Weak approximation rates for integral functionals of Markov processes

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Abstract   We obtain weak rates for approximation of an integral functional of a Markov process by integral sums. An assumption on the process is formulated only in terms of its transition probability density, and, therefore, our approach is not strongly dependent on the structure of the process. Applications to the estimates of the rates of approximation of the Feynman–Kac semigroup and of the price of "occupation-time options" are provided.

Keywords   Markov processes, integral functional, weak approximation rates, Feynman-Kac formula, occupation-time option
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1   Introduction and main results

Let X_t, $t \geq 0$, be a Markov process with values in $\mathbb{R}^d$. We consider the following objects:

1) the integral functional

$$I_T(h) = \int_0^T h(X_t) \, dt$$

of this process;

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2) the sequence of integral sums

$$I_{T,n}(h) = \frac{T}{n} \sum_{k=0}^{n-1} h(X_{(kT)/n}), \quad n \geq 1.$$

The problem we are focused on is obtaining upper bounds on the accuracy of approximation of the integral functional $I_T(h)$ by the integral sums $I_{T,n}(h)$ without any regularity assumption on the function h. The function h is only assumed to be measurable and bounded. Therefore, the class of functionals $I_T(h)$ contains, for example, the *occupation time* of the process X in a set $A \subset \mathcal{B}(\mathbb{R}^d)$ (in this case, $h = \mathbb{I}_A$).

The problem of estimating the expectation of expressions that contain both the value of a process and the value of an integral functional of this process arises naturally in a wide range of probabilistic problems. Two of them related with the Feynman–Kac semigroup and the price of an occupation-time option are discussed in Section 3. An exact calculation of such expressions, if possible, can be performed only under substantial assumptions on the structure of functionals and processes; see, for example, [3], where the price of an occupation-time option is precisely calculated for a Lévy process with only negative jumps. For more complicated models, it is natural to use approximative methods, which naturally require estimates of approximation errors. This motivates the main aim of the paper to evaluate the error bounds for discrete approximations of the integral functional $I_T(h)$.

In what follows, P_x denotes the law of the Markov process X conditioned by $X_0 = x$, and $\mathbb{E}_x$ denotes the expectation with respect to this law. Both the absolute value of a real number and the Euclidean norm in $\mathbb{R}^d$ are denoted by $|\cdot|$; $\|\cdot\|$ denotes the sup-norm in L_∞.

The following result was obtained in [2] as a part of the proof of a more general statement (see Theorem 2.5 in [2]).

Proposition 1. *Suppose that X is a multidimensional diffusion process with bounded Hölder continuous coefficients and that its diffusion coefficient satisfies the uniform ellipticity condition*

$$\big(a(x)\theta, \theta\big)_{\mathbb{R}^d} \geq c|\theta|^2, \quad x, \theta \in \mathbb{R}^d, \; c > 0.$$

Then there exists a positive constant C such that

$$\big|\mathbb{E}_x I_T(h) - \mathbb{E}_x I_{T,n}(h)\big| \leq C\|h\|\frac{\log n}{n}. \tag{1}$$

The scheme of the proof of this result can be extended straightforwardly to the case of arbitrary Markov process that satisfies the following assumption (see Proposition 2.1 [1]):

X. The process X possesses a transition probability density $p_t(x, y)$ that is differentiable with respect to t and satisfies

$$\big|\partial_t p_t(x, y)\big| \leq C_T t^{-1} q_{t,x}(y), \quad t \leq T, \; C_T \geq 1, \tag{2}$$

for some measurable function q such that for any fixed t and x, the function $q_{t,x}$ is a distribution density.

Remark 1. A diffusion process satisfies condition **X** with

$$q_{t,x}(y) = c_1 t^{-d/2} \exp\left(-c_2 t^{-1}|x - y|^2\right)$$

and properly chosen c_1, c_2. The other examples of the processes satisfying condition (2) are provided in [1]. Among them, we should mention an α-stable process.

