussian jump structure of the noise, extending the study by [1] beyond the standard Laplacian. Specifically, we investigate the following novel SPDE:

$$\begin{cases} \left(\dfrac{\partial u}{\partial t} - \mathcal{L}u\right)(t,x) & = \dot{L}^d_x, \quad t > 0, \\ u(0,x) & = 0, \end{cases} \qquad x \in \mathbb{R}. \tag{3}$$

The primary contribution of this paper lies in providing a rigorous mathematical framework for the SPDE (3). By utilizing the L^2-isometry for Lévy integrals and the stochastic Fubini theorem, we establish the well-posedness of the solution, demonstrate the equivalence between mild and weak formulations, and provide an explicit characterization of the covariance function. This reveals how the second-order structure is affected by the transmission conditions across the interface $x = 0$.

Furthermore, we conduct a comprehensive study of the sample path regularity in both space and time. By employing a meticulous scaling analysis of the fractionalized Green's profile, we prove that the spatial trajectories $x \mapsto u(t,x)$ are almost surely Hölder continuous of order $\gamma < d$. Simultaneously, we demonstrate a significant temporal smoothing effect, proving that the temporal paths $t \mapsto u(t,x)$ exhibit a higher regularity of order $\beta < \frac{2d+1}{4}$. These results show that the spatial regularity is robustly governed by the fractional parameter d, even in the presence of the singular interface and the jump structure of the noise.

The paper is organized as follows: Section 2 provides necessary preliminaries. In Section 3, we define the mild and weak solution concepts and prove their existence and equivalence. Section 4 focuses on the second-order structure and the pathwise Hölder regularity in space and time. Technical results regarding the scaling and properties of the fractional Green's kernel are provided in the Appendix.

2 Preliminaries

Throughout this work, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space on which all stochastic processes are defined, with $\mathbb{E}$ representing the corresponding expectation operator.

Let $L = \{L(t), t \geqslant 0\}$ be a *Lévy process* in $\mathbb{R}$ without a Brownian component, characterized by the Lévy-Khintchine triplet $(\gamma, 0, \nu)$. Here, $\gamma \in \mathbb{R}$ is the drift and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$, the standard integrability condition

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty,$$

and the second-order moment condition:

$$\int_{|x|>1} x^2 \nu(dx) < \infty. \tag{4}$$

Its distribution is uniquely determined by the Lévy-Khintchine formula:

$$\mathbb{E}[\exp\{iuL(t)\}] = \exp\{t\psi(u)\}, \quad u \in \mathbb{R}, \tag{5}$$

where the characteristic exponent $\psi(u)$ is given by:

$$\psi(u) = i\gamma u + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leqslant 1\}}\right) \nu(dx). \tag{6}$$

Under the centering condition $\mathbb{E}[L(1)] = 0$, which is well-defined under (4), the drift and characteristic exponent simplify to:

$$\gamma = -\int_{|x|>1} x\nu(dx) \quad \text{and} \quad \psi(u) = \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \nu(dx). \tag{7}$$

Consequently, $L(t)$ is a centered square-integrable martingale with variance:

$$\mathrm{Var}(L(t)) = t\,\mathcal{C}_L^2, \quad t \geqslant 0; \qquad \mathcal{C}_L^2 := \int_{\mathbb{R}} x^2\nu(dx) < \infty. \tag{8}$$

Having characterized the one-sided process, we extend it to the entire real line. Following [12], we define the *two-sided Lévy process* $L = \{L(t), t \in \mathbb{R}\}$ as:

$$L(t) := L^1(t)\mathbb{1}_{\{t\geqslant 0\}} - L^2((-t)-)\mathbb{1}_{\{t<0\}}, \tag{9}$$

where L^1 and L^2 are independent, identically distributed copies of the one-sided process defined above.

Crucially, under the centering and second-moment conditions (4)-(7), the two-sided process L is a centered, square-integrable martingale with respect to its natural filtration.

While L is uniquely determined by its characteristic exponent (7), its role as a spatial noise is best operationalized through its spectral representation. For any Borel set $B \in \mathcal{B}(\mathbb{R})$, the Lévy–Itô decomposition allows us to represent the process as:

$$L(B) = \int_B \int_{\mathbb{R}\setminus\{0\}} w\,\bar{\mu}(dy, dw), \tag{10}$$

where $\bar{\mu}(dy, dw) = \mu(dy, dw) - dy\nu(dw)$ is the compensated Poisson random measure associated with the jumps of L, and dy denotes the Lebesgue measure on $\mathbb{R}$. This representation provides the explicit random measure structure necessary for the isometric framework and the analytical interchanges used throughout this work.

Remark 1. Consider a stochastic process $U = \{U(x); x \in \mathbb{R}\}$ defined by the integral:

$$U(x) := \int_{\mathbb{R}} f(y, x)dL(y), \tag{11}$$

where $f(\cdot, x) \in L^2(\mathbb{R})$ for each $x \in \mathbb{R}$. According to [12] and [17], this integral exists as an $L^2(\Omega, \mathbb{P})$ limit of approximating step functions. The semimartingale structure of L guarantees that this integral is well-defined and satisfies the following L^2-isometry:

$$\mathbb{E}\left[U^2(x)\right] = \mathcal{C}_L^2 \|f(\cdot, x)\|^2_{L^2(\mathbb{R})}. \tag{12}$$

Furthermore, the law of $U(x)$ is infinitely divisible with characteristic function:

$$\mathbb{E}[\exp(i\theta U(x))] = \exp\left(\int_{\mathbb{R}} \psi(\theta f(y, x))dy\right), \tag{13}$$

where ψ is defined in (7).

In this work, we specifically study the SPDE (3) driven by $\dot{L}^d$, the fractional Lévy noise, which is formally related to the process defined below.

Definition 2.1. For a fractional integration parameter $d \in (0, 1/2)$, the stochastic process $L^d = \{L^d(t), t \in \mathbb{R}\}$ given by the moving average representation

$$L^d(t) := \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d\right] L(ds), \tag{14}$$

is called a *fractional Lévy process (fLp)*, where L is the two-sided driving Lévy process defined previously. Here, $x_+ = \max(x, 0)$ denotes the positive part of x.

Remark 2. The fractional Lévy process $L^d(t)$ defined in (14) is a particular instance of the stochastic integral introduced in Remark 1. Specifically, the kernel

$$f_t(s) = \frac{1}{\Gamma(d+1)} \left[(t-s)_+^d - (-s)_+^d\right], \tag{15}$$

belongs to $L^2(\mathbb{R})$ for each $t \in \mathbb{R}$ when $d \in (0, 1/2)$.

The following proposition summarizes essential properties of the fLp that will be used subsequently.

Proposition 2.2. *Let $L^d = \{L^d(t), t \in \mathbb{R}\}$ be a fractional Lévy Process (fLp) with fractional integration parameter $d \in \left(0, \frac{1}{2}\right)$, driven by the two-sided Lévy process L with finite variance ($\mathbb{E}[L(1)^2] < \infty$) and characteristic exponent ψ. Then, the following properties hold:*

(i) L^d is a centered process ($\mathbb{E}[L^d(t)] = 0$) and possesses a continuous modification given by the improper Riemann integral:

$$L^d(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \left[(t-s)_+^{d-1} - (-s)_+^{d-1}\right] L(s)\, ds, \quad t \in \mathbb{R}. \tag{16}$$

Furthermore, L^d is locally β-Hölder continuous for all $\beta < d$.

(ii) The process L^d is not Gaussian (unless L is a Brownian motion), but the law of $L^d(t)$ is infinitely divisible for all $t \in \mathbb{R}$ (cf. [12]). L^d is also not self-similar.

(iii) L^d has stationary increments, and for $s, t \in \mathbb{R}$, the covariance function is:

$$\mathrm{Cov}(L^d(t), L^d(s)) = \frac{K_d}{2} \left[|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right], \tag{17}$$

where $K_d = \dfrac{\mathbb{E}[L(1)^2]}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}$.

(iv) If L is symmetric, then L^d is symmetric in distribution. The finite-dimensional distributions are given by:

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^n u_j L^d(t_j)\right)\right] = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^n u_j f_{t_j}(s)\right) ds\right\}, \tag{18}$$

where $f_{t_j}(s)$ is defined in (15).

(v) For all $\alpha > d + \frac{1}{2}$, the process satisfies the almost sure growth condition:

$$\lim_{|y|\to\infty} \frac{|L^d(y)|}{|y|^\alpha} = 0, \quad a.s. \tag{19}$$

(vi) Let I_-^d denote the right-sided Riemann-Liouville fractional integral operator defined for a measurable function φ by:

$$I_-^d(\varphi)(y) = \frac{1}{\Gamma(d)} \int_y^{+\infty} \varphi(v)(v-y)^{d-1} dv. \tag{20}$$

We denote by $\mathcal{H}_d$ the space of measurable functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $I_-^d(\varphi) \in L^2(\mathbb{R})$. For any $\varphi \in \mathcal{H}_d$, the following integral identity holds in $L^2(\Omega, \mathbb{P})$:

$$\int_{\mathbb{R}} \varphi(y)\, dL^d(y) = \int_{\mathbb{R}} I_-^d(\varphi)(y)\, dL(y), \tag{21}$$

and the process satisfies the L^2-isometry:

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} \varphi(y) dL^d(y)\right)^2\right] = \mathbb{E}[L(1)^2] \int_{\mathbb{R}} \left|I_-^d(\varphi)(y)\right|^2 dy. \tag{22}$$

For detailed proofs of all the above assertions, we refer the reader to [12] and [7].

Remark 3. For every $x, y \in \mathbb{R}$, the covariance of the fLp L^d is directly proportional to that of a standard fractional Brownian motion (fBm). Specifically:

$$\mathrm{Cov}(L^d(x), L^d(y)) = K_d \cdot \mathrm{Cov}(B_x^{H_d}, B_y^{H_d}), \tag{23}$$

where K_d is the constant defined in (17), and $\{B_t^{H_d}, t \in \mathbb{R}\}$ is a two-sided fBm with Hurst parameter

$$H_d = d + \frac{1}{2} \in (1/2, 1). \tag{24}$$

This relationship highlights that while L^d may possess non-Gaussian distributions, its second-order dependence structure is proportional to that of a fractional Brownian motion.

Henceforth, when referring to the fLp L^d, we will always consider its continuous modification given by (16).

3 Analysis of the existence and characterization of solutions

In this section, we rigorously examine the well-posedness of the stochastic partial differential equation (SPDE) defined in (3) by establishing two key solution concepts: the *mild solution* and the *weak solution*. The mild solution is constructed using the fundamental solution, or Green's function, G of the associated deterministic parabolic

equation $\dfrac{\partial u}{\partial t} = \mathcal{L}u$. For the SPDE driven by the fractional Lévy noise $\dot{L}_x^d$ with zero initial condition, the mild solution is defined by:

$$u(t,x) = \int_0^t \left(\int_{\mathbb{R}} G(t-s,x,y)\, dL^d(y) \right) ds. \tag{25}$$

Establishing the existence of this integral is contingent upon the properties of the fundamental solution G. Following the characterization of the mild solution, we will introduce the concept of the weak solution to establish a rigorous equivalence between the two solution frameworks. Let us first introduce and characterize the main tool used in our study, the fundamental solution G.

3.1 Fundamental solution and properties

One fundamental interest in our study is the explicit expression of the fundamental solution $G(t,x,z)$ for the operator $\mathcal{L}$. This solution, which is central to the analysis of the SPDE, is given by the following proposition, as established in [24].

Proposition 3.1. *There exists a unique fundamental solution $G(t,x,z)$ for the deterministic operator (1). It is explicitly expressed as:*

$$\begin{aligned} G(t,x,z) = \frac{\mathbb{1}_{\{t>0\}}}{\sqrt{2\pi a(z)t}} \Bigg\{ & \exp\left(-\frac{(f(x)-f(z))^2}{2t} \right) \\ & + \beta\ \mathrm{sign}(z) \exp\left(-\frac{(|f(x)|+|f(z)|)^2}{2t} \right) \Bigg\}, \end{aligned} \tag{26}$$

where $a(z)$ is defined in (2), and the parameters are:

$$f(z) = \frac{z}{\sqrt{a(z)}} \quad \text{and} \quad \beta = \frac{\rho_2\sqrt{a_2} - \rho_1\sqrt{a_1}}{\rho_2\sqrt{a_2} + \rho_1\sqrt{a_1}}, \tag{27}$$

with the sign function defined piecewise:

$$\mathrm{sign}(z) = \begin{cases} -1 & \text{if } z \in \mathcal{S}_1 = (-\infty, 0] \\ 1 & \text{if } z \in \mathcal{S}_2 = (0, +\infty). \end{cases}$$

Remark 4. The fundamental solution $G(t,x,y)$ associated with the operator $\mathcal{L}$ satisfies the following essential properties (see, e.g., [15] and the references therein):

1. For $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$, G is a solution to the homogeneous parabolic equation:
$$\frac{\partial G}{\partial t}(t,x,y) = \mathcal{L}_x G(t,x,y).$$
2. The fundamental solution G respects the flux continuity condition across the interface $x = 0$. That is, for $t > 0$ and fixed y:
$$\rho_1 a_1 \frac{\partial G(t,0^-,y)}{\partial x} = \rho_2 a_2 \frac{\partial G(t,0^+,y)}{\partial x}.$$

Throughout the remainder of this work, we denote by $p(t,z)$ the fundamental solution to the standard heat equation (the heat kernel), defined for $t > 0$ and $z \in \mathbb{R}$ by:

$$p(t,z) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right). \tag{28}$$

Lemma 1. *The fundamental solution G associated with the operator $\mathcal{L}$ admits the following representation for $y \neq 0$:*

$$G(t,x,y) = \frac{1}{\sqrt{a(y)}} \left[p(t, A(x,y)) + \beta \operatorname{sign}(y) p(t, B(x,y))\right], \tag{29}$$

where

$$A(x,y) = f(x) - f(y) \quad \text{and} \quad B(x,y) = |f(x)| + |f(y)|. \tag{30}$$

Consequently, the partial derivative of G with respect to the spatial variable y is given by:

$$\frac{\partial G(t,x,y)}{\partial y} = \frac{1}{a(y)} \left[-\frac{\partial p}{\partial z}(t, A(x,y)) + \beta \frac{\partial p}{\partial z}(t, B(x,y))\right]. \tag{31}$$

Proof. The representation (29) for $G(t,x,y)$ follows directly from the expression (26); hence, we focus on proving the identity for its spatial derivative.

