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Terminal entropy identities for Brownian–Poisson additive processes on product filtrations. I. Product representations and martingale entropy balance

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Abstract We establish an exact martingale Φ-entropy balance on a Brownian–Poisson product filtration with deterministic jump compensator. For terminal variables with positive lower and finite upper bounds, the balance separates the Gaussian energy from the nonlinear jump remainder on an explicit representation domain. We formulate the Brownian–Poisson predictable representation and L^2-Clark projection formulas in the form needed for this identity. Marked predictable projections provide compatible versions of the add-one difference and preserve the positivity required by the entropy remainder; finite-window arguments justify the passage between integrable and square-integrable representations. This first part supplies the entropy-balance framework used for the evolution-family formulas and concentration estimates in Part II.

Keywords Additive process, predictable representation, entropy identity, Malliavin calculus, evolution family, product filtration

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1 Introduction

The entropy of a terminal random variable can be expressed through the predictable integrands of its conditional-expectation martingale. We study this identity on a product filtration generated by a Brownian motion and an independent Poisson random measure with deterministic compensator. The main result is the martingale Φ-entropy balance in Lemma 3.2, with separate Gaussian and jump contributions. Part II [11] identifies these contributions for terminal additive functionals $f(X_T)$ through the evolution family $P_{t,T}$, and derives functional inequalities and concentration estimates.

The representation theory builds on the Malliavin–Skorohod calculus of Di Nunno and Vives [6]. That work treats nonstationary additive processes, gives L^2 and L^1 Clark–Ocone–Haussmann (COH) formulas, and identifies the Poisson derivative with the raw add-one difference; see Sections 3, 5 and 6 therein. The Brownian–Poisson representation and derivative formulas used here belong to this established theory. Our contribution is their formulation as an entropy-balance framework on the completed product space, with the representation domain and the nonlinear energy terms treated together. The use of martingale representation to prove entropy inequalities follows the path-space approach of Capitaine, Hsu and Ledoux [3]; the convexity of the energy integrands is part of the Φ-entropy framework of Chafaï [4].

The passage from a linear representation to the entropy balance requires several compatibility checks. The jump remainder $A^{\Phi}(u,v) = \Phi(u+v) - \Phi(u) - \Phi'(u)v$ is defined for $u > 0$ and $u + v > 0$. Thus its predictable arguments must be identified on the same $\nu \otimes \mathbb{P}$-class. Lemmas 2.21–2.24 construct the marked projection and show that $M_{t-} + {}^{p}(\Psi F)_{t,z} = {}^{p}(F \circ \varepsilon^{+})_{t,z}$, preserving a positive lower bound. Lemma 2.15 then identifies the integrands of an integrable representation of a bounded martingale with its L^2 integrands. This permits the use of the stochastic-integral isometry for the linear terms, while the nonlinear jump remainder is handled by an L^1 compensation argument. These are the entropy-specific steps that connect the additive-process calculus to the exact identity; the final calculation uses Itô's formula.

Section 2 sets out the product realization, derivative domains, representation formulas and predictable projection calculus. Section 3 proves the entropy balance under the stated positivity and energy assumptions. The concluding remarks describe its use in Part II.

2 Product Brownian–Poisson additive noise spaces

We write $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$, and fix the $\mathbb{R}^d$-valued product noise space below. The norms $\|\cdot\|_{\mathrm{HS}}$ and $\|\cdot\|_{\mathrm{op}}$ are the Hilbert–Schmidt and operator norms. Background on additive and Lévy processes, Gaussian Malliavin calculus, and Poisson add-one calculus is taken from [1, 2, 6–9, 12, 13, 15, 17, 18].

Assumption 2.1 (Product Brownian–Poisson realization)**.** Fix a finite horizon $[0,T]$. The underlying filtration is the usual augmentation of the product filtration generated by a k-dimensional Brownian motion B and an independent Poisson random measure N on $[0,T] \times \mathbb{R}_0^d$ with deterministic compensator $\nu(\mathrm{d}t, \mathrm{d}z)$. The Gaussian covariance of the additive process is absolutely continuous in time and is represented by a

deterministic measurable field $\sigma : [0,T] \to \mathbb{R}^{d\times k}$ satisfying

$$C_t = \int_0^t \sigma_s \sigma_s^\top \, \mathrm{d}s, \qquad \int_0^T \|\sigma_s\|_{\mathrm{HS}}^2 \, \mathrm{d}s < \infty.$$