Under assumption **X**, Proposition 1 and Proposition 2.1 in [1] give bounds for the rate of approximation of *expectations* of the integral functionals of the process X. Such approximation rates are called *weak*. Strong L_p-rates, that is, the bounds for

$$E_x\left|I_T(h) - I_{T,n}(h)\right|^p,$$

have been recently obtained in [4] for diffusion processes and in [1] without restrictions on the structure of the processes. In this paper, we provide another generalization of the weak rate (1), namely, the rates of approximation for expectations of more complicated functionals. Let us formulate the main result of this paper.

Theorem 1. *Suppose that X holds. Then for each* $k \in \mathbb{N}$ *and any bounded function* f,

$$\left|E_x\left(I_T(h)\right)^k f(X_T) - E_x\left(I_{T,n}(h)\right)^k f(X_T)\right| \leq 6k^2 C_T T^k \|h\|^k \left(\frac{\log n}{n}\right)\|f\|.$$

Clearly, Proposition 2.1 in [1] is a particular case of Theorem 1. The latter statement is a substantial extension of the former one: it contains both the moments of any order of the integral functional and the value of the process in the final time moment. Using the Taylor expansion, we obtain the following corollary of Theorem 1.

Consider any analytic function g defined in a neighborhood of 0 and constants D_g, $R_g > 0$ such that $\left|\frac{g^{(m)}(0)}{m!}\right| \leq D_g \left(\frac{1}{R_g}\right)^m$ for any natural m.

Corollary 1. *Suppose that X holds. Then for any bounded function* f *and a function* h *such that* $T\|h\| < R_g$, *we have:*

$$\left|E_x g\left(I_T(h)\right) f(X_T) - E_x g\left(I_{T,n}(h)\right) f(X_T)\right| \leq C_{T,h,D_g,R_g} \left(\frac{\log n}{n}\right)\|f\|, \quad (3)$$

where

$$C_{T,h,D_g,R_g} = 6 D_g C_T \frac{T\|h\|}{R_g}\left(1 + \frac{T\|h\|}{R_g}\right)\frac{1}{(1 - \frac{T\|h\|}{R_g})^3}.$$

We provide the proof of Theorem 1 in Section 2. In Section 3, we give an application of Theorem 1 to estimates of the rates of approximation of the Feynman–Kac semigroup and of the price of an occupation-time option.

2 Proof of Theorem 1

Denote

$$S_{k,a,b} := \left\{(s_1, s_2, \ldots, s_k) \in \mathbb{R}^k | a \leq s_1 < s_2 < \cdots < s_k \leq b\right\}, \quad k \in \mathbb{N}, \ a, b \in \mathbb{R},$$

and for each $t \in [kT/n, (k+1)T/n)$, put $\eta_n(t) := \frac{kT}{n}$.

We have:

$$\frac{1}{k!}\Big(E_x\big[\big(I_T(h)\big)^k - \big(I_{T,n}(h)\big)^k\big]f(X_T)\Big)$$

$$= E_x \int_{S_{k,0,T}} \Big[\prod_{i=1}^{k} h(X_{s_i}) - \prod_{i=1}^{k} h(X_{\eta_n(s_i)})\Big] f(X_T) \prod_{i=1}^{k} ds_i$$

$$= \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \Big(\prod_{i=1}^{k} h(y_i)\Big) f(z) \Big(\prod_{i=1}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$\times dz \prod_{j=1}^{k} dy_j \prod_{i=1}^{k} ds_i - \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \Big(\prod_{i=1}^{k} h(y_i)\Big) f(z)$$

$$\times \Big(\prod_{i=1}^{k} p_{\eta_n(s_i) - \eta_n(s_{i-1})}(y_{i-1}, y_i)\Big) p_{T-\eta_n(s_k)}(y_k, z) dz \prod_{j=1}^{k} dy_j \prod_{i=1}^{k} ds_i,$$

where $s_0 = 0$, $y_0 = x$.

