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Asymptotic properties of the heat equation driven by stochastic measure

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Abstract For the heat equations driven by a stochastic measure on $[-L, L]$ with the Dirichlet boundary condition, we prove that solutions tend to the solution of the heat equations defined on $\mathbb{R}$ as $L \to \infty$. The estimate of the convergence rate is obtained. For a stochastic measure, we assume the σ-additivity in probability only.

Keywords Stochastic measure, stochastic heat equation, Dirichlet boundary conditions, asymptotic properties of the solution
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1 Introduction

In this paper, we consider the stochastic heat equation with the Dirichlet boundary condition, which can formally be written as

$$\begin{aligned} &\frac{\partial u^L}{\partial t} = \frac{\partial^2 u^L}{\partial x^2} + f\big(t, x, u^L(t,x)\big) + \sigma(t,x)\frac{\partial \mu}{\partial x}, \\ &u^L(t,-L) = u^L(t,L) = 0, \quad u^L(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $(t,x) \in [0,T] \times [-L,L]$, and μ is a stochastic measure defined on $\mathcal{B}(\mathbb{R})$ (Borel σ-algebra in $\mathbb{R}$). For μ we assume only σ-additivity in probability, assumptions for f, σ and u_0 are given in Section 3. We study the solution to the formal equation (1.1) in the mild form (see (3.1) below).

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We study the convergence as $L \to \infty$, and prove that u^L tends to the solution to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f\big(t, x, u(t,x)\big) + \sigma(t,x)\frac{\partial \mu}{\partial x}, \quad u(0,x) = u_0(x), \tag{1.2}$$

where $(t,x) \in [0,T] \times \mathbb{R}$. The solution to equation (1.2) is also considered in the mild form (see (3.2) below).

As the main result of the paper, we obtain an exponential rate for the pointwise convergence $u^L \to u$. This is a natural result that follows from the form of the Green's functions. Zero Dirichlet conditions represent absorbing boundaries whose effect decays exponentially as the domain expands.

For an equation driven by space-time white noise, this problem was studied in [2, 3], where the Neumann and mixed boundary conditions were also considered.

The difference of the form $|u^L - u|$ is often called a localization error and, in different settings, has been studied in [1, 6]. The impact of localization on some computational problems in financial mathematics is considered in [8].

Theory of equations driven by stochastic measures is developed in [15]. Existence and uniqueness of mild solutions to (1.1) and (1.2), their boundedness and Hölder regularity, are proved in [18] and [17] respectively. Also, generalization of equation (1.2) is considered in [16], where the L_2-continuity of the solution was proved. For the heat equation on a bounded domain in $\mathbb{R}^d$, we can refer to [10].

Different classes of equations driven by space-time white noise are considered in detail in [4], those driven by spatially colored Gaussian noise in [5], and those driven by a Lévy process in [12].

The rest of the paper is organized as follows. Section 2 contains the basic information concerning the integrals with respect to stochastic measures. In Section 3, we formulate the problem and our main assumptions. In Section 4 we obtain an estimate for the difference of the stochastic integrals from our equations. Section 5 contains the main result of the paper.

2 Preliminaries

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and $\mathcal{B}$ a σ-algebra of subsets of X.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

We do not assume the moment existence or martingale properties for SM. We can say that μ is an L_0-valued measure.

Important examples of SMs are random orthogonal measure with finite structural function (see [7, Section 5.3]), α-stable random measures defined on a σ-algebra for $\alpha \in (0,1) \cup (1,2]$ (see [19, Chapter 3]). Many examples of the SMs on $\mathcal{B}(\mathbb{R})$ may be given by the Wiener-type integral

$$\mu(A) = \int_{\mathbb{R}} \mathbf{1}_A(x) f(x)\, dY_x, \tag{2.1}$$

where f is a deterministic function, integrable on $\mathbb{R}$. For instance, we can take in (2.1) any square integrable martingale Y_x, or $Y_x = W_x^H$ – the fractional Brownian motion with Hurst index $H > 1/2$ (see Corollary 1.9.4 in [11]).

Other examples of SMs can be found in Section 1.2.1 [15].

For deterministic measurable functions $f : \mathsf{X} \to \mathbb{R}$, an integral of the form $\int_{\mathsf{X}} f\, d\mu$ is defined and studied in [9, Chapter 7], [15, Chapter 1]. In particular, every bounded measurable f is integrable with respect to any μ.

Below, we will use the following statement.