The jump compensator satisfies the finite-horizon Lévy condition

$$\int_{[0,T]\times\mathbb{R}_0^d} (1 \wedge |z|^2)\, \nu(\mathrm{d}t, \mathrm{d}z) < \infty.$$

It is diffuse in time: $\nu(\{t\} \times \mathbb{R}_0^d) = 0$ for every deterministic $t \in [0,T]$. This temporal regularity is used in the predictable add-one invariance arguments. We write $\widetilde{N} = N - \nu$. All predictable projections, martingale representations, Clark projections, and stochastic integrals are taken with respect to this product filtration.

The product space is

$$\Omega = \Omega_B \times \Omega_J, \qquad \Omega_B = C_0([0,T];\mathbb{R}^k),$$

where Ω_B carries the canonical Brownian path. Put $E_T := [0,T] \times \mathbb{R}_0^d$ and $A_n := [0,T] \times \{n^{-1} < |z| \le n\}$. Then $\nu(A_n) < \infty$ and $A_n \uparrow E_T$. We take Ω_J to be the space of counting measures η on E_T such that $\eta(A_n) < \infty$ for every n, equipped with the Poisson law of intensity ν. We write

$$\mathcal{F} := \mathcal{B}(\Omega_B) \otimes \mathcal{J},$$

where $\mathcal{J}$ is generated by $\eta \mapsto \eta(A)$ for all $A \in \mathcal{B}(E_T)$, with values in $\mathbb{N}_0 \cup \{\infty\}$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$. A finite-$\nu$ window is a Borel set $A \subset E_T$ with $\nu(A) < \infty$; for each such fixed set, $\eta(A) < \infty$ almost surely. The law $\mathbb{P}$ is the product of Wiener measure and this Poisson law. For $\omega = (\omega^B, \eta)$,

$$B_t(\omega) = \omega^B(t), \qquad N_\omega(A) = \eta(A), \quad A \in \mathcal{B}(E_T),$$

and the add-one map is

$$\varepsilon^+_{t,z}(\omega^B, \eta) := (\omega^B, \eta + \delta_{(t,z)}), \qquad (t,z) \in E_T.$$

The add-one map is measurable. We use $\mathbb{P}$ also on the completion $\mathcal{F}_T$ of $\mathcal{F}$; Borel representatives refer to $\mathcal{F}$. The filtration $(\mathcal{F}_t)$ is the usual augmentation in Assumption 2.1. Predictable fields are measurable with respect to its predictable sigma-field, with marks adjoined. Martingales are taken in their càdlàg versions, with $M_{0-} = M_0$. In the case of Remark 2.4, the filtration coincides with the natural path filtration.

Lemma 2.2 (Canonical add-one representatives)**.** *Let F, G be $\mathcal{F}_T$-measurable real-valued functionals. If $F = G$ $\mathbb{P}$-a.s., then*

$$F(\varepsilon^+_{t,z}\omega) = G(\varepsilon^+_{t,z}\omega) \quad \textit{for } \nu \otimes \mathbb{P}\textit{-a.e. } (t,z,\omega).$$

Consequently, after choosing any Borel representatives, $\Psi_{t,z}F := F(\varepsilon^+_{t,z}\omega) - F(\omega)$ is a well-defined $L^0(\nu \otimes \mathbb{P})$-class independent of that choice.

Proof. Choose $\mathcal{F}$-measurable representatives of F, G and put $D := \{F \neq G\}$, so $\mathbb{P}(D) = 0$. For a finite-ν window A_0, apply Mecke's identity on the Poisson coordinate and integrate over the Brownian coordinate to obtain

$$\int_{A_0} \mathbb{P}(\varepsilon^+_{t,z}\omega \in D)\, \nu(\mathrm{d}t, \mathrm{d}z) = \mathbb{E} \int_{A_0} \mathbf{1}_D(\varepsilon^+_{t,z}\omega)\, \nu(\mathrm{d}t, \mathrm{d}z) = \mathbb{E}[\mathbf{1}_D N(A_0)] = 0.$$

Choose an increasing sequence of finite-ν-measure sets $A_n \uparrow E_T$; applying the preceding identity to each A_n and then passing to the union proves the claim. This is the canonical add-one convention used for Poisson random measures; compare [6, Section 5] and [18, Section 4]. □

