Modern Stochastics: Theory and Applications 14 (1) (2027) 107–134
https://doi.org/10.15559/26-VMSTA314

Exact variational cost for small-noise exit time in a two-slope autoregressive model

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Received: 1 August 2026, Revised: 26 October 2026, Accepted: 26 October 2026,
Published online: 4 November 2026

Abstract We determine the quadratic variational cost that yields the large-deviation upper bound for the mean exit time from $(-1, 1)$ in the small-noise regime for an autoregressive process driven by a continuous two-slope map f lying strictly below the diagonal on $(0, 1)$. The Klebaner–Liptser reduction expresses this constant as an infimum, over all finite horizons, of quadratic path actions. Högnäs and Jung studied several nonlinear and piecewise-linear autoregressive maps and evaluated the corresponding variational problem for the two-slope family in the boundary cases in which one of the slopes vanishes; the case in which both slopes p and q are nonzero was left open. We solve this remaining case over the full admissible parameter range, including the finite-horizon attainment question, and give an exact parameter classification of the cost. We also settle the remaining boundary regimes $q = 1$ and $p = 0$, $q > 1$, as well as the limiting case $f(1) = 1$. Together with the earlier boundary formulas, this completes the exact variational analysis of continuous two-slope maps satisfying $f(x) < x$ for $0 < x < 1$.

Keywords Exit time, nonlinear autoregressive process, large deviation principle, discrete quasipotential, piecewise-linear map, breakpoint-constrained variational problem
2020 MSC 60F10, 60J05, 39A50, 49J21

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1 Introduction

Small random perturbations of stable deterministic systems often produce a characteristic metastable behaviour: the process remains close to an attracting state for a long time and then leaves its neighbourhood through a rare fluctuation. In population models this mechanism is related to persistence near a carrying capacity and to the onset of extinction [5–7]. This motivation leads naturally to the study of small-noise autoregressive recursions.

The model considered here is

$$X_{n+1}^{\varepsilon} = h(X_n^{\varepsilon}) + \varepsilon\xi_{n+1}, \qquad X_0^{\varepsilon} = 0, \tag{1.1}$$

where the innovations are independent standard Gaussian random variables and $\varepsilon > 0$ is small. We study the first exit time $\tau_\varepsilon := \min\{n \geq 1 : |X_n^{\varepsilon}| \geq 1\}$ from $(-1, 1)$. The large-deviation method of Klebaner and Liptser [12] (see [17] for the general theory) reduces an exponential upper bound for $\mathbb{E}_0\tau_\varepsilon$ to an infimum, over finite horizons, of quadratic path actions. The resulting constant is a discrete quasipotential: it measures the least noise energy required to drive the deterministic system from the stable fixed point to the boundary. The present paper evaluates this variational constant exactly for a continuous two-slope map.

For the linear autoregression $h(x) = px$, $|p| < 1$, the variational problem is explicit and gives the sharp value $(1 - p^2)/2$. The same large-deviation framework has been developed for past-dependent recursions, moderate deviations, multivariate autoregressions, ARMA processes, and non-Gaussian noise [12, 13, 11, 14, 9]. Complementary approaches to first-passage and level-crossing questions use martingale and analytic methods [16, 15, 4, 2, 3]. Piecewise-linear and threshold autoregressions are classical nonlinear time-series models [1], but their exit variational problems are much less explicit than in the linear case.

Högnäs and Jung [10] studied several nonlinear and piecewise-linear autoregressive maps and obtained explicit bounds in a number of special cases; see also [8] for exit times and invariant distributions of a related stochastic difference equation. For the continuous two-slope family considered below, their formulas cover the boundary cases in which one of the two slopes vanishes. The case in which both slopes are nonzero was not evaluated. This is the remaining case in which the breakpoint constraints genuinely enter the variational minimization.

We solve this remaining case over the full admissible parameter range. The analysis separates two classes of candidate minimizing paths: paths having a vertex at the breakpoint and paths with no internal vertex at the breakpoint. We compute the least cost in both classes, determine exactly when the stationary candidate in the second class satisfies the required inequalities, and compare it with the best breakpoint path. This yields a complete parameter classification involving an interior stationary value, a breakpoint value, or the linear value $(1 - p^2)/2$. We also treat the remaining boundary regimes $q = 1$ and $p = 0$, $q > 1$, as well as the limiting case $f(1) = 1$, and determine whether the variational infimum is attained at a finite horizon.

The condition studied in this paper is that the graph of the positive branch lies strictly below the diagonal, $f(x) < x$ for $0 < x < 1$. It guarantees deterministic attraction toward the origin from every interior initial state, but it does not require

both slopes to be smaller than one; in particular, an expanding second slope is allowed when the affine branch still remains below the diagonal. The regimes in which

the graph meets or rises above the diagonal inside $(0, 1)$ have different deterministic and variational structures and are therefore left for separate treatment.

The paper is organized as follows. Section 2 formulates the exit-time variational problem, introduces the two-slope family and the monotone reduction, recalls the previously known boundary cases, and states the main classification. Section 3 proves the affine one-segment minimization used throughout, and Sections 4 and 5 analyze, respectively, paths that hit the breakpoint and paths with no internal vertex there. Section 6 states the key admissibility and comparison results needed for the constrained classification, Section 7 proves the main theorem, and Section 8 treats the remaining boundary regimes. The detailed proofs of the admissibility and comparison results are given in Section 9.

2 Variational formulation and proof architecture

2.1 *Exit-time variational problem*

Let $h\colon [-1,1] \to [-1,1]$ be continuous, with $h(0) = 0$ and $|h(x)| < |x|$ for $0 < |x| < 1$. For a finite path $\mathbf{y} = (y_0, \dots, y_N)$, define the quadratic action

$$I_N(\mathbf{y}; h) := \frac{1}{2} \sum_{j=1}^{N} \big(y_j - h(y_{j-1})\big)^2. \tag{2.1}$$

The associated exit-time variational cost is

$$S(h) := \inf_{N \geq 1} \min_{\substack{y_0 = 0,\ |y_N| = 1 \\ |y_j| < 1\ (1 \leq j < N)}} I_N(\mathbf{y}; h). \tag{2.2}$$

The large-deviation reduction gives the following statement, which is the only probabilistic result from [12, 10] used in the variational analysis below. We emphasize that no matching lower bound is asserted here for the nonlinear maps of this paper.

Theorem 2.1. *For the process* (1.1)*,*

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E}_0 \tau_\varepsilon \leq S(h).$$

For $h(x) = px$, $|p| < 1$, direct minimization gives $S(h) = (1 - p^2)/2$, recovering the classical linear value. The problem in the nonlinear setting is thus to compute $S(h)$, keeping this exact variational value distinct from a matching lower bound for the mean exit-time exponent, which is not presently available for the maps considered here.

2.2 *Two-slope family and monotone reduction*

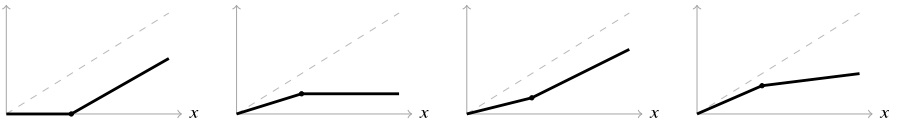
We now specialize the map h to the two-slope family and fix the notation. Let $\mathcal{F}$ be the family of continuous two-slope functions $f : [0,1] \to [0,1]$ of the form

$$f(x) = \begin{cases} px, & 0 \leq x \leq c, \\ qx + c(p - q), & c \leq x \leq 1, \end{cases} \qquad 0 \leq p < 1, \quad q \geq 0, \quad 0 < c < 1, \tag{2.3}$$

which additionally satisfy

$$f(x) < x \qquad (0 < x < 1). \tag{2.4}$$

Continuity at c is built into (2.3), because the slope-p and slope-q pieces both take the value pc there.



(a) $p = 0$ (b) $q = 0$ (c) $q > p$ (d) $q < p$

Fig. 1. Representative graphs in the two-slope family: (a) a flat first piece $p = 0$; (b) a flat second piece $q = 0$; (c) $q > p$; (d) $q < p$. The dashed line is $y = x$ and the dot marks the breakpoint (c, pc)

For $f \in \mathcal{F}$, the exit-time problem for (1.1) is driven by the symmetric extension of f to $[-1, 1]$ defined in (2.5) below, whose quasipotential (2.2) is the object to be evaluated. We record, in the notation used throughout, the reduction of Högnäs and Jung [10] of this two-sided problem to a one-sided minimization over monotone paths. Take arbitrary $f, g \in \mathcal{F}$ and define a function on $[-1, 1]$ by

$$h_{f,g}(x) = \begin{cases} f(x), & 0 \le x \le 1, \\ -g(-x), & -1 \le x \le 0. \end{cases} \tag{2.5}$$

Because $f(0) = g(0) = 0$, this function is continuous at the origin; moreover it is increasing and satisfies $|h_{f,g}(x)| < |x|$ for $x \in (-1, 1) \setminus \{0\}$. Write, for the one-sided monotone problem,

$$S_N(f) := \min_{0=y_0\le y_1\le\cdots\le y_N=1} I_N(\mathbf{y}; f), \qquad S(f) := \inf_{N\ge 1} S_N(f). \tag{2.6}$$

Using Lemmas 1 and 3 of [10], the two-sided N-step problem decomposes into these one-sided monotone problems:

$$\min_{\substack{y_0=0,\ |y_N|=1 \\ |y_j|<1,\ 1\le j<N}} I_N(\mathbf{y}; h_{f,g}) = \min\{S_N(f), S_N(g)\}. \tag{2.7}$$

In particular, for the symmetric extension $h_{f,f}$, the two-sided N-step problem has value $S_N(f)$. Taking the infimum over N in (2.7) identifies the quasipotential (2.2) of the introduction with these one-sided costs, $S(h_{f,g}) = \min\{S(f), S(g)\}$ and $S(h_{f,f}) = S(f)$, so that Theorem 2.1 applies with the constant $S(f)$ evaluated below.

2.3 *Previously known boundary cases*

Högnäs and Jung [10] evaluated $S(f)$ in the two boundary cases of the family (2.3), where one of the slopes vanishes. We restate their results in the present notation.

Proposition 2.2 (Högnäs–Jung [10]). *Let $f \in \mathcal{F}$ have $p = 0$, with $0 \le q \le 1$. For $0 \le q < 1$,*

$$S_1(f) = \frac{1}{2}, \qquad S_N(f) = \frac{1-q^2}{2} \frac{\left(1 + \dfrac{q-q^N}{1-q} c\right)^2}{1-q^{2N}}, \quad N \ge 2, \tag{2.8}$$

while for $q = 1$,

$$S_N(f) = \frac{(1+(N-1)c)^2}{2N}, \qquad N \ge 1. \tag{2.9}$$

Proposition 2.3 (Högnäs–Jung [10]). *Let $f \in \mathcal{F}$ have $q = 0$. Then*

$$S(f) = \begin{cases} \dfrac{1}{2}\big((1-pc)^2 + (1-p^2)c^2\big), & c \le p, \\ \dfrac{1}{2}(1-p^2), & c \ge p. \end{cases} \tag{2.10}$$

These two propositions cover the boundary cases $p = 0$, $0 \le q \le 1$ and $q = 0$ of the family (2.3); the case in which both slopes are nonzero is covered by neither, and it is the subject of this paper.

2.4 Main classification and boundary results

We now state the complete classification of the variational problem (2.6) for a general continuous two-slope map. A minimizing path belongs to one of two classes according to whether it has a vertex at the breakpoint c: a path that hits c has a cost $\widehat{S}$, and a path with no internal vertex at c has a cost S_m, and the theorem compares the two. We record these values and the parameters that govern them. Set

$$\Delta := 1 - q + c(q-p), \qquad c_0 := \frac{p+q}{1+q}, \qquad \lambda := \frac{q^2-p^2}{1-p^2}. \tag{2.11}$$

For $m \in \mathbb{Z}_{\ge 0}$, the stationary calculation for paths with no internal vertex at c gives the limiting algebraic quantities

$$S_m := \frac{(1+q)\big(\Delta - c(q-p)q^m\big)^2}{2(1-q)\big(1-\lambda q^{2m}\big)}, \qquad Y_m := \frac{(1+q)\,q^m\big(\Delta - c(q-p)q^m\big)}{1-\lambda q^{2m}}, \tag{2.12}$$

with $S_0 = (1-p^2)/2$ and $Y_0 = 1-p^2$. An index satisfies the limiting feasibility condition if $c(1-p^2) < Y_m < c(1-p^2)/p$, and we write $\mathcal{M} := \{m \in \mathbb{Z}_{\ge 0} : c(1-p^2) < Y_m < c(1-p^2)/p\}$ for this limiting feasibility set; the full nondecreasing-path admissibility of the indices used in the theorem is established later. We also introduce the real number $m_0 := \log_q\big(c(1-p^2)/((p+q)\Delta)\big)$ and, for $n \ge 1$, the breakpoint values

$$\widehat{S}_n := \frac{1-p^2}{2}c^2 + \frac{(1+q)\big(\Delta - c(1-p)q^n\big)^2}{2(1-q)\big(1-q^{2n}\big)}, \tag{2.13}$$

and, as proved in Section 4, the actual infimum over paths that hit the breakpoint is

$$\widehat{S} := \min\{\widehat{S}_{n^*}, \widehat{S}_{n^*+1}\}, \qquad n^* := \max\left\{1, \left\lfloor \log_q \frac{c(1-p)}{\Delta} \right\rfloor\right\}. \tag{2.14}$$

The following is our main result.