Rewrite the expression under the integral

$$\Big(\prod_{i=1}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$- \Big(\prod_{i=1}^{k} p_{\eta_n(s_i) - \eta_n(s_{i-1})}(y_{i-1}, y_i)\Big) p_{T-\eta_n(s_k)}(y_k, z)$$

in the form

$$\Big(\prod_{i=1}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$\mp p_{\eta_n(s_1)}(x, y_1) \Big(\prod_{i=2}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$- \Big(\prod_{i=1}^{k} p_{\eta_n(s_i) - \eta_n(s_{i-1})}(y_{i-1}, y_i)\Big) p_{T-\eta_n(s_k)}(y_k, z)$$

$$= \Big(\prod_{i=1}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$\mp p_{\eta_n(s_1)}(x, y_1) \Big(\prod_{i=2}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$\mp p_{\eta_n(s_1)}(x, y_1) p_{s_2 - \eta_n(s_1)}(y_1, y_2) \Big(\prod_{i=3}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i)\Big) p_{T-s_k}(y_k, z)$$

$$- \Big(\prod_{i=1}^{k} p_{\eta_n(s_i) - \eta_n(s_{i-1})}(y_{i-1}, y_i)\Big) p_{T-\eta_n(s_k)}(y_k, z)$$

$$\vdots$$
$$= J_1 + J_2 + \cdots + J_{2k-1} + J_{2k},$$

where

$$J_1 = \left(p_{s_1}(x, y_1) - p_{\eta_n(s_1)}(x, y_1)\right)\left(\prod_{i=2}^{k} p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)p_{T-s_k}(y_k, z),$$

$$J_2 = p_{\eta_n(s_1)}(x, y_1)\left(p_{s_2-s_1}(y_1, y_2) - p_{s_2-\eta_n(s_1)}(y_1, y_2)\right)$$
$$\times \left(\prod_{i=3}^{k} p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)p_{T-s_k}(y_k, z),$$

$$J_3 = p_{\eta_n(s_1)}(x, y_1)\left(p_{s_2-\eta_n(s_1)}(y_1, y_2) - p_{\eta_n(s_2)-\eta_n(s_1)}(y_1, y_2)\right)$$
$$\times \left(\prod_{i=3}^{k} p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)p_{T-s_k}(y_k, z),$$

$$J_4 = p_{\eta_n(s_1)}(x, y_1)p_{\eta_n(s_2)-\eta_n(s_1)}(y_1, y_2)\left(p_{s_3-s_2}(y_2, y_3) - p_{s_3-\eta_n(s_2)}(y_2, y_3)\right)$$
$$\times \left(\prod_{i=4}^{k} p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)p_{T-s_k}(y_k, z),$$

$$J_5 = p_{\eta_n(s_1)}(x, y_1)p_{\eta_n(s_2)-\eta_n(s_1)}(y_1, y_2)\left(p_{s_3-\eta_n(s_2)}(y_2, y_3) - p_{\eta_n(s_3)-\eta_n(s_2)}(y_2, y_3)\right)$$
$$\times \left(\prod_{i=4}^{k} p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)p_{T-s_k}(y_k, z),$$

$$\vdots$$

$$J_{2k-1} = \left(\prod_{i=1}^{k-1} p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i)\right)$$
$$\times \left(p_{s_k-\eta_n(s_{k-1})}(y_{k-1}, y_k) - p_{\eta_n(s_k)-\eta_n(s_{k-1})}(y_{k-1}, y_k)\right)p_{T-s_k}(y_k, z),$$

$$J_{2k} = \left(\prod_{i=1}^{k} p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i)\right)\left(p_{T-s_k}(y_k, z) - p_{T-\eta_n(s_k)}(y_k, z)\right).$$

Therefore,

$$\frac{1}{k!}\left(E_x\left[(I_T(h))^k - (I_{T,n}(h))^k\right]f(X_T)\right)$$
$$= \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i)\right)f(z)(J_1 + J_2 + \cdots + J_{2k-1} + J_{2k})dz \prod_{j=1}^{k} dy_j \prod_{i=1}^{k} ds_i.$$
$$(4)$$

Our way to estimate each of $2k$ terms in (4) is mostly the same, but its realization is different for the first, the last, and the intermediate terms. Let us estimate the first term in (4):

$$\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=1}^{k} ds_i \right|$$

$$= \left| \int_0^T \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1 \right|$$

$$\leq \left| \int_0^{T/n} \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1 \right|$$

$$+ \left| \int_{T/n}^T \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1 \right|$$