By symmetry, it suffices to prove the case $y < 0$, where $a(y) = a_1$ and $\operatorname{sign}(y) = -1$. In this region, $f(y) = y/\sqrt{a_1}$, which implies $|f(y)| = -y/\sqrt{a_1}$. Consequently, the arguments A and B simplify to:

$$A(x,y) = f(x) - \frac{y}{\sqrt{a_1}} \quad \text{and} \quad B(x,y) = |f(x)| - \frac{y}{\sqrt{a_1}}.$$

Both arguments share the same derivative with respect to y:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial y} = -\frac{1}{\sqrt{a_1}}.$$

Differentiating the expression for $G(t,x,y)$ using the chain rule yields:

$$\begin{aligned}\frac{\partial G}{\partial y}(t,x,y) &= \frac{1}{\sqrt{a_1}} \left[\frac{\partial p}{\partial z}(t,A) \cdot \left(-\frac{1}{\sqrt{a_1}}\right) - \beta \frac{\partial p}{\partial z}(t,B) \cdot \left(-\frac{1}{\sqrt{a_1}}\right)\right] \\ &= \frac{1}{a_1} \left[-\frac{\partial p}{\partial z}(t,A) + \beta \frac{\partial p}{\partial z}(t,B)\right].\end{aligned}$$

Recalling that $a_1 = a(y)$ for $y < 0$, the identity is established. □

We now establish Gaussian upper bounds for G to facilitate the subsequent analysis of the mild solution.

Lemma 2. *The fundamental solution G satisfies the following estimates:*

1. *For every $x, v \in \mathbb{R}$ and $\tau > 0$:*

$$|G(\tau,x,v)| \leq \frac{C_1}{\sqrt{\tau}} \exp\left(-\frac{|f(x) - f(v)|^2}{2\tau}\right), \tag{32}$$

where $C_1 = \frac{1+|\beta|}{\sqrt{2\pi a_{\min}}}$ and $a_{\min} = \min(a_1, a_2)$.

2. *For every $x \in \mathbb{R}$ and $\tau > 0$, G is bounded by a centered Gaussian kernel in v:*

$$|G(\tau, x, v)| \leq C(x, \tau) \exp\left(-\frac{v^2}{4a_{\max}\tau}\right), \tag{33}$$

where $a_{\max} = \max(a_1, a_2)$ and the prefactor is:

$$C(x, \tau) = \frac{C_1}{\sqrt{\tau}} \exp\left(\frac{|f(x)|^2}{2\tau}\right).$$

Proof. 1. From Remark 4, we have:

$$|G(\tau, x, v)| \leq \frac{1}{\sqrt{a_{\min}}} \left(|p(\tau, A(x, v))| + |\beta||p(\tau, B(x, v))|\right).$$

Since $|B(x, v)| = |f(x)| + |f(v)| \geq |f(x) - f(v)| = |A(x, v)|$, it follows that $p(\tau, B) \leq p(\tau, A)$. Thus:

$$|G(\tau, x, v)| \leq \frac{1 + |\beta|}{\sqrt{a_{\min}}} \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{|f(x) - f(v)|^2}{2\tau}\right),$$

which confirms (32).

2. Starting from (32), we use the inequality $|f(x) - f(v)|^2 \geq \frac{1}{2}|f(v)|^2 - |f(x)|^2$ to shift the exponent:

$$\exp\left(-\frac{|f(x) - f(v)|^2}{2\tau}\right) \leq \exp\left(\frac{|f(x)|^2}{2\tau}\right) \exp\left(-\frac{|f(v)|^2}{4\tau}\right).$$

Since $|f(v)|^2 = v^2/a(v)$ and $a(v) \leq a_{\max}$, we have

$$\exp(-|f(v)|^2/4\tau) \leq \exp(-v^2/4a_{\max}\tau).$$

Combining these estimates yields:

$$|G(\tau, x, v)| \leq \frac{C_1}{\sqrt{\tau}} \exp\left(\frac{|f(x)|^2}{2\tau}\right) \exp\left(-\frac{v^2}{4a_{\max}\tau}\right),$$

establishing (33).

□

3.2 $L^2(\Omega)$-Existence of the mild solution and moment bound

The construction of the mild solution $u(t, x)$ defined in (25) relies on the theory of stochastic integration with respect to the fractional Lévy process L^d. By the fractional duality established in Proposition 2.2, the well-posedness of the stochastic integral is governed by the space-time integrability of the fractionalized kernel $I_-^d G$. Specifically, it requires that the $L^2(\mathbb{R})$-norm of the kernel be square-integrable over the time interval $[0, t]$. The following theorem establishes that the Riemann-Liouville fractional integral of the fundamental solution satisfies this requirement by exhibiting a precise temporal scaling for all $d \in (0, 1/2)$.

Theorem 3.2. *Let $\tau > 0$ and $d \in (0, 1/2)$. For any fixed $x \in \mathbb{R}$, the right-sided Riemann-Liouville fractional integral of the fundamental solution $I_-^d(G(\tau, x, \cdot))$, as defined in Proposition 2.2-(vi), is well-defined in $L^2(\mathbb{R})$ and satisfies the following temporal scaling bound:*

$$\|I_-^d(G(\tau, x, \cdot))\|_{L^2(\mathbb{R})} \leqslant C\, \tau^{\frac{2d-1}{4}}, \tag{34}$$

where C is a positive constant independent of τ.

Proof. By invoking the Gaussian upper bound established in Lemma 2-(1) and substituting it into the definition of the right-sided Riemann-Liouville fractional integral, we obtain:

$$\begin{aligned} |I_-^d(G(\tau, x, \cdot))(y)| &\leqslant \frac{1}{\Gamma(d)} \int_y^\infty |G(\tau, x, v)|(v - y)^{d-1}\, dv \\ &\leqslant \frac{C_1 a_{\min}^{\frac{d-1}{2}}}{\Gamma(d)\sqrt{\tau}} \int_y^\infty \exp\left(-\frac{(f(x) - f(v))^2}{2\tau}\right)(f(v) - f(y))^{d-1}\, dv. \end{aligned}$$

Applying the change of variables $w = \frac{f(v)-f(y)}{\sqrt{\tau}}$ yields:

$$\begin{aligned} |I_-^d(G(\tau, x, \cdot))(y)| &\leqslant \frac{C_1 a_{\min}^{\frac{d-1}{2}} \sqrt{a_{\max}}}{\Gamma(d)} \tau^{\frac{d-1}{2}} \int_0^\infty \exp\left(-\frac{(\frac{f(x)-f(y)}{\sqrt{\tau}} - w)^2}{2}\right) w^{d-1}\, dw \\ &= C_2 \tau^{\frac{d-1}{2}} \Psi_d\left(\frac{f(x) - f(y)}{\sqrt{\tau}}\right), \end{aligned}$$

where $C_2 = \dfrac{C_1 a_{\min}^{\frac{d-1}{2}} \sqrt{a_{\max}}}{\Gamma(d)}$ and $\Psi_d(z) = \displaystyle\int_0^\infty \exp\left(-\frac{(z-w)^2}{2}\right) w^{d-1} dw$.

To establish the L^2-integrability and temporal scaling, we analyze the asymptotic behavior of $\Psi_d(z)$ as $|z| \to \infty$.

Case 1: $z \to -\infty$ (The $y \to +\infty$ regime). Note that $z = -|z|$. The exponent in the integral expands as $-(|z| + w)^2/2 = -|z|^2/2 - w|z| - w^2/2$. Thus,

$$\Psi_d(-|z|) \leqslant e^{-\frac{|z|^2}{2}} \int_0^\infty e^{-w|z|} w^{d-1} dw = \Gamma(d)|z|^{-d} e^{-\frac{|z|^2}{2}}.$$

The dominant Gaussian decay $O(|z|^{-d} e^{-|z|^2/2})$ ensures that $\Psi_d(z)$ is square-integrable as $y \to +\infty$.

Case 2: $z \to +\infty$ (The $y \to -\infty$ regime). For large z, the Gaussian kernel peaks at $w = z$, far from the power-law singularity at the origin. We decompose the integral into two regions:

- If $w \in [0, z/2]$, then $(z - w)^2 \geqslant z^2/4$. Thus,

$$\begin{aligned} \Psi_{d,1}(z) = \int_0^{z/2} \exp\left(-\frac{(z-w)^2}{2}\right) w^{d-1}\, dw &\leqslant e^{-z^2/8} \int_0^{z/2} w^{d-1}\, dw \\ &= \frac{(z/2)^d}{d}\, e^{-z^2/8}. \end{aligned}$$

- If $w \in [z/2, \infty)$, we substitute $w = z + \xi$:

$$\Psi_{d,2}(z) = \int_{-z/2}^{\infty} e^{-\frac{\xi^2}{2}} (z+\xi)^{d-1} d\xi = z^{d-1} \int_{-z/2}^{\infty} e^{-\frac{\xi^2}{2}} \left(1 + \frac{\xi}{z}\right)^{d-1} d\xi.$$

By the Dominated Convergence Theorem, as $z \to \infty$, the integral converges to $\int_{-\infty}^{\infty} e^{-\xi^2/2} d\xi = \sqrt{2\pi}$.

Combining these, we find $\Psi_d(z) \sim \sqrt{2\pi} z^{d-1}$ as $z \to +\infty$. Since $d < 1/2$, $\Psi_d^2(z) \sim 2\pi z^{2d-2}$ is integrable at infinity because $2d - 2 < -1$.

Utilizing these bounds with the spatial change of variables $z = (f(x) - f(y))/\sqrt{\tau}$, we obtain:

$$\|I_-^d G(\tau, x, \cdot)\|_{L^2}^2 \leqslant \int_{\mathbb{R}} \left(C_2 \tau^{\frac{d-1}{2}} \Psi_d(z)\right)^2 \sqrt{a_{\max}} \sqrt{\tau} dz = C_3 \tau^{d-1/2},$$

where $C_3 = C_2^2 \sqrt{a_{\max}} \int_{\mathbb{R}} \Psi_d^2(z) dz$.

Taking the square root yields $\|I_-^d G(\tau, x, \cdot)\|_{L^2} \leqslant C \tau^{\frac{2d-1}{4}}$. □

Building upon the integrability properties established in Theorem 3.2, we now prove the existence and second-moment boundedness of the mild solution.

Corollary 3.3. *The mild solution $u = \{u(t,x)\}_{t \geq 0, x \in \mathbb{R}}$ defined by (25) is well-defined in $L^2(\Omega, \mathbb{P})$ for all $t > 0$ and $x \in \mathbb{R}$. Furthermore, its second moment satisfies the following growth estimate:*

$$\mathbb{E}[u(t,x)^2] \leq C \frac{t^{d+3/2}}{d + 1/2}, \tag{35}$$

where C is a positive constant depending on the system parameters and the variance of the driving Lévy process $\sigma_L^2 = \mathbb{E}[L(1)^2]$.

Proof. By definition, the mild solution is given by:

$$u(t,x) = \int_0^t I(t-s, x)\, ds, \quad \text{where } I(\tau, x) = \int_{\mathbb{R}} G(\tau, x, y)\, dL^d(y).$$

By the L^2-isometry for fractional Lévy integrals established in Proposition 2.2-(vi), the second moment of the inner spatial integral is given by:

$$\mathbb{E}[I(\tau,x)^2] = \sigma_L^2 \|I_-^d(G(\tau, x, \cdot))\|_{L^2(\mathbb{R})}^2. \tag{36}$$

Theorem 3.2 ensures that $I_-^d(G(\tau, x, \cdot)) \in L^2(\mathbb{R})$ for all $\tau > 0$, confirming that $I(\tau, x)$ is a well-defined $L^2(\Omega, \mathbb{P})$ process. By applying the Cauchy-Schwarz inequality to u(t,x) and successively employing (36) and (34), it follows that:

$$\begin{aligned} \mathbb{E}[u(t,x)^2] &\leq t \int_0^t \mathbb{E}[I(s,x)^2]\, ds = \sigma_L^2 t \int_0^t \|I_-^d(G(s,x,\cdot))\|_{L^2(\mathbb{R})}^2\, ds \\ &\leq \sigma_L^2 t \int_0^t C^2 s^{d-1/2} ds = \frac{\sigma_L^2 C^2}{d+1/2} t^{d+3/2}. \end{aligned} \tag{37}$$

This completes the proof of Corollary 3.3. □

Building upon the existence results, we establish the fundamental scaling properties of the kernel using the following operator definitions.

Definition 3.4. Let $\varphi(z) := \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ for $z \in \mathbb{R}$. We define the integral operators $K_\pm : \mathbb{R} \to \mathbb{R}$ and, for a fixed parameter $L > 0$, the truncated operator $K_-^{(L)} : \mathbb{R} \to \mathbb{R}$ as follows:

$$K_-(\eta) := \frac{1}{\Gamma(d)} \int_0^\infty \varphi(\eta - w) w^{d-1}\, dw, \quad K_+(\eta) := \frac{1}{\Gamma(d)} \int_0^\infty \varphi(\eta + w) w^{d-1}\, dw, \tag{38}$$

$$K_-^{(L)}(\eta) := \frac{1}{\Gamma(d)} \int_0^L \varphi(\eta - w) w^{d-1}\, dw. \tag{39}$$

Lemma 3. *Let $y \in \mathbb{R}$. The fractional integral $(I_-^d G)(\tau, x, \cdot)(y)$ satisfies the following scaling identities:*

1. *If $y > 0$:*

$$(I_-^d G)(\tau, x, \cdot)(y) = a_2^{\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \left[K_- \left(\frac{A(x,y)}{\sqrt{\tau}} \right) + \beta K_+ \left(\frac{B(x,y)}{\sqrt{\tau}} \right) \right]. \tag{40}$$

2. *If $y < 0$:*

$$\begin{aligned} (I_-^d G)(\tau, x, \cdot)(y) = {} & a_1^{\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \left[K_-^{(L)} \left(\frac{A(x,y)}{\sqrt{\tau}} \right) - \beta K_-^{(L)} \left(\frac{B(x,y)}{\sqrt{\tau}} \right) \right] \\ & + a_2^{\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \left[\frac{1}{\Gamma(d)} \int_\lambda^\infty \varphi(\alpha - w)\, w^{d-1}\, dw \right. \\ & \left. + \beta \frac{1}{\Gamma(d)} \int_\lambda^\infty \varphi(\mu + w)\, w^{d-1}\, dw \right]. \end{aligned} \tag{41}$$

where $L := \frac{-y}{\sqrt{a_1}\sqrt{\tau}} > 0$, $\lambda := \frac{-y}{\sqrt{a_2}\sqrt{\tau}} > 0$, $\alpha := \frac{f(x) - y/\sqrt{a_2}}{\sqrt{\tau}}$, and $\mu := \frac{|f(x)| + y/\sqrt{a_2}}{\sqrt{\tau}}$.