Lemma 1 (Lemma 3.1 [14], Lemma 2.3 [15]). *Let μ be an SM on $(\mathsf{X}, \mathcal{B})$, $f_k : \mathsf{X} \to \mathbb{R}$, $k \geq 1$, be measurable functions such that $\hat{f}(x) = \sum_{k=1}^{\infty} |f_k(x)|$ is integrable with respect to μ. Then*

$$\sum_{k=1}^{\infty} \left(\int_{\mathsf{X}} f_k \, d\mu \right)^2 < \infty \quad a.s.$$

For any $j \in \mathbb{Z}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(j)} = \big(d_{(k-1)n}^{(j)}, d_{kn}^{(j)}\big], \quad 1 \leq k \leq 2^n.$$

The following lemma is a key tool for estimating a version of the stochastic integral. In the sequel, we assume that SM μ is defined on Borel subsets of $\mathbb{R}$.

Lemma 2 (Lemma 2.8 [15]). *Let Z be an arbitrary set, and the function $q(z,x) : Z \times [j, j+1] \to \mathbb{R}$ be such that all paths $q(z,\cdot)$ are continuous on $[j, j+1]$, $\mathsf{A} \subset (j, j+1]$ be a Borel set. Denote*

$$q_n(z,x) = \sum_{1 \leq k \leq 2^n} q\big(z, d_{(k-1)n}^{(j)}\big) \mathbf{1}_{\Delta_{kn}^{(j)}}(x).$$

Then the random function

$$\eta(z) = \int_{\mathsf{A}} q(z,x)\, d\mu(x), \quad z \in Z,$$

has a version

$$\widetilde{\eta}(z) = \int_{\mathsf{A}} q_0(z,x)\, d\mu(x) + \sum_{n \geq 1} \left(\int_{\mathsf{A}} q_n(z,x)\, d\mu(x) - \int_{\mathsf{A}} q_{n-1}(z,x)\, d\mu(x) \right) \tag{2.2}$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned} \big|\widetilde{\eta}(z)\big| &\leq \big|q(z,j)\mu(\mathsf{A})\big| \\ &\quad + \left\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} \big| q\big(z, d_{kn}^{(j)}\big) - q\big(z, d_{(k-1)n}^{(j)}\big) \big|^2 \right\}^{1/2} \\ &\quad \times \left\{ \sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} \big| \mu\big(\Delta_{kn}^{(j)} \cap \mathsf{A}\big) \big|^2 \right\}^{1/2}. \end{aligned} \tag{2.3}$$

In the sequel, by C and $C(\omega)$ we denote the constants and random constants, respectively, whose exact values are not important. Notations of the kind C_λ, C_δ means that these constants may depend of λ, δ respectively.

3 The problem

We consider (1.1) in the mild form

$$u^L(t,x) = \int_{(-L,L]} G^L(t,x,y)u_0(y)\,dy + \int_0^t \int_{(-L,L]} G^L(t-s,x,y) f\big(s,y,u^L(s,y)\big)\,dy\,ds + \int_{(-L,L]} \int_0^t G^L(t-s,x,y)\sigma(s,y)\,ds\,d\mu(y), \tag{3.1}$$

where equality holds a.s. for each $(t,x) \in [0,T] \times [-L,L]$. Here $G^L(t,x,y)$ is the fundamental solution of the heat equation with the Dirichlet boundary condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (t,x) \in [0,T] \times [-L,L], \quad u(t,-L) = u(t,L) = 0.$$

It is known that

$$G^L(t,x,y) = \sum_{n=-\infty}^{\infty} \big(p(t,x-y+4nL) - p\big(t,x+y+(4n+2)L\big)\big),$$

where

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}$$

is a standard heat kernel (see, for example, [13, Section 8.4.2], [4, Lemma 1.4.1]).

We will refer to the following assumptions on elements of (3.1).

Assumption A1. $u_0(y) = u_0(y,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable and $|u_0(y)| \le C_{u_0}(\omega)$ for some random constant $C_{u_0}(\omega)$.

Assumption A2. $f(s,y,v) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable, and bounded:

$$\big|f(s,y,v)\big| \le C_f$$

for some constant C_f.

Assumption A3. $f(s,y,v)$ is globally Lipschitz in v, y:

$$\big|f(s,y_1,v_1) - f(s,y_2,v_2)\big| \le L_f\big(|v_1 - v_2| + |y_1 - y_2|\big), \quad y_i,\ v_i \in \mathbb{R},$$

for some constant L_f.

Assumption A4. $\sigma(s,y) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable, and

$$\big|\sigma(s,y)\big| \le C_\sigma, \quad \big|\sigma(s,y_1) - \sigma(s,y_2)\big| \le L_\sigma |y_1 - y_2|^{\beta(\sigma)}$$

for some $1/2 < \beta(\sigma) < 1$ and constants $C_\sigma,\ L_\sigma$.

Assumption A5. $|y|^\tau$ is integrable w.r.t. μ on $\mathbb{R}$ for some $\tau > 1/2$.

Assumptions A1–A4 and Theorem 5.1 1) [18] imply that (3.1) has a unique bounded solution.