Let us consider each term in detail:

$$\left| \int_0^{T/n} \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1 \right|$$

$$\leq \int_0^{T/n} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left| \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 \right| dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1$$

$$\leq \|h\|^k \|f\| \int_0^{T/n} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} |J_1| dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1$$

$$\leq \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \int_0^{T/n} \int_{\mathbb{R}^d} \left| p_{s_1}(x, y_1) - p_{\eta_n(s_1)}(x, y_1) \right| dy_1 ds_1$$

$$\leq \frac{2}{(k-1)!} \|h\|^k \|f\| T^k \frac{1}{n}.$$

Next, we have

$$\left| \int_{T/n}^T \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) J_1 dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i ds_1 \right|$$

$$= \left| \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \partial_u p_u(x, y_1) \right.$$

$$\left. \times \left(\prod_{i=2}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i) \right) p_{T-s_k}(y_k, z) dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i du ds_1 \right|$$

$$\leq \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left| \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \partial_u p_u(x, y_1) \right.$$

$$\left. \times \left(\prod_{i=2}^{k} p_{s_i - s_{i-1}}(y_{i-1}, y_i) \right) p_{T-s_k}(y_k, z) \right| dz \prod_{j=1}^{k} dy_j \prod_{i=2}^{k} ds_i \times du ds_1$$

$$\leq \|h\|^k \|f\| \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left|\partial_u p_u(x, y_1)\right| \left(\prod_{i=2}^k p_{s_i-s_{i-1}}(y_{i-1}, y_i)\right)$$

$$\times\, p_{T-s_k}(y_k, z) dz \prod_{j=1}^k dy_j \prod_{i=2}^k ds_i du ds_1.$$

Integrating over $z, y_k, y_{k-1}, \ldots, y_2$ and then over $s_k, s_{k-1}, \ldots, s_2$, we derive

$$\left| \int_{T/n}^T \int_{S_{k-1,s_1,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i)\right) f(z) J_1 dz \prod_{j=1}^k dy_j \prod_{i=2}^k ds_i ds_1 \right|$$

$$\leq \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} \int_{\mathbb{R}^d} \left|\partial_u p_u(x, y_1)\right| dy_1 du ds_1$$

$$\leq C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} \int_{\mathbb{R}^d} u^{-1} q_{u,x}(y_1) dy_1 du ds_1$$

$$= C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \int_{T/n}^T \int_{\eta_n(s_1)}^{s_1} u^{-1} du ds_1$$

$$= C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{s_1} u^{-1} du ds_1$$

$$= C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^{k-1} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_u^{(i+1)T/n} u^{-1} ds_1 du$$

$$\leq C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{1}{n} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} u^{-1} du$$

$$= C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{1}{n} \int_{T/n}^T u^{-1} du$$

$$= C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}.$$

Therefore,

$$\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i)\right) f(z) J_1 dz \prod_{j=1}^k dy_j \prod_{i=1}^k ds_i \right|$$

$$\leq 3 C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}.$$

Now we are ready to estimate the last summand in (4):

$$\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i)\right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^k ds_i \right|$$

$$= \left| \int_0^T \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

$$\leq \left| \int_0^{T-T/n} \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

$$+ \left| \int_{T-T/n}^T \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

Let us estimate each term separately. We get

$$\left| \int_{T-T/n}^T \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

$$\leq \int_{T-T/n}^T \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left| \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} \right| dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k$$

$$\leq \|h\|^k \|f\| \int_{T-T/n}^T \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} |J_{2k}| dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k$$

$$\leq \frac{2}{(k-1)!} \|h\|^k \|f\| T^k \frac{1}{n}.$$

For the other term, we obtain:

$$\left| \int_0^{T-T/n} \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

$$= \left| \int_0^{T-T/n} \int_{\eta_n(s_k)}^{s_k} \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) \right.$$

$$\left. \times \left(\prod_{i=1}^k p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i) \right) \partial_{T-u} p_{T-u}(y_k, z) dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i du ds_k \right|$$

$$\leq \int_0^{T-T/n} \int_{\eta_n(s_k)}^{s_k} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left| \left(\prod_{i=1}^k h(y_i) \right) f(z) \right.$$

$$\left. \times \left(\prod_{i=1}^k p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i) \right) \partial_{T-u} p_{T-u}(y_k, z) \right| dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i du ds_k$$