Theorem 2.4. *Assume* $0 < p < 1$, $q > 0$, $q \neq p, 1$, *and* $\Delta > 0$. *Then*

$$S(f) = \begin{cases} S_{m^*}, & q > p,\ c < c_0, \\ S_{m^*}, & q < p,\ c < c_0,\ \mathcal{M} = \{m^*\}, \\ \widehat{S}, & q < p,\ c < c_0,\ \mathcal{M} = \emptyset, \\ (1-p^2)/2, & c \geq c_0. \end{cases}$$

Here $m^* \in \{\lfloor m_0 \rfloor, \lceil m_0 \rceil\}$*: when* $q > p$ *it is whichever of the two gives the smaller value of* S_m*, and when* $q < p$ *it is whichever of the two is feasible—at most one is—so that* $\mathcal{M} = \{m^*\}$ *if such an index exists and* $\mathcal{M} = \emptyset$ *otherwise. None of the displayed values of* $S(f)$ *is attained at a finite horizon.*

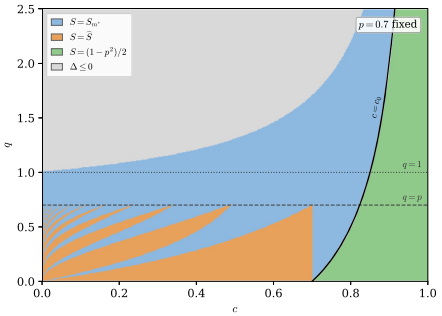


Fig. 2. The classification of Theorem 2.4 in the (c, q)-plane at a fixed first slope p (here $p = 0.7$), computed exactly from the closed forms for S_m, Y_m, and $\widehat{S}$. Each colour marks which value $S(f)$ takes: the interior stationary value S_{m^*}, the breakpoint value $\widehat{S}$, the linear value $(1-p^2)/2$ (arising for $c \geq c_0$), and the excluded area $\Delta \leq 0$

The cases $q = p$, $q = 0$, and $p = 0, q \leq 1$ were described above through the results of Klebaner–Liptser and Högnäs–Jung. For completeness, we state here the three remaining boundary results proved in this paper.

Proposition 2.5. *Assume* $0 < p < 1$, $q = 1$, *and* $0 < c < 1$, *and put* $d := c(1-p)$, $a := 1/(1-p^2)$. *If* $c \geq (1+p)/2$, *then* $S(f) = (1-p^2)/2$. *If* $c < (1+p)/2$, *then*

$$S(f) = \min_{m \in \mathbb{Z}_{\geq 0}} \frac{(1+dm)^2}{2(m+a)},$$

the minimum being attained at one of the two integers adjacent to $m_1 := 1/d - 2a$. *This value is not attained at any finite horizon.*

Proposition 2.6. *Assume* $p = 0$, $q > 1$, *and* $\Delta = 1 - q + cq > 0$. *Then, with* $n_0 := \max\{1, \lfloor \log_q(c/\Delta) \rfloor\}$,

$$S(f) = \min_{n\in\{n_0,n_0+1\}} \frac{(1+q)(\Delta - cq^n)^2}{2(1-q)(1-q^{2n})},$$

and the minimum is attained at a finite horizon.

Proposition 2.7. *Assume* $0 \le p < 1$, $0 < c < 1$, *and* $\Delta = 0$, *so that* $q = (1-pc)/(1-c) > 1$. *Then* $c < c_0$ *and*

$$S(f) = \lim_{m\to\infty} S_m = \frac{(1-p^2)(q^2-1)}{2(q^2-p^2)}.$$

This value is not attained at a finite horizon. In particular, for $p = 0$, $S(f) = c - c^2/2$.

Proof architecture. Every admissible path belongs to exactly one of two classes: (I) at least one vertex satisfies $y_j = c$; (II) every internal vertex satisfies $y_j \neq c$. These cases require different arguments. In Case I, fixing a vertex at c splits the action into two independent minimizations; we therefore analyze this case first and denote its optimal value by $\widehat{S}$. In Case II, there is a unique index k with $y_k < c < y_{k+1}$; we first compute the stationary solution of the quadratic equations, and only afterwards verify the breakpoint inequalities, monotonicity, and comparison with Case I. The final theorem compares the best feasible limiting value with $\widehat{S}$. Both the decomposition of Case I and the boundary analyses reduce to a single affine minimization, isolated as the one-segment minimization of Section 3; the structural results needed for the Case II comparison are stated in Section 6 and proved in Section 9.

3 A one-segment minimization

The analysis of Case I and of the boundary regimes repeatedly reduces to minimizing the action over a single affine piece, and we isolate that minimization here. Let $a < b$ and $g(x) = \alpha x + \gamma$, where $\alpha > 0$ and $\alpha \neq 1$; for $N \ge 1$, denote the constrained and unconstrained minima, respectively, by

$$I_N(a,b;g) := \min_{a=x_0\le\cdots\le x_N=b} \frac{1}{2}\sum_{i=1}^{N}\big(x_i - g(x_{i-1})\big)^2,$$

$$J_N(a,b;g) := \min_{\substack{x_0=a,\ x_N=b\\ x_1,\dots,x_{N-1}\in\mathbb{R}}} \frac{1}{2}\sum_{i=1}^{N}\big(x_i - g(x_{i-1})\big)^2.$$

Lemma 3.1. *Suppose that* $g(x) \le x$ *on* $[a,b]$, *and set* $d_a := a - g(a)$, $d_b := b - g(b)$. *Then, for every* $N \ge 1$, *the unrestricted minimum is*

$$J_N(a,b;g) = \frac{1-\alpha^2}{2(1-\alpha)^2}\,\frac{\big(d_b - d_a\alpha^N\big)^2}{1-\alpha^{2N}}. \tag{3.1}$$

If $d_a = 0$, *then necessarily* $0 < \alpha < 1$, *and*

$$\inf_{N\ge1} I_N(a,b;g) = \frac{1+\alpha}{2(1-\alpha)}d_b^2. \tag{3.2}$$

If $d_b = 0$, then necessarily $\alpha > 1$, and

$$\inf_{N\geq 1} I_N(a,b;g) = \frac{1+\alpha}{2(\alpha-1)} d_a^2. \tag{3.3}$$

Finally, if $d_a d_b \neq 0$, define $N_0 := \max\{1, \lfloor \log_\alpha(d_a/d_b)\rfloor\}$. Then

$$\inf_{N\geq 1} I_N(a,b;g) = \min\{J_{N_0}(a,b;g), J_{N_0+1}(a,b;g)\}. \tag{3.4}$$

Proof. Fix $N \geq 1$ and write $e_i := x_i - g(x_{i-1}) = x_i - \alpha x_{i-1} - \gamma$ for $i = 1, \dots, N$. Multiplying by α^{N-i} and summing gives the telescoping identity

$$\sum_{i=1}^{N} \alpha^{N-i} e_i = b - \alpha^N a - \gamma \sum_{r=0}^{N-1} \alpha^r = \frac{d_b - d_a\alpha^N}{1-\alpha}, \tag{3.5}$$

so that, by the Cauchy–Schwarz inequality,

$$\sum_{i=1}^{N} e_i^2 \geq \frac{\left(\sum_{i=1}^N \alpha^{N-i} e_i\right)^2}{\sum_{i=1}^N \alpha^{2(N-i)}} = \frac{1-\alpha^2}{(1-\alpha)^2} \frac{\left(d_b - d_a\alpha^N\right)^2}{1-\alpha^{2N}},$$

with equality exactly when the error vector is proportional to $(\alpha^{N-1}, \dots, \alpha, 1)$. The endpoint identity (3.5) fixes the proportionality constant, so the unrestricted minimizer satisfies

$$e_i = x_i - g(x_{i-1}) = \beta\alpha^{N-i}, \quad i = 1, \dots, N, \qquad \beta = \frac{1-\alpha^2}{1-\alpha} \frac{d_b - d_a\alpha^N}{1-\alpha^{2N}}, \tag{3.6}$$

and it is unique, the map from the intermediate vertices to the error vector being invertible. Halving the bound proves (3.1).

We next determine when this minimizer is nondecreasing. Since $g(x) \leq x$ on $[a,b]$, the endpoint gaps d_a and d_b are nonnegative, and

$$d_b - d_a = (1-\alpha)(b-a); \tag{3.7}$$

hence $d_a = 0$ forces $0 < \alpha < 1$, and $d_b = 0$ forces $\alpha > 1$. Put $L_N := (d_b - d_a\alpha^N)/(1-\alpha^{2N})$. Iterating (3.6) from $x_0 = a$ and subtracting successive coordinates,

$$x_i - x_{i-1} = -d_a\alpha^{i-1} + L_N\left(\alpha^{N-i} + \alpha^{N+i-1}\right), \qquad i = 1, \dots, N. \tag{3.8}$$

If $d_a = 0$, then $0 < \alpha < 1$ and $L_N = d_b/(1-\alpha^{2N}) > 0$, so every increment is positive. If $d_b = 0$, then $\alpha > 1$ and $L_N = d_a\alpha^N/(\alpha^{2N}-1) > 0$; after division by α^{i-1} the right-hand side of (3.8) decreases with i, and its last value $x_N - x_{N-1} = d_a\alpha^{N-1}(1+\alpha)/(\alpha^{2N}-1)$ is positive. Thus in both endpoint cases the minimizer is strictly increasing.

Assume now $d_a d_b > 0$. If $0 < \alpha < 1$, the normalized increments $(x_i - x_{i-1})/\alpha^{i-1}$ increase with i, so only $i = 1$ matters: substituting L_N into $x_1 - x_0 = -d_a + L_N(\alpha^{N-1} + \alpha^N) \geq 0$ and clearing the positive denominator $1-\alpha^{2N}$, the condition becomes

$$d_b\alpha^{N-1}(1+\alpha) \geq d_a\left(1+\alpha^{2N-1}\right). \tag{3.9}$$

If $\alpha > 1$, the same normalized increments decrease, so $i = N$ is decisive, and $x_N - x_{N-1} = -d_a\alpha^{N-1} + L_N(1+\alpha^{2N-1}) \geq 0$ with $L_N = (d_a\alpha^N - d_b)/(\alpha^{2N}-1)$ becomes

$$d_a(1+\alpha) \geq d_b\big(\alpha^N + \alpha^{1-N}\big). \tag{3.10}$$

Thus, when $d_a d_b > 0$, the unrestricted minimizer is admissible for a given N precisely when the relevant one of (3.9) and (3.10) holds. Neither condition is automatic, and we do not check them yet: we first carry out the minimization over N, which singles out two candidate integers, and then return at the end of the proof to verify the appropriate condition at those two integers only.

We now minimize over N. If $d_a = 0$, then (3.1) reads $J_N(a,b;g) = \frac{1+\alpha}{2(1-\alpha)}\frac{d_b^2}{1-\alpha^{2N}}$, which decreases strictly to the value in (3.2); if $d_b = 0$, then $\alpha > 1$ and

$$J_N(a,b;g) = \frac{1+\alpha}{2(\alpha-1)}d_a^2\frac{\alpha^{2N}}{\alpha^{2N}-1} \downarrow \frac{1+\alpha}{2(\alpha-1)}d_a^2.$$

In both cases the unrestricted minimizer is nondecreasing for every N, so $I_N(a,b;g) = J_N(a,b;g)$; this proves (3.2) and (3.3) together with the nonattainment statements.