Proof. We utilize the Green kernel representation (29) and distinguish two cases based on the initial position y.
Case 1: $y > 0$. Since $v \geq y > 0$, the medium is homogeneous with $a(v) = a_2$, $\operatorname{sign}(v) = 1$, and $f(v) = v/\sqrt{a_2}$. Applying the change of variables $v = y + \sqrt{a_2 \tau} w$ ($w \in [0, \infty)$), we have:

- $v - y = \sqrt{a_2 \tau} w$ and $dv = \sqrt{a_2 \tau}\, dw$.
- $A(x, v) = A(x, y) - \sqrt{\tau} w$ and $B(x, v) = B(x, y) + \sqrt{\tau} w$.

Substituting these relations into the integral and utilizing the Gaussian density $p(\tau, \xi) = \frac{1}{\sqrt{\tau}} \varphi(\xi/\sqrt{\tau})$ allows us to factor out the scaling constant $a_2^{\frac{d-1}{2}} \tau^{\frac{d-1}{2}}$, yielding (40).
Case 2: $y < 0$. The integration interval $[y, \infty)$ crosses the interface at $v = 0$. We partition the domain into $J_1 = \int_y^0 (\dots)\, dv$ and $J_2 = \int_0^\infty (\dots)\, dv$.

- *For J_1:* In this region, $a(v) = a_1$, $\operatorname{sign}(v) = -1$, and $|f(v)| = |f(y)| - \sqrt{\tau}w$. Using $v = y + \sqrt{a_1\tau}w$, the upper limit $v = 0$ corresponds to $w = L = -y/(\sqrt{a_1\tau})$. This results in the truncated operator $K_-^{(L)}$ applied to the shifted spatial arguments.
- *For J_2:* In this region, $a(v) = a_2$, $\operatorname{sign}(v) = 1$, and $f(v) = y/\sqrt{a_2} + \sqrt{\tau}w$. Using $v = y + \sqrt{a_2\tau}w$, the lower limit $v = 0$ corresponds to $w = \lambda = -y/(\sqrt{a_2\tau})$. The spatial parameters α and μ emerge naturally as the image of the transmission across the interface $v = 0$.

Summing J_1 and J_2 completes the proof of the piecewise representation (41). □

Corollary 3.5. *For every $\tau > 0$ and $x \in \mathbb{R}$, there exists a constant $C > 0$ such that the temporal derivative of the fractionalized kernel satisfies:*

$$\|\partial_\tau (I_-^d G)(\tau, x, \cdot)\|_{L^2(\mathbb{R})} \leq C\tau^{\frac{2d-5}{4}}, \tag{42}$$

Proof. The regularity estimate follows from differentiating the piecewise representation obtained in Lemma 3. Let $\Psi_\pm(\eta) := |\mathcal{K}_\pm(\eta)| + |\eta\mathcal{K}'_\pm(\eta)|$. As $\varphi \in \mathcal{S}(\mathbb{R})$, asymptotic analysis ensures $\Psi_\pm \in L^2(\mathbb{R})$ for $d < 1/2$.

1. Case $y > 0$: Applying the chain rule to (40), note that $\partial_\tau(\xi/\sqrt{\tau}) = -\dfrac{1}{2\tau}(\xi/\sqrt{\tau})$. The time derivative scales as $\tau^{\frac{d-3}{2}}$:

$$|\partial_\tau (I_-^d G)| \leq C\tau^{\frac{d-3}{2}} \left[\Psi_- \left(\frac{A}{\sqrt{\tau}}\right) + \beta\Psi_+ \left(\frac{B}{\sqrt{\tau}}\right)\right].$$

Squaring and integrating over $(0, \infty)$ with the change of variables $z = y/(\sqrt{a_2\tau})$ yields:

$$\|\partial_\tau (I_-^d G)\|^2_{L^2(0,\infty)} \leq C\tau^{d-3} \cdot \sqrt{\tau} \int_0^\infty |\Psi|^2 dz = C\tau^{d-5/2}.$$

2. Case $y < 0$: We partition the fractional integral $(I_-^d G) = J_1 + J_2$ over the intervals $(y, 0)$ and $(0, \infty)$, respectively. Define the normalized spatial argument $\eta := \dfrac{A(x,y)}{\sqrt{\tau}}$, which satisfies the scaling relation $\partial_\tau \eta = -\dfrac{1}{2\tau}\eta$. Applying Leibniz's rule to the truncated operator $K_-^{(L)}(\eta)$ with $L(\tau) = \dfrac{-y}{\sqrt{a_1\tau}}$ and noting $\partial_\tau L = -\frac{1}{2\tau}L$, we obtain:

$$\partial_\tau K_-^{(L)}(\eta) = \frac{1}{\Gamma(d)} \int_0^L \varphi'(\eta - w)\partial_\tau \eta\, w^{d-1} dw + \frac{1}{\Gamma(d)} \varphi(\eta - L) L^{d-1} \partial_\tau L.$$

Substituting the relations $\partial_\tau \eta = -\dfrac{1}{2\tau}\eta$ and $\partial_\tau L = -\dfrac{1}{2\tau}L$, the second term simplifies to the boundary residual $\mathcal{R}_1(z_1) = \dfrac{z_1^d}{2\Gamma(d)}\varphi(\eta - L)$, where $z_1 = \frac{-y}{\sqrt{a_1\tau}}$. A symmetric application of Leibniz's rule to J_2 yields the residual

$$\mathcal{R}_2(z_2) = \frac{z_2^d}{2\Gamma(d)} [\varphi(\alpha - z_2) + \beta\varphi(\mu + z_2)],$$

with $z_2 = \frac{-y}{\sqrt{a_2\tau}}$. Since the residuals $\mathcal{R}_i \in L^2(0,\infty)$ exhibit Gaussian decay, the temporal derivative is bounded by:

$$|\partial_\tau(I_-^d G)| \leq C\tau^{\frac{d-3}{2}}\left[\Psi_-\left(\frac{A(x,y)}{\sqrt{\tau}}\right) + |\beta|\Psi_-\left(\frac{B(x,y)}{\sqrt{\tau}}\right) + \Psi_-(\alpha) + |\beta|\Psi_+(\mu) + \mathcal{R}_1(z_1) + \mathcal{R}_2(z_2)\right].$$

Integrating over $(-\infty, 0)$ via the substitution $z = -y/(\sqrt{a_i\tau})$, the residuals $\mathcal{R}_i$ and the kernel derivatives contribute uniformly to the bound:

$$\|\partial_\tau(I_-^d G)\|_{L^2(-\infty,0)} \leq C\tau^{\frac{2d-5}{4}}.$$

This completes the proof. □

3.3 Alternative representation and pathwise equivalence

To facilitate the pathwise analysis of the mild solution, we establish an alternative representation of the stochastic integral that avoids the direct use of Young integration theory in favor of a Lebesgue integral formulation.

Lemma 4. *For all $x \in \mathbb{R}$, $t > 0$, and $i \in \{1, 2\}$, the Young integral*

$$\int_0^{+\infty} p\left(t - s, x \pm \frac{y}{\sqrt{a_i}}\right) dL^d(y)$$

exists almost surely for $0 \leqslant s < t$, and its value is given by:

- $\frac{1}{\sqrt{a_i}}\int_0^{+\infty} L^d(y)\frac{\partial p}{\partial z}\left(t - s, x - \frac{y}{\sqrt{a_i}}\right) dy$, *if the inner sign is minus;*
- $-\frac{1}{\sqrt{a_i}}\int_0^{+\infty} L^d(y)\frac{\partial p}{\partial z}\left(t - s, x + \frac{y}{\sqrt{a_i}}\right) dy$, *if the inner sign is plus.*

Furthermore, for each path ω of L^d, there exists a positive constant $C_{x,t,\omega}$ such that:

$$\left|\int_0^{+\infty} L^d(y,\omega)\frac{\partial p}{\partial z}\left(t - s, x \pm \frac{y}{\sqrt{a_i}}\right) dy\right| \leqslant C_{x,t,\omega}(t - s)^{-1/2}. \tag{43}$$

Proof. We prove the result for the case of the negative inner sign; the case for the positive sign follows by symmetry. For a fixed time horizon $0 \leqslant s < t$ and $x \in \mathbb{R}$, consider the integral over an arbitrary compact interval $[a, b] \subset (0, +\infty)$. Let

$$f(y) = p\left(t - s, x - \frac{y}{\sqrt{a_i}}\right).$$

Since $t - s > 0$, the heat kernel f is continuously differentiable on $[a, b]$, which implies it is 1-Hölder continuous. Given that the sample paths of L^d possess β-Hölder regularity for any $\beta < d$, and $d > 0$, the Young condition $(1 + \beta > 1)$ is satisfied.

Consequently, by Young's Integrability Theorem [28], the integral is well-defined pathwise and satisfies the following integration by parts formula:

$$\int_a^b f(y)dL^d(y) = f(b)L^d(b) - f(a)L^d(a) - \int_a^b L^d(y)df(y). \tag{44}$$

Substituting the differential $df(y) = f'(y)dy = -\frac{1}{\sqrt{a_i}}\frac{\partial p}{\partial z}\left(t-s, x - \frac{y}{\sqrt{a_i}}\right)dy$ into (44), we obtain the almost sure identity:

$$\begin{aligned}
&\int_a^b p\left(t-s, x-\frac{y}{\sqrt{a_i}}\right)dL^d(y) \\
&= p\left(t-s, x-\frac{b}{\sqrt{a_i}}\right)L^d(b) - p\left(t-s, x-\frac{a}{\sqrt{a_i}}\right)L^d(a) \\
&\quad + \frac{1}{\sqrt{a_i}}\int_a^b L^d(y)\frac{\partial p}{\partial z}\left(t-s, x-\frac{y}{\sqrt{a_i}}\right)dy.
\end{aligned}$$

Next, we extend this to the improper integral by evaluating the boundary terms. At the origin, the almost sure continuity of L^d and $L^d(0) = 0$ imply that $\lim_{a\to 0^+} f(a)L^d(a) = 0$.

For the limit as $b \to +\infty$, we invoke the asymptotic growth property of the fractional Lévy process (see Proposition 2.2). Specifically, for almost every $\omega \in \Omega$, there exists a path-dependent constant $C_\omega > 0$ such that the realization $L^d(\cdot, \omega)$ satisfies:

$$|L^d(y,\omega)| \leqslant C_\omega |y|^\alpha \quad \text{for all } |y| \geqslant 1, \tag{45}$$

where $\alpha \in (d, d+1)$. Since the heat kernel $p(t-s,\cdot)$ and its derivatives exhibit exponential decay as $|y| \to \infty$, the product $p(t-s, x-b/\sqrt{a_i})L^d(b,\omega)$ vanishes as $b \to \infty$.

Finally, we establish the absolute integrability and the singularity bound (43). We recall the scaling property of the heat kernel derivative:

$$\left|\frac{\partial p}{\partial z}(\tau, z)\right| = \tau^{-1}\left|\Psi\left(\frac{z}{\sqrt{\tau}}\right)\right|, \quad \text{where } \Psi(u) = \frac{u}{\sqrt{2\pi}}e^{-u^2/2}.$$

- *Near the origin ($y \in [0,c]$):* Since L^d is continuous, $|L^d(y,\omega)|$ is bounded by M_ω on $[0,c]$. Setting $\tau = t-s$ and using the change of variables $z = (x - y/\sqrt{a_i})/\sqrt{\tau}$, we have $dy = -\sqrt{a_i\tau}dz$. Then:

$$\begin{aligned}
\int_0^c |L^d(y,\omega)|\left|\frac{\partial p}{\partial z}\right| dy &\leqslant M_\omega\sqrt{a_i\tau}\int_{-\infty}^{+\infty}\tau^{-1}|\Psi(z)|\,dz \\
&= M_\omega\sqrt{a_i}\,(t-s)^{-1/2}\int_{\mathbb{R}}|\Psi(z)|\,dz.
\end{aligned}$$

As $\Psi \in L^1(\mathbb{R})$, this portion is $O((t-s)^{-1/2})$.

- *Near infinity ($y \in [c, +\infty)$):* For $y \geq c$, we utilize the growth bound $|L^d(y,\omega)| \leqslant C_\omega y^\alpha$ for some $\alpha > d + 1/2$. Let I_∞ denote the tail integral. Using the scaling identity for $|\partial_z p|$ with $\tau = t - s$, we have:

$$I_\infty \leqslant \frac{C_\omega}{\sqrt{a_i}\tau} \int_c^{+\infty} y^\alpha \left| \Psi\left(\frac{x - y/\sqrt{a_i}}{\sqrt{\tau}} \right) \right| dy.$$

By the change of variables $z = \frac{y/\sqrt{a_i} - x}{\sqrt{\tau}}$, we get:

$$I_\infty \leqslant C_\omega \tau^{-1/2} a_i^{\alpha/2} \int_{z_c}^{+\infty} |x + z\sqrt{\tau}|^\alpha |\Psi(z)|\, dz,$$

where $z_c = (c/\sqrt{a_i} - x)/\sqrt{\tau}$. Applying the inequality

$$(x + y)^\alpha \leqslant \max(1, 2^{\alpha-1})(x^\alpha + y^\alpha),$$

which holds for any $x, y \geq 0$ and $\alpha > 0$, we obtain:

$$I_\infty \leqslant C'_{\omega,\alpha} \tau^{-1/2} \int_{z_c}^{+\infty} \left(|x|^\alpha + |z|^\alpha \tau^{\alpha/2} \right) |\Psi(z)|\, dz,$$

where $C'_{\omega,\alpha} = C_\omega a_i^{\alpha/2} \max(1, 2^{\alpha-1})$.

Since $\Psi(z)$ is a Schwartz function, its moments $\int_{\mathbb{R}} |z|^k |\Psi(z)|\, dz$ are finite for all $k \geqslant 0$. Noting that $z_c \to \infty$ as $\tau \to 0^+$ (for c chosen sufficiently large such that $c/\sqrt{a_i} > x$), and that τ is bounded by the time horizon T, the integral remains uniformly bounded by a constant $D_\alpha(x, T, \omega)$. Thus, we obtain the sharp estimate:

$$I_\infty \leqslant D_\alpha(x, T, \omega)(t - s)^{-1/2}.$$

The combination of these local and asymptotic estimates confirms that the integral converges absolutely and is dominated by the $(t - s)^{-1/2}$ singularity. □

Building on the pathwise equivalence established in Lemma 4, the following theorem provides a global representation of the mild solution for any $t > 0$. This result simplifies the study of the solution by expressing it as a standard Lebesgue integral of the localized process L^d.