$$\leq \|h\|^k \|f\| \int_0^{T-T/n} \int_{\eta_n(s_k)}^{s_k} \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i) \right)$$

$$\times \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i du ds_k.$$

Let us rewrite this expression in the form

$$\|h\|^k \|f\| \int_{S_{k-1,0,T}} \int_{(\mathbb{R}^d)^{k-1}} \left(\prod_{i=1}^{k-1} p_{\eta_n(s_i)-\eta_n(s_{i-1})}(y_{i-1}, y_i) \right)$$

$$\times \int_0^{T-T/n} \int_{\eta_n(s_k)}^{s_k} \int_{(\mathbb{R}^d)^2} p_{\eta_n(s_k)-\eta_n(s_{k-1})}(y_{k-1}, y_k)$$

$$\times \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz dy_k du ds_k \prod_{j=1}^{k-1} dy_j \prod_{i=1}^{k-1} ds_i$$

and consider the inner integral

$$\int_0^{T-T/n} \int_{\eta_n(s_k)}^{s_k} \int_{(\mathbb{R}^d)^2} p_{\eta_n(s_k)-\eta_n(s_{k-1})}(y_{k-1}, y_k) \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz dy_k du ds_k$$

$$= \sum_{i=0}^{n-2} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{s_k} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{k-1})}(y_{k-1}, y_k)$$

$$\times \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz dy_k du ds_k$$

$$= \sum_{i=0}^{n-2} \int_{iT/n}^{(i+1)T/n} \int_u^{(i+1)T/n} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{k-1})}(y_{k-1}, y_k)$$

$$\times \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz dy_k ds_k du$$

$$\leq \frac{T}{n} \sum_{i=0}^{n-2} \int_{iT/n}^{(i+1)T/n} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{k-1})}(y_{k-1}, y_k) \left| \partial_{T-u} p_{T-u}(y_k, z) \right| dz dy_k du$$

$$\leq C_T \frac{T}{n} \sum_{i=0}^{n-2} \int_{iT/n}^{(i+1)T/n} \int_{\mathbb{R}^d} p_{iT/n-\eta_n(s_{k-1})}(y_{k-1}, y_k)(T-u)^{-1} dy_k du$$

$$= C_T \frac{T}{n} \sum_{i=0}^{n-2} \int_{iT/n}^{(i+1)T/n} (T-u)^{-1} du = C_T \frac{T}{n} \int_0^{T-T/n} (T-u)^{-1} du = T C_T \frac{\log n}{n}.$$

Therefore, we have:

$$\left| \int_0^{T-T/n} \int_{S_{k-1,0,s_k}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^{k-1} ds_i ds_k \right|$$

$$\leq C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}$$

and

$$\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) J_{2k} dz \prod_{j=1}^k dy_j \prod_{i=1}^k ds_i \right|$$

$$\leq 3 C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}.$$

To complete the proof, we should additionally consider the following terms in (4):

$$
\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \right.
$$

$$
\times \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)
$$

$$
\left. \times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z) dz \prod_{q=1}^{k} dy_q \prod_{r=1}^{k} ds_r \right| \tag{5}
$$

and

$$
\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \right.
$$

$$
\times \left(p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) - p_{\eta_n(s_j)-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)
$$

$$
\left. \times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z) dz \prod_{q=1}^{k} dy_q \prod_{r=1}^{k} ds_r \right|, \tag{6}
$$

where $j = \overline{2, k}$.

Consider (5) in more detail. We rewrite it in the form

$$
\left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_{j-2}}^{s_j} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \right.
$$

$$
\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)
$$

$$
\times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)
$$

$$
\left. \times dz \prod_{q=1}^{k} dy_q \prod_{r=j+1}^{k} ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \right|
$$

$$
\leq \left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_{j-2}}^{s_j-T/n} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \right.
$$

$$
\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)
$$

$$
\times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)
$$

$$
\left. \times dz \prod_{q=1}^{k} dy_q \prod_{r=j+1}^{k} ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \right|
$$

$$+ \left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_j-T/n}^{s_j} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) \right.$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)$$

$$\times \left(\prod_{m=j+1}^k p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)$$

$$\left. \times dz \prod_{q=1}^k dy_q \prod_{r=j+1}^k ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \right|.$$