Suppose finally that $d_a d_b > 0$. Extend the right-hand side of (3.1) to real $x > 0$ and set $z = \alpha^x$; apart from its constant prefactor it becomes $(d_b - d_a z)^2/(1-z^2)$, with

$$\frac{d}{dz}\frac{(d_b - d_a z)^2}{1-z^2} = \frac{2(d_b - d_a z)(d_b z - d_a)}{(1-z^2)^2}. \tag{3.11}$$

For $0 < \alpha < 1$, (3.7) gives $0 < d_a < d_b$, and $z = d_a/d_b$ is the only critical point in the admissible range $0 < z < 1$; for $\alpha > 1$ one has $d_a > d_b > 0$, and the same point is the only critical point in the admissible range $z > 1$. Taking into account the direction of $z = \alpha^x$ and, for $\alpha > 1$, the sign of the prefactor in (3.1), the real-variable cost is in both cases strictly decreasing before $N_{\mathrm{r}} = \log_\alpha(d_a/d_b)$ and strictly increasing afterwards. Its integer minimum is therefore among N_0 and $N_0 + 1$, with N_0 as in Lemma 3.1; if $N_{\mathrm{r}} < 1$, it occurs at $N = 1$.

It remains to check the ordering constraint at these two integers. Let $0 < \alpha < 1$ and $N_{\mathrm{r}} \geq 1$, so that $\alpha^{N_0+1} \leq d_a/d_b \leq \alpha^{N_0}$. For $N = N_0$,

$$d_a\big(1+\alpha^{2N-1}\big) \leq d_b\alpha^N\big(1+\alpha^{2N-1}\big) \leq d_b\alpha^{N-1}(1+\alpha),$$

while for $N = N_0 + 1$ the same bound, written $d_a/d_b \leq \alpha^{N-1}$, gives $d_a(1+\alpha^{2N-1}) \leq d_b\alpha^{N-1}(1+\alpha)$ directly; thus (3.9) holds at both. Let now $\alpha > 1$ and $N_{\mathrm{r}} \geq 1$, so that $\alpha^{N_0} \leq d_a/d_b \leq \alpha^{N_0+1}$. For $N = N_0$ we get $d_a(1+\alpha) \geq d_b\alpha^N(1+\alpha) \geq d_b(\alpha^N + \alpha^{1-N})$, and for $N = N_0 + 1$, using $d_a/d_b \geq \alpha^{N-1}$, $d_a(1+\alpha) \geq d_b\alpha^{N-1}(1+\alpha) \geq d_b(\alpha^N + \alpha^{1-N})$; thus (3.10) holds at both. If $0 < N_{\mathrm{r}} < 1$, the optimal one-step path has no internal vertex and is automatically nondecreasing.

Therefore an unrestricted minimizer realizing the integer minimum is admissible for the monotone problem, and since $J_N(a,b;g) \leq I_N(a,b;g)$ for every N, this proves (3.4). □

We keep throughout the distinction just established: for fixed N and $d_a d_b > 0$, the unrestricted value (3.1) need not equal the constrained value $I_N(a,b;g)$; the two problems agree only after optimization over N, because at least one of the optimal adjacent integers is admissible.

4 Case I: paths that hit the breakpoint

Assume throughout this section that $0 < p < 1$, $q > 0$, $q \neq 1$, and $\Delta > 0$. We compute the infimum over nondecreasing paths that hit the breakpoint c; fixing the breakpoint vertex separates the action into two affine one-segment problems.

Lemma 4.1. *Fix $N \geq 2$ and $j \in \{1, \dots, N-1\}$. Among paths satisfying $y_j = c$, the two parts of the action can be minimized independently:*

$$\min_{\substack{0=y_0\leq\cdots\leq y_N=1\\ y_j=c}} I_N(\mathbf{y}; f) = \min_{0=y_0\leq\cdots\leq y_j=c} \frac{1}{2}\sum_{i=1}^{j}(y_i - f(y_{i-1}))^2 + \min_{c=y_j\leq\cdots\leq y_N=1} \frac{1}{2}\sum_{i=j+1}^{N}(y_i - f(y_{i-1}))^2. \tag{4.1}$$

Proof. With $y_0 = 0$, $y_j = c$, and $y_N = 1$ fixed, write $A_j(y_1, \dots, y_{j-1}) := \frac{1}{2}\sum_{i=1}^{j}(y_i - f(y_{i-1}))^2$ and $B_{N-j}(y_{j+1}, \dots, y_{N-1}) := \frac{1}{2}\sum_{i=j+1}^{N}(y_i - f(y_{i-1}))^2$. The variables occurring in A_j and B_{N-j} are disjoint, and every admissible full path with $y_j = c$ satisfies $I_N(\mathbf{y}; f) = A_j + B_{N-j}$; hence its action is at least $\min A_j + \min B_{N-j}$. Conversely, the two admissible sets are compact and the two actions continuous, so minimizing left and right subpaths exist, and concatenating them at their common endpoint c gives an admissible full path of action exactly $\min A_j + \min B_{N-j}$. □

Denote by

$$\mathcal{S}_{\mathrm{b}}(f) := \inf_{\substack{N\geq 2,\ 1\leq j<N\\ 0=y_0\leq\cdots\leq y_N=1,\ y_j=c}} I_N(\mathbf{y}; f)$$

the infimum over the class of paths that hit the breakpoint, put $n := N - j$, and set $g_p(x) := px$, $g_q(x) := qx + c(p - q)$. Since the two positive lengths j and n vary independently, the decomposition lemma yields

$$\mathcal{S}_{\mathrm{b}}(f) = \inf_{j\geq 1} I_j(0, c; g_p) + \inf_{n\geq 1} I_n(c, 1; g_q). \tag{4.2}$$

For the left one-segment problem the endpoint gaps are $d_0 = 0$ and $d_c = c - g_p(c) = c(1 - p)$; the fixed-$j$ minimizer is increasing, and Lemma 3.1 gives

$$I_j(0, c; g_p) = \frac{1 - p^2}{2(1 - p^{2j})}c^2 \downarrow \frac{1 - p^2}{2}c^2 \qquad (j \to \infty), \tag{4.3}$$

the limit being strict and not attained at a finite left-hand length. For the right one-segment problem the endpoint gaps $d_c = c - g_q(c) = c(1 - p)$ and $d_1 = 1 - g_q(1) = \Delta$ are both positive, and the unconstrained n-step value from Lemma 3.1 is

$$J_n(c, 1; g_q) = \frac{(1 + q)\big(\Delta - c(1 - p)q^n\big)^2}{2(1 - q)(1 - q^{2n})}. \tag{4.4}$$

We compare with the sequence $\widehat{S}_n$ of (2.13), whose second term is, at an arbitrary fixed n, an unrestricted algebraic value whose stationary path need not be nondecreasing;

thus $\widehat{S}_n$ is not to be read as an admissible path cost for every n. The minimizer of the real-variable extension of the right one-segment problem is $x_{\mathrm{b}} := \log_q\big(c(1-p)/\Delta\big) > 0$, and by Lemma 3.1 the constrained right one-segment minimum is attained at one of the two adjacent integer lengths n^* and n^*+1 of (2.14), at which the unrestricted minimizers are themselves nondecreasing. Consequently the algebraic values in (2.13) are attained by admissible paths hitting the breakpoint for the two lengths that determine the minimum.

Proposition 4.2. *The infimum over all paths that hit the breakpoint is*

$$\mathcal{S}_{\mathrm{b}}(f) = \widehat{S}, \tag{4.5}$$

with $\widehat{S}$ as in (2.14), and it is not attained by any finite path.

Proof. Combining (4.2) with (4.3) and the integer-minimization part of Lemma 3.1,

$$\mathcal{S}_{\mathrm{b}}(f) = \frac{1-p^2}{2}c^2 + \min\{J_{n^*}(c,1;g_q), J_{n^*+1}(c,1;g_q)\} = \min\{\widehat{S}_{n^*}, \widehat{S}_{n^*+1}\},$$

which is (4.5). The right one-segment problem has an optimal finite length, but the left contribution in (4.3) approaches its infimum strictly as $j \to \infty$; therefore no finite full path attains $\widehat{S}$. □

5 Case II: paths with no vertex at the breakpoint

Throughout this section we work under the assumptions of Theorem 2.4: $0 < p < 1$, $q > 0$, $q \neq p, 1$, and $\Delta > 0$, and we assume that no internal vertex equals c. Since the path is nondecreasing with $y_0 = 0 < c < 1 = y_N$, there is a unique index $k \in \{0, \dots, N-1\}$ such that

$$y_k < c < y_{k+1}; \tag{5.1}$$

we write $m := N-k-1$ for the number of steps on the right piece $[c,1]$. The following lemmas compute the corresponding algebraic stationary path and then determine when it satisfies (5.1). For $m \in \mathbb{Z}_{\geq 0}$, put

$$N_m := 1 - c(p-q)\sum_{j=0}^{m-1} q^j, \qquad D_m := \sum_{j=0}^{m-1} q^{2j} + \frac{q^{2m}}{1-p^2}, \tag{5.2}$$

where an empty sum is zero, so $N_0 = 1$ and $D_0 = (1-p^2)^{-1}$. The quantities S_m and Y_m of (2.12) then have the equivalent form

$$S_m = \frac{N_m^2}{2D_m}, \qquad Y_m = \frac{N_m q^m}{D_m}, \tag{5.3}$$

which defines them for every $m \geq 0$; the value S_m is a Case II action only for indices satisfying the feasibility condition (5.8).

5.1 Quadratic minimization and exact coordinates

For fixed m and k, we first derive a Cauchy–Schwarz lower bound, as in the linear case, and then use its equality case to compute y_k and y_{k+1}; the ordering and breakpoint

constraints are not imposed in the inequality itself, but verified afterwards as feasibility (Corollary 5.2). Throughout, the terminal-side slope products are

$$\alpha_i := \begin{cases} p, & i \le k+1, \\ q, & i \ge k+2, \end{cases} \qquad W_j := \prod_{i=j+1}^{N} \alpha_i \quad (0 \le j \le N), \qquad D_{m,k} := \sum_{j=1}^{N} W_j^2.$$

Lemma 5.1. *Fix $N \ge 1$ and an index $k \in \{0, \dots, N-1\}$, and put $m = N-k-1$. Consider paths $0 = y_0 \le \cdots \le y_N = 1$ for which $y_0, \dots, y_k < c < y_{k+1}, \dots, y_N$, so that $f(y_{j-1}) = p y_{j-1}$ for $1 \le j \le k+1$ and $f(y_{j-1}) = q y_{j-1} + c(p-q)$ for $k+2 \le j \le N$. Then $D_{m,k} = \sum_{j=0}^{m-1} q^{2j} + q^{2m} \sum_{j=0}^{k} p^{2j}$, and every such path obeys*

$$I_N(\mathbf{y}; f) \ge \frac{N_m^2}{2D_{m,k}}, \tag{5.4}$$

with equality if and only if the errors $e_j = y_j - f(y_{j-1})$ satisfy $e_j = \alpha_{j+1} e_{j+1}$; equivalently,

$$y_j - f(y_{j-1}) = f'(y_j)\big(y_{j+1} - f(y_j)\big), \qquad 1 \le j \le N-1. \tag{5.5}$$

In particular, the equality case in (5.4) determines a unique path. For this path, (5.5) holds, the action equals $S(m,k) = N_m^2/(2D_{m,k})$, and the two coordinates adjacent to c are

$$y_k = \frac{p(1-p^{2k})}{1-p^2} \frac{N_m q^m}{D_{m,k}}, \tag{5.6}$$

$$y_{k+1} = \frac{1-p^{2(k+1)}}{1-p^2} \frac{N_m q^m}{D_{m,k}}. \tag{5.7}$$

Proof. Write $e_j = y_j - f(y_{j-1})$, so that $e_j = y_j - \alpha_j y_{j-1}$ for $1 \le j \le k+1$ and $e_j = y_j - \alpha_j y_{j-1} - c(p-q)$ for $k+2 \le j \le N$. The weights satisfy $W_{j-1} = \alpha_j W_j$ and $W_N = 1$, so the terms $y_j - \alpha_j y_{j-1}$ telescope to $\sum_{j=1}^{N} W_j (y_j - \alpha_j y_{j-1}) = W_N y_N - W_0 y_0 = 1$. Since $W_j = q^{N-j}$ for $j \ge k+1$, the constant $c(p-q)$, present only for $j \ge k+2$, contributes $\sum_{j=1}^{N} W_j e_j = 1 - c(p-q) \sum_{j=k+2}^{N} W_j = 1 - c(p-q) \sum_{i=0}^{m-1} q^i = N_m$, and the Cauchy–Schwarz inequality yields $N_m^2 = (\sum_j W_j e_j)^2 \le (\sum_j W_j^2)(\sum_j e_j^2) = 2D_{m,k} I_N(\mathbf{y}; f)$, which is (5.4). Equality holds precisely when $e_j = \lambda W_j$ for a common scalar λ; since $W_j / W_{j+1} = \alpha_{j+1} = f'(y_j)$, this proportionality is exactly the recurrence $e_j = \alpha_{j+1} e_{j+1}$, that is, (5.5). Evaluating $\sum_j W_j^2$ on the slope-p and slope-q blocks gives the stated closed form for $D_{m,k}$. The equality profile forces $\lambda D_{m,k} = \sum_j W_j e_j = N_m$, so $\lambda = N_m / D_{m,k}$ and $I_N(\mathbf{y}; f) = \frac{1}{2}\lambda^2 D_{m,k} = N_m^2/(2D_{m,k})$.