Also, we consider (1.2) in the mild form

$$u(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)f\big(s,y,u(s,y)\big)\,dy\,ds + \int_{\mathbb{R}}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y). \tag{3.2}$$

The existence and uniqueness of the bounded solution to (3.2) follows from Theorem 5.1 1) [17].

This paper aims to prove that

$$u^L(t,x) \to u(t,x), \quad L \to \infty,$$

for each $(t,x) \in [0,T] \times \mathbb{R}$. Mainly, we will consider $L = i\delta$ for some fixed $\delta > 0$, $i \geq 1$, and obtain the estimate of $|u^L(t,x) - u(t,x)|$ for some versions of the solutions.

In the estimates below, we will use that

$$\left|\frac{\partial p}{\partial y}(t-s,x-y)\right| \leq \frac{C_\lambda}{t-s} e^{-\frac{\lambda(x-y)^2}{t-s}} \tag{3.3}$$

for any $\lambda \in (0,1/4)$ and some constant C_λ.

4 Auxiliary lemma

In the sequel, we will use the notations $D_L = (-L,L]$, $D_L^c = \mathbb{R}\setminus(-L,L]$, $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$.

Consider the stochastic terms

$$\vartheta^L(t,x) = \int_{D_L}\int_0^t G^L(t-s,x,y)\sigma(s,y)\,ds\,d\mu(y),$$
$$\vartheta(t,x) = \int_{\mathbb{R}}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y).$$

In the following statement, we fix any $\delta > 0$, and consider $L = i\delta$, $i \in \mathbb{Z}_+$.

Lemma 3. *Let Assumptions A4 and A5 hold. Then, for each $\delta > 0$ there exist versions of ϑ^L, ϑ such that for all $\omega \in \Omega$ for each $x \in D_L$, $L = i\delta$, $i \in \mathbb{Z}_+$, $0 < t \leq T$, for any $0 < \lambda < \frac{1}{4}$ holds*

$$\big|\vartheta^L(t,x) - \vartheta(t,x)\big| \leq C_\delta(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}. \tag{4.1}$$

Proof. For $x \in D_L$, we will estimate

$$\begin{aligned}\vartheta^L(t,x) - \vartheta(t,x) &= \int_{D_L}\int_0^t G^L(t-s,x,y)\sigma(s,y)\,ds\,d\mu(y)\\ &\quad - \int_{\mathbb{R}}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y)\\ &= \int_{D_L}\int_0^t \big(G^L(t-s,x,y) - p(t-s,x-y)\big)\sigma(s,y)\,ds\,d\mu(y)\\ &\quad - \int_{D_L^c}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y) := I_1 - I_2.\end{aligned}$$

First, consider I_1. For $x, y \in D_L$,

$$G^L(t-s,x,y) - p(t-s,x-y) = \sum_{n\in\mathbb{Z}^*} p(t-s,x-y+4nL) - \sum_{n\in\mathbb{Z}^*} p\big(t-s,x+y+(4n+2)L\big) - p(t-s,x+y+2L). \tag{4.2}$$

For $x, y \in D_L$ we have that $|x+y+2L| \geq L - |x|$.

We will take i large enough such that $|x-y+4nL|, |x+y+(4n+2)L| > T \geq t-s$, $n \in \mathbb{Z}^*$ here. (By Lemma 4.2 [17], ϑ is bounded. By (4.9) of [18], each $\vartheta^{i\delta}$ is bounded. Thus, excluding the finite number of i does not influence (4.1).)

For fixed t, x set

$$q(z,y) = \int_0^t p(t-s,x-y)\sigma(s,y)\,ds, \quad z = (t,x),\ y \in D_L. \tag{4.3}$$

Our considerations will imply that this function is continuous in y; thus, the conditions of Lemma 2 hold (for this purpose, we can use estimate (4.6) below).

Consider

$$q(z,y+h) - q(z,y) = \int_0^t p(t-s,x-y)\big(\sigma(s,y+h) - \sigma(s,y)\big)\,ds + \int_0^t \big(p(t-s,x-y-h) - p(t-s,x-y)\big)\sigma(s,y+h)\,ds := I_{11} + I_{12}.$$

We will use the following simple estimate for $b/t \geq 1$.

$$\int_0^t \frac{1}{r} e^{-\frac{b}{r}} dr \overset{b/r=z}{=} \int_{b/t}^\infty \frac{1}{z} e^{-z} dz \leq \int_{b/t}^\infty e^{-z} dz = e^{-b/t}. \tag{4.4}$$