We estimate each term separately:

$$\left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_j-T/n}^{s_j} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) \right.$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)$$

$$\times \left(\prod_{m=j+1}^k p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)$$

$$\left. \times dz \prod_{q=1}^k dy_q \prod_{r=j+1}^k ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \right|$$

$$\leq \|h\|^k \|f\|$$

$$\times \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_j-T/n}^{s_j} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^j} \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right)$$

$$\times \left| p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right|$$

$$\times \prod_{q=1}^j dy_q \prod_{r=j+1}^k ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2}$$

$$\leq \frac{2}{(k-1)!} \|h\|^k \|f\| T^k \frac{1}{n}.$$

For the other term, we have

$$\left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_{j-2}}^{s_j-T/n} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^k h(y_i) \right) f(z) \right.$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)$$

$$\times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)$$

$$\times dz \prod_{q=1}^{k} dy_q \prod_{r=j+1}^{k} ds_r ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \Bigg|$$

$$= \left| \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_{j-2}}^{s_j-T/n} \int_{\eta_n(s_{j-1})}^{s_{j-1}} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \right.$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j)$$

$$\times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z)$$

$$\times dz \prod_{q=1}^{k} dy_q \prod_{r=j+1}^{k} ds_r du ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2} \Bigg|$$

$$\leq \|h\|^k \|f\|$$

$$\times \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{s_{j-2}}^{s_j-T/n} \int_{\eta_n(s_{j-1})}^{s_{j-1}} \int_{S_{k-j,s_j,T}} \int_{(\mathbb{R}^d)^j} \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right|$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \prod_{q=1}^{j} dy_q \prod_{r=j+1}^{k} ds_r du ds_{j-1} ds_j \prod_{v=1}^{j-3} ds_v ds_{j-2}$$

$$\leq \|h\|^k \|f\|$$

$$\times \int_0^T \int_{S_{j-3,0,s_{j-2}}} \int_{s_{j-2}}^T \int_{S_{k-j,s_j,T}} \int_0^{s_j-T/n} \int_{\eta_n(s_{j-1})}^{s_{j-1}} \int_{(\mathbb{R}^d)^j} \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right|$$

$$\times \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \prod_{q=1}^{j} dy_q du ds_{j-1} \prod_{r=j+1}^{k} ds_r ds_j \prod_{v=1}^{j-3} ds_r ds_{j-2}.$$

Again, we consider the inner integral:

$$\int_0^{\eta_n(s_j)-T/n} \int_{\eta_n(s_{j-1})}^{s_{j-1}} \int_{(\mathbb{R}^d)^2} p_{\eta_n(s_{j-1})-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})$$

$$\times \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right| dy_j dy_{j-1} du ds_{j-1}$$

$$= \sum_{i=0}^{\eta_n(s_j)n/T-2} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{s_{j-1}} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})$$

$$\times \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right| dy_j dy_{j-1} du ds_{j-1}$$

$$
= \sum_{i=0}^{\eta_n(s_j)n/T-2} \int_{iT/n}^{(i+1)T/n} \int_{u}^{(i+1)T/n} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})
$$

$$
\times \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right| dy_j dy_{j-1} ds_{j-1} du
$$

$$
\leq \frac{T}{n} \sum_{i=0}^{\eta_n(s_j)n/T-2} \int_{iT/n}^{(i+1)T/n} \int_{(\mathbb{R}^d)^2} p_{iT/n-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})
$$

$$
\times \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right| dy_j dy_{j-1} du
$$

$$
\leq C_T \frac{T}{n} \sum_{i=0}^{\eta_n(s_j)n/T-2} \int_{iT/n}^{(i+1)T/n} \int_{\mathbb{R}^d} p_{iT/n-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})(s_j - u)^{-1} dy_{j-1} du
$$

$$
= C_T \frac{T}{n} \sum_{i=0}^{\eta_n(s_j)n/T-2} \int_{iT/n}^{(i+1)T/n} (s_j - u)^{-1} du = C_T \frac{T}{n} \int_{0}^{\eta_n(s_j)-T/n} (s_j - u)^{-1} du.
$$