It remains to read off y_k and y_{k+1}. Since $W_1 = p^k q^m$, the equality profile gives $e_1 = \lambda W_1 = N_m p^k q^m / D_{m,k}$ and $e_j = e_1 / p^{j-1}$ for $1 \le j \le k+1$. On the left piece $y_j = p y_{j-1} + e_j$, so induction yields $y_i = e_1(1-p^{2i}) / \big(p^{i-1}(1-p^2)\big)$ for $1 \le i \le k+1$: the identity holds at $i = 1$, and if it holds at i, then

$$y_{i+1} = p\, e_1 \frac{1-p^{2i}}{p^{i-1}(1-p^2)} + \frac{e_1}{p^i} = e_1 \frac{1-p^{2(i+1)}}{p^i(1-p^2)}.$$

Taking $i = k$ and $i = k+1$ and substituting $e_1 = N_m p^k q^m / D_{m,k}$ gives (5.6)–(5.7). □

Corollary 5.2. *Call the pair* (m, k) feasible *if the coordinates* (5.6)–(5.7) *satisfy* $y_k < c < y_{k+1}$. *For fixed* m, $S(m, k)$ *decreases strictly to* S_m. *Moreover the limit of these coordinates satisfies* $y_k < c < y_{k+1}$ *exactly when*

$$c(1 - p^2) < Y_m < \frac{c(1 - p^2)}{p}. \tag{5.8}$$

Proof. Since $0 < p < 1$, we have $\sum_{j=0}^{k} p^{2j} = (1-p^{2(k+1)})/(1-p^2) \nearrow (1-p^2)^{-1}$, so $D_{m,k} \nearrow D_m$ while N_m does not depend on k; consequently $S(m, k) = N_m^2/(2D_{m,k}) \downarrow S_m$, strictly. Equations (5.6)–(5.7) tend to

$$y_k \longrightarrow \frac{pY_m}{1 - p^2}, \qquad y_{k+1} \longrightarrow \frac{Y_m}{1 - p^2}, \tag{5.9}$$

so the feasibility condition $y_k < c < y_{k+1}$ is precisely (5.8). □

5.2 Continuous optimization of the algebraic value

Corollary 5.2 leaves us to minimize the limiting values S_m over $m \in \mathbb{Z}_{\geq 0}$; we first analyze this sequence as a real-variable function and then use the result in the four cases of Theorem 2.4. Throughout we assume $q \neq 1$. It follows directly from the definitions of N_m, D_m, and $S_m = N_m^2/(2D_m)$ that

$$N_m = \frac{\Delta + c(p - q)q^m}{1 - q} = \frac{\Delta - c(q - p)q^m}{1 - q}, \tag{5.10}$$

so that S_m has the equivalent closed form

$$S_m = \frac{1 + q}{2(1 - q)} \frac{(\Delta - c(q - p)q^m)^2}{1 - \lambda q^{2m}}. \tag{5.11}$$

To optimize over m we relax it to a real variable: setting $t = q^m > 0$ and

$$\delta := c(q - p), \tag{5.12}$$

we minimize the smooth function $\mathcal{S}(t) = \frac{1+q}{2(1-q)}(\Delta - \delta t)^2/(1 - \lambda t^2)$, for which $S_m = \mathcal{S}(q^m)$, and round to an adjacent integer afterwards (the rounding is performed in Theorem 2.4 and Lemma 6.1).

Lemma 5.3. *Let* $q \neq 1$. *On the half-line* $m \geq 0$, *the derivative of the continuous extension of* $S_m = \mathcal{S}(q^m)$ *can vanish only at the real point* m_0 *defined in Section 2.4, equivalently at* $t_0 := q^{m_0} = c(1 - p^2)/((p + q)\Delta)$. *If* $m_0 > 0$, *this point is a strict minimum when* $q > p$ *and a strict maximum when* $q < p$. *If* $m_0 \leq 0$, *there is no critical point on* $m > 0$.

Proof. The constant factor in $\mathcal{S}$ does not affect the critical points, and by the quotient rule

$$\frac{d}{dt} \frac{(\Delta - \delta t)^2}{1 - \lambda t^2} = \frac{-2\delta(\Delta - \delta t)(1 - \lambda t^2) + 2\lambda t(\Delta - \delta t)^2}{(1 - \lambda t^2)^2} = \frac{2(\Delta - \delta t)(-\delta + \lambda \Delta t)}{(1 - \lambda t^2)^2}.$$

The numerator vanishes at $t = \Delta/\delta$ and at $t = \delta/(\lambda\Delta)$.

The first root lies outside the domain $t = q^m$, $m \geq 0$. If $q < p$, then $\delta < 0$ and $\Delta/\delta < 0$. If $p < q < 1$, then $\Delta - \delta = 1 - q > 0$, so $\Delta/\delta > 1$, whereas $t \leq 1$. If $q > 1$, then $0 < \Delta/\delta < 1$, whereas $t \geq 1$. Hence the sole possible critical point is

$$t_0 = \frac{\delta}{\lambda\Delta} = \frac{c(1-p^2)}{(p+q)\Delta}. \tag{5.13}$$

To read off its type, restore the prefactor $(1+q)/(2(1-q))$ and factor $-\delta + \lambda\Delta t = \lambda\Delta(t - t_0)$, which gives $\mathcal{S}'(t) = \frac{1+q}{1-q}(\Delta - \delta t)\lambda\Delta(t-t_0)/(1-\lambda t^2)^2$. Here $(1-\lambda t^2)^2 > 0$; the factor $\Delta - \delta t = (1-q)N_m$ carries the sign of $1-q$; and λ carries the sign of $q - p$ (as $\Delta > 0$). Since $(1+q)/(1-q)$ also carries the sign of $1-q$, the two factors $1-q$ cancel and $\operatorname{sgn}\mathcal{S}'(t) = \operatorname{sgn}(q-p) \cdot \operatorname{sgn}(t-t_0)$. Thus $\mathcal{S}'$ passes from negative to positive at t_0 when $q > p$, and from positive to negative when $q < p$; as $t = q^m$ is a monotone reparametrization, m_0 is correspondingly the unique minimizer of S_m when $q > p$ and its unique maximizer when $q < p$. This applies inside the domain only when $m_0 > 0$; if $m_0 \leq 0$, the critical point lies outside $m > 0$. □

Lemma 5.4. *Let c_0 be as in (2.11). Then $m_0 > 0$ if and only if $c < c_0$.*

Proof. By (5.13), $t_0 = q^{m_0} = c(1-p^2)/((p+q)\Delta) > 0$. If $0 < q < 1$, then $m \mapsto q^m$ is decreasing, so $m_0 > 0$ is equivalent to $t_0 < 1$, that is, since $(p+q)\Delta > 0$, to $c(1-p^2) < (p+q)\Delta = (p+q)(1-q) + c(p+q)(q-p)$, or $c\big(1-p^2-(p+q)(q-p)\big) < (p+q)(1-q)$. The bracket equals $1-q^2 = (1-q)(1+q)$, and dividing by $1-q > 0$ gives $c(1+q) < p+q$, that is, $c < c_0$. If $q > 1$, then q^m is increasing, so $m_0 > 0$ is equivalent to $t_0 > 1$, i.e. $c(1-p^2) > (p+q)\Delta$; the same algebra gives $c(1-q)(1+q) > (p+q)(1-q)$, and division by $1-q < 0$ reverses it into $c(1+q) < p+q$ once again. Thus $m_0 > 0$ if and only if $c < c_0$. □

Remark 5.5. *For $q > 1$, the assumption $\Delta > 0$ already supplies a lower bound on c: since $\Delta = 1 - q + c(q-p)$ and $q - p > 0$, one has $\Delta > 0$ if and only if $c > c_1 := (q-1)/(q-p)$. Hence, for $q > 1$, the condition $m_0 > 0$ describes the interval $c_1 < c < c_0$.*

Corollary 5.6. *Recall $S_0 = (1-p^2)/2$. Over the nonnegative integers m: (a) if $c < c_0$, then $m_0 > 0$ and S_m is unimodal, with an interior extremum near m_0—a minimum when $q > p$, in which case the integer optimum is one of the two integers adjacent to m_0, and a maximum when $q < p$; (b) if $c \geq c_0$, then $m_0 \leq 0$ and S_m is monotone for $m \geq 0$: increasing when $q > p$, so that $\inf_m S_m = S_0 = (1-p^2)/2$, and decreasing when $q < p$.*

Proof. Combine Lemmas 5.3 and 5.4. If $c < c_0$, the unique real extremum m_0 is positive, which gives the unimodal alternative (a), the adjacent-integer statement being the standard rounding of a unimodal sequence. If $c \geq c_0$, then $m_0 \leq 0$, so the real extremum lies at or below 0 and $\mathcal{S}$ is monotone for $m \geq 0$; translating through $t = q^m$ as in the proof of Lemma 5.3 gives the stated monotonicity of S_m, with value $S_0 = (1-p^2)/2$ at the endpoint $m = 0$. □

Lemma 5.7. *For $q \neq 1$, the real number m_0 satisfies the feasibility condition (5.8).*

Proof. Rewriting the closed form of D_m as $D_m = (1-\lambda q^{2m})/(1-q^2)$ and using (5.10) gives $Y_m = N_m q^m / D_m = (1+q)q^m(\Delta - \delta q^m)/(1-\lambda q^{2m})$, with $\delta = c(q-p)$

as in (5.12). At m_0 the critical relation $\delta = \lambda\Delta q^{m_0}$ (proof of Lemma 5.3) gives $\Delta - \delta q^{m_0} = \Delta(1 - \lambda q^{2m_0})$, so, with $q^{m_0} = t_0 = c(1-p^2)/((p+q)\Delta)$, one gets $Y_{m_0} = (1+q)\Delta q^{m_0} = c(1+q)(1-p^2)/(p+q)$. Dividing (5.8), that is $c(1-p^2) < Y_{m_0} < c(1-p^2)/p$, by $c(1-p^2) > 0$ turns it into $1 < (1+q)/(p+q) < 1/p$, and each of the two inequalities is equivalent to $p < 1$, which holds. □

Lemma 5.8. *For every $n \geq 1$, $\widehat{S}_n \geq S_n$. Equality holds if and only if $Y_n = c(1-p^2)$, that is, the limiting value of y_{k+1} equals c (as $k \to \infty$ with $m = n$ fixed).*

Proof. A common-denominator reduction of $\widehat{S}_n$ from (2.13) and S_n from (5.11) gives

$$\widehat{S}_n - S_n = -\frac{\big[(1-q^2)D_n\big(c(1-p^2) - Y_n\big)\big]^2}{2(q^{2n}-1)(1-p^2-(q^2-p^2)q^{2n})}, \tag{5.14}$$

where D_n, Y_n denote D_m, Y_m at $m = n$. Writing the second denominator factor as $1 - p^2 - (q^2 - p^2)q^{2n} = (1-q^2)(1-p^2)D_n$ with $D_n > 0$, the denominator equals $2(q^{2n}-1)(1-q^2)(1-p^2)D_n$, which is negative because $q^{2n} - 1$ and $1 - q^2$ have opposite signs for every $q \neq 1$; as the numerator is the negative of a square, $\widehat{S}_n \geq S_n$. Finally $(1-q^2)D_n \neq 0$, so the numerator vanishes precisely when $Y_n = c(1-p^2)$, i.e. when the limiting value of y_{k+1} equals c. The boundary case $q = 1$ is excluded from (5.14) but obeys the same bound: there, with $d = c(1-p)$ and $a = 1/(1-p^2)$ (Proposition 2.5),

$$S_n = \frac{(1+dn)^2}{2(n+a)}, \qquad \widehat{S}_n = \frac{1-p^2}{2}c^2 + \frac{(1-c+nd)^2}{2n},$$
$$\widehat{S}_n - S_n = \frac{\big(1-c-cnp(1-p)\big)^2}{2n\big(1+(1-p^2)n\big)} \geq 0,$$

again with equality precisely when $1 - c - cnp(1-p) = 0$, i.e. when $Y_n = c(1-p^2)$, equivalently the limiting value of y_{k+1} equals c. □

Remark 5.9. *A feasible index has $c(1-p^2) < Y_n$ by (5.8), so $Y_n \neq c(1-p^2)$ and the inequality is strict, $\widehat{S}_n > S_n$. For an infeasible index (5.14) still gives $\widehat{S}_n \geq S_n$, now for the algebraic value S_n, which is not a path cost there.*

6 Feasibility and comparison lemmas

The proof of Theorem 2.4 rests on the lemmas collected below, grouped by the regime that uses them and proved in Section 9. For $q > p$ and $c < c_0$ the sequence S_m has an interior minimum, and the integer nearest that minimum is itself feasible.