Theorem 3.6. *For any $t > 0$ and $x \in \mathbb{R}$, the mild solution $u := \{u(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ defined by (25) is almost surely equivalent to the integrated-by-parts form:*

$$u(t,x) = -\int_0^t \left(\int_{-\infty}^{+\infty} L^d(y) \frac{\partial G(t - s, x, y)}{\partial y}\, dy \right) ds. \tag{46}$$

Proof. The proof relies on applying the pathwise results established in Lemma 4 to the specific structure of the Green's function.

Step 1: For fixed $x \in \mathbb{R}$ and $0 \leqslant s < t$, the inner integral of the mild solution is the Young integral $\int_{\mathbb{R}} G(t - s, x, y)\, dL^d(y)$, which exists almost surely. Using the explicit

representation of G from (29), we decompose this integral over the intervals $(-\infty, 0)$ and $(0, +\infty)$.

On each interval, $G(t-s, x, y)$ is a linear combination of terms of the form $p(t-s, x' \pm y/\sqrt{a_i})$. By applying the integration by parts result established in Lemma 4 to each component, we immediately obtain the almost sure identity:

$$\int_{-\infty}^{+\infty} G(t-s, x, y) dL^d(y) = -\int_{-\infty}^{+\infty} L^d(y) \frac{\partial G(t-s, x, y)}{\partial y} dy.$$

Substituting this identity into the definition (25) yields the representation (46).

Step 2: To ensure that (46) defines a valid pathwise solution, we examine the absolute integrability of the resulting Lebesgue form. By summing the individual bounds provided by Equation (43) in Lemma 4 for each component of the Green's function, there exists a positive path-dependent constant $C_{x,t,\omega}$ such that for all $0 \leq s < t$:

$$\left| \int_{-\infty}^{+\infty} L^d(y, \omega) \frac{\partial G(t-s, x, y)}{\partial y} dy \right| \leqslant C_{x,t,\omega} (t-s)^{-1/2}.$$

The time integrability of this bound over the interval $[0, t]$ follows immediately:

$$\int_0^t \left| \int_{-\infty}^{+\infty} L^d(y) \frac{\partial G(t-s, x, y)}{\partial y} dy \right| ds \leqslant C_{x,t,\omega} \int_0^t (t-s)^{-1/2} ds = 2C_{x,t,\omega} \sqrt{t} < \infty.$$

This absolute convergence confirms the almost sure equivalence between the mild solution and its integrated-by-parts form, concluding the proof. □

3.4 Weak solution

Let us suppose formally for the moment that Equation (3) admits a classical solution $u \colon \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ that is a function belonging to $C^2([0, \infty) \times \mathbb{R})$ and satisfying Equation (3) on a set $\Omega' \in \Omega$ of probability 1.

Let $T > 0.$ be fixed. For a test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R})$, we multiply the equation by $\varphi(s, x)$ and $\rho(x)$ and integrate over $[0, T] \times \mathbb{R}$. This gives us:

$$\underbrace{\int_{[0,T]\times\mathbb{R}} \frac{\partial u(s,x)}{\partial s} \varphi(s,x)\rho(x)\, dx\, ds}_{I} - \underbrace{\int_{[0,T]\times\mathbb{R}} \mathcal{L}u(s,x)\varphi(s,x)\rho(x)\, dx\, ds}_{J}$$
$$= \int_{[0,T]\times\mathbb{R}} \varphi(s,x)\rho(x)\, dL^d(x) ds.$$

We use integration by parts with respect to the time variable s.

$$I = \int_{[0,T]\times\mathbb{R}} \frac{\partial u(s,x)}{\partial s} \varphi(s,x)\rho(x)\, dx\, ds = \int_{\mathbb{R}} \left(\int_0^T \frac{\partial u(s,x)}{\partial s} \varphi(s,x)\, ds \right) \rho(x)\, dx.$$

Applying the integration by parts formula for the inner integral, we get

$$\int_0^T \frac{\partial u(s,x)}{\partial s} \varphi(s,x)\, ds = [u(s,x)\varphi(s,x)]_{s=0}^{s=T} - \int_0^T u(s,x) \frac{\partial \varphi(s,x)}{\partial s}\, ds$$

$$= u(T,x)\varphi(T,x) - u(0,x)\varphi(0,x) - \int_0^T u(s,x)\frac{\partial\varphi(s,x)}{\partial s}\,ds.$$

Since $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$, we have $\varphi(0,x) = 0$ and $\varphi(T,x) = 0$. Also, the initial condition is $u(0,x) = 0$. This simplifies the expression to:

$$\int_0^T \frac{\partial u(s,x)}{\partial s}\varphi(s,x)\,ds = -\int_0^T u(s,x)\frac{\partial\varphi(s,x)}{\partial s}\,ds.$$

Substituting this back into the expression for I, we get:

$$I = -\int_{\mathbb{R}}\left(\int_0^T u(s,x)\frac{\partial\varphi(s,x)}{\partial s}\,ds\right)\rho(x)dx = -\int_{[0,T]\times\mathbb{R}} u(s,x)\frac{\partial\varphi(s,x)}{\partial s}\rho(x)dxds.$$

As for $J = \int_{[0,T]\times\mathbb{R}} \mathcal{L}u(s,x)\varphi(s,x)\rho(x)\,dxds$, substituting the definition of $\mathcal{L}$, we have:

$$J = \frac{1}{2}\int_{[0,T]\times\mathbb{R}} \frac{\partial}{\partial x}\left(\rho(x)a(x)\frac{\partial u(s,x)}{\partial x}\right)\varphi(s,x)\,dx\,ds.$$

Applying the definition of the distributional derivative we get:

$$J = -\frac{1}{2}\int_0^T\int_{\mathbb{R}} \rho(x)a(x)\frac{\partial u(s,x)}{\partial x}\,\frac{\partial\varphi(s,x)}{\partial x}\,dx\,ds.$$

Now, to handle the discontinuity of the coefficients $a(x)$ and $\rho(x)$ at $x = 0$, we split the spatial integral into two parts: one over $(-\infty,0)$ and one over $(0,+\infty)$.

$$\begin{aligned} J &= -\frac{1}{2}\int_0^T\int_{\mathbb{R}^\star} \rho(x)a(x)\frac{\partial u(s,x)}{\partial x}\frac{\partial\varphi(s,x)}{\partial x}\,dx\,ds \\ &= -\frac{1}{2}\left(\int_0^T\int_0^{+\infty} \rho_2 a_2\frac{\partial u(s,x)}{\partial x}\frac{\partial\varphi(s,x)}{\partial x}\,dx\,ds\right. \\ &\qquad \left. + \int_0^T\int_{-\infty}^0 \rho_1 a_1\frac{\partial u(s,x)}{\partial x}\frac{\partial\varphi(s,x)}{\partial x}\,dx\,ds\right). \end{aligned}$$

We, now, perform a second integration by parts on each of these two integrals. For the integral over $(0,+\infty)$:

$$\begin{aligned} &\int_{(0,+\infty)} \rho_2 a_2\frac{\partial u(s,x)}{\partial x}\,\frac{\partial\varphi(s,x)}{\partial x}\,dx \\ &= \rho_2 a_2\left(\left[u(s,x)\frac{\partial\varphi(s,x)}{\partial x}\right]_{x=0^+}^{x\to+\infty} - \int_{(0,+\infty)} u(s,x)\,\frac{\partial^2\varphi(s,x)}{\partial^2 x}\,dx\right) \\ &= \rho_2 a_2\left(0 - u(s,0)\frac{\partial\varphi(s,0)}{\partial x} - \int_{(0,+\infty)} u(s,x)\,\frac{\partial^2\varphi(s,x)}{\partial^2 x}\,dx\right) \\ &= -\rho_2 a_2 u(s,0)\frac{\partial\varphi(s,0)}{\partial x} - \int_{(0,+\infty)} u(s,x)\rho_2 a_2\frac{\partial^2\varphi(s,x)}{\partial^2 x}\,dx. \end{aligned}$$

For the integral over $(-\infty, 0)$:

$$\begin{aligned}
&\int_{(-\infty,0)} \rho_1 a_1 \frac{\partial u(s,x)}{\partial x} \frac{\partial \varphi(s,x)}{\partial x}\, dx \\
&= \rho_1 a_1 \left(\left[u(s,x) \frac{\partial \varphi(s,x)}{\partial x} \right]_{x \to -\infty}^{x=0^-} - \int_{(-\infty,0)} u(s,x) \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx \right) \\
&= \rho_1 a_1 \left(u(s,0) \frac{\partial \varphi(s,0)}{\partial x} - 0 - \int_{(-\infty,0)} u(s,x) \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx \right) \\
&= \rho_1 a_1 u(s,0) \frac{\partial \varphi(s,0)}{\partial x} - \int_{(-\infty,0)} u(s,x) \rho_1 a_1 \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx.
\end{aligned}$$

Substituting these two results back into the expression for J and combining the terms:

$$\begin{aligned}
J &= -\frac{1}{2} \int_0^T \left(-\rho_2 a_2 u(s,0) \frac{\partial \varphi(s,0)}{\partial x} - \int_{(0,+\infty)} u(s,x) \rho_2 a_2 \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx \right) ds \\
&\quad - \frac{1}{2} \int_0^T \left(\rho_1 a_1 u(s,0) \frac{\partial \varphi(s,0)}{\partial x} - \int_{(-\infty,0)} u(s,x) \rho_1 a_1 \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx \right) ds \\
&= \frac{1}{2} \int_0^T (\rho_2 a_2 - \rho_1 a_1) u(s,0) \frac{\partial \varphi(s,0)}{\partial x}\, ds \\
&\quad + \frac{1}{2} \int_0^T ds \left(\int_{(-\infty,0)} u(s,x) \rho_1 a_1 \frac{\partial^2 \varphi(s,x)}{\partial^2 x} dx + \int_{(0,+\infty)} u(s,x) \rho_2 a_2 \frac{\partial^2 \varphi(s,x)}{\partial^2 x} dx \right) \\
&= \frac{1}{2} \int_0^T (\rho_2 a_2 - \rho_1 a_1) u(s,0) \frac{\partial \varphi(s,0)}{\partial x}\, ds \\
&\quad + \frac{1}{2} \int_{[0,T]\times\mathbb{R}^\star} u(s,x) \rho(x) a(x) \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx ds.
\end{aligned}$$

By the definition of the operator $\mathcal{L}$, we can write the last integral as:

$$\frac{1}{2} \int_{[0,T]\times\mathbb{R}^\star} u(s,x) \rho(x) a(x) \frac{\partial^2 \varphi(s,x)}{\partial^2 x}\, dx ds = \int_{[0,T]\times\mathbb{R}^\star} u(s,x) \mathcal{L}\varphi(s,x) \rho(x)\, dx ds.$$

Thus, the final expression for J is:

$$J = \frac{1}{2} \int_0^T (\rho_2 a_2 - \rho_1 a_1) u(s,0) \frac{\partial \varphi(s,0)}{\partial x}\, ds + \int_{[0,T]\times\mathbb{R}^\star} u(s,x) \mathcal{L}\varphi(s,x)\, \rho(x)\, dx ds.$$

Combining the results for terms I and J, we get the weak formulation of the SPDE (3):

$$\begin{aligned}
&- \int_{[0,T]\times\mathbb{R}^\star} u(s,x) \left[\frac{\partial \varphi(s,x)}{\partial s} + \mathcal{L}\varphi(s,x) \right] \rho(x)\, dx\, ds \\
&\quad = \int_{[0,T]\times\mathbb{R}} \varphi(s,x) \rho(x)\, dL^d(x) ds \\
&\qquad + \frac{1}{2} \int_0^T (\rho(0^+) a(0^+) - \rho(0^-) a(0^-)) u(s,0) \frac{\partial \varphi(s,0)}{\partial x} ds \quad \text{a.s.}
\end{aligned}$$

Here, $\rho(0^+) = \rho_2$ and $a(0^+) = a_2$, while $\rho(0^-) = \rho_1$ and $a(0^-) = a_1$.

This leads to the formal definition of a weak solution for the SPDE (3):

Definition 3.7. A stochastic process $u := \{u(t,x)\}_{\{t\geq 0, x\in\mathbb{R}\}}$ is a *weak solution* to SPDE (3) if $u \in L^1_{Loc}(\mathbb{R}_+ \times \mathbb{R}, dx\, dt)$, and for all $T > 0$ and every $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$, the process u satisfies the following equation almost surely:

$$\begin{aligned} & -\int_{[0,T]\times\mathbb{R}^\star} u(s,x)\left[\frac{\partial\varphi(s,x)}{\partial s} + \mathcal{L}\varphi(s,x)\right]\rho(x)\,dx\,ds \\ = & \int_{[0,T]\times\mathbb{R}} \varphi(s,x)\rho(x)\,dL^d(x)ds \\ + & \frac{1}{2}\int_0^T (\rho(0^+)a(0^+) - \rho(0^-)a(0^-))u(s,0)\frac{\partial\varphi(s,0)}{\partial x}ds. \end{aligned} \tag{47}$$

3.5 Equivalence between mild and weak solutions

The following theorem confirms that the well-posedness of the mild solution u guarantees it satisfies the weak formulation derived in the previous section.

Theorem 3.8. *The mild solution $u := \{u(t,x)\}_{\{t\geq 0, x\in\mathbb{R}\}}$ defined by Equation (25) is an almost surely well-defined weak solution in the sense of formulation given in* (47).

Proof. Let u be the mild solution, $T > 0$ and $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$ be a test function. Introduce the notation $H(\varphi) := -\dfrac{\partial\varphi}{\partial s} - \mathcal{L}\varphi$. The Left-Hand Side (LHS) of the weak formulation is:

$$\begin{aligned} LHS &= \int_{[0,T]\times\mathbb{R}^\star} u(s,x)\,H(\varphi)(s,x)\,\rho(x)\,dx\,ds \\ &= \int_{[0,T]\times\mathbb{R}^\star}\left(\int_0^s\int_{\mathbb{R}} G(s-r,x,y)\,dL^d(y)dr\right)H(\varphi)(s,x)\rho(x)\,dx\,ds. \end{aligned} \tag{48}$$

We break the proof down into two steps.