We get for $|x-y|, |x-y-h| \geq kL - |x| > T$, $h < L$

$$|I_{12}| \overset{A4}{\leq} C_\sigma \int_0^t \big|p(t-s,x-h-y) - p(t-s,x-y)\big| ds \leq C_\sigma \int_0^t \left| \int_{x-h}^x \frac{\partial p(t-s,v-y)}{\partial v}\, dv \right| ds \overset{(3.3)}{\leq} C_\sigma \int_0^t \left(\frac{C_\lambda}{t-s} \int_{x-h}^x e^{-\frac{\lambda|v-y|^2}{t-s}} dv \right) ds$$
$$\overset{t-s=r}{=} C_\sigma C_\lambda \int_{x-h}^x dv \int_0^t \frac{1}{r} e^{-\frac{\lambda|v-y|^2}{r}} dr \overset{(4.4)}{\leq} C_\sigma C_\lambda \int_{x-h}^x e^{-\frac{\lambda|v-y|^2}{t}} dv \leq C_\sigma C_\lambda |h| e^{-\frac{\lambda(kL-|x|)^2}{t}}.$$

Also,

$$|I_{11}| \leq \int_0^t p(t-s,x-y)\big|\sigma(s,y+h) - \sigma(s,y)\big|\,ds \overset{A4}{\leq} C L_\sigma h^{\beta(\sigma)} \int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{4(t-s)}} ds \leq C h^{\beta(\sigma)} \int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac{(kL-|x|)^2}{4(t-s)}} ds$$
$$\leq C h^{\beta(\sigma)} e^{-\frac{(kL-|x|)^2}{4t}} \int_0^t \frac{1}{\sqrt{t-s}} ds \leq C h^{\beta(\sigma)} e^{-\frac{(kL-|x|)^2}{4t}}. \tag{4.5}$$

(The constants C_λ and C may also depend on σ, though this plays no role in our considerations.) Thus, for any $0 < \lambda < \frac{1}{4}$

$$\left|q(z, y+h) - q(z,y)\right| \le C_\lambda h^{\beta(\sigma)} e^{-\frac{\lambda(kL-|x|)^2}{t}}. \tag{4.6}$$

Similarly, consider $p(t-s, x+y+(4n+2)L)$, $n \in \mathbb{Z}^*$, $p(t-s, x+y+2L)$, and for

$$\widetilde{q}(z,y) = \int_0^t \Bigg(\sum_{n\in\mathbb{Z}^*} p(t-s, x-y+4nL) - \sum_{n\in\mathbb{Z}^*} p\big(t-s, x+y+(4n+2)L\big) - p(t-s, x+y+2L) \Bigg) \sigma(s,y)\, ds$$

we get

$$\left|\widetilde{q}(z, y+h) - \widetilde{q}(z,y)\right| \le CC_\lambda h^{\beta(\sigma)} e^{-\frac{\lambda(L-|x|)^2}{t}}, \quad z=(t,x),\ y,\ y+h \in D_L. \tag{4.7}$$

As in (4.5), we obtain

$$\left|q(z,y)\right| \le CC_\lambda e^{-\frac{\lambda(kL-|x|)^2}{t}}.$$

Therefore,

$$\left|\widetilde{q}(z,y)\right| \le CC_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}, \quad z=(t,x),\ y \in D_L. \tag{4.8}$$

We take $\tau > 1/2$ from A5 and some $\beta > 0$ such that $\beta + 1 - 2\beta(\sigma) < 0$. From (2.3), for the version (2.2) of the stochastic integral (taking intervals $[j\delta, (j+1)\delta]$ instead of $[j, j+1]$ and $d_{kn}^{(j)} = (j + k2^{-n})\delta$), we get

$$\begin{aligned}
|I_1| &= \left| \int_{D_L} \int_0^t \big(G^L(t-s,x,y) - p(t-s,x-y)\big)\sigma(s,y)\, ds\, d\mu(y) \right| \\
&= \left| \int_{D_L} \widetilde{q}(z,y)\, d\eta(y) \right| \le \sum_{j\in\mathbb{Z}} \left| \int_{(j\delta,(j+1)\delta]\cap D_L} \widetilde{q}(z,y)\, d\mu(y) \right| \\
&\overset{(2.3),L=i\delta}{\le} \sum_{j:(j\delta,(j+1)\delta]\subset D_L} \left|\widetilde{q}(z, j\delta)\mu\big((j\delta,(j+1)\delta]\big)\right| \\
&\quad + \left\{ \sum_{n\ge 1} 2^{n\beta} \sum_{1\le k\le 2^n} \left|\widetilde{q}\big(z, d_{kn}^{(j)}\big) - \widetilde{q}\big(z, d_{(k-1)n}^{(j)}\big)\right|^2 \right\}^{1/2} \\
&\quad \times \left\{ \sum_{n\ge 1} 2^{-n\beta} \sum_{1\le k\le 2^n} \left|\mu\big(\Delta_{kn}^{(j)}\big)\right|^2 \right\}^{1/2} \\
&\overset{(4.8),(4.7)}{\le} CC_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}} \Bigg[\sum_{j:(j\delta,(j+1)\delta]\subset D_L} \left|\mu\big(j\delta,(j+1)\delta]\big)\right| \\
&\quad + \sum_{j:(j\delta,(j+1)\delta]\subset D_L} \left\{ \sum_{n\ge 1} 2^{n(\beta+1-2\beta(\sigma))} \sum_{1\le k\le 2^n} \left|\mu\big(\Delta_{kn}^{(j)}\big)\right|^2 \right\}^{1/2} \Bigg]
\end{aligned}$$