Remark 2.3 (Choice of Borel representatives). Whenever a positive functional is specified only up to $\mathbb{P}$-null sets, we choose a Borel representative on a full-measure set where the required pointwise bounds hold. On the exceptional set we assign a fixed value in the same positive interval whenever a two-sided bound is assumed, for instance a value in $[\varepsilon, M]$ when $\varepsilon \le F \le M$; if only strict positivity is needed, we assign the value 1. With this convention, $\Psi_{t,z}F$ is a well-defined $L^0(\nu \otimes \mathbb{P})$-class independent of the representative, and Lemma 2.2 transfers the convention to $F(\varepsilon^+_{t,z}\omega)$ for $\nu \otimes \mathbb{P}$-a.e. (t, z, ω). This convention is used whenever ΨF and the positivity conditions for entropy remainders are written pointwise.

The symbol D^B denotes the Malliavin derivative with respect to the Brownian driver. With

$$W_t := \int_0^t \sigma_s\, \mathrm{d}B_s, \qquad C_t := \int_0^t \sigma_s \sigma_s^\top\, \mathrm{d}s,$$

we use the shorthand

$$\langle v_t, \mathrm{d}C_t\, v_t \rangle := |\sigma_t^\top v_t|^2\, \mathrm{d}t$$

for $\mathbb{R}^d$-valued vector fields v_t. Brownian stochastic integrals are written in driver form $\int_0^t \langle \theta_s, \mathrm{d}B_s \rangle$, with predictable $\mathbb{R}^k$-valued integrand θ. We take deterministic characteristics (Γ, C, ν) relative to the truncation $h(z) = z\mathbf{1}_{\{|z| \le 1\}}$.

2.1 *Canonical additive processes and evolution families*

By an additive process we mean a càdlàg process with independent increments, stochastic continuity, and initial value zero. On $[0, T]$, let X be the $\mathbb{R}^d$-valued additive process with deterministic characteristics (Γ_t, C_t, ν), where Γ is continuous of finite variation, $C_t = \int_0^t \sigma_s \sigma_s^\top\, \mathrm{d}s$, and ν satisfies the Lévy condition of Assumption 2.1. We use the Lévy–Itô decomposition

$$X_t = \Gamma_t + W_t + J_t, \qquad W_t = \int_0^t \sigma_s\, \mathrm{d}B_s,$$

where

$$J_t = \int_{(0,t]\times\{|z|\le 1\}} z\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_{(0,t]\times\{|z|>1\}} z\, N(\mathrm{d}s, \mathrm{d}z).$$

The compensator is non-atomic at deterministic times, so replacing $(0, t]$ by $[0, t]$ in predictable representations causes no ambiguity.

Remark 2.4 (Comparison with the natural filtration). The main results are formulated on the product Brownian–Poisson filtration. Under the additional nondegeneracy condition $d = k$, with σ_t invertible for Lebesgue-a.e. t and measurable $t \mapsto \sigma_t^{-1}$, the usual augmentation of the natural filtration of X coincides with it: the jumps of X recover the Poisson coordinate, and the Brownian coordinate is recovered from $\int_0^t \sigma_s^{-1}\, \mathrm{d}W_s$ after subtracting drift and jumps.

Lemma 2.5 (Add-one identity for the additive path)**.** *For each fixed $r \in [0,T]$,*

$$X_r(\varepsilon^+_{t,z}\omega) = X_r(\omega) + z\mathbf{1}_{\{t\le r\}} \quad \text{for } \nu\otimes\mathbb{P}\text{-a.e. } (t,z,\omega).$$

Consequently, for every bounded Borel function f and each fixed $r \in [0,T]$,

$$f(X_r(\varepsilon^+_{t,z}\omega)) = f(X_r(\omega) + z\mathbf{1}_{\{t\le r\}}) \quad \nu\otimes\mathbb{P}\text{-a.e.}$$

Proof. Fix $r \in [0,T]$ and set $E_n := \{z \in \mathbb{R}^d_0 : n^{-1} < |z| \le n\}$. Let $X_r^{(n)}$ be the truncated Lévy–Itô representative obtained by retaining only jumps with marks in E_n in the jump part. For every n, the set $[0,T]\times E_n$ has finite ν-measure. The finite-window add-one identity holds for the chosen representatives:

$$X_r^{(n)}(\varepsilon^+_{t,z}\omega) = X_r^{(n)}(\omega) + z\mathbf{1}_{\{t\le r\}}\mathbf{1}_{E_n}(z) \tag{2.1}$$

for $\nu\otimes\mathbb{P}$-a.e. (t,z,ω), after intersecting the corresponding countable family of conull sets. Indeed, on a finite window the identity is pathwise for the Poisson configuration, and Lemma 2.2 transfers it to the fixed canonical representatives.