Lemma 6.1. *Assume $q > p$, $\Delta > 0$, and $c < c_0$. Let m^* be whichever of the two integers adjacent to m_0 yields the smaller value of S_m. Then m^* satisfies the feasibility condition (5.8); equivalently, $m^* \in \mathcal{M}$.*

For $q < p$ and $c < c_0$, instead, S_m has an interior maximum, so feasibility is decided at the two ends of a short interval. The next three lemmas control the length of that interval, compare the feasible value of S_m with the breakpoint value, and bound the cost of every path whose number of steps on the right-hand piece is not feasible.

Lemma 6.2. *Assume $0 < q < p < 1$ and $c < c_0$. Then the real solutions m of the feasibility condition (5.8) form an interval of length less than one, so $\mathcal{M}$ contains at most one integer.*

Lemma 6.3. *Assume $0 < q < p < 1$ and $c < c_0$. If $\mathcal{M} = \{m^*\}$, then $S_{m^*} < \widehat{S}$.*

Lemma 6.4. *Assume $0 < q < p < 1$ and $c < c_0$. For every $m \notin \mathcal{M}$, every path with $y_k < c < y_{k+1}$ and m steps on the right piece has $I_N(\mathbf{y}; f) \geq \widehat{S}$.*

Finally, for $q < p$ and $c \geq c_0$ the sequence S_m is monotone and only $m = 0$ is feasible; the last lemma excludes every $m \geq 1$.

Lemma 6.5. *Assume $0 < q < p < 1$ and $c \geq c_0$. Then for every $m \geq 1$ and $k \geq 0$ the coordinates (5.6)–(5.7) satisfy $y_{k+1} < c$; hence no index $m \geq 1$ is feasible.*

7 Proof of the classification theorem

Proof of Theorem 2.4. Every admissible nondecreasing path belongs to one of the two classes studied above: it either has an internal vertex at c, or it has none. The first class is governed by Proposition 4.2, the second by Corollary 5.2 and the lemmas of Section 6. We separate the four cases.

Case 1: $q > p$, $c < c_0$. We prove $S(f) = S_{m^*}$, where m^* is the better of the two integers adjacent to m_0; three things must be checked: that m^* is the integer minimizer, that it is feasible, and that S_{m^*} is smaller than the breakpoint value $\widehat{S}$. By Corollary 5.6(a), $m_0 > 0$ and S_m is unimodal with an interior minimum, so its minimum over $m \in \mathbb{Z}_{\geq 0}$ is attained at one of the two integers adjacent to m_0; call it m^*. By Lemma 6.1 it satisfies the feasibility condition (5.8), so S_{m^*} is the relevant limiting value: the finite costs $S(m^*, k)$ decrease strictly to it as $k \to \infty$ (Corollary 5.2). For the comparison with the breakpoint value, Lemma 5.8 gives $\widehat{S}_n \geq S_n \geq S_{m^*}$ for each threshold index $n \in \{n^*, n^* + 1\}$, and this is strict: if $S_n > S_{m^*}$, then $\widehat{S}_n \geq S_n > S_{m^*}$; and if $S_n = S_{m^*}$, then n minimizes S_m and so is feasible (Lemma 6.1), whence $\widehat{S}_n > S_n = S_{m^*}$ by Remark 5.9. In either case $\widehat{S}_n > S_{m^*}$, so $S_{m^*} < \widehat{S}$. Since the finite costs decrease strictly to S_{m^*} and this value is smaller than every breakpoint value, $S(f) = S_{m^*}$, attained at no finite horizon. This is the first line of the theorem.

Case 2: $q < p$, $c < c_0$. Here $m_0 > 0$ is a *maximum* of S_m (Corollary 5.6(a)): S_m increases on $(0, m_0)$ and decreases on (m_0, ∞). It has no interior minimum, so among the feasible indices the cost decreases toward the two ends of the feasible interval, not toward the interior maximum m_0. By Lemma 5.7, m_0 lies strictly inside that interval, and by Lemma 6.2 the interval has length less than one; hence the only integers it can contain are $\lfloor m_0 \rfloor$ and $\lceil m_0 \rceil$, and at most one of them is feasible. Its endpoints, where $y_k = c$ or $y_{k+1} = c$, are breakpoint configurations of Case I and are excluded from Case II. Thus $\mathcal{M} = \{m^*\}$ or $\mathcal{M} = \emptyset$, giving the second and third lines of the theorem.

If $\mathcal{M} = \{m^*\}$, the only feasible limiting value from Case II is S_{m^*}; by Lemma 6.3 $S_{m^*} < \widehat{S}$, and by Lemma 6.4 every path whose number of steps on the right-hand piece lies outside $\mathcal{M}$ costs at least $\widehat{S}$, so no other path can do better. Since $S(m^*, k)$ decreases strictly to S_{m^*} (Corollary 5.2), $S(f) = S_{m^*}$, attained at no finite horizon. If $\mathcal{M} = \emptyset$, there is no feasible Case II limiting value, and by Lemma 6.4 every path costs at least $\widehat{S}$; the breakpoint class of Case I therefore supplies the infimum, whose value $\widehat{S}$ is realized in the limit but at no finite horizon, so $S(f) = \widehat{S}$.

Case 3: $q > p$**,** $c \geq c_0$**.** Here $m_0 \leq 0$, so by Corollary 5.6(b) S_m is increasing for $m \geq 0$ and $S_m \geq S_0 = (1-p^2)/2$. The finite algebraic costs exceed this, $S(m,k) > S_m \geq S_0$ (Corollary 5.2), and so do the breakpoint values, $\widehat{S}_n \geq S_n \geq S_0$ (Lemma 5.8); thus $(1-p^2)/2$ is a lower bound for every admissible cost. It is approached along the feasible $m = 0$ configurations,

$$S(0,k) = \frac{1-p^2}{2(1-p^{2(k+1)})} \downarrow \frac{1-p^2}{2}, \tag{7.1}$$

each term lying strictly above the limit. Therefore $S(f) = (1-p^2)/2$, attained at no finite horizon.

Case 4: $q < p$**,** $c \geq c_0$**.** Here $m_0 \leq 0$, so by Corollary 5.6(b) the formula S_m is *decreasing* for $m \geq 0$, sinking toward its infimum as $m \to \infty$. None of these small values is feasible, however: by Lemma 6.5 every index $m \geq 1$ has $y_{k+1} < c$. The only feasible index is $m = 0$, admissible because $c \geq c_0 > p$ (indeed $c_0 - p = q(1-p)/(1+q) > 0$); as in Case 3 it gives (7.1). For the matching lower bound, $q < p$ gives $f(x) \leq px$ on $[0,1]$, so every nondecreasing path has $0 \leq y_j - py_{j-1} \leq y_j - f(y_{j-1})$, and the linear Cauchy–Schwarz estimate yields

$$I_N(\mathbf{y}; f) \geq \frac{1}{2}\sum_{j=1}^{N}(y_j - py_{j-1})^2 \geq \frac{1-p^2}{2(1-p^{2N})} > \frac{1-p^2}{2}. \tag{7.2}$$

Together with the preceding $m = 0$ paths this gives $S(f) = (1-p^2)/2$, attained at no finite horizon.

The four cases prove the formula, and the strict finite-k inequalities above prove nonattainment. □

8 Boundary regimes

This section proves the three remaining boundary results, Propositions 2.5, 2.6 and 2.7.

Proof of Proposition 2.5. Let $d = c(1-p)$, so $f(x) = x - d$ on $[c,1]$. The finite and limiting costs for paths with no internal vertex at c are

$$S(m,k) = \frac{(1+dm)^2}{2(m+C_k)}, \quad C_k = \frac{1-p^{2(k+1)}}{1-p^2}, \qquad S_m = \frac{(1+dm)^2}{2(m+a)}, \quad a = \frac{1}{1-p^2},$$

and a direct differentiation gives $S'_m = (1+dm)(dm+2da-1)/\big(2(m+a)^2\big)$; thus $m_1 = 1/d - 2a$, and $m_1 > 0$ if and only if $c < (1+p)/2$. As in Corollary 5.2, the limiting coordinates satisfy $y_k < c < y_{k+1}$ exactly when the feasibility condition (5.8) holds; at $q = 1$ the quantity appearing there is

$$Y_m = \frac{(1-p^2)(1+dm)}{1+(1-p^2)m}, \tag{8.1}$$

so feasibility reads $c(1-p^2) < Y_m < c(1-p^2)/p$. Both denominators are positive and $d > 0$; clearing them turns these two inequalities into $L < m < U$, where $L := (p-c)/d$ and $U := (1-c)/(pd)$. The real minimizer m_1 always lies strictly

inside this interval, since $m_1 - L = (1+p-c)/\big(c(1+p)\big) > 0$ and $U - m_1 = (1+p-c)/\big(cp(1+p)\big) > 0$.

By the sign of S'_m, the map $m \mapsto S_m$ is unimodal, strictly decreasing before m_1 and strictly increasing after. When $m_1 > 0$ its integer minimizer over $m \geq 0$ is therefore one of the neighbours $m^* \in \{\lfloor m_1 \rfloor, \lceil m_1 \rceil\}$, and this m^* is feasible by the same floor/ceiling contrapositive as in Lemma 6.1, in the simpler form available here. Indeed, a common-denominator subtraction gives $\operatorname{sgn}(S_m - S_{m+1}) = \operatorname{sgn} F(m)$ with $F(m) = 1 - d^2m^2 - d^2m - 2ad^2m - ad^2 - 2ad$, which plays the role of P in (9.2) but is now simply strictly decreasing for $m \geq 0$, with endpoint values $F(L) = (c-1)(p-1)(1+p-c) > 0$ and $F(U-1) = -(c-1)(p-1)(1+p-c)/p^2 < 0$. So if the floor $n = \lfloor m_1 \rfloor$ were the minimizer yet $n \leq L$, then $F(n) \geq F(L) > 0$, i.e. $S_n > S_{n+1}$, contradicting minimality; and if the ceiling $n+1 = \lceil m_1 \rceil$ were the minimizer yet $n+1 \geq U$, then $F(n) \leq F(U-1) < 0$, i.e. $S_n < S_{n+1}$, again a contradiction. Since $n < m_1 < U$ and $n+1 > m_1 > L$ are automatic, $m^* \in (L, U)$ in every case.

It remains to compare with paths having an internal vertex at c: such a path has cost $\widehat{S}_n$ for some $n \geq 1$, and $\widehat{S}_n \geq S_n$ by Lemma 5.8, while every finite path with no internal vertex at c is represented by a pair (m,k) with $S(m,k) > S_m$, since $C_k < a$. If $c < (1+p)/2$, then $m_1 > 0$ and the integer minimizer m^* is feasible, so $S(m^*,k) \downarrow S_{m^*}$; for every threshold index $n \geq 1$ we have $\widehat{S}_n \geq S_n \geq S_{m^*}$, and this is strict, since $S_n > S_{m^*}$ gives $\widehat{S}_n > S_{m^*}$ while $S_n = S_{m^*}$ makes n a minimizer of S_m, hence feasible, whence $\widehat{S}_n > S_n = S_{m^*}$ by Remark 5.9. Thus $S_{m^*} < \widehat{S}$, while every Case II path has cost $S(m,k) > S_m \geq S_{m^*}$; hence $S(f) = S_{m^*}$. If $c \geq (1+p)/2$, then $m_1 \leq 0$, so S_m increases on $m \geq 0$ and its integer minimizer is $m = 0$; moreover $c > p$ makes $m = 0$ feasible, with $S_0 = (1-p^2)/2$ and $S(0,k) \downarrow S_0$. Every threshold index gives $\widehat{S}_n \geq S_n \geq S_0$, and every Case II path gives $S(m,k) > S_m \geq S_0$; hence $S(f) = (1-p^2)/2$. In both cases the finite costs decrease strictly to the infimum, which is therefore attained at no finite horizon. Collecting the two cases,

$$S(f) = \begin{cases} \dfrac{(1+dm^*)^2}{2(m^*+a)}, & c < (1+p)/2, \\[2ex] \dfrac{1-p^2}{2}, & c \geq (1+p)/2. \end{cases}$$

□

Proof of Proposition 2.6. Since $p = 0$, the map is $f(x) = 0$ on $[0,c]$ and $f(x) = q(x-c)$ on $[c,1]$. As $f(x) < x$ on $(0,1]$—for $x \geq c$, $x - f(x) \geq \Delta > 0$—a minimizing path may be taken nondecreasing, $0 = y_0 \leq \cdots \leq y_N = 1$.