Step 1: We first justify the almost sure Fubini interchange property for the integral LHS:

$$LHS \overset{\text{a.s.}}{=} \int_{[0,T]}\int_{\mathbb{R}}\left[\int_r^T\int_{\mathbb{R}^\star} G(s-r,x,y)H(\varphi)(s,x)\rho(x)dsdx\right]dL^d(y)dr. \tag{49}$$

Let $\mathcal{H}(s,x) = H(\varphi)(s,x)\rho(x)$. The inner stochastic integral of the mild solution is converted into a deterministic Lebesgue integral using the Integration by Parts Identity (46):

$$LHS = -\int_{[0,T]\times\mathbb{R}^*}\left(\int_0^s\int_{\mathbb{R}} L^d(y)\frac{\partial G(s-r,x,y)}{\partial y}dydr\right)\mathcal{H}(s,x)dxds.$$

The resulting expression is a deterministic Lebesgue integral. The required absolute integrability condition for the deterministic Fubini theorem holds because:

1. The test function term $\mathcal{H}(s,x)$ is bounded on the compact support of φ.

2. The derivative $\dfrac{\partial G}{\partial y}$ decays exponentially as $|y| \to \infty$.

3. The fractional Lévy process path $L^d(y)$ has sub-polynomial growth at infinity ($|L^d(y)| \leqslant C_\omega |y|^\alpha$).

Since the exponential decay of $\dfrac{\partial G}{\partial y}$ dominates the growth of $L^d(y)$, the overall integrand is absolutely integrable. Therefore,

$$LHS \overset{\text{a.s.}}{=} -\int_0^T \left(\int_{\mathbb{R}} L^d(y) \left[\int_r^T \int_{\mathbb{R}^\star} \frac{\partial G(s-r,x,y)}{\partial y} \mathcal{H}(s,x) dx ds \right] dy \right) dr.$$

Now, we aim to convert the inner deterministic integral back into a stochastic integral. For fixed $r \in [0,T]$, let $\mathbf{K}_{r,T}(y)$ be the inner deterministic function defined by:

$$\mathbf{K}_{r,T}(y) = \int_r^T \int_{\mathbb{R}^\star} G(s-r,x,y) \mathcal{H}(s,x) dx ds.$$

Because of the smoothness of G on $\mathbb{R}^\star$ and φ, $\mathbf{K}_{r,T}$ is derivable and for any $y \neq 0$:

$$\frac{\partial \mathbf{K}_{r,T}(y)}{\partial y} = \int_r^T \int_{\mathbb{R}^\star} \frac{\partial G(s-r,x,y)}{\partial y} \mathcal{H}(s,x) dx ds.$$

Substituting this back into the expression for LHS:

$$LHS \overset{\text{a.s.}}{=} \int_0^T \left(-\int_{\mathbb{R}} L^d(y) \frac{\partial \mathbf{K}_{r,T}(y)}{\partial y} dy \right) dr = \int_0^T \left(-\int_{\mathbb{R}} L^d(y) \frac{\partial \mathbf{K}_{r,T}(y)}{\partial y} dy \right) dr.$$

To validate the almost sure conversion back to the stochastic integral, we observe that $\mathbf{K}_{r,T}$ inherits the regularity and asymptotic properties of the fundamental solution G. Specifically, the smoothness of G and φ ensures $\mathbf{K}_{r,T} \in C^1(\mathbb{R}^\star)$, while the exponential decay of the heat kernel ensures that $|L^d(y)\mathbf{K}_{r,T}(y)| \to 0$ as $|y| \to \infty$ almost surely. Consequently, the boundary terms in the integration by parts formula vanish, allowing us to apply the pathwise identity:

$$-\int_{\mathbb{R}} L^d(y) \frac{\partial \mathbf{K}_{r,T}(y)}{\partial y} dy \overset{\text{a.s.}}{=} \int_{\mathbb{R}} \mathbf{K}_{r,T}(y)\, dL^d(y). \tag{50}$$

Substituting this back into the expression for LHS yields:

$$\begin{aligned} LHS &\overset{\text{a.s.}}{=} \int_0^T \left(\int_{\mathbb{R}} \mathbf{K}_{r,T}(y)\, dL^d(y) \right) dr \\ &= \int_{[0,T]} \int_{\mathbb{R}} \left[\int_r^T \int_{\mathbb{R}^\star} G(s-r,x,y)\, \mathcal{H}(s,x)\, ds\, dx \right] dL^d(y)\, dr, \end{aligned}$$

which establishes the identity (49) and completes Step 1.

Step 2: The inner deterministic integral from the Fubini interchange (49) is denoted $D(\varphi,r,y)$:

$$D(\varphi,r,y) = \int_r^T \int_{\mathbb{R}^\star} G(s-r,x,y) H(\varphi)(s,x) \rho(x)\, ds\, dx.$$

Substituting $H(\varphi) = -\frac{\partial \varphi}{\partial s} - \mathcal{L}\varphi$, we decompose D into two main components:

$$D(\varphi, r, y) = \underbrace{\int_r^T \int_{\mathbb{R}^\star} -G(s-r, x, y) \frac{\partial \varphi}{\partial s} \rho(x)\, dx\, ds}_{\text{Term 1}} + \underbrace{\int_r^T \int_{\mathbb{R}^\star} -G(s-r, x, y)(\mathcal{L}\varphi)\, \rho(x)\, dx\, ds}_{\text{Term 2}}.$$

Analysis of Term 1: We first integrate by parts with respect to the time variable s. Given the boundary condition $\varphi(T, x) = 0$ and the fact that the Green's function satisfies $\lim_{s \to r^+} G(s-r, x, y) = \delta(x-y)$, we obtain:

$$\int_r^T -G(s-r, x, y) \frac{\partial \varphi}{\partial s}\, ds = [-G(s-r, x, y)\varphi(s, x)]_{s=r}^{s=T} + \int_r^T \varphi(s, x) \frac{\partial G(s-r, x, y)}{\partial s}\, ds.$$

Integrating over space with the weight $\rho(x)$ and applying the Dirac property at the lower bound $s = r$, Term 1 becomes:

$$\text{Term 1} = \varphi(r, y)\rho(y) + \int_r^T \int_{\mathbb{R}^\star} \frac{\partial G(s-r, x, y)}{\partial s} \varphi(s, x)\rho(x)\, dx\, ds.$$

Analysis of Term 2: For a fixed $s \in [r, T]$, we consider the inner spatial integral I_s defined over the partitioned domain $\mathbb{R}^\star = (-\infty, 0) \cup (0, \infty)$:

$$I_s = \int_{\mathbb{R}^\star} -G(s-r, x, y)(\mathcal{L}\varphi)(s, x)\rho(x)\, dx. \tag{51}$$

Recalling the definition of the operator $\mathcal{L}\varphi = \frac{1}{2\rho} \frac{\partial}{\partial x} \left(\rho a \frac{\partial \varphi}{\partial x} \right)$, and noting that the product $\rho(x)a(x)$ assumes the constant values $\rho_1 a_1$ on $(-\infty, 0)$ and $\rho_2 a_2$ on $(0, \infty)$, we perform the first integration by parts:

$$\begin{aligned} I_s &= -\frac{1}{2} \int_0^{+\infty} G(s-r, x, y) \frac{\partial}{\partial x} \left(\rho_2 a_2 \frac{\partial \varphi(s, x)}{\partial x} \right) dx \\ &\quad - \frac{1}{2} \int_{-\infty}^0 G(s-r, x, y) \frac{\partial}{\partial x} \left(\rho_1 a_1 \frac{\partial \varphi(s, x)}{\partial x} \right) dx \\ &= -\frac{1}{2} \left[G(s-r, x, y) \rho_2 a_2 \frac{\partial \varphi(s, x)}{\partial x} \right]_{x=0^+}^{x \to +\infty} - \frac{1}{2} \left[G(s-r, x, y) \rho_1 a_1 \frac{\partial \varphi(s, x)}{\partial x} \right]_{x \to -\infty}^{x=0^-} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^\star} \frac{\partial G(s-r, x, y)}{\partial x} \rho(x) a(x) \frac{\partial \varphi(s, x)}{\partial x} dx. \end{aligned}$$

Since φ has compact support in $\mathbb{R}$, the terms at $x \to \pm\infty$ vanish. Evaluating the boundary terms at the interface $x = 0$, we obtain:

$$I_s = \frac{1}{2} G(s-r, 0, y)(\rho_2 a_2 - \rho_1 a_1) \frac{\partial \varphi(s, 0)}{\partial x}$$

$$\underbrace{+\frac{1}{2}\int_{\mathbb{R}^\star}\rho(x)a(x)\frac{\partial G(s-r,x,y)}{\partial x}\frac{\partial\varphi(s,x)}{\partial x}dx}_{R_s}.$$

Performing a second integration by parts on R_s over the partitioned intervals $(-\infty,0)$ and $(0,\infty)$, we obtain:

$$\begin{aligned}R_s &= \left[\varphi(s,x)\frac{\rho_2 a_2}{2}\frac{\partial G(s-r,x,y)}{\partial x}\right]_{x=0^+}^{x\to+\infty} + \frac{1}{2}\left[\varphi(s,x)\rho_1 a_1\frac{\partial G(s-r,x,y)}{\partial x}\right]_{x\to-\infty}^{x=0^-}\\ &\quad -\int_{\mathbb{R}^\star}\varphi(s,x)\left[\frac{1}{2}\frac{\partial}{\partial x}\left(\rho(x)a(x)\frac{\partial G(s-r,x,y)}{\partial x}\right)\right]dx\\ &= -\frac{1}{2}\varphi(s,0)\left(\rho_2 a_2\frac{\partial G(s-r,0^+,y)}{\partial x}-\rho_1 a_1\frac{\partial G(s-r,0^-,y)}{\partial x}\right)\\ &\quad -\int_{\mathbb{R}^\star}\varphi(s,x)\mathcal{L}G(s-r,x,y)\rho(x)\,dx.\end{aligned}$$

By substituting the expressions for I_s and R_s into the space-time integral, we obtain the final form for Term 2:

$$\begin{aligned}\text{Term 2} = &-\int_r^T\int_{\mathbb{R}^\star}\varphi(s,x)\mathcal{L}G(s-r,x,y)\rho(x)\,dx\,ds\\ &-\frac{1}{2}(\rho_2 a_2-\rho_1 a_1)\int_r^T G(s-r,0,y)\frac{\partial\varphi(s,0)}{\partial x}\,ds\\ &-\frac{1}{2}\int_r^T\varphi(s,0)\left(\rho_2 a_2\frac{\partial G(s-r,0^+,y)}{\partial x}-\rho_1 a_1\frac{\partial G(s-r,0^-,y)}{\partial x}\right)ds.\end{aligned}$$

By the construction of the fundamental solution G, it satisfies the governing equation $\partial_s G=\mathcal{L}G$ on $\mathbb{R}^\star\times(r,T]$, which implies:

$$\int_r^T\int_{\mathbb{R}^\star}\left(\frac{\partial G(s-r,x,y)}{\partial s}-\mathcal{L}G(s-r,x,y)\right)\varphi(s,x)\rho(x)\,dx\,ds=0.$$

Furthermore, by construction, the fundamental solution G honors the transmission condition across the interface $x=0$:

$$\rho_1 a_1\frac{\partial G(s-r,0^-,y)}{\partial x}=\rho_2 a_2\frac{\partial G(s-r,0^+,y)}{\partial x}. \tag{52}$$

This property ensures that the boundary integral involving $\varphi(s,0)$ in the expansion of R_s vanishes identically. To simplify the final expression for the kernel $D(\varphi,r,y)$, we define the jump constant $[\rho a]_0:=\rho_2 a_2-\rho_1 a_1$. Consequently, the kernel simplifies to:

$$D(\varphi,r,y)=\varphi(r,y)\rho(y)-\frac{1}{2}[\rho a]_0\int_r^T G(s-r,0,y)\frac{\partial\varphi(s,0)}{\partial x}\,ds. \tag{53}$$

Substituting the simplified expression for $D(\varphi,r,y)$ back into the LHS of (48) and interchanging the order of integration via the almost sure Fubini theorem established in Step 1, we obtain:

$$LHS\overset{\text{a.s.}}{=}\int_0^T\left[\int_{\mathbb{R}}\varphi(r,y)\rho(y)\,dL^d(y)\right]dr$$

$$+\frac{1}{2}[\rho a]_0 \int_0^T \frac{\partial \varphi(s,0)}{\partial x}\left[\int_0^s \int_{\mathbb{R}} G(s-r,0,y)\, dL^d(y)\, dr\right] ds$$
$$\overset{\text{a.s.}}{=} \int_{[0,T]\times\mathbb{R}} \varphi(s,x)\rho(x)\, dL^d(x)\, ds + \frac{1}{2}[\rho a]_0 \int_0^T u(s,0)\frac{\partial \varphi(s,0)}{\partial x}\, ds.$$

In the final equality, we identify the expression within the brackets as the mild solution evaluated at the interface, $u(s,0)$, and relabel the integration variables (r,y) as (s,x) to match the standard weak notation. This identity exactly matches the Right-Hand Side of the weak formulation (47), which completes the proof. □

We now establish the converse result to demonstrate the full equivalence between the weak and mild solution concepts for Equation (3).

Theorem 3.9. *Every weak solution of the SPDE* (3) *is almost surely equivalent to the mild solution.*

Proof. Having established in Theorem 3.8 that every mild solution satisfies the weak formulation, uniqueness reduces to proving the triviality of the difference $w := u_1 - u_2$ in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$ almost surely. Linearity of the weak formulation ensures w satisfies the homogeneous adjoint system:

$$\int_0^T \int_{\mathbb{R}^\star} w(s,x)\,[-\partial_s\varphi - \mathcal{L}\varphi]\,\rho(x)\,dx\,ds = \frac{1}{2}(\rho_2 a_2 - \rho_1 a_1)\int_0^T w(s,0)\partial_x\varphi(s,0)\,ds, \tag{54}$$

for all $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$. To establish $w = 0$ a.e., fix $\psi \in C_c^\infty\big((0,T)\times(0,+\infty)\big)$ and let φ be the unique solution to the dual backward problem

$$\partial_t\varphi + \frac{1}{2}a_2\partial^2_{xx}\varphi = \psi.$$

To eliminate the interface term, define $\tilde{\varphi} := \varphi\chi$, where $\chi \in C_c^\infty\big((0,T)\times(0,\infty)\big)$ is a cutoff function such that $\chi \equiv 1$ on $\operatorname{supp}\psi$. We apply Friedrichs' space-time mollification $\varphi_\varepsilon := \tilde{\varphi} * \rho_\varepsilon$. For $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\chi, \partial((0,T)\times(0,+\infty)))$, the support of φ_ε is strictly separated from $x = 0$, enforcing vanishing boundary conditions $\varphi_\varepsilon(s,0) = 0$ and $\partial_x\varphi_\varepsilon(s,0) = 0$.