The truncated variables converge to the fixed Lévy–Itô representative X_r in probability. The omitted small compensated part has squared $L^2(\mathbb{P})$ norm equal to

$$\int_{[0,r]\times\{|z|\le n^{-1}\}} |z|^2 \nu(\mathrm{d}s,\mathrm{d}z) \to 0,$$

and the omitted large-jump part is absent with probability tending to one because $\nu([0,r]\times\{|z|>n\}) \downarrow 0$. Choose a subsequence n_j such that $X_r^{(n_j)} \to X_r$ $\mathbb{P}$-a.s., and let A be the exceptional set. By Lemma 2.2,

$$\varepsilon^+_{t,z}\omega \notin A \quad \text{for } \nu\otimes\mathbb{P}\text{-a.e. } (t,z,\omega).$$

Thus, on a common $\nu\otimes\mathbb{P}$-conull set,

$$X_r^{(n_j)}(\omega) \to X_r(\omega), \qquad X_r^{(n_j)}(\varepsilon^+_{t,z}\omega) \to X_r(\varepsilon^+_{t,z}\omega).$$

Passing to the limit in (2.1) and using $\mathbf{1}_{E_{n_j}}(z) \to 1$ for every $z \neq 0$ gives

$$X_r(\varepsilon^+_{t,z}\omega) = X_r(\omega) + z\mathbf{1}_{\{t\le r\}} \quad \nu\otimes\mathbb{P}\text{-a.e.}$$

The deterministic drift and Brownian coordinate are unchanged by $\varepsilon^+_{t,z}$, so this is the add-one identity for the full additive path. The assertion for bounded Borel functions follows by composition. □

For $0 \le s < t \le T$, we write

$$P_{s,t} f(x) := \mathbb{E}[f(x + X_t - X_s)], \qquad x \in \mathbb{R}^d,$$

for bounded Borel $f : \mathbb{R}^d \to \mathbb{R}$, and denote by $\mu_{s,t}$ the law of $X_t - X_s$. In the stationary Lévy case, $\Gamma_t = t\gamma$, $C_t = tQ$ and $\nu(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t\, \nu_0(\mathrm{d}z)$, with constant $\gamma \in \mathbb{R}^d$ and symmetric positive semidefinite Q, the evolution family reduces to a one-parameter convolution semigroup.

2.2 *Malliavin operators, entropy functionals, and admissible classes*

We next introduce the analytic classes used in the entropy identities. The class $\mathcal{C}$ is the standard Chafaï class for conditional Jensen arguments, and the admissible core records the positivity and integrability hypotheses entering the Itô entropy computation.

The assumptions are expressed in terms of the Brownian-driver derivative D^B and the raw add-one difference defined by the add-one map. In the L^2 theory these derivatives are the graph-closure derivatives of Definition 2.7. The L^1 Clark–Ocone–Haussmann hypotheses below specify derivative integrability and identification of the representation integrands; Part II verifies them for concrete terminal classes. When a reference uses a mark-normalized jump derivative, the conversion to the present notation is obtained by multiplying by the mark; see Part II for the stationary normalization.

For a measurable functional $F : \Omega \to \mathbb{R}$, we define the raw jump difference operator by

$$\Psi_{t,z} F(\omega) := F(\varepsilon^+_{t,z}\omega) - F(\omega), \qquad (t, z) \in [0, T] \times \mathbb{R}^d_0,$$

where $\varepsilon^+_{t,z}\omega$ denotes the path obtained by adding a jump of size z at time t on the jump coordinate. Lemma 2.2 makes ΨF an intrinsic $L^0(\nu \otimes \mathbb{P})$-class. We write $D^{\mathcal{B}}_t F$ for the Gaussian Malliavin derivative, whenever it exists.