$$\le CC_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}\Bigg[\bigg(\sum_{j\in\mathbb{Z}}\big(|j|+1\big)^{2\tau}\Big(\mu\big(\big(j\delta,(j+1)\delta\big]\big)\Big)^2\bigg)^{1/2}\bigg(\sum_{j\in\mathbb{Z}}\big(|j|+1\big)^{-2\tau}\bigg)^{1/2}$$
$$+\bigg(\sum_{j\in\mathbb{Z}}\big(|j|+1\big)^{2\tau}\sum_{n\ge1}2^{n(\beta+1-2\beta(\sigma))}\sum_{1\le k\le 2^n}\Big|\mu\big(\Delta_{kn}^{(j)}\big)\Big|^2\bigg)^{1/2}\bigg(\sum_{j\in\mathbb{Z}}\big(|j|+1\big)^{-2\tau}\bigg)^{1/2}\Bigg]. \tag{4.9}$$

Here, the sums with μ have a form $\sum_{l=1}^{\infty}(\int_{\mathbb{R}} f_l\,d\mu)^2$, where

$$\big\{f_l(y),\ l\ge1\big\}=\big\{\big(|j|+1\big)^{\tau}\mathbf{1}_{(j\delta,(j+1)\delta]}(y),\ j\in\mathbb{Z}\big\},$$
$$\big\{f_l(y),\ l\ge1\big\}=\big\{\big(|j|+1\big)^{\tau}2^{n(\beta+1-2\beta(\sigma))/2}\mathbf{1}_{\Delta_{kn}^{(j)}}(y),\ j\in\mathbb{Z},\ n\ge1,\ 1\le k\le 2^n\big\}.$$

We have $\sum_{l=1}^{\infty}|f_l(y)|\le C(|y|+1)^{\tau}$ in both cases. From Lemma 1 it follows that

$$\sum_{l=1}^{\infty}\bigg(\int_{\mathsf{X}} f_l\,d\mu\bigg)^2<+\infty\ \text{a.s.}$$

We take a set $\Omega_1\subset\Omega$ such that for $\omega\in\Omega_1$, this series is finite for both sequences $\{f_l\}$. Thus $\mathsf{P}(\Omega_1)=1$ and

$$|I_1|\le C_\delta(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}} \tag{4.10}$$

holds for all $L=i\delta$, $i\in\mathbb{Z}_+$, and $\omega\in\Omega_1$. The set Ω_1 depends only on μ, δ and β.

In I_2 we have $|x-y|\ge L-|x|$. Again we consider $q(z,y)$ from (4.3), now with $|y|\ge L$. Repeating the consideration from (4.3) to (4.6), we get here

$$\big|q(z,y+h)-q(z,y)\big|\le C_\lambda h^{\beta(\sigma)}e^{-\frac{\lambda(L-|x|)^2}{t}},$$
$$\big|q(z,y)\big|\le CC_\lambda e^{-\frac{\lambda(kL-|x|)^2}{t}},\quad |y|\ge L.$$

Similarly to (4.9), we get

$$|I_2|=\bigg|\int_{D_L^c}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y)\bigg|$$
$$=\bigg|\int_{D_L^c}q(z,y)\,d\eta(y)\bigg|\le\sum_{j\in\mathbb{Z}}\bigg|\int_{(j\delta,(j+1)\delta]\cap D_L^c}q(z,y)\,d\mu(y)\bigg|$$
$$\overset{(2.3),L=i\delta}{\le}\sum_{j:(j\delta,(j+1)\delta]\subset D_L^c}\Big|q(z,j\delta)\mu\big(\big(j\delta,(j+1)\delta\big]\big)\Big|$$
$$+\bigg\{\sum_{n\ge1}2^{n\beta}\sum_{1\le k\le 2^n}\Big|q\big(z,d_{kn}^{(j)}\big)-q\big(z,d_{(k-1)n}^{(j)}\big)\Big|^2\bigg\}^{1/2}$$
$$\times\bigg\{\sum_{n\ge1}2^{-n\beta}\sum_{1\le k\le 2^n}\Big|\mu\big(\Delta_{kn}^{(j)}\big)\Big|^2\bigg\}^{1/2}$$
$$\overset{(4.8),(4.7),L=i\delta}{\le}CC_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}\Bigg[\sum_{j:(j\delta,(j+1)\delta]\subset D_L^c}\Big|\mu\big(j\delta,(j+1)\delta\big]\big)\Big|$$