We have

$$
\int_{0}^{s_j-T/n} \int_{\eta_n(s_{j-1})}^{s_{j-1}} \int_{(\mathbb{R}^d)^2} p_{\eta_n(s_{j-1})-\eta_n(s_{j-2})}(y_{j-2}, y_{j-1})
$$

$$
\times \left| \partial_{s_j-u} p_{s_j-u}(y_{j-1}, y_j) \right| dy_j dy_{j-1} du ds_{j-1}
$$

$$
\leq C_T \frac{T}{n} \int_{0}^{s_j-T/n} (s_j - u)^{-1} du \leq T C_T \frac{\log n}{n}.
$$

Therefore, we obtain

$$
\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \right.
$$

$$
\times \left(p_{s_j-s_{j-1}}(y_{j-1}, y_j) - p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right) \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right)
$$

$$
\left. \times p_{T-s_k}(y_k, z) dz \prod_{q=1}^{k} dy_q \prod_{r=1}^{k} ds_r \right|
$$

$$
\leq 3C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}.
$$

Analogously, we also have:

$$
\left| \int_{S_{k,0,T}} \int_{(\mathbb{R}^d)^{k+1}} \left(\prod_{i=1}^{k} h(y_i) \right) f(z) \left(\prod_{l=1}^{j-1} p_{\eta_n(s_l)-\eta_n(s_{l-1})}(y_{l-1}, y_l) \right) \right.
$$

$$
\times \left(p_{s_j-\eta_n(s_{j-1})}(y_{j-1}, y_j) - p_{\eta_n(s_j)-\eta_n(s_{j-1})}(y_{j-1}, y_j) \right)
$$

$$
\left. \times \left(\prod_{m=j+1}^{k} p_{s_m-s_{m-1}}(y_{m-1}, y_m) \right) p_{T-s_k}(y_k, z) dz \prod_{q=1}^{k} dy_q \prod_{r=1}^{k} ds_r \right|
$$

$$\leq 3C_T \frac{1}{(k-1)!} \|h\|^k \|f\| T^k \frac{\log n}{n}.$$

Therefore, we finally obtain

$$\left| E_x\big[\big(I_T(h)\big)^k - \big(I_{T,n}(h)\big)^k\big] f(X_T) \right| \leq 6k^2 C_T T^k \|h\|^k \left(\frac{\log n}{n}\right) \|f\|,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3   Applications

3.1   Discrete approximation of the Feynman–Kac semigroup

Let X be a Brownian motion with values in $\mathbb{R}^d$. Then condition **X** holds with

$$q_{t,x}(y) = c_1 t^{-d/2} \exp\big(-c_2 t^{-1} |x-y|^2\big),$$

where c_1, c_2 are some positive constants.

Let h be a bounded measurable function. Then, it is known (see, e.g., [6], Chapter 1) that the family of operators

$$R_t^h f(x) = E_x\big[f(X_t) \exp\{\lambda I_t(h)\}\big]$$

forms a semigroup on $L_p(\mathbb{R}^d)$, $p \geq 1$, and its generator equals

$$\mathcal{A}_h f = \frac{1}{2}\Delta f + \lambda h f.$$

This semigroup is called the Feynman–Kac semigroup.

Denote

$$R_{t,n}^h f(x) = E_x\big[f(X_t) \exp\{\lambda I_{t,n}(h)\}\big].$$

Then, using the Taylor expansion of the exponential function and Theorem 1, we have the following statement.

Corollary 2. *For any bounded functions f, h and real positive number λ, we have:*

$$\left| R_t^h f(x) - R_{t,n}^h f(x) \right| \leq C_{T,\lambda,h} \left(\frac{\log n}{n}\right) \|f\|,$$

where

$$C_{T,\lambda,h} = 6 C_T \lambda \|h\| T \big(1 + \lambda \|h\| T\big) \exp\big\{\lambda \|h\| T\big\}.$$

Therefore, the main result of this paper provides an approximation of the Feynman–Kac semigroup with accuracy $(\log n)/n$.