Let $j_0 = \min\{j \geq 1 : y_j \geq c\}$. The vertices before j_0 lie in $[0,c)$, where $f \equiv 0$, so

$$\frac{1}{2}\sum_{j=1}^{j_0}\big(y_j - f(y_{j-1})\big)^2 = \frac{1}{2}\sum_{j=1}^{j_0} y_j^2 \geq \frac{1}{2}y_{j_0}^2,$$

which is the cost of a single step from c to y_{j_0} under $g(x) = q(x-c)$, since $g(c) = 0$. As the remaining vertices lie in $[c,1]$, where $f = g$, the path costs at least the l-step value $J_l(c,1)$ with $l = N - j_0 + 1$; conversely each admissible $c = x_0 \leq \cdots \leq x_l = 1$ gives the admissible path $0, x_1, \ldots, x_l$ of equal cost. Hence $S(f) = \inf_{l \geq 1} J_l(c,1)$.

We apply Lemma 3.1 on $[c, 1]$ with slope $\alpha = q$. The endpoint gaps $d_c = c$ and $d_1 = \Delta$ are positive, and $d_c - d_1 = (q-1)(1-c) > 0$, so $\log_q(c/\Delta) > 0$. By (3.1) and (3.4), the l-step value $J_l(c, 1) = (1+q)(\Delta - cq^l)^2/[2(1-q)(1-q^{2l})]$ is unimodal in l, with integer minimum at one of $n_0, n_0 + 1$, where $n_0 = \max\{1, \lfloor \log_q(c/\Delta) \rfloor\}$. Thus $S(f)$ is the stated minimum of $J_l(c, 1)$ over $l \in \{n_0, n_0 + 1\}$, realised by the minimizing path itself; in contrast with the preceding cases, the infimum is attained at a finite horizon. □

Proof of Proposition 2.7. Since $\Delta = 0$, we have $c(q-p) = q-1$ and $q > 1$. Moreover, $c_0 - c = (1-p^2)/[(1+q)(q-p)] > 0$. For paths hitting c, affine minimization gives the left and right block infima $(1-p^2)c^2/2$ and $(1+q)c^2(1-p)^2/[2(q-1)]$; their sum is $\widehat{S} = c(1-p)$.

In Case II, write $y_k < c < y_{k+1}$ and $m = N - k - 1$. The calculation of Lemma 5.1 gives $I_N(\mathbf{y}; f) \geq S(m, k) \geq S_m$, where

$$S_m = \left(\frac{2(1-q^{-2m})}{q^2-1} + \frac{2}{1-p^2} \right)^{-1} \downarrow \frac{(1-p^2)(q^2-1)}{2(q^2-p^2)} \quad (m \to \infty).$$

For $0 < p < 1$, $S(m, k) \downarrow S_m$ as $k \to \infty$; for $p = 0$, $S(m, 0) = S_m$.

For $0 < p < 1$, the equality coordinates (5.6)–(5.7), as $k \to \infty$ and then $m \to \infty$, approach $p(q^2-1)/(q^2-p^2) < c$ and $c < (q^2-1)/(q^2-p^2) < 1$, respectively. Thus, for sufficiently large m and then sufficiently large k, the crossing constraints hold. Both affine blocks are increasing by Lemma 3.1, so these equality paths are admissible and their actions approach $\lim_m S_m$. For $p = 0$, take $k = 0$: then $y_1 = 2S_m \in (c, 1)$ for $m \geq 1$, and the right block is increasing by Lemma 3.1. Hence these paths are also admissible and approach the same limit.

Finally, $c = (q-1)/(q-p)$ gives

$$\widehat{S} - \lim_{m \to \infty} S_m = \frac{(1-p)^2(q-1)^2}{2(q^2-p^2)} > 0.$$

Thus $S(f) = \lim_m S_m$, with $S_m < \widehat{S}$ for all sufficiently large m. Both classes have strictly larger finite-path costs, so the infimum is not attained. For $p = 0$, substituting $q = 1/(1-c)$ yields $S(f) = c - c^2/2$. □

9 Proofs of the feasibility and comparison lemmas

We prove the lemmas of Section 6 in the order in which the cases of Theorem 2.4 invoke them.

Proof of Lemma 6.1. Throughout, $t = q^m$, and $\delta = c(q-p) > 0$, $\lambda = (q^2-p^2)/(1-p^2) > 0$ as in (2.11) and (5.12). As in Lemma 5.7, put

$$Y(t) := Y_m = \frac{(1+q)t(\Delta - \delta t)}{1 - \lambda t^2}, \qquad L := c(1-p^2), \quad U := \frac{c(1-p^2)}{p},$$

so that (5.8) reads $L < Y(t) < U$. By Lemma 5.7 the real minimizer m_0 is strictly feasible; hence the floor can only fall too far to the left and the ceiling only too far to

the right. We treat the two failures separately, each time showing that the supposedly chosen integer is beaten by its neighbour.

Subtracting (5.11) at m and $m+1$ gives

$$S_m - S_{m+1} = \frac{-(1+q)tP(t)}{2(1-\lambda t^2)(1-\lambda q^2 t^2)}, \qquad P(t) = 2\delta\Delta - (1+q)(\delta^2+\lambda\Delta^2)t + 2q\lambda\delta\Delta t^2. \tag{9.1}$$

For $p < q < 1$ we have $0 < \lambda < q^2 < 1$ and $0 < t \le 1$, while for $q > 1$ we have $\lambda > q^2 > 1$ and $t \ge 1$; in both ranges the two denominator factors carry the same sign, so the denominator is positive and

$$P(q^m) < 0 \Rightarrow S_m > S_{m+1}, \qquad P(q^m) > 0 \Rightarrow S_m < S_{m+1}. \tag{9.2}$$

The quadratic P opens upward, $P(0) = 2\delta\Delta > 0$, and at $t_* := \lambda^{-1/2}$,

$$P(t_*) = -\frac{(1+q)(\delta - \sqrt{\lambda}\,\Delta)^2}{\sqrt{\lambda}} \le 0. \tag{9.3}$$

We also record the upward-opening quadratic

$$Q(c) = (q-p)(1+q)c^2 + 2(p-q)(p+q)c + (p+q)(q-1), \tag{9.4}$$

whose vertex is $c_0 = (p+q)/(1+q)$, with $Q(c_0) = (p^2-1)(p+q)/(1+q) < 0$. If $p < q < 1$, the lemma allows $0 < c < c_0$ and $Q(0) = (p+q)(q-1) < 0$; if $q > 1$, then $\Delta > 0$ forces $c > c_1 := (q-1)/(q-p)$, the lemma allows $c_1 < c < c_0$, and $Q(c_1) = -(1-p^2)(q-1)/(q-p) < 0$. In both cases the admissible range lies to the left of the vertex, where Q decreases towards $Q(c_0)$; hence

$$Q(c) < 0 \quad \text{throughout the range allowed by the lemma.} \tag{9.5}$$

Case $m^* = \lfloor m_0 \rfloor$. If $m_0 \in \mathbb{Z}_{\ge 0}$ there is nothing to prove; otherwise put $n = \lfloor m_0 \rfloor$ and $t_n = q^n$. If the floor is the better adjacent integer then $S_n \le S_{n+1}$, so by (9.2) it suffices to derive $P(t_n) < 0$ from the contrary assumption that n is infeasible on the left, i.e. $n \le m_-$, where m_- is the left endpoint of the feasible component containing m_0. There the upper inequality is sharp, $Y(t_-) = U$ with $t_- = q^{m_-}$, equivalently $G_-(t_-) = 0$, where $G_-(t) := (1+q)(\Delta t - \delta t^2) - U(1-\lambda t^2) = (1-\lambda t^2)(Y(t) - U)$.

Claim 1: $P(t_-) < 0$. The leading coefficient of G_- is $U\lambda - (1+q)\delta = cq(1-p)(q-p)/p > 0$ and its constant term is $-U < 0$, so G_- has a unique positive root, namely t_-, and $G_-(s) < 0$ for $0 < s < t_-$. Dividing the quadratic P by the quadratic G_- leaves a linear remainder, so evaluation at the root gives

$$P(t_-) = ut_- + v, \quad v = \frac{2c\Delta(1+q)(q-p)}{1-p} > 0, \quad u = -\frac{(q+1)(q-p)}{(1-p)^2}R(c), \tag{9.6}$$

where $R(c) = (q-p)(1+q^2-2p)c^2 + 2(q-p)(p+q)(1-q)c + (p+q)(1-q)^2$. Grant for a moment that $u < 0$. Then $s \mapsto us + v$ is negative exactly for $s > -v/u$, while G_- is negative exactly to the left of t_-; hence $P(t_-) < 0 \iff t_- > -v/u \iff G_-(-v/u) < 0$. That $u < 0$, i.e. $R(c) > 0$: for $p < q < 1$ every coefficient of R is positive, since $1+q^2-2p > 1+q^2-2q = (1-q)^2$; for $q > 1$, the upward-opening R has

vertex $c_R = (p+q)(q-1)/(1+q^2-2p)$ with $c_R - c_1 = -(1-p)^2(q-1)/\big((q-p)(1+q^2-2p)\big) < 0$, so R increases on (c_1, ∞) and $R(c) > R(c_1) = (1-p)^2(q-1)^2/(q-p) > 0$. Finally, substituting (9.6) and factoring gives the identity

$$u^2 G_-(-v/u) = \kappa\, c(1-c)(q-p)^3(q-1)^2(q+1)^2\, \Lambda(c)\, Q(c), \tag{9.7}$$

where $\Lambda(c) := c(q+1-2p) - (q-1)$ and $\kappa = 1/\big(p(1-p)^2\big) > 0$. Here $\Lambda > 0$ in both parameter ranges: for $p < q < 1$ this is immediate from $q - 1 < 0$, and for $q > 1$ it follows from $c > c_1$ and $q + 1 - 2p > q - p$. Since $Q(c) < 0$ by (9.5), the right-hand side of (9.7) is negative, so $G_-(-v/u) < 0$ and therefore

$$P(t_-) < 0. \tag{9.8}$$

It remains to move this sign to t_n. If $q < 1$, then $t_* > 1$ and $n \le m_-$ gives $t_- \le t_n \le 1 < t_*$; if $q > 1$, then $t_* < 1$ and the same assumption gives $t_* < 1 \le t_n \le t_-$. Either way t_n lies between t_* and t_-, where P is nonpositive at both ends by (9.3) and (9.8); convexity yields $P(t_n) < 0$, hence $S_n > S_{n+1}$ by (9.2), a contradiction. Thus the floor, whenever it is chosen, is feasible.

Case $m^* = \lceil m_0 \rceil$. With $n = \lfloor m_0 \rfloor$, the ceiling $n+1$ satisfies its left inequality automatically, so it can only fail at the right endpoint, where $Y(t) > L$ is lost. If the ceiling is the better adjacent integer then $S_{n+1} \le S_n$, so by (9.2) it suffices to derive $P(t_n) > 0$ from the contrary assumption that the ceiling is infeasible at that right endpoint. The right boundary is a root $t_+ = q^{m_+}$ of $G_+(t) := (1+q)(\Delta t - \delta t^2) - L(1-\lambda t^2) = (1-\lambda t^2)(Y(t) - L)$, whose leading coefficient is $L\lambda - (1+q)\delta = c(q-p)(p-1) < 0$, so G_+ opens downward; whenever it has real roots $\rho_1 < \rho_2$, Vieta's formulas $\rho_1\rho_2 = -L/\big(L\lambda - (1+q)\delta\big) > 0$ and $\rho_1 + \rho_2 = -(1+q)\Delta/\big(L\lambda - (1+q)\delta\big) > 0$ show that both are positive, and $G_+ > 0$ exactly on (ρ_1, ρ_2). Dividing $P(t/q)$ by $G_+(t)$ again leaves a linear remainder, so at each root

$$P(\rho_i/q) = u_+\rho_i + v_+, \quad v_+ = -\frac{2cp\Delta(1+q)(q-p)}{q(1-p)} < 0,$$
$$u_+ = \frac{(q-p)(1+q)}{q(1-p)^2} R_+(c), \tag{9.9}$$

where $R_+(c) = (p+q)(1-q)^2 + 2(q-p)(1-q)(p+q)c - (q-p)\big(1-q^2-2p(1-p)\big)c^2$.