$$+\sum_{j:(j\delta,(j+1)\delta]\subset D_L^c}\left\{\sum_{n\geq 1}2^{n(\beta+1-2\beta(\sigma))}\sum_{1\leq k\leq 2^n}\left|\mu\left(\Delta_{kn}^{(j)}\right)\right|^2\right\}^{1/2}\Bigg]$$

$$\leq CC_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}\Bigg[\left(\sum_{j\in\mathbb{Z}}\left(|j|+1\right)^{2\tau}\left(\mu\left(\left(j\delta,(j+1)\delta\right]\right)\right)^2\right)^{1/2}\left(\sum_{j\in\mathbb{Z}}\left(|j|+1\right)^{-2\tau}\right)^{1/2}$$

$$+\left(\sum_{j\in\mathbb{Z}}\left(|j|+1\right)^{2\tau}\sum_{n\geq 1}2^{n(\beta+1-2\beta(\sigma))}\sum_{1\leq k\leq 2^n}\left|\mu\left(\Delta_{kn}^{(j)}\right)\right|^2\right)^{1/2}\left(\sum_{j\in\mathbb{Z}}\left(|j|+1\right)^{-2\tau}\right)^{1/2}\Bigg]. \tag{4.11}$$

As in (4.10), we get that

$$|I_2|\leq C_\delta(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}, \tag{4.12}$$

(4.10) and (4.12) imply that (4.1) holds. □

Remark 1. Let SM μ be such that $\mu((a,b])\leq C(\omega)$, $a,\ b\in\mathbb{R}$. Then we can obtain (4.1) for arbitrary $L\in\mathbb{R}_+$, and get

$$\left|\vartheta^L(t,x)-\vartheta(t,x)\right|\leq C(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}},$$

where $C(\omega)$ is independent of L. To prove this, we can fix $\delta=1$, and note that in this case in (4.9) we get additional summands

$$\left|\mu((j_0,L]\cap D_L)\right|+\left|\mu\left((-L,-j_0]\right)\right|$$
$$+\sum_{n\geq 1}2^{-n\beta}\left|\mu\left((d_{k_nn}^{(j_0)},L]\right)\right|+\sum_{n\geq 1}2^{-n\beta}\left|\mu\left(\left(-L,d_{k_nn}^{(-j_0)}\right]\right)\right|$$

for some j_0, k_n that depend on L (and similar summands in (4.11)). If $\mu((a,b])\leq C(\omega)$, then these sums are uniformly bounded.

5 Main result

Theorem 1. *Let Assumptions A1–A5 hold, u^L is the solution to (3.1), u is the solution to (3.2). Then, for each $\delta>0$ there exist versions of u^L, u such that for all $\omega\in\Omega$ for each $x\in D_L$, $L=i\delta$, $i\in\mathbb{Z}_+$, $0<t\leq T$, for any $0<\lambda<\frac{1}{4}$ holds*

$$\left|u^L(t,x)-u(t,x)\right|\leq C_\delta(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}. \tag{5.1}$$

Proof. Consider

$$u^L(t,x)-u(t,x)=\left(\int_{D_L}G^L(t,x,y)u_0(y)\,dy-\int_{\mathbb{R}}p(t,x-y)u_0(y)\,dy\right)$$
$$+\left(\int_0^t\int_{D_L}G^L(t-s,x,y)f\left(s,y,u^L(s,y)\right)dy\,ds\right.$$
$$\left.-\int_0^t\int_{\mathbb{R}}p(t-s,x-y)f\left(s,y,u(s,y)\right)dy\,ds\right)$$

$$+\left(\int_{D_L}\int_0^t G^L(t-s,x,y)\sigma(s,y)\,ds\,d\mu(y)\right.$$
$$\left.-\int_{\mathbb{R}}\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,d\mu(y)\right)$$
$$:= J_1+J_2+J_3,$$

by Lemma 3, for some versions of the integrals, holds

$$|J_3|\le C_\delta(\omega)C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}}.$$

For J_1, we have

$$|J_1|\le\int_{D_L}\left|G^L(t,x,y)-p(t,x-y)\right|\left|u_0(y)\right|dy+\int_{D_L^c}p(t,x-y)\left|u_0(y)\right|dy$$
$$\overset{A1}{\le} C_{u_0}(\omega)\int_{D_L}\left|G^L(t,x,y)-p(t,x-y)\right|dy+C_{u_0}(\omega)\int_{D_L^c}p(t,x-y)\,dy$$
$$:= J_{11}+J_{12}.$$

We recall the Chernoff inequality (see, for instance, Example 2.1 [20]). For random variable $X\sim N(\mu,\sigma^2)$ and $r\ge 0$ holds

$$\mathsf{P}\{X\ge\mu+r\}\le\exp\left\{-\frac{r^2}{2\sigma^2}\right\}. \tag{5.2}$$

We have that $p(t,y-x)$, as a function of y, is the density function of the distribution $N(x,2t)$.