Definition 2.6 (Bounded finite-window product core)**.** Let $\mathscr{C}_{\mathrm{prod}}$ be the linear span of bounded random variables of the form

$$F = \varphi\Bigg(\int_0^T \langle h_1(t), \mathrm{d}B_t \rangle, \dots, \int_0^T \langle h_m(t), \mathrm{d}B_t \rangle, \int_{E_T} \rho_1(t, z)\, N(\mathrm{d}t, \mathrm{d}z), \dots, \int_{E_T} \rho_\ell(t, z)\, N(\mathrm{d}t, \mathrm{d}z) \Bigg),$$

where $m, \ell \ge 0$, $h_i \in L^2([0, T]; \mathbb{R}^k)$ are deterministic, each ρ_j is a bounded Borel function supported on a set of finite ν-measure, and $\varphi \in C^1_b(\mathbb{R}^{m+\ell})$. Throughout, C^1_b means bounded with bounded first derivative. If vector-valued mark integrals are used, each component is listed as a separate coordinate, so the dimension of φ is the total number of displayed scalar coordinates. Here $X^{(n)}_T$ denotes the Lévy–Itô variable obtained by retaining marks $n^{-1} < |z| \le n$ in both jump integrals and leaving the drift and Brownian terms unchanged, as in Lemma 2.5. Thus $f(X^{(n)}_T)$, $f \in C^1_b(\mathbb{R}^d)$, belongs to the core by expressing the retained jump part through bounded mark integrals.

Definition 2.7 (L^2 Brownian–Poisson graph domain)**.** Let

$$\|F\|^2_{1,2;B,N} := \|F\|^2_{L^2(\mathbb{P})} + \mathbb{E}\int_0^T |D^B_t F|^2 \, \mathrm{d}t + \mathbb{E}\int_{E_T} |\Psi_{t,z}F|^2 \, \nu(\mathrm{d}t, \mathrm{d}z).$$

We denote by $\mathbb{D}^{1,2}_{B,N}$ the completion, modulo equality in $L^2(\mathbb{P})$, of the core $\mathscr{C}_{\mathrm{prod}}$ from Definition 2.6 under this graph norm. Thus every element is represented by an L^2 limit F, a Brownian derivative $D^B F \in L^2([0,T]\times\Omega;\mathbb{R}^k, \mathrm{d}t\otimes\mathbb{P})$, and a jump difference $\Psi F \in L^2(\nu\otimes\mathbb{P})$ obtained as graph-norm limits of core approximants. The closedness statement for the raw Poisson add-one operator, recalled in Lemma 2.8, identifies this limit with $F(\varepsilon^+_{t,z}\omega) - F(\omega)$ for a Borel representative of F. The pair (D^B, Ψ) is jointly closable on the core: if $F_n \to 0$ in $L^2(\mathbb{P})$ and $D^B F_n \to U$, $\Psi F_n \to V$ in the two derivative spaces, then the closability of each component gives $U = 0$ and $V = 0$.

Lemma 2.8 (Closedness of the Poisson add-one difference)**.** *The operator*

$$\Psi : \mathscr{C}_{\mathrm{prod}} \subset L^2(\mathbb{P}) \longrightarrow L^2(\nu\otimes\mathbb{P}), \qquad \Psi_{t,z}F = F(\varepsilon^+_{t,z}\cdot) - F,$$

is closable. Its closed domain is understood with the graph norm induced by this operator. If F_m belongs to this closed domain, $F_m \to F$ in $L^2(\mathbb{P})$, and $\Psi F_m \to U$ in $L^2(\nu\otimes\mathbb{P})$, then F belongs to the closed domain and admits a Borel representative, still denoted by F, such that

$$U_{t,z}(\omega) = F(\varepsilon^+_{t,z}\omega) - F(\omega) \quad \text{in } L^2(\nu\otimes\mathbb{P}).$$

Consequently the Ψ component of the graph-norm completion in Definition 2.7 is the raw add-one difference of the chosen representative.