The range $p < q < 1$. Here $t_0 = q^{m_0}$, so $Y(t_0) > L$ by the strict feasibility of m_0 (Lemma 5.7), and $1 - \lambda t_0^2 > 0$ because $0 < t_0 < 1$ and $0 < \lambda < 1$; hence $G_+(t_0) = (1 - \lambda t_0^2)(Y(t_0) - L) > 0$ and $t_0 \in (\rho_1, \rho_2)$. The ceiling lies to the right of m_0, and $q < 1$ makes $t = q^m$ decrease as m increases; so t leaves (ρ_1, ρ_2) through its left end, and the right endpoint of the component is $t_+ = \rho_1$, not ρ_2. Infeasibility of the ceiling means $n + 1 \ge m_+$, i.e. $q^{n+1} \le t_+$, i.e. $t_n \le t_+/q$. Since P opens upward with $P(0) > 0$ and $P(t_*) \le 0$, it has a first zero $r_1 \in (0, t_*]$, and on $(0, t_*]$ positivity of P occurs exactly to the left of r_1; moreover $t_+ < t_0 < 1$ and $\lambda < q^2$ give $t_+/q < 1/q < t_*$. So it suffices to prove that $P(t_+/q) > 0$, for then t_+/q lies to the left of r_1, and so does t_n.

By (9.9) with $\rho_1 = t_+$, and since $v_+ < 0$, the inequality $P(t_+/q) > 0$ is equivalent to $u_+ > 0$ together with $t_+ > -v_+/u_+$. Positivity of u_+ is positivity of R_+: its derivative $R_+'(c) = 2(q-p)\big((1-q)(p+q) - c(1-q^2-2p(1-p))\big)$ has its linear factor

positive at both $c = 0$ and $c = c_0$, so R_+ increases on $[0, c_0]$ and $R_+(c) > R_+(0) = (p+q)(1-q)^2 > 0$. Next, the same computation that gave (9.7) yields the identical product for G_+, now with $\kappa = 1/\big(q^2(1-p)^2\big)$ in place of $1/\big(p(1-p)^2\big)$; hence $u_+^2 G_+(-v_+/u_+) < 0$ by the same two signs $\Lambda > 0$ and $Q < 0$. This places $-v_+/u_+$ outside $[\rho_1, \rho_2]$; to exclude the far side, compare it with t_0. From (9.9),

$$-\frac{v_+}{u_+} = \frac{2cp\Delta(1-p)}{R_+(c)}, \qquad t_0 = \frac{\delta}{\lambda\Delta} = \frac{c(1-p^2)}{(p+q)\Delta},$$

so $-v_+/u_+ < t_0$ is equivalent to $R_+(c) > 2p(p+q)\Delta^2/(1+p)$, which is exactly the identity $R_+(c) - 2p(p+q)\Delta^2/(1+p) = -(p-1)(q-1)Q(c)/(p+1) > 0$. As $t_0 \in (\rho_1, \rho_2)$, the point $-v_+/u_+$ must lie to the left of $\rho_1 = t_+$. Hence $P(t_+/q) > 0$, so $P(t_n) > 0$ and $S_n < S_{n+1}$ by (9.2), contradicting $S_{n+1} \le S_n$.

The range $q > 1$. Here $c_1 < c < c_0$, and the discriminant of G_+ is $\mathcal{D} = (1+q)^2\Delta^2 - 4c^2(q-p)(1-p)^2(1+p)$. If $\mathcal{D} < 0$, then the downward-opening G_+ is negative everywhere; since $t \ge 1$ and $\lambda > q^2 > 1$ give $1 - \lambda t^2 < 0$, the identity $G_+(t) = (1-\lambda t^2)(Y(t) - L)$ forces $Y(t) > L$ for every integer m, so the ceiling is automatically feasible. Assume therefore $\mathcal{D} \ge 0$. Now $1 - \lambda t^2 < 0$, so $Y(t) > L$ means $G_+(t) < 0$, i.e. t lies outside $[\rho_1, \rho_2]$, and $t = q^m$ increases with m. Hence a component contained in (ρ_2, ∞) is unbounded to the right and has no finite endpoint, whereas on $(0, \rho_1)$ the point t moves right towards ρ_1 and leaves the component there. So again $t_+ = \rho_1$, and infeasibility of the ceiling gives $q^{n+1} \ge t_+$, i.e. $t_n \ge t_+/q$. Because $\lambda > q^2$ we have $t_* < 1$. Since P opens upward with $P(t_*) \le 0$ by (9.3), its larger zero r_2 satisfies $r_2 \ge t_*$, and for $t > t_*$ one has $P(t) > 0$ exactly when $t > r_2$; the integer range $t \ge 1 > t_*$ lies in that regime. Moreover $m_+ > m_0 > 0$ gives $t_+ > 1$, and $qt_* = q/\sqrt{\lambda} < 1$ gives $t_* < 1/q$, whence $t_+/q > 1/q > t_*$. Thus it again suffices to prove

$$P(t_+/q) > 0, \tag{9.10}$$

for then $t_+/q > r_2$, and $t_n \ge t_+/q$ gives $t_n > r_2$, i.e. $P(t_n) > 0$.

We evaluate $P(\cdot/q)$ at both roots at once. By (9.9) and Vieta's formulas,

$$\begin{aligned} P(\rho_1/q)P(\rho_2/q) &= u_+^2\rho_1\rho_2 + u_+v_+(\rho_1+\rho_2) + v_+^2 \\ &= \frac{(c-1)(p-q)^2(q-1)^2(q+1)^2}{q^2(p-1)^3} M_1(c)Q(c), \end{aligned}$$

with $M_1(c) = 2cp - cq - c + q - 1$. The prefactor is positive, M_1 is linear and negative at both endpoints of (c_1, c_0), and $Q < 0$ by (9.5); hence

$$P(\rho_1/q)P(\rho_2/q) > 0, \tag{9.11}$$

so the two values share a sign. To identify it, take their sum:

$$\begin{aligned} P(\rho_1/q) + P(\rho_2/q) &= u_+(\rho_1+\rho_2) + 2v_+ = \frac{(q+1)\big(\delta - (q-1)\big)}{cq(1-p)^3}\,\mathcal{H}(c), \\ \mathcal{H}(c) &= c_0\mathcal{D} + \frac{(p-1)^2(q-p)(2p+q-1)(2p+1-q)}{q+1}c^2. \end{aligned} \tag{9.12}$$

The prefactor is positive, because $c > c_1$ is equivalent to $\delta > q - 1$. If $q < 2p + 1$, the second term in (9.12) is strictly positive, so $\mathcal{H}(c) > 0$ whenever $\mathcal{D} \geq 0$. If $q \geq 2p + 1$, the case $\mathcal{D} \geq 0$ is empty: there $\mathcal{D}$ is an upward-opening quadratic in c with $\mathcal{D}(c_1) = 4(p-1)^2(p+1)(q-1)^2/(p-q) < 0$ (negative for every $q > p$) and $\mathcal{D}(c_0) = (1-p)^2(1+p)\big[(1+p) - 4(p+q)^2(q-p)/(1+q)^2\big] < 0$, the latter because $q \geq 2p+1$ gives $q - p \geq 1 + p$ while $q > 1$ gives $4(p+q)^2 > (1+q)^2$; so by convexity $\mathcal{D} < 0$ throughout (c_1, c_0). In every nonempty case the sum is therefore positive, which together with (9.11) gives $P(\rho_i/q) > 0$ for $i = 1, 2$. Since $t_+ = \rho_1$, this proves (9.10).

Both alternatives are now covered, so the adjacent integer that minimizes S_m satisfies (5.8), as claimed. □

Proof of Lemma 6.2. We keep the notation of Lemma 6.1: $t = q^m$, with δ and λ as in (5.12) and (2.11) – both negative in the present regime – and $Y(t) = (1+q)t(\Delta - \delta t)/(1 - \lambda t^2)$. Since $0 < q < 1$, increasing m moves t to the left. The limit feasibility condition is $c(1-p^2) < Y(t) < c(1-p^2)/p$; let t_+ and t_- be the two boundary points of the feasible interval, so that $Y(t_+) = c(1-p^2)$ and $Y(t_-) = c(1-p^2)/p$. The feasible interval in the m-variable has length less than one exactly when

$$m_+ - m_- < 1 \quad \Longleftrightarrow \quad \frac{t_+}{t_-} > q \quad \Longleftrightarrow \quad qt_- < t_+,$$

and this last inequality is what we prove.

Claim 1: Y is strictly increasing on $(0, t_-]$; in particular $t_+ \in (0, t_-)$. The upper boundary equation $Y(t) = c(1-p^2)/p$ is equivalent to $G_-(t) = 0$, where $G_-(t) := \frac{c(1-p^2)}{p}(1-\lambda t^2) - (1+q)t(\Delta - \delta t)$, the negative of the G_- of Lemma 6.1 in the opposite regime. Its leading coefficient is $-c(1-p^2)\lambda/p + (1+q)\delta = c(p-q)q(1-p)/p > 0$, while $G_-(0) = c(1-p^2)/p > 0$ and $G_-(1/q) = -(c-1)(q-1)(q+1)/q < 0$, so t_- is the smaller positive root of this upward-opening quadratic and $G_-'(t_-) < 0$. Differentiating the identity $G_-(t) = -(1-\lambda t^2)\big(Y(t) - c(1-p^2)/p\big)$ at t_- therefore gives $Y'(t_-) > 0$. Now Y' has numerator $\Delta - 2\delta t + \lambda \Delta t^2$ and positive denominator $(1-\lambda t^2)^2$, so this numerator is > 0 at t_-. It is a concave quadratic, its t^2-coefficient $\lambda\Delta$ being negative, with value $\Delta > 0$ at $t = 0$; positive at both ends of $[0, t_-]$, it is positive throughout, so Y is strictly increasing on $(0, t_-]$. The same monotonicity and $Y(0) = 0 < c(1-p^2) = Y(t_+) < Y(t_-)$ place $t_+ \in (0, t_-)$.

By Claim 1 both qt_- and t_+ lie in $(0, t_-]$, where Y is strictly increasing, so

$$qt_- < t_+ \quad \Longleftrightarrow \quad Y(qt_-) < Y(t_+) = c(1-p^2) = pY(t_-).$$

The lemma is thus reduced to the following claim.

Claim 2: $Y(qt_-) < pY(t_-)$. Reducing to a common denominator gives

$$pY(t) - Y(qt) = \frac{(1+q)t}{(1-\lambda t^2)(1-\lambda q^2 t^2)}\, W(t),$$
$$W(t) = p(\Delta - \delta t)(1 - \lambda q^2 t^2) - q(\Delta - \delta qt)(1 - \lambda t^2).$$

Since $\lambda < 0$ and $t > 0$, the factor in front is positive, so $Y(qt) < pY(t)$ if and only if $W(t) > 0$; Claim 2 is thus the single sign $W(t_-) > 0$. Dividing the cubic W by the

quadratic G_- leaves a linear remainder, so at the root t_-,

$$W(t_-) = ut_- + v, \qquad u = -\frac{p(q+1)}{c(p-1)^2}R(c), \qquad v = \frac{2p(q+1)(cp-cq+q-1)}{p-1},$$

where $R(c) := (p-q)(2p-q^2-1)c^2 + 2(q-1)(p-q)(p+q)c + (p+q)(q-1)^2$. Here $v > 0$, because $cp - cq + q - 1 < 0$ and $p - 1 < 0$. Moreover $R(c) > 0$ on $[0, c_0]$, whence $u < 0$. Indeed, the endpoint values are

$$R(0) = (p+q)(q-1)^2 > 0, \qquad R(c_0) = \frac{(p-1)^2(p+q)(2p+q^2+1)}{(q+1)^2} > 0;$$

if the coefficient $(p-q)(2p-q^2-1)$ of c^2 is non-positive, then R is concave or linear and attains its minimum on $[0, c_0]$ at an endpoint, while if $2p-q^2-1 > 0$, then R opens upward with vertex $c_R = (1-q)(p+q)/(2p-q^2-1)$, and $0 < 2p-q^2-1 < 1-q^2$ gives $c_R/c_0 = (1-q^2)/(2p-q^2-1) > 1$, so R decreases on $[0, c_0]$ with minimum $R(c_0) > 0$.