Thus, for $x\in D_L$, we obtain

$$\int_{D_L}p(t,x-y+4nL)\,dy\overset{v=y-4nL}{=}\int_{(-(4n+1)L,-(4n-1)L]}p(t,x-v)\,dv$$
$$\overset{(5.2)}{\le}\exp\left\{-\frac{((4|n|-1)L-|x|)^2}{4t}\right\}.$$

In a similar way, we consider $p(t,x+y+(4n+2)L)$, $p(t,x+y+2L)$ in (4.2), and obtain that

$$J_{11}\le C(\omega)\exp\left\{-\frac{(L-|x|)^2}{4t}\right\}. \tag{5.3}$$

Also, (5.2) implies that

$$J_{12}\le 2C_{u_0}(\omega)\exp\left\{-\frac{(L-|x|)^2}{4t}\right\}. \tag{5.4}$$

For J_2, we get

$$|J_2|\le\int_0^t\int_{D_L}\left|G^L(t-s,x,y)-p(t-s,x-y)\right|\left|f\left(s,y,u^L(s,y)\right)\right|dy\,ds$$

$$
\begin{aligned}
&+ \int_0^t \int_{D_L} p(t-s,x-y)\big|f\big(s,y,u^L(s,y)\big) - f\big(s,y,u(s,y)\big)\big|\,dy\,ds \\
&+ \int_0^t \int_{D_L^c} p(t-s,x-y)\big|f\big(s,y,u^L(s,y)\big)\big|\,dy\,ds \\
&:= J_{21} + J_{22} + J_{23}.
\end{aligned}
$$

Similarly to (5.3), we obtain

$$
\begin{aligned}
J_{21} &\overset{\text{A2}}{\leq} \int_0^t C_f \int_{D_L} \big|G^L(t-s,x,y) - p(t-s,x-y)\big|\,dy\,ds \\
&\leq C(\omega) \int_0^t \exp\left\{-\frac{(L-|x|)^2}{4(t-s)}\right\} ds \leq C(\omega) \exp\left\{-\frac{(L-|x|)^2}{4t}\right\}.
\end{aligned}
$$

As in (5.4), we get

$$
J_{23} \overset{\text{A2}}{\leq} \int_0^t C_f \int_{D_L^c} p(t-s,x-y)\,dy\,ds \leq C(\omega) \exp\left\{-\frac{(L-|x|)^2}{4t}\right\}.
$$

Also

$$
J_{22} \overset{\text{A3}}{\leq} L_f \int_0^t \int_{D_L} p(t-s,x-y)\big|u^L(s,y) - u(s,y)\big|\,dy\,ds.
$$

Gathering together all estimates of J_1, J_2, and J_3, we obtain that for any $\lambda < \frac{1}{4}$

$$
\begin{aligned}
&\big|u^L(t,x) - u(t,x)\big|\mathbf{1}_{\{x\in D_L\}} \\
&\quad \leq C_\delta(\omega) C_\lambda \exp\left\{-\frac{\lambda(L-|x|)^2}{t}\right\} \\
&\qquad + C(\omega) \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\big|u^L(s,y) - u(s,y)\big|\mathbf{1}_{\{y\in D_L\}}\,dy\,ds.
\end{aligned}
$$

For $|u^L(t,x)) - u(t,x)|\mathbf{1}_{|x|\leq L}$, we will use the space-time Gronwall-type inequality (see Lemma C.1.7 (d) [4]), and obtain

$$
\big|u^L(t,x) - u(t,x)\big| \leq \xi_{\delta,\lambda,L}(t,x) + C(\omega) \int_0^t \int_{\mathbb{R}} \mathcal{K}(t-s,x-y)\xi_{\delta,\lambda,L}(s,y)\,dy\,ds \tag{5.5}
$$

where $\mathcal{K}$ denotes the series with iterated space-time convolutions

$$
\mathcal{K}(t,x) = \sum_{k=1}^{\infty} p^{*k}(t,x)
$$

(see more details in Section C.1.3 [4]) and

$$
\xi_{\delta,\lambda,L}(t,x) = C_\delta(\omega) C_\lambda \exp\left\{-\frac{\lambda(L-|x|)^2}{t}\right\}\mathbf{1}_{\{x\in D_L\}}.
$$