Proof. First consider a core sequence. Let $G_m \in \mathscr{C}_{\mathrm{prod}}$ satisfy $G_m \to G$ in $L^2(\mathbb{P})$ and $\Psi G_m \to V$ in $L^2(\nu\otimes\mathbb{P})$. Choose a subsequence m_j such that $G_{m_j} \to G$ $\mathbb{P}$-a.s. and, after passing to a further subsequence if necessary, $\Psi G_{m_j} \to V$ $\nu\otimes\mathbb{P}$-a.e. Let A be the exceptional $\mathbb{P}$-null set for the first convergence. By the Mecke null-set transfer in Lemma 2.2, $\varepsilon^+_{t,z}\omega \notin A$ for $\nu\otimes\mathbb{P}$-a.e. (t,z,ω). Hence, on a common $\nu\otimes\mathbb{P}$-conull set,

$$G_{m_j}(\varepsilon^+_{t,z}\omega) \to G(\varepsilon^+_{t,z}\omega), \qquad G_{m_j}(\omega) \to G(\omega).$$

For core variables, $\Psi_{t,z}G_{m_j} = G_{m_j}(\varepsilon^+_{t,z}\omega) - G_{m_j}(\omega)$ pointwise outside the fixed representative null set. Taking limits along the subsequence gives

$$V_{t,z} = G(\varepsilon^+_{t,z}\omega) - G(\omega) \quad \nu\otimes\mathbb{P}\text{-a.e.}$$

If $G = 0$ and $G_m \to 0$, this identity gives $V = 0$, so Ψ is closable. It also identifies the graph limit of every core graph-convergent sequence with the raw add-one representative.

Now let F_m be a sequence in the closed domain such that $F_m \to F$ in $L^2(\mathbb{P})$ and $\Psi F_m \to U$ in $L^2(\nu\otimes\mathbb{P})$. For each m, choose $G_{m,j} \in \mathscr{C}_{\mathrm{prod}}$ with $G_{m,j} \to F_m$ and $\Psi G_{m,j} \to \Psi F_m$ in graph norm. Choose $j(m)$ so large that

$$\|G_{m,j(m)} - F_m\|_{L^2(\mathbb{P})} + \|\Psi G_{m,j(m)} - \Psi F_m\|_{L^2(\nu\otimes\mathbb{P})} < m^{-1}.$$

Then the core diagonal sequence $G_m := G_{m,j(m)}$ satisfies $G_m \to F$ in $L^2(\mathbb{P})$ and $\Psi G_m \to U$ in $L^2(\nu \otimes \mathbb{P})$. Applying the core case just proved identifies U with the raw add-one difference of a Borel representative of F, and shows that F belongs to the closed domain. □

Three domains are used. The graph domain $\mathbb{D}^{1,2}_{B,N}$ is the L^2 domain for the Brownian–Poisson projection formula and for terminal observables. The analytic entropy core $\mathscr{A}^{\varepsilon}_{T,\Phi}$ combines bounded positivity with the L^1 jump-difference and entropy-remainder assumptions needed in derivative inequalities. The COH domain $\mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$ is the representation class on which the L^1 Clark–Ocone–Haussmann identification of predictable integrands is imposed.

For a convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ of class C^1 on $(0,\infty)$, define

$$A^{\Phi}(u,v) := \Phi(u+v) - \Phi(u) - \Phi'(u)v, \qquad u > 0, \quad u+v > 0.$$

For nonnegative F with $F, \Phi(F) \in L^1(\mathbb{P})$, define the Φ-entropy by

$$\mathrm{Ent}_{\Phi}(F) := \mathbb{E}[\Phi(F)] - \Phi(\mathbb{E}[F]).$$

For $\Phi(u) = u \log u$, with $0 \log 0 := 0$, we write

$$\mathrm{Ent}(F) := \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F].$$

The exact identities below use the integrability assumptions in their stated domains.

Definition 2.9 (Chafaï class $\mathcal{C}$)**.** Let $\mathcal{C}$ denote the class of functions $\Phi : \mathbb{R}_+ \to \mathbb{R}$ such that Φ is continuous on $\mathbb{R}_+$, convex, of class C^2 on $(0,\infty)$, and either affine or else $\Phi''(u) > 0$ for all $u > 0$ and $1/\Phi''$ is concave on $(0,\infty)$.

Remark 2.10 (Non-negativity of Φ-entropy). For convex Φ, Jensen's inequality gives $\mathrm{Ent}_{\Phi}(F) \geq 0$ whenever the terms are integrable; if Φ is strictly convex on the convex hull of the essential range of F, equality holds only for a.s. constant F.