Since $u < 0$ and $v > 0$, the inequality $W(t_-) > 0$ is equivalent to $t_- < t^* := -v/u > 0$. Substituting t^* into the boundary quadratic and multiplying by $u^2 > 0$,

$$u^2 G_-(t^*) = \frac{p(q+1)^2(c-1)(p-q)(q-1)^2}{c(p-1)^2} F_1(c)F_2(c),$$
$$F_1(c) = 2cp - cq - c + q - 1,$$
$$F_2(c) = -(p-q)(q+1)c^2 + 2(p-q)(p+q)c + (p+q)(q-1).$$

Both factors are negative for $0 < c < c_0$: if $2p - q - 1 \le 0$ then $F_1 < 0$ outright, and otherwise F_1 increases in c with $F_1(c_0) = (p-1)(2p+1+q)/(q+1) < 0$; and F_2 opens downward with vertex c_0, where $F_2(c_0) = (p-1)(p+1)(p+q)/(q+1) < 0$. As the prefactor is negative, $G_-(t^*) < 0$. By Claim 1 the upward-opening G_- is negative precisely between its two positive roots, of which t_- is the smaller; hence $t^* > t_-$ and $W(t_-) > 0$. This proves Claim 2, and with it the lemma. □

Proof of Lemma 6.3. We keep the notation of Lemma 6.2 and write $n_0 = c(1-p)/\Delta$. Assume $m \in \mathcal{M}$; we must prove $S_m < \widehat{S}$. The comparison is governed by the real minimizer $\log_q n_0$ of the breakpoint sequence $\widehat{S}_n$, rather than by the critical point m_0 of S_m. Feasibility of m links the two: it forces n_0/q^m to lie between q and 1, so that $\log_q n_0$ lies between m and $m+1$; hence $\widehat{S}$ is decided by the two adjacent breakpoint values $\widehat{S}_m$ and $\widehat{S}_{m+1}$ (Claim 1), each of which we then show exceeds S_m (Claim 2).

Claim 1: $m < \log_q n_0 < m+1$; consequently $\widehat{S} = \min\{\widehat{S}_m, \widehat{S}_{m+1}\}$ when $m \ge 1$, and $\widehat{S} = \widehat{S}_{m+1}$ when $m = 0$. Write $t = q^m$. Feasibility of m reads $c(1-p^2) < Y(t) < c(1-p^2)/p$; substituting $c = n_0\Delta/(1-p)$ and clearing positive denominators gives $\frac{p(1+q)}{p+1+(p-q)qt^2} < \frac{n_0}{t} < \frac{1+q}{p+1-(p-q)t^2}$. The upper bound is at most 1, because $p+1-(p-q)t^2-(1+q) = (p-q)(1-t^2) \ge 0$, while the lower bound exceeds q, because its denominator is positive and $\frac{p(1+q)}{p+1+(p-q)qt^2} - q = \frac{(p-q)(1-q^2t^2)}{p+1+(p-q)qt^2} > 0$ by $p > q$ and $q^2t^2 = q^{2(m+1)} < 1$. Hence $q < n_0/t < 1$, that is, $q^{m+1} < n_0 < q^m$, equivalently $m < \log_q n_0 < m+1$. By Lemma 3.1 the real minimizer of $\widehat{S}_n$ is $\log_q n_0$. For $m \ge 1$

it lies in $(m, m+1)$, so the integer minimizer is $n^* = m$ and $\widehat{S} = \min\{\widehat{S}_m, \widehat{S}_{m+1}\}$. For $m = 0$ it lies in $(0, 1)$, at or below the first admissible index $n = 1$; by the unimodality in Lemma 3.1 the sequence $\widehat{S}_n$ is then increasing from $n = 1$, so $\widehat{S} = \widehat{S}_1 = \widehat{S}_{m+1}$.

Claim 2: $S_m < \widehat{S}_{m+1}$, and if $m \geq 1$, also $S_m < \widehat{S}_m$. For $m \geq 1$ the second inequality is exactly Lemma 5.8: since $m \in \mathcal{M}$, the lower feasibility inequality $c(1-p^2) < Y_m$ in (5.8) makes it strict (Remark 5.9), so $\widehat{S}_m > S_m$. For the first inequality, write $t = q^m$ and $A_{m+1} = \sum_{j=0}^{m} q^{2j}$. Since $\Delta - c(1-p)q^{m+1} = (1-q)(N_{m+1} - cq^{m+1})$ and $1 - q^{2(m+1)} = (1-q^2)A_{m+1}$, the right-hand term of $\widehat{S}_{m+1}$ in (2.13) equals $(N_{m+1} - cq^{m+1})^2/(2A_{m+1})$, and the recurrence $N_{m+1} = N_m - c(p-q)q^m$ collapses its numerator to $N_{m+1} - cq^{m+1} = N_m - cpt$. Thus $\widehat{S}_{m+1} = \frac{1}{2}(1-p^2)c^2 + (N_m - cpt)^2/(2A_{m+1})$, and a common-denominator reduction against $S_m = N_m^2/(2D_m)$, using $(1-p^2)A_{m+1} = (1-p^2)D_m - p^2t^2$, gives

$$\widehat{S}_{m+1} - S_m = \frac{\big(c(1-p^2)D_m - pN_mq^m\big)^2}{2(1-p^2)D_m\, A_{m+1}} \geq 0. \tag{9.13}$$

The denominator is positive, and the upper feasibility inequality $pY_m < c(1-p^2)$ forces $c(1-p^2)D_m \neq pN_mq^m$; hence (9.13) is strict, $\widehat{S}_{m+1} > S_m$.

By Claim 1 every breakpoint value entering $\widehat{S}$ is $\widehat{S}_m$ (only for $m \geq 1$) or $\widehat{S}_{m+1}$, and by Claim 2 both exceed S_m. Hence $S_m < \widehat{S}$ for every $m \in \mathcal{M}$; in particular $S_{m^*} < \widehat{S}$. □

Proof of Lemma 6.4. First assume $m \geq 1$, fix such a path, and set $u := y_k \in [0, c)$, $v := y_{k+1} \in (c, 1]$. Splitting the action at these two vertices: the k steps in $[0, c]$ contribute at least $\frac{1-p^2}{2}u^2$ (trivially when $k = 0$, and by Lemma 3.1 with $g_p(x) = px$ when $k \geq 1$), the step from u to v contributes $\frac{1}{2}(v - pu)^2$, and the m steps in $[c, 1]$ contribute at least $J_m(v, 1; g_q)$ (Lemma 3.1, with $g_q(x) = qx + c(p-q)$). Hence $I_N(\mathbf{y}; f) \geq \Phi_m(u, v) := \frac{1-p^2}{2}u^2 + \frac{1}{2}(v - pu)^2 + J_m(v, 1; g_q)$, so $I_N(\mathbf{y}; f) \geq \bar{S}_m := \min_R \Phi_m$ with $R := \{(u, v) : 0 \leq u \leq c \leq v \leq 1\}$; we prove $\bar{S}_m \geq \widehat{S}$ for $m \notin \mathcal{M}$.

The quadratic Φ_m is strictly convex. Minimizing first in u, $\partial_u\Phi_m = u - pv = 0$ gives $u = pv$ and $\Phi_m(pv, v) = \frac{1-p^2}{2}v^2 + J_m(v, 1; g_q)$, a strictly convex function of v with unique minimizer $v_* := Y_m/(1-p^2)$ and value S_m (Corollary 5.2). Thus the unconstrained minimizer of Φ_m is (pv_*, v_*); by (5.8) it lies in the interior of R precisely when $c < v_* < c/p$, that is, when $m \in \mathcal{M}$. Hence for $m \notin \mathcal{M}$ the minimizer over R lies on the boundary, and two of the four sides are excluded, since at every boundary point

$$\partial_u\Phi_m(0, v) = -pv < 0, \quad \partial_v\Phi_m(u, 1) = (1 - pu) - \frac{(1+q)q^m\Delta}{1+q^m} \geq (1 - pc) - q\Delta > 0,$$

the last inequality using $(1+q)q^m/(1+q^m) \leq q$ for $m \geq 1$ and $(1-pc) - q\Delta = (1-p)(1-q) + (1-c)(p(1-q) + q^2) > 0$. The first sign shows that Φ_m strictly decreases as one moves off the side $u = 0$ into R, the second that it strictly increases toward the side $v = 1$; so the minimum lies on $u = c$ or on $v = c$. On the side $v = c$,

$$\min_{0 \leq u \leq c} \Phi_m(u, c) = \min_{0 \leq u \leq c} \left\{ \frac{1-p^2}{2}u^2 + \frac{1}{2}(c - pu)^2 \right\} + J_m(c, 1; g_q)$$

$$= \frac{1-p^2}{2}c^2 + J_m(c,1;g_q) = \widehat{S}_m \geq \widehat{S},$$

the first minimum being attained at $u = pc$; and on the side $u = c$, concatenating the transition from c to v with the unrestricted m-step fixed-endpoint path from v to 1,

$$\min_{c\leq v\leq 1} \Phi_m(c,v) = \frac{1-p^2}{2}c^2 + \min_{c\leq v\leq 1}\left\{\frac{1}{2}(v-pc)^2 + J_m(v,1;g_q)\right\}$$
$$\geq \frac{1-p^2}{2}c^2 + J_{m+1}(c,1;g_q) = \widehat{S}_{m+1} \geq \widehat{S}.$$

Thus every boundary minimum of Φ_m is at least $\widehat{S}$, and therefore $\bar{S}_m \geq \widehat{S}$. For $m = 0$ the right block is empty and $v = y_N = 1$, so the cost is a function of u alone; minimizing $\frac{1-p^2}{2}u^2 + \frac{1}{2}(1-pu)^2$ over $0 \leq u \leq c$, with the minimum at $u = c$ (forced by $0 \notin \mathcal{M}$, i.e. $c \leq p$), gives $\bar{S}_0 = \frac{1-p^2}{2}c^2 + \frac{1}{2}(1-pc)^2 = \widehat{S}_1 \geq \widehat{S}$. Thus $I_N(\mathbf{y};f) \geq \widehat{S}$ for every such path when $m \notin \mathcal{M}$. □

Proof of Lemma 6.5. Fix $m \geq 1$ and write $t = q^m \in (0,q]$. By Corollary 5.2 the coordinate y_{k+1} increases to $Y_m/(1-p^2)$ as $k \to \infty$, and in the limit $y_k < c < y_{k+1}$ holds exactly under the feasibility condition (5.8). Since $y_{k+1} < Y_m/(1-p^2)$ at every finite k, a violation of the lower bound of (5.8),

$$Y_m < c(1-p^2), \tag{9.14}$$

already forces $y_{k+1} < c$ for all $k \geq 0$; it therefore suffices to prove (9.14) for every $t = q^m \in (0,q]$. Expressing N_m and D_m through t turns $Y_m = N_m q^m/D_m$ into $Y_m = (1+q)t(\Delta + c(p-q)t)\big/\left(1 - \frac{q^2-p^2}{1-p^2}t^2\right)$, whose denominator is positive because $q < p$. Clearing it, (9.14) is equivalent to $h(t) > 0$, where

$$h(t) = c\left[(1-p^2) - (q^2-p^2)t^2\right] - (1+q)t(\Delta + c(p-q)t)$$
$$= c(1-p^2) - (1+q)\Delta t - c(p-q)(1-p)t^2.$$

The quadratic h is concave in t, its t^2-coefficient $-c(p-q)(1-p)$ being negative, so on $[0,q]$ it attains its minimum at an endpoint. At $t = 0$ we have $h(0) = c(1-p^2) > 0$, while at $t = q$

$$h(q) = (1-q^2)\big(c(1-p^2+pq) - q\big) \geq (1-q^2)\big(c_0(1-p^2+pq) - q\big)$$
$$= (1-q^2)\frac{(1-p)(p^2+p-q^2)}{1+q} > 0,$$

where the first inequality uses $c \geq c_0$ together with $1 - p^2 + pq > 0$, and the last uses $p^2 + p - q^2 > p^2 > 0$. Hence $h > 0$ on $[0,q]$, which is (9.14); therefore $y_{k+1} < c$ for all k, and no pair (m,k) with $m \geq 1$ is feasible. □

Acknowledgements

The authors thank the anonymous reviewers for their careful reading and constructive comments, which helped improve the manuscript.

Declaration of AI use

The authors used OpenAI's Codex for text editing and polishing throughout the manuscript. For Proposition 2.7 and Lemmas 6.1–6.5, the proof strategies were supplied by the authors, while Codex assisted with algebraic calculations and simplifications and with writing the corresponding arguments. The authors reviewed and corrected the AI-assisted material and take full responsibility for the paper.

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