Substituting φ_ε into (54), the transmission term vanishes. By the uniform convergence of φ_ε and its derivatives to $\tilde{\varphi}$ and the L^2-integrability of w, the Dominated Convergence Theorem as $\varepsilon \downarrow 0$ yields:

$$\int_0^T \int_0^{+\infty} w(s,x)\left[-\partial_s\tilde{\varphi}(s,x) - \frac{1}{2}a_2\partial^2_{xx}\tilde{\varphi}(s,x)\right]\rho_2\,dx\,ds = 0 \quad \text{a.s.}$$

Since $\tilde{\varphi} = \varphi$ on $\operatorname{supp}\psi$ and φ satisfies the dual PDE, the bracketed term reduces to $-\psi(s,x)$. Thus, the orthogonality condition becomes:

$$\int_{(0,T)\times(0,+\infty)} w(s,x)\,\psi(s,x)\,d\lambda_2(s,x) = 0 \quad \text{a.s.} \tag{55}$$

Exploiting the topological separability of $C_c^\infty\big((0,T)\times(0,+\infty)\big)$, this holds for all ψ outside a null set $N_+ \subset \Omega$. By Theorem 1.2.5 in [8] (which asserts that if f, g

are locally integrable on an open domain X satisfying $\int_X f\phi\,dx = \int_X g\phi\,dx$ for all $\phi \in C_c^\infty(X)$, then $f = g$ almost everywhere), it follows that $w = 0$ λ_2-a.e. on the positive half-line $(0,T)\times(0,+\infty)$.

A symmetric argument on the negative domain $(-\infty,0)$ yields $w = 0$ λ_2-a.e. for $\omega \notin N_-$. Since the interface $\Gamma_0 = (0,T)\times\{0\}$ has Lebesgue measure zero, defining $N = N_+ \cup N_-$ ensures $w = 0$ λ_2-a.e. on $(0,T)\times\mathbb{R}$ for all $\omega \notin N$. This establishes pathwise uniqueness, completing the proof. □

Remark 5. For the purposes of the regularity analysis presented in the following sections, and by utilizing (21) and (25), we can express the mild solution as:

$$u(t,x) = \int_0^t \int_{\mathbb{R}} (I_-^d G)(t-r,x,y)dL(y)\,dr = \int_0^t \left(\int_{\mathbb{R}} (I_-^d G)(\tau,x,y)dL(y) \right) d\tau. \tag{56}$$

4 Solution properties

Beyond well-posedness, a fundamental characterization of the mild solution lies in its second-order structure. The following results provide explicit kernels for the covariance, which serve as a bridge between the fractional noise geometry and the spatial regularity established in the sequel.

Lemma 5. *For any $t > 0$ and $x, z \in \mathbb{R}$, the covariance of the mild solution $u(t,x)$ is given in terms of the heat kernel derivatives by:*

$$\mathbb{E}[u(t,x)u(t,z)] = -\frac{K_d}{2} \int_0^t \int_0^t \iint_{\mathbb{R}^2} \frac{\partial G(t-s_1,x,y_1)}{\partial y_1} \frac{\partial G(t-s_2,z,y_2)}{\partial y_2} \times |y_1-y_2|^{2d+1}\,dy_1\,dy_2\,ds_1\,ds_2, \tag{57}$$

where K_d is the constant associated with the covariance of L^d as defined in (17). Equivalently, this covariance admits the following Riesz kernel representation:

$$\mathbb{E}[u(t,x)u(t,z)] = \alpha_d \int_0^t \int_0^t \iint_{\mathbb{R}^2} G(t-s_1,x,y_1)G(t-s_2,z,y_2) \times |y_1-y_2|^{2d-1}\,dy_1\,dy_2\,ds_1\,ds_2, \tag{58}$$

where the constant α_d is defined as $\alpha_d = d(2d+1)K_d$.

Proof. From Theorem 3.6, the mild solution $u(t,x)$ admits the almost sure representation:

$$u(t,x) = -\int_0^t \int_{\mathbb{R}} L^d(y_1) \frac{\partial G(t-s_1,x,y_1)}{\partial y_1} dy_1 ds_1.$$

Consequently, the product of the solution at two points (x,z) is:

$$\mathbb{E}[u(t,x)u(t,z)] = \int_0^t \int_0^t \iint_{\mathbb{R}^2} \frac{\partial G_1}{\partial y_1} \frac{\partial G_2}{\partial y_2} \mathbb{E}[L^d(y_1)L^d(y_2)] dy_1 dy_2 ds_1 ds_2, \tag{59}$$

where $G_1 = G(t-s_1,x,y_1)$ and $G_2 = G(t-s_2,z,y_2)$.

To justify the application of Fubini's theorem, we verify the absolute integrability of the integrand. By the Cauchy-Schwarz inequality and the variance structure established in Proposition 2.2, we have:

$$\mathbb{E}\left[|L^d(y_1)L^d(y_2)|\right] \leqslant \sqrt{K_d|y_1|^{2d+1} \cdot K_d|y_2|^{2d+1}} = K_d|y_1|^{H_d}|y_2|^{H_d},$$

where $H_d = d + 1/2$. Substituting this into (59) allows us to decouple the integrals. Using the standard Gaussian upper bound for the heat kernel derivative, the spatial integral $\mathcal{I}_s := \int_{\mathbb{R}} |y|^{H_d}|\partial_y G|dy$ satisfies:

$$\mathcal{I}_s \leqslant \mathcal{C}_1(x)(t-s)^{-1/2} + \mathcal{C}_2(t-s)^{-1/2+H_d/2}.$$

As $s \to t$, the singularity is of order $(t-s)^{-1/2}$, which is integrable over $[0,t]$. This confirms the absolute integrability required to invoke Fubini's theorem.

By substituting the explicit covariance of L^d from Proposition 2.2 into (59), we obtain:

$$\begin{aligned}\mathbb{E}[u(t,x)u(t,z)] = \frac{K_d}{2}\int_0^t\int_0^t\iint_{\mathbb{R}^2}\frac{\partial G_1}{\partial y_1}\frac{\partial G_2}{\partial y_2} \\ \times\left(|y_1|^{2d+1} + |y_2|^{2d+1} - |y_1-y_2|^{2d+1}\right)dy_1\,dy_2\,ds_1\,ds_2.\end{aligned} \tag{60}$$

We split the above into three components, I_1, I_2, and I_3. For I_1, associated with $|y_1|^{2d+1}$, the separability of the integrand yields:

$$I_1 = \frac{K_d}{2}\left(\int_0^t\int_{\mathbb{R}}|y_1|^{2d+1}\frac{\partial G_1}{\partial y_1}\,dy_1ds_1\right)\left(\int_0^t\int_{\mathbb{R}}\frac{\partial G_2}{\partial y_2}\,dy_2ds_2\right).$$

By the Fundamental Theorem of Calculus, the second factor vanishes because $\int_{\mathbb{R}}\partial_{y_2}G_2\,dy_2 = 0$. By symmetry, $I_2 = 0$. Thus, the covariance reduces to the derivative form (57):

$$\mathbb{E}[u(t,x)u(t,z)] = I_3 = -\frac{K_d}{2}\int_0^t\int_0^t\iint_{\mathbb{R}^2}\frac{\partial G_1}{\partial y_1}\frac{\partial G_2}{\partial y_2}|y_1-y_2|^{2d+1}\,dy_1dy_2ds_1ds_2.$$

To derive the Riesz form (58), we decompose the spatial domain into $D_+ = \{y_1 > y_2\}$ and $D_- = \{y_1 < y_2\}$. On D_+, the kernel is classically differentiable. Applying integration by parts with respect to y_1:

$$\begin{aligned}\int_{y_2}^{\infty}\partial_{y_1}G_1(y_1-y_2)^{2d+1}dy_1 = \left[G_1(y_1-y_2)^{2d+1}\right]_{y_1=y_2}^{y_1=\infty} \\ -(2d+1)\int_{y_2}^{\infty}G_1(y_1-y_2)^{2d}dy_1.\end{aligned}$$

The boundary term at the diagonal vanishes since $2d+1 > 0$, and the term at infinity vanishes by Gaussian decay. A symmetric argument for D_- and a subsequent integration by parts with respect to y_2 utilizes the identity:

$$\frac{\partial^2}{\partial y_1\partial y_2}|y_1-y_2|^{2d+1} = -(2d+1)(2d)|y_1-y_2|^{2d-1},$$

which holds pointwise in $D_+ \cup D_-$. Since the diagonal $y_1 = y_2$ is a null set, and accounting for the negative signs from the IPP formula and the original expression, we arrive at the representation:

$$\mathbb{E}[u(t,x)u(t,z)] = \alpha_d \int_0^t \int_0^t \iint_{\mathbb{R}^2} G_1 G_2 |y_1 - y_2|^{2d-1}\, dy_1 dy_2 ds_1 ds_2.$$

The integral is finite as the Riesz kernel $|y_1 - y_2|^{2d-1}$ is locally integrable for $d > 0$ (since $2d - 1 > -1$), and the global convergence is ensured by the exponential spatial decay and temporal integrability of the heat kernels. □

Remark 6. It is instructive to compare our covariance structure with the results obtained in [18] for the standard Laplacian. In a homogeneous medium, where $a_1 = a_2$ and $\rho_1 = \rho_2$, the transmission parameter β defined in (27) vanishes identically. In this limit, the fundamental solution $G(t,x,y)$ collapses to the standard Gaussian heat kernel $p(a_1 t, x - y) = \dfrac{1}{\sqrt{2\pi a_1 t}} \exp\left(-\dfrac{(x-y)^2}{2a_1 t}\right)$. Consequently, the representation (58) reduces to the classical covariance of the heat equation driven by space-only fractional noise on the real line. The structural novelty of (58) thus resides in the interplay between the spatial correlation of the fractional Lévy noise and the interface-induced reflections. The parameters a_i and ρ_i effectively "refract" the noise correlation as it propagates across the origin, a phenomenon analytically captured by the skew-diffusion terms in our Green's function profile.

In addition to the general representations established above, the specific geometry of the Green's function allows for a more explicit characterization of the covariance kernel. By performing the temporal integration analytically, the covariance is reduced from a complex space-time iterated integral to a standard double spatial representation involving effective diffusion paths. Based on this analytical reduction, we obtain the following result.

Corollary 4.1. *For any fixed $t > 0$ and $x, z \in \mathbb{R}$, the covariance $\mathbb{E}[u(t,x)u(t,z)]$ admits the following spatial representation:*

$$\mathbb{E}[u(t,x)u(t,z)] = -\frac{K_d}{2} \iint_{\mathbb{R}^2} h(t,x,y_1) h(t,z,y_2) |y_1 - y_2|^{2d+1}\, dy_1 dy_2, \tag{61}$$

where for all $(x,y) \in \mathbb{R}^2$, the kernel h is defined by:

$$h(t,x,y) = \frac{1}{a(y)} \left[\operatorname{sign}(f(x) - f(y)) \operatorname{erfc}\left(\frac{|f(x) - f(y)|}{\sqrt{2t}}\right) - \beta \operatorname{sign}(|f(x)| + |f(y)|) \operatorname{erfc}\left(\frac{|f(x)| + |f(y)|}{\sqrt{2t}}\right)\right].$$

Proof. By Lemma 5, the covariance is expressed through the spatial derivatives of the Green's function. Applying Fubini's theorem to decouple the temporal and spatial integrals, we obtain (61) with:

$$h(t,x,y) = \int_0^t \frac{\partial G(s,x,y)}{\partial y} ds.$$

According to Lemma 1, for $y \neq 0$, the partial derivative of G is given by:

$$\frac{\partial G(s,x,y)}{\partial y} = \frac{1}{a(y)} \left[-\frac{\partial p}{\partial z}(s, A(x,y)) + \beta \frac{\partial p}{\partial z}(s, B(x,y)) \right], \tag{62}$$

where A and B are defined in (30). To evaluate the time integral, we utilize the following identity for the derivative of the standard heat kernel $p(s,z)$, which holds for all $z \in \mathbb{R}$ and $s > 0$:

$$\int_0^t \frac{\partial p}{\partial z}(s,z)ds = -\text{sign}(z)\,\text{erfc}\left(\frac{|z|}{\sqrt{2t}}\right). \tag{63}$$

We now integrate each term of (62) over $s \in [0,t]$:

- First term: Substituting $z = A(x,y)$ into the identity (63) yields:

$$\begin{aligned}\int_0^t -\frac{\partial p}{\partial z}(s,A)ds &= -\left[-\text{sign}(A)\,\text{erfc}\left(\frac{|A|}{\sqrt{2t}}\right)\right] \\ &= \text{sign}(f(x)-f(y))\,\text{erfc}\left(\frac{|f(x)-f(y)|}{\sqrt{2t}}\right).\end{aligned}$$

- Second term: Substituting $z = B(x,y)$ into the identity (63) yields:

$$\begin{aligned}\int_0^t \beta\frac{\partial p}{\partial z}(s,B)ds &= \beta\left[-\text{sign}(B)\,\text{erfc}\left(\frac{|B|}{\sqrt{2t}}\right)\right] \\ &= -\beta\,\text{sign}(|f(x)|+|f(y)|)\,\text{erfc}\left(\frac{|f(x)|+|f(y)|}{\sqrt{2t}}\right).\end{aligned}$$

Combining these two components results in the expression for $h(t,x,y)$ provided in the statement. Note that while the derivative $\partial_y G$ is formally defined for $y \neq 0$, the resulting kernel h is well-defined and continuous across the interface for all $(x,y) \in \mathbb{R}^2$, confirming the validity of the representation. □

Remark 7. The explicit representation in Corollary 4.1 constitutes a significant analytical refinement over the general covariance structure established in preceding results. By performing the temporal integration analytically, the covariance is reduced from a complex space-time iterated integral to a standard double spatial integral involving the closed-form kernel $h(t,x,y)$. This characterization is particularly beneficial as it provides an explicit mapping of how the transmission parameter β and the diffusion coefficients $a(y)$ locally distort the fractional noise correlation. While previous representations define the covariance implicitly through the fundamental solution, this result isolates the structural influence of the interface into a single, computable spatial profile defined by the complementary error function.

4.1 Spatial sample path regularity

In this section, we establish the Hölder continuity of the spatial sample paths of the mild solution. The regularity is determined by the competition between the roughness of the Lévy noise and the smoothing effect of the fractionalized heat kernel.