Definition 2.11 (Derivative core $\mathscr{A}^{\varepsilon}_{T,\Phi}$)**.** Fix $\Phi \in \mathcal{C}$ and $\varepsilon > 0$. We denote by $\mathscr{A}^{\varepsilon}_{T,\Phi}$ the class of all $\mathcal{F}_T$-measurable functionals $F : \Omega \to (0,\infty)$ such that:

$$\varepsilon \leq F \leq M^{\star} \quad \text{a.s. for some } M^{\star} < \infty,$$

$$F \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \text{ and admits a Brownian derivative representative } D^B F,$$

$$\Psi F \text{ is the raw add-one difference, with } \Psi F \in L^1(E_T \times \Omega, \nu \otimes \mathbb{P}),$$

$$\mathbb{E}\left[\int_0^T |D^B_t F|^2 \, \mathrm{d}t\right] < \infty,$$

$$\mathbb{E}\left[\int_0^T \Phi''(F)|D^B_t F|^2 \, \mathrm{d}t\right] < \infty,$$

$$A^{\Phi}(F, \Psi F) \in L^1(E_T \times \Omega, \nu \otimes \mathbb{P}).$$

If Φ is affine, both the Φ''-term and the A^{Φ}-term vanish. Otherwise, the bound $\varepsilon \leq F \leq M^{\star}$ and the unweighted Gaussian energy condition imply the weighted Gaussian condition, since $\Phi''(F)|D^B_t F|^2 \, \mathrm{d}t \leq \sup_{u \in [\varepsilon, M^{\star}]} \Phi''(u)|D^B_t F|^2 \, \mathrm{d}t$ as measures. We

keep the weighted term in the definition because it is the Gaussian contribution appearing in the derivative inequality.

The class $\mathscr{A}^{\varepsilon}_{T,\Phi}$ supports the derivative inequalities. Its assumptions encode the Gaussian energy, the jump remainder, and the positivity of the arguments of Φ and A^{Φ}. The main entropy identity is formulated on a representation domain stated directly in terms of the predictable integrands of the conditional-expectation martingale.

Definition 2.12 (Representation domains under $\mathbb{P}$)**.** Fix $\Phi \in \mathcal{C}$ and $\varepsilon > 0$.

(i) We denote by $\mathscr{R}^{\varepsilon}_{T,\Phi}(\mathbb{P})$ the class of all $\mathcal{F}_T$-measurable functionals $F : \Omega \to (0,\infty)$ such that $\varepsilon \le F \le M^{\star}$ a.s. for some $M^{\star} < \infty$, and the martingale

$$M_t := \mathbb{E}[F \mid \mathcal{F}_t], \qquad 0 \le t \le T,$$

admits a predictable representation

$$M_t = \mathbb{E}[F] + \int_0^t \langle \theta_s, \mathrm{d}B_s \rangle + \int_{[0,t]\times\mathbb{R}_0^d} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z)$$

for a predictable $\mathbb{R}^k$-valued process θ, identified up to $\mathrm{d}t \otimes \mathbb{P}$-null sets, and a predictable real-valued function ξ on $[0,T]\times\mathbb{R}_0^d\times\Omega$, identified up to $\nu\otimes\mathbb{P}$-null sets, where the stochastic integrals are integrable martingales on $[0,T]$. We require these integrands to satisfy

$$M_{t-} + \xi_{t,z} > 0 \quad \text{for } \nu\otimes\mathbb{P}\text{-a.e. } (t,z,\omega),$$

and

$$\mathbb{E}\left[\int_0^T \Phi''(M_{t-})|\theta_t|^2 \,\mathrm{d}t\right] < \infty,$$
$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0^d} A^{\Phi}(M_{t-}, \xi_{t,z})\, \nu(\mathrm{d}t, \mathrm{d}z)\right] < \infty.$$

(ii) We denote by $\mathscr{R}_{T,2}(\mathbb{P})$ the class of all $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that

$$F - \mathbb{E}[F] = \int_0^T \langle \theta_t, \mathrm{d}B_t \rangle + \int_{[0,T]\times\mathbb{R}_0^d} \xi_{t,z}\, \widetilde{N}(\mathrm{d}t, \mathrm{d}z)$$

for some predictable integrands, with θ $\mathbb{R}^k$-valued and ξ real-valued, satisfying

$$\mathbb{E}\left[\int_0^T |\theta_t|^2 \,\mathrm{d}t\right] < \infty, \qquad \xi \in L^2([0,T]\times\mathbb{R}_0^d\times\Omega, \nu(\mathrm{d}t,\mathrm{d}z)\otimes\mathbb{P}).$$