3.2 Approximation of the price of an occupation-time option

Let the price of an asset $S = \{S_t, t \geq 0\}$ be of the form

$$S_t = S_0 \exp(X_t),$$

where X is a one-dimensional Markov process satisfying condition **X**. The time spent by S in a defined set $J \subset \mathbb{R}$ (or the time spent by X in a set $J' = \{x : e^x \in J\}$) from time 0 to time T is given by

$$\int_0^T \mathbb{I}_{\{S_t \in J\}} dt = \int_0^T \mathbb{I}_{\{X_t \in J'\}} dt.$$

We consider an occupation-time option (see [5]) whose price depends on the time spent by the process S in a set J. In contrast to the traditional barrier options, which are activated or canceled when the process S hits a defined level (barrier), the payoff of an occupation-time option depends on the time spent by the price of the asset above/below this level.

For the strike price K, the barrier L, and the knock-out rate ρ, the payoff of a down-and-out call occupation-time option is given by

$$\exp\left(-\rho \int_0^T \mathbb{I}_{\{S_t \leq L\}} dt\right)(S_T - K)_+.$$

Then, for the risk-free interest rate r, its price is given by

$$\mathbf{C}(T) = \exp(-rT)E\left[\exp\left(-\rho \int_0^T \mathbb{I}_{\{S_t \leq L\}} dt\right)(S_T - K)_+\right].$$

Denote

$$\mathbf{C}_n(T) = \exp(-rT)E\left[\exp\left(-\rho T/n \sum_{k=0}^{n-1} \mathbb{I}_{\{S_{kT/n} \leq L\}} dt\right)(S_T - K)_+\right].$$

We provide the following corollary of Theorem 1.

Proposition 2. *Suppose that* **X** *holds and there exists* $u > 1$ *such that* $G := E \exp(uX_T) = E S_T^u < +\infty$. *Then*

$$\left|\mathbf{C}_n(T) - \mathbf{C}(T)\right| \leq 3 \max\{C_{T,\rho}, G\} \exp(-rT)\left(\frac{\log n}{n^{1-1/u}}\right),$$

where $C_{T,\rho} = 6C_T \rho T (1 + \rho T) \exp(\rho T)$.

Proof. For some $N > 0$, we denote

$$\mathbf{C}^N(T) = \exp(-rT)E\left[\exp\left(-\rho \int_0^T \mathbb{I}_{\{S_t \leq L\}} dt\right)((S_T - K)_+ \wedge N)\right],$$

$$\mathbf{C}_n^N(T) = \exp(-rT)E\left[\exp\left(-\rho T/n \sum_{k=0}^{n-1} \mathbb{I}_{\{S_{kT/n} \leq L\}} dt\right)((S_T - K)_+ \wedge N)\right].$$

Then

$$\big|\mathbf{C}_n(T) - \mathbf{C}(T)\big| \le \big|\mathbf{C}_n^N(T) - \mathbf{C}^N(T)\big| + \big|\mathbf{C}(T) - \mathbf{C}^N(T)\big| + \big|\mathbf{C}_n(T) - \mathbf{C}_n^N(T)\big|.$$

We estimate each term separately. According to Corollary 2,

$$\big|\mathbf{C}_n^N(T) - \mathbf{C}^N(T)\big| \le N C_{T,\rho} \exp(-rT)\left(\frac{\log n}{n}\right).$$

For other terms, we have:

$$
\begin{aligned}
\big|\mathbf{C}(T) - \mathbf{C}^N(T)\big| &+ \big|\mathbf{C}_n(T) - \mathbf{C}_n^N(T)\big| \\
&\le 2\exp(-rT)E\big[(S_T - K)_+ - (S_T - K)_+ \wedge N\big] \le 2\exp(-rT)E[S_T \mathbb{I}_{\{S_T > N\}}] \\
&= 2\exp(-rT)E\left[\frac{S_T N^{u-1}\mathbb{I}_{\{S_T > N\}}}{N^{u-1}}\right] \le \frac{2G}{N^{u-1}}\exp(-rT).
\end{aligned}
$$

Now, putting $N = n^{1/u}$ completes the proof. $\square$

Therefore, the main result of this paper provides the approximate value $\mathbf{C}_n(T)$ of the price of an occupation-time option $\mathbf{C}(T)$ with accuracy of order $(\log n)/n^{1-1/u}$ for the class of processes X satisfying $\mathbf{X}$ and the condition $E\exp(uX_T) < +\infty$ for some $u > 1$.

Acknowledgments

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