Theorem 4.2. *Let $u(t,x)$ be the mild solution to the fractional SPDE (3). For any fixed $t \in (0,T]$: The stochastic process $x \mapsto u(t,x)$ possesses a modification which is almost surely locally Hölder continuous on any compact set $K \subset \mathbb{R}$ with exponent γ for all:*

$$\gamma < d. \tag{64}$$

Proof. The proof proceeds by establishing a sharp $L^2(\Omega)$ estimate for the spatial increments of the mild solution, followed by the application of the Kolmogorov-Chentsov continuity criterion.
Step 1: Applying the L^2-isometry established in (12) to the stochastic integral representation (56), we evaluate the second moment of the spatial increment as:

$$\mathbb{E}|\Delta_{x,z}u(t)|^2 = \mathcal{C}_L^2 \int_{\mathbb{R}} \left| \int_0^t g(\tau,x,z,y)\, d\tau \right|^2 dy, \tag{65}$$

where $\mathcal{C}_L^2$ is the variance constant defined in (8). To decouple the temporal and spatial dependencies, we apply Minkowski's integral inequality, which yields:

$$\begin{aligned} \left(\int_{\mathbb{R}} \left| \int_0^t g(\tau,x,z,y)\, d\tau \right|^2 dy \right)^{1/2} &\leqslant \int_0^t \left(\int_{\mathbb{R}} |g(\tau,x,z,y)|^2\, dy \right)^{1/2} d\tau \\ &= \int_0^t \|g(\tau,x,z,\cdot)\|_{L^2(\mathbb{R})}\, d\tau. \end{aligned} \tag{66}$$

By squaring (66) and applying the Cauchy-Schwarz inequality to the resulting temporal integral over $[0,t]$, we conclude:

$$\mathbb{E}|\Delta_{x,z}u(t)|^2 \leqslant \mathcal{C}_L^2 \cdot t \int_0^t \|g(\tau,x,z,\cdot)\|_{L^2(\mathbb{R})}^2\, d\tau. \tag{67}$$

Step 2: We decompose the kernel difference as $g(\tau,x,z,y) = g_{\text{main}} + \Delta_{x,z}\mathcal{R}$;

$$g_{\text{main}} = \frac{\tau^{(d-1)/2}}{\sqrt{a(y)}} \left[\Delta_{x,z}K_- \left(\frac{f(y)}{\sqrt{\tau}} \right) + \beta\, \text{sign}(y) \Delta_{x,z}K_+ \left(\frac{|f(y)|}{\sqrt{\tau}} \right) \right], \tag{68}$$

where the kernel differences are defined as:

$$\Delta_{x,z}K_- := \int_0^\infty \left[\varphi\left(\frac{f(y)-f(x)}{\sqrt{\tau}} - w \right) - \varphi\left(\frac{f(y)-f(z)}{\sqrt{\tau}} - w \right) \right] \frac{w^{d-1}}{\Gamma(d)} dw,$$

$$\Delta_{x,z}K_+ := \int_0^\infty \left[\varphi\left(\frac{|f(y)|-|f(x)|}{\sqrt{\tau}} + w \right) - \varphi\left(\frac{|f(y)|-|f(z)|}{\sqrt{\tau}} + w \right) \right] \frac{w^{d-1}}{\Gamma(d)} dw,$$

and

$$\Delta_{x,z}\mathcal{R} := \begin{cases} 0 & \text{if } y > 0 \\ \mathcal{R}(x,y,\tau) - \mathcal{R}(z,y,\tau) & \text{if } y < 0, \end{cases} \tag{69}$$

with

$$\mathcal{R}(x,y,\tau) = \frac{z_{1,x}^d}{2\Gamma(d)} \varphi\left(\frac{A(x,y)}{\sqrt{\tau}} - z_{1,x} \right) + \frac{z_{2,x}^d}{2\Gamma(d)} \left[\varphi(\alpha_x - z_{2,x}) + \beta\varphi(\mu_x + z_{2,x}) \right].$$

Here, $z_{i,x} = -y/\sqrt{a_i\tau}$, $\alpha_x = (f(x) - y/\sqrt{a_2})/\sqrt{\tau}$, and $\mu_x = (|f(x)| + y/\sqrt{a_2})/\sqrt{\tau}$. These expressions are derived directly from Lemma 3 by applying the Fundamental Theorem of Calculus to the truncated integrals: the difference $(I_-^d G)(x) - (I_-^d G)(z)$ isolates the bulk behavior (g_{main}) from the boundary contributions evaluated at the interface-dependent limits $L(\tau) = -y/\sqrt{a_1\tau}$ and $\lambda(\tau) = -y/\sqrt{a_2\tau}$.

Applying the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$, we obtain

$$\|g_{\text{main}}\|^2_{L^2(\mathbb{R})} \leqslant C\tau^{d-1}\delta J(\theta),$$

with

$$J(\theta) := \sqrt{\theta}\|(K_- + \beta\,\text{sign}(y)K_+)(\cdot + \theta^{-1/2}) - (K_- + \beta\,\text{sign}(y)K_+)(\cdot)\|^2_{L^2}, \tag{70}$$

where $\delta = |f(x) - f(z)|$ and $\theta = \tau/\delta^2$.

To bound the residual increment $\Delta_{x,z}\mathcal{R}$, we perform the change of variables $\xi = -y/\sqrt{\tau}$, which maps the integration domain $(-\infty, 0)$ to $(0, \infty)$.

Linearizing the increment $\Delta_{x,z}\varphi$ via the Mean Value Theorem introduces the spatial scaling factor $\delta/\sqrt{\tau}$. The residual bound is then:

$$\int_{-\infty}^{0} |\Delta_{x,z}\mathcal{R}|^2 dy = \sqrt{\tau}\int_0^{\infty} |\Delta_{x,z}\mathcal{R}|^2 d\xi \leqslant C\sqrt{\tau}\left(\frac{\delta}{\sqrt{\tau}}\right)^2 \int_0^{\infty} \xi^{2d}|\varphi'(\dots)|^2 d\xi.$$

Given that $\xi^{2d}|\varphi'|^2$ is integrable, the integral yields a constant. Substituting $\delta/\sqrt{\tau} = \theta^{-1/2}$, we obtain:

$$\int_{-\infty}^{0} |\Delta_{x,z}\mathcal{R}|^2 dy \leqslant C\delta J(\theta).$$

Combining the main kernel scaling and residual bounds, $\|g\|^2_{L^2(\mathbb{R})} \leqslant C'\tau^{d-1}\delta J(\theta)$ holds for all y, where $C' = 12\sqrt{a_{\max}}/a_{\min}$. Substituting this into the isometry (67) and setting $\tau = \delta^2\theta$ yields:

$$\mathbb{E}|\Delta_{x,z}u(t)|^2 \leqslant \delta^{2d+1}\frac{C'C_L^2 t\sqrt{a_{\max}}}{a_{\min}}\int_0^{t/\delta^2} \theta^{d-1}J(\theta)\,d\theta. \tag{71}$$

Step 3: We evaluate the integral $\mathcal{I}(\delta) = \displaystyle\int_0^{t/\delta^2} \theta^{d-1}J(\theta)\,d\theta$ by analyzing the asymptotic behavior of the integrand across two distinct regimes. Throughout this step, we denote

$$\mathcal{K} := K_- + \beta\,\text{sign}(y)K_+.$$

1. The Small-Scale Regime ($\theta \to 0$): Applying Properties 1 and 3 of Lemma 6 to the composite kernel $\mathcal{K}$, we have

$$\|\mathcal{K}(\cdot + h) - \mathcal{K}(\cdot)\|^2_{L^2} \leqslant 2\|\mathcal{K}\|^2_{L^2}.$$

This implies:

$$J(\theta) = \sqrt{\theta}\|\mathcal{K}(\cdot + \theta^{-1/2}) - \mathcal{K}(\cdot)\|^2_{L^2} \leqslant 2\|\mathcal{K}\|^2_{L^2}\sqrt{\theta}. \tag{72}$$

Consequently, near the origin, the integrand satisfies $\theta^{d-1}J(\theta) \leqslant C\theta^{d-1/2}$. Since $d > 0$, the exponent $d - 1/2 > -1$, which ensures that the integral $\int_0^1 \theta^{d-1}J(\theta)d\theta$ converges to a finite constant C_1 independent of δ.

2. The Large-Scale Regime ($\theta \to \infty$): For large θ, the spatial shift $h = \theta^{-1/2}$ vanishes. Invoking Property 4 of Lemma 6 for $\mathcal{K}$, we get:

$$J(\theta) \leqslant \sqrt{\theta}\left(\|\mathcal{K}'\|^2_{L^2}(\theta^{-1/2})^2\right) = \|\mathcal{K}'\|^2_{L^2}\theta^{-1/2}. \tag{73}$$

The integrand for the tail behavior is thus bounded by $\theta^{d-1} \cdot \theta^{-1/2} = \theta^{d-3/2}$. For $d < 1/2$, we have $d - 3/2 < -1$, which guarantees the convergence of the tail integral:

$$\int_1^{t/\delta^2} \theta^{d-1}J(\theta)d\theta \leqslant \|\mathcal{K}'\|^2_{L^2}\int_1^{\infty} \theta^{d-3/2}d\theta = \frac{\|\mathcal{K}'\|^2_{L^2}}{1/2 - d} < \infty.$$

Combining these results, the integral $\mathcal{I}(\delta)$ remains bounded by a constant $\tilde{C}$ that is independent of δ as $\delta \to 0$. Substituting this into (71), we conclude:

$$\mathbb{E}|\Delta_{x,z}u(t)|^2 \leqslant C(t)\delta^{2d+1}\mathcal{I}(\delta) \leqslant \tilde{C}(t)\delta^{2d+1}. \tag{74}$$

Step 4: From (74), we have:

$$\mathbb{E}|\Delta_{x,z}u(t)|^2 \leqslant \tilde{C}(t)|f(x) - f(z)|^{2d+1}. \tag{75}$$

Note that the mapping f is globally Lipschitz continuous with

$$|f(x) - f(z)| \leqslant a_{\min}^{-1/2}|x - z|.$$

Substituting this Lipschitz bound into (75) yields the final moment estimate in the Euclidean metric:

$$\mathbb{E}|u(t,x) - u(t,z)|^2 \leqslant C^*(t)|x - z|^{2d+1}, \tag{76}$$

where $C^*(t) = \tilde{C}(t)a_{\min}^{-(d+1/2)}$.

Applying the Kolmogorov-Chentsov theorem, there exists a modification of the process $x \mapsto u(t,x)$ that is almost surely Hölder continuous with exponent γ for any $\gamma < d$. □

Remarks 4.3. Theorem 4.2 prompts three main structural and physical observations regarding the fine-scale behavior of the mild solution:

1. It is noteworthy that the presence of the interface at $x = 0$ and the discontinuity of the diffusion coefficient $a(x)$ do not degrade the Hölder regularity of the solution. The smoothing effect of the generalized fractional operator $(I_-^d G)$ remains robust across the entire domain, ensuring that the sample path continuity is precisely governed by the fractional order of the driving noise.

2. From an analytical and physical viewpoint, the fact that the spatial exponent γ remains completely independent of the transmission parameter β reflects a fundamental separation of scales. Hölder continuity is an intrinsically microscopic, high-frequency property that characterises the infinitesimal fluctuations of the random field. Conversely, the parameter β governs a macroscopic transmission condition, maintaining the global mass-flux balance and regulating how diffusion paths refract or reflect across the origin. As the spatial increment scale shrinks ($|x - z| \to 0$), the localized regularizing mechanism of the fractionalized heat kernel dominates the path dynamics entirely. The boundary reflections induced by β merely alter the directional amplitude and the localized variance profile of the noise propagation, acting as a bounded residual factor that leaves the intrinsic fractal dimension of the sample paths unaffected.

3. While Theorem 4.2 guarantees that the sample paths remain Hölder continuous everywhere up to order d (including at the interface), a distinct and challenging question arises regarding the mathematical proof of *sharpness* exactly at $x_0 = 0$. At this transmission boundary, the localization technique deployed in the proof of Theorem 4.2 no longer applies because the boundary residual increment $\Delta_{x,z}\mathcal{R}$ does not vanish; rather, the direct singular kernel and the reflected components interact non-trivially, balanced by the transmission coefficient β. Investigating whether the exponent d remains strictly optimal at $x_0 = 0$ or if the interface induces a fine-scale smoothing effect constitutes a perspective for future research.

   Regarding the general case, the exponent $\gamma < d$ represents the sharpest upper bound derivable from our current stochastic integral representation. Establishing its strict optimality remains a delicate open question. Specifically, establishing a rigorous pathwise lower bound would require two main analytical steps: first, deriving a sharp local lower estimate for the incremental variance of the form $\mathbb{E}|u(t,x) - u(t,z)|^2 \geq C(x_0)|x - z|^{2d+1}$ on compact subdomains $K(x_0) \subset \mathbb{R}^*$ centered at a fixed spatial reference point $x_0 \neq 0$ away from the interface $x_0 = 0$ by showing that the direct singular kernel strictly dominates boundary reflection perturbations; and second, applying sample-path transfer mechanisms—such as demonstrating the divergence of the fractional Sobolev norm $\mathbb{E}[[u(t,\cdot)]^2_{W^{d+1/2,2}(K(x_0))}] = +\infty$ or utilizing anti-concentration bounds for non-Gaussian noise—to convert this second-moment behavior into almost sure sample-path non-regularity. Such fine-scale pathwise lower estimates are beyond the scope of this work.

4.2 Temporal sample path regularity

In this section, we establish the Hölder continuity of the mild solution with respect to the temporal variable.

Theorem 4.4. *Let $u(t,x)$ be the mild solution to (3) driven by the fractional Lévy noise L^d with $d \in (0, 1/2)$. For any fixed $x \in \mathbb{R}$ and $T > 0$: The process $t \mapsto u(t,x)$ admits a modification which is almost surely locally Hölder continuous on $[0,T]$ with exponent β for all: $\beta < \dfrac{2d+1}{4}$.*

Proof. Fix $x \in \mathbb{R}$ and let $0 \leqslant s < t \leqslant T$. We define the temporal increment $\Delta_{s,t}u(x) := u(t,x) - u(s,x)$. Using the representation in Remark 5, we decompose the increment as:

$$\begin{aligned}\Delta_{s,t}u(x) &= \int_s^t \left(\int_{\mathbb{R}} (I_-^d G)(t-r,x,y)\, dL(y) \right) dr \\ &\quad + \int_0^s \left(\int_{\mathbb{R}} \left[(I_-^d G)(t-r,x,y) - (I_-^d G)(s-r,x,y) \right] dL(y) \right) dr \\ &:= \mathcal{T}_A(s,t) + \mathcal{T}_B(s,t).\end{aligned}$$

By the triangle inequality in $L^2(\Omega)$, we estimate each term independently.