It is well-known that

$$\int_{\mathbb{R}} p(t_1, x-y)p(t_2, y)\,dy = p(t_1+t_2, x),$$

therefore,

$$p^{*2}(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s, x-y)p(s,y)\,dy\,ds = \int_0^t p(t,x)\,ds = tp(t,x),$$

and applying induction, we easily obtain that

$$p^{*k}(t,x) = \frac{t^{k-1}}{(k-1)!}p(t,x) \quad \Rightarrow \quad \mathcal{K}(t,x) = e^t p(t,x).$$

Further, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \mathcal{K}(t-s, x-y)\xi_{\delta,\lambda,L}(s,y)\,dy\,ds \\
&= C(\omega)\int_0^t \int_{\mathbb{R}} e^{t-s} p(t-s, x-y)\exp\left\{-\frac{\lambda(L-|y|)^2}{s}\right\}\mathbf{1}_{\{y\in D_L\}}\,dy\,ds \\
&= C(\omega)\int_0^t \int_{\mathbb{R}} e^{t-s}\frac{1}{2\sqrt{\pi(t-s)}}\exp\left\{-\frac{(x-y)^2}{4(t-s)}\right\}\exp\left\{-\frac{\lambda(L-|y|)^2}{s}\right\}\mathbf{1}_{\{y\in D_L\}}dy\,ds \\
&\le C(\omega)\frac{2}{\sqrt{\lambda}}\int_0^t \frac{\sqrt{\pi s}}{\sqrt{\lambda}}\int_{\mathbb{R}} \frac{\sqrt{\lambda}}{\sqrt{\pi(t-s)}}\exp\left\{-\frac{\lambda(x-y)^2}{t-s}\right\}\frac{\sqrt{\lambda}}{\sqrt{\pi s}}\left(\exp\left\{-\frac{\lambda(y-L)^2}{s}\right\}\right. \\
&\quad \left.+\exp\left\{-\frac{\lambda(y+L)^2}{s}\right\}\right)dy\,ds.
\end{aligned}$$

From the properties of the convolution of density functions of normal distributions, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\sqrt{\lambda}}{\sqrt{\pi(t-s)}}\exp\left\{-\frac{\lambda(x-y)^2}{t-s}\right\}\frac{\sqrt{\lambda}}{\sqrt{\pi s}}\exp\left\{-\frac{\lambda(y-L)^2}{s}\right\}dy \\
&\quad = \frac{\sqrt{\lambda}}{\sqrt{\pi t}}\exp\left\{-\frac{\lambda(x-L)^2}{t}\right\}, \\
&\int_{\mathbb{R}} \frac{\sqrt{\lambda}}{\sqrt{\pi(t-s)}}\exp\left\{-\frac{\lambda(x-y)^2}{t-s}\right\}\frac{\sqrt{\lambda}}{\sqrt{\pi s}}\exp\left\{-\frac{\lambda(y+L)^2}{s}\right\}dy \\
&\quad = \frac{\sqrt{\lambda}}{\sqrt{\pi t}}\exp\left\{-\frac{\lambda(x+L)^2}{t}\right\},
\end{aligned}$$

therefore

$$\int_0^t \int_{\mathbb{R}} \mathcal{K}(t-s, x-y)\xi_{\delta,\lambda,L}(s,y)\,dy\,ds \le C_{\lambda,\delta}(\omega)\exp\left\{-\frac{\lambda(L-|x|)^2}{t}\right\},$$

and (5.5) implies the result of our Theorem. □

Remark 2. Let SM μ be such that $\mu((a,b]) \le C(\omega),\ a,\ b \in \mathbb{R}$. Then we can obtain (5.1) for arbitrary $L \in \mathbb{R}_+$, i.e.

$$\left|u^L(t,x) - u(t,x)\right| \le C(\omega) C_\lambda e^{-\frac{\lambda(L-|x|)^2}{t}},$$

where $C(\omega)$ is independent of L. This follows from Remark 1, because we used form $L = i\delta$ only in Lemma 3.

Remark 3. For the heat equation with a multiplicative noise driven by the Wiener process, under some assumptions, Theorem 3.1 [2] states that for all $(t,x) \in [0,T] \times D_L$, $p \ge 1$ holds

$$\begin{aligned}&\left\|u^L(t,x) - u(t,x)\right\|_{\mathsf{L}_p(\Omega)}\\ &\quad\le C\left(1 + \sup_{x\in\mathbb{R}}\left|u_0(x)\right|\right)\left(\exp\left\{-\frac{(L-x)^2}{8t}\right\} + \exp\left\{-\frac{(L+x)^2}{8t}\right\}\right),\end{aligned}$$

where C is independent of L, x, t, u_0. In our paper, we have a similar convergence rate for $u^L \to u$.

Also, note that Theorem 3.1 [2] includes the cases of the mixed boundary conditions and the Neumann boundary conditions for the equation.

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