Step 1: Estimation of $\mathcal{T}_A$. Proceeding as in the spatial regularity analysis, we apply the Stochastic Fubini Theorem and the Itô isometry for Lévy martingales. Utilizing Minkowski's integral inequality, we have:

$$\|\mathcal{T}_A(s,t)\|_{L^2(\Omega)} \leqslant \mathcal{C}_L \int_s^t \|(I_-^d G)(t-r,x,\cdot)\|_{L^2(\mathbb{R})}\, dr.$$

By invoking the $L^2(\mathbb{R})$ temporal scaling bound established in Theorem 3.2, the integral becomes:

$$\|\mathcal{T}_A(s,t)\|_{L^2(\Omega)} \leqslant C \int_0^{t-s} \sigma^{\frac{2d-1}{4}}\, d\sigma = C(t-s)^{\frac{2d+3}{4}}.$$

Step 2: Estimation of $\mathcal{T}_B$. Let $\delta = t - s > 0$. Following the same procedure as in Step 1, we utilize the linearity of the stochastic integral and the isometry property to write:

$$\|\mathcal{T}_B(s,t)\|_{L^2(\Omega)} \leqslant \mathcal{C}_L \int_0^s \|(I_-^d G)(t-r,x,\cdot) - (I_-^d G)(s-r,x,\cdot)\|_{L^2(\mathbb{R})}\, dr.$$

By performing the change of variables $\theta = s - r$, we obtain:

$$\begin{aligned}\|\mathcal{T}_B(s,t)\|_{L^2(\Omega)} &\leqslant \mathcal{C}_L \int_0^s \|(I_-^d G)(\theta+\delta,x,\cdot) - (I_-^d G)(\theta,x,\cdot)\|_{L^2(\mathbb{R})}\, d\theta \\ &= \mathcal{C}_L\,(J_1 + J_2)\,;\end{aligned}$$

$$\begin{aligned}J_1 &= \int_0^\delta \|(I_-^d G)(\theta+\delta,x,\cdot) - (I_-^d G)(\theta,x,\cdot)\|_{L^2(\mathbb{R})}\, d\theta, J_2 \\ &= \int_\delta^s \|(I_-^d G)(\theta+\delta,x,\cdot) - (I_-^d G)(\theta,x,\cdot)\|_{L^2(\mathbb{R})}\, d\theta.\end{aligned}$$

Using the triangle inequality and the L^2-scaling bound from Theorem 3.2:

$$J_1 \leqslant C \int_0^\delta \left((\theta+\delta)^{\frac{2d-1}{4}} + \theta^{\frac{2d-1}{4}} \right) d\theta \leqslant C \int_0^{2\delta} \tau^{\frac{2d-1}{4}}\, d\tau = C\delta^{\frac{2d+3}{4}}.$$

By applying the Mean Value Theorem and using Corollary 3.5, we obtain:

$$J_2 = \mathcal{C}_L \int_\delta^s \|(I_-^d G)(\theta+\delta,x,\cdot) - (I_-^d G)(\theta,x,\cdot)\|_{L^2(\mathbb{R})}\, d\theta$$

$$\leqslant C_L \int_\delta^s \left(\int_\theta^{\theta+\delta} \left\| \frac{\partial}{\partial \phi} (I_-^d G)(\phi, x, \cdot) \right\|_{L^2(\mathbb{R})} d\phi \right) d\theta$$
$$\leqslant C \int_\delta^s \left(\int_\theta^{\theta+\delta} \phi^{\frac{2d-5}{4}} d\phi \right) d\theta.$$

Since the function $\phi \mapsto \phi^{\frac{2d-5}{4}}$ is decreasing for $\phi > 0$, the inner integral is bounded by $\delta\theta^{\frac{2d-5}{4}}$. Consequently:

$$J_2 \leqslant C\delta \int_\delta^s \theta^{\frac{2d-5}{4}} d\theta \leqslant C\delta \left[\theta^{\frac{2d-1}{4}}\right]_\delta^s \leqslant C\delta \cdot \delta^{\frac{2d-1}{4}} = C\delta^{\frac{2d+3}{4}}.$$

Combining the estimates for J_1 and J_2, we conclude that

$$\|\mathcal{T}_B(s,t)\|_{L^2(\Omega)} \leqslant C|t-s|^{\frac{2d+3}{4}}.$$

Step 3: Conclusion. Combining the estimates for $\mathcal{T}_A$ and $\mathcal{T}_B$, we establish the $L^2(\Omega)$ moment bound:

$$\mathbb{E}|u(t,x) - u(s,x)|^2 \leqslant C|t-s|^{\frac{2d+3}{2}}.$$

By the Kolmogorov-Chentsov continuity criterion, the sample paths are almost surely Hölder continuous with exponent $\beta < \frac{\frac{2d+3}{2}-1}{2} = \frac{2d+1}{4}$. □

Remark 8. Theorem 4.4 raises critical considerations regarding the boundary behavior and the sharpness of the temporal Hölder exponent β:

1. When the observation point lies exactly on the transmission interface ($x_0 = 0$), the geometric configuration alters the asymptotic balance. Specifically, the reflected component of the kernel no longer exhibits the exponential damping effect that prevails in the bulk of the domain. Consequently, the reflected term cannot be treated as a higher-order non-singular perturbation of order $\mathcal{O}\left(|t-s|^2\right)$. Instead, the direct and reflected kernel components merge at the same asymptotic scale. This constructive interaction modifies the structural constant $C(s)$ through the transmission coefficient β, highlighting the complex role of the interface matching conditions on local sample path dynamics.

2. Regarding the optimality of the exponent $\beta < (2d+1)/4$, this bound represents the sharpest upper limit derivable from our current stochastic integral representation. Establishing strict pathwise optimality remains a delicate open question. A rigorous lower bound proof would necessitate a precise variance estimate that accounts for the intricate interplay between the direct and reflected kernel components under the interface matching conditions described in item (i). Such an analysis is beyond the scope of this work and constitutes an interesting perspective for future investigation.

4.3 Illustrative example: truncated symmetric α-stable fractional noise

To illustrate the concrete applicability of our abstract framework, we consider the case where the driving fractional Lévy noise L^d is constructed from a truncated symmetric

α-stable process $L(t) = Z_{\alpha,R}(t)$ on $\mathbb{R}$, with stability index $\alpha \in (1,2)$ and a fixed maximum jump size $R > 0$.

This localized jump process perfectly satisfies every structural hypothesis formulated in our main framework:

1. The process $Z_{\alpha,R}$ possesses no Brownian component and its Lévy measure $\nu_R(dx)$ is obtained by restricting the standard symmetric α-stable measure to the compact domain $[-R, R]$:

$$\nu_R(dx) = A_\alpha |x|^{-(1+\alpha)} \mathbb{1}_{\{|x| \leqslant R\}}\, dx, \quad \text{where } A_\alpha = \frac{\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi}. \quad (77)$$

   Here, A_α is the standard density normalization constant for the Lévy measure of a symmetric α-stable process (see Sato [20, Example 14.7]). We note that A_α is directly related to the tail-asymptotic constant $C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$ given in Samorodnitsky and Taqqu [19, Property 1.2.15] via the identity $A_\alpha = \frac{\alpha}{2} C_\alpha$.

   Because ν_R supported on a compact set, the outer second-moment condition (4) is holds trivially:

$$\int_{|x|>1} x^2 \nu_R(dx) = \int_{1<|x|\leqslant R} x^2 \nu_R(dx) < \infty. \quad (78)$$

2. Due to the strict spatial symmetry of the truncated measure $\nu_R(dx)$ around the origin, the integrated drift component satisfies $\gamma = -\int_{|x|>1} x\nu_R(dx) = 0$. Consequently, the centering condition $\mathbb{E}[L(1)] = 0$ holds, and the characteristic exponent $\psi(u)$ defined in (7) simplifies directly to the real-valued form:

$$\psi(u) = \int_{-R}^{R} (\cos(ux) - 1)\, \nu_R(dx). \quad (79)$$

3. The micro-structural variance constant $\mathcal{C}_L^2$ required by the noise L^2-isometry (8) is strictly finite. Since $\alpha < 2$, direct integration yields the explicit value:

$$\mathcal{C}_L^2 = \int_{\mathbb{R}} x^2 \nu_R(dx) = 2A_\alpha \int_0^R x^{1-\alpha}\, dx = \frac{2A_\alpha R^{2-\alpha}}{2-\alpha} < +\infty. \quad (80)$$

This explicit expression for $\mathcal{C}_L^2$ can be directly substituted into the variance upper bounds of Theorems 4.2 and 4.4, thereby providing a fully explicit parameterization of the Hölder regularity constants for non-Gaussian jump models.

5 Appendix

Lemma 6. *Let $d \in (0, 1/2)$. Let $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ be the kernel defined by the right-sided Riemann-Liouville fractional integral of the standard Gaussian density $\phi_0(x) = (2\pi)^{-1/2} e^{-x^2/2}$:*

$$\mathcal{K}(\eta) := \frac{1}{\Gamma(d)} \int_\eta^\infty \phi_0(\zeta)(\zeta - \eta)^{d-1} d\zeta. \quad (81)$$

The following properties hold and uniquely justify the spatial regularity of the mild solution:

1. *$\mathcal{K} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. It is non-negative, symmetric ($\mathcal{K}(\eta) = \mathcal{K}(-\eta)$), and unimodal.*

2. *The function $\phi(h) := \|\mathcal{K}(\cdot + h) - \mathcal{K}(\cdot)\|^2_{L^2(\mathbb{R})}$ is strictly non-decreasing for $h \in [0, \infty)$.*

3. *For all $h \in \mathbb{R}$, $\phi(h) \leqslant 2\|\mathcal{K}\|^2_{L^2}$.*

4. *$\mathcal{K}$ is globally Lipschitz continuous with $\mathcal{K}' \in L^2(\mathbb{R})$, satisfying*

$$\phi(h) \leqslant h^2 \|\mathcal{K}'\|^2_{L^2}.$$

Proof. 1. By the substitution $u = \zeta - \eta$, we can write the kernel as a convolution $\mathcal{K} = \phi_0 * \rho_d$, where $\rho_d(u) = \dfrac{1}{\Gamma(d)} u^{d-1} \mathbb{1}_{\{u>0\}}$ denotes the weighted power-law kernel.

As the convolution of a symmetric unimodal Gaussian density and a decreasing power-law kernel, $\mathcal{K}$ inherits non-negativity and unimodality. Since $\phi_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and ρ_d is locally integrable with power-law decay at the origin and infinity, Young's inequality for convolutions ensures $\mathcal{K} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Symmetry follows from the properties of the Riemann-Liouville integral of the Gaussian.

2. Expanding the square, we obtain:

$$\phi(h) = \int_{\mathbb{R}} \mathcal{K}(\xi + h)^2 d\xi + \int_{\mathbb{R}} \mathcal{K}(\xi)^2 d\xi - 2 \int_{\mathbb{R}} \mathcal{K}(\xi + h)\mathcal{K}(\xi) d\xi = 2\|\mathcal{K}\|^2_{L^2} - 2R_{\mathcal{K}}(h), \tag{82}$$

where $R_{\mathcal{K}}(h) = \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi+h)\mathcal{K}(\xi) d\xi$ is the autocorrelation of $\mathcal{K}$. We now invoke Theorem 1.6 in [6], which establishes that the class of symmetric unimodal distributions is closed under convolution. Since $\mathcal{K}$ is symmetric and unimodal, its convolution with its reflection is also symmetric and unimodal about the origin. Consequently, $R_{\mathcal{K}}$ attains its maximum at $h = 0$ and is non-increasing for $h > 0$. It follows from (82) that $\phi(h)$ is non-decreasing for $h \in [0, \infty)$.

3. Since the kernel $\mathcal{K}$ is non-negative (as the convolution of the Gaussian ϕ_0 and the power-law kernel ρ_d), its autocorrelation $R_{\mathcal{K}}(h)$ is non-negative for all $h \in \mathbb{R}$. Applying thisto (82), we immediately obtain the third assertion.

4. The derivative is given by the convolution $\mathcal{K}' = \phi_0' * \rho_d$. Since $\phi_0'(\zeta) = -\zeta\phi_0(\zeta)$ is a smooth, rapidly decreasing function and $\rho_d(u) = \frac{1}{\Gamma(d)} u^{d-1} \mathbb{1}_{\{u>0\}}$ is locally integrable for $d > 0$, the convolution is well-defined and locally bounded. To justify the square-integrability over $\mathbb{R}$, we first examine the behavior for $\eta > 0$ using the substitution $u = \zeta - \eta$:

$$\begin{aligned} |\mathcal{K}'(\eta)| &= \frac{1}{\sqrt{2\pi}\Gamma(d)} \left| \int_0^\infty (\eta + u) e^{-\frac{(\eta+u)^2}{2}} u^{d-1} \, du \right| \\ &\leqslant \frac{e^{-\eta^2/2}}{\sqrt{2\pi}\Gamma(d)} \int_0^\infty (\eta + u) e^{-u^2/2} u^{d-1} \, du. \end{aligned}$$

Splitting the integral into two components yields:

$$|\mathcal{K}'(\eta)| \leqslant \frac{e^{-\eta^2/2}}{\sqrt{2\pi}\Gamma(d)} \left[\eta \int_0^\infty e^{-u^2/2} u^{d-1} du + \int_0^\infty e^{-u^2/2} u^d du \right].$$

Since $d \in (0, 1/2)$, both integrals in the brackets converge to finite constants, implying the bound

$$|\mathcal{K}'(\eta)| \leqslant C(1 + |\eta|)e^{-\eta^2/2}.$$

This exponential decay ensures that $\mathcal{K}'$ is square-integrable on $(0, \infty)$. By the symmetry of $\mathcal{K}$, its derivative $\mathcal{K}'$ is anti-symmetric, meaning $|\mathcal{K}'(\eta)|^2$ is symmetric; thus, square-integrability extends to the negative axis. Combined with its local boundedness near the origin, we conclude $\mathcal{K}' \in L^2(\mathbb{R})$.

Finally, by the Fundamental Theorem of Calculus, the Cauchy-Schwarz inequality, and the translation invariance of the L^2 norm, we obtain:

$$\begin{aligned} \phi(h) = \int_{\mathbb{R}} \left| \int_0^h \mathcal{K}'(\xi + s) ds \right|^2 d\xi &\leqslant h \int_{\mathbb{R}} \int_0^h |\mathcal{K}'(\xi + s)|^2 ds \, d\xi \\ &= h \int_0^h \left(\int_{\mathbb{R}} |\mathcal{K}'(\xi + s)|^2 d\xi \right) ds \\ &= h \int_0^h \|\mathcal{K}'\|_{L^2}^2 ds = h^2 \|\mathcal{K}'\|_{L^2}^2. \end{aligned}$$

This concludes the proof. □

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