Remark 2.13 (Representation-domain convention). Definition 2.12(i) records the positivity and integrability hypotheses used in the martingale entropy balance. For bounded functionals the product L^2 predictable representation property (PRP) gives a square-integrable representation, and Lemma 2.15 identifies its integrands with those of any finite-horizon integrable representation. Thus all representation energies depend only on the corresponding $\mathrm{d}t\otimes\mathbb{P}$- and $\nu\otimes\mathbb{P}$-classes. The condition $M_{t-}+\xi_{t,z} > 0$ is included because $A^{\Phi}(u,v)$ requires $u > 0$ and $u + v > 0$. When $\xi = {}^{p}(\Psi F)$, Lemma 2.24 gives $M_{t-} + \xi_{t,z} = {}^{p}(F(\varepsilon^{+}_{t,z}\cdot))$, so positivity of the canonical representative of F transfers to the compensator class.

The following results give the representation inputs for the entropy balance. The Brownian projection part is the classical Clark–Ocone mechanism [5, 12, 14], while the Poisson projection and add-one duality are based on the martingale representation and Mecke calculus for Poisson random measures [9, 10]. The finite-window localization, density, and closedness arguments are given in the proofs below.

Proposition 2.14 (Brownian–Poisson L^2 predictable representation)**.** *Let $M = (M_t)_{0\le t\le T}$ be a square-integrable martingale on the usual augmented Brownian–Poisson product filtration. Then there exist predictable integrands θ and ξ such that*

$$M_t = M_0 + \int_0^t \langle \theta_s, \mathrm{d}B_s\rangle + \int_{[0,t]\times\mathbb{R}_0^d} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z),$$

with

$$\theta \in L^2([0,T]\times\Omega; \mathbb{R}^k, \mathrm{d}t\otimes\mathbb{P}), \qquad \xi \in L^2(\nu\otimes\mathbb{P}).$$

The two stochastic-integral martingales are orthogonal, and the Itô isometry gives

$$\mathbb{E}[(M_T - M_0)^2] = \mathbb{E}\int_0^T |\theta_t|^2\, \mathrm{d}t + \mathbb{E}\int_{[0,T]\times\mathbb{R}_0^d} |\xi_{t,z}|^2 \nu(\mathrm{d}t, \mathrm{d}z).$$

The Brownian integrand θ and the jump integrand ξ are unique in these L^2 equivalence classes.

Proof. The Brownian and compensated-Poisson stochastic integrals are continuous and purely discontinuous martingales, respectively, and hence are orthogonal. Their isometries show that the map sending a pair of square-integrable predictable integrands to the sum of its terminal integrals is an isometry. Its range $\mathcal{H}$ is therefore a closed subspace of centered $L^2(\mathcal{F}_T)$.

Fix a finite-ν window A. Let U be a bounded Brownian-coordinate variable and V a bounded variable measurable with respect to $N|_A$. In their coordinate filtrations, the classical Brownian and Poisson representation theorems give

$$U_t = \mathbb{E}[U] + \int_0^t \langle h_s, \mathrm{d}B_s\rangle, \qquad V_t = \mathbb{E}[V] + \int_{A\cap([0,t]\times\mathbb{R}_0^d)} k_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z),$$

where U_t and V_t are the coordinate conditional-expectation martingales; see [12] and [10, Theorem 3.1]. Independence of the coordinates and of the Poisson restrictions to A and A^c gives $\mathbb{E}[UV \mid \mathcal{F}_t] = U_tV_t$. The coordinate martingales retain their martingale property in the product filtration. Since $[U, V] = 0$, integration by parts yields

$$U_tV_t - \mathbb{E}[U]\mathbb{E}[V] = \int_0^t V_{s-}\langle h_s, \mathrm{d}B_s\rangle + \int_{A\cap([0,t]\times\mathbb{R}_0^d)} U_{s-}k_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

The integrands are square-integrable because U_t and V_t are bounded. Thus $UV - \mathbb{E}[UV] \in \mathcal{H}$. Finite linear combinations of these products are dense in the finite-window product L^2-space: bounded coordinate indicators generate its sigma-field, and the monotone-class theorem followed by truncation gives density. Closedness of $\mathcal{H}$ therefore gives the representation on each finite window.

Choose increasing finite-ν windows $A_n \uparrow E_T$, and let $\mathcal{G}_n$ be the sigma-field generated by the full Brownian coordinate and $N|_{A_n}$. For $Y \in L^2(\mathcal{F}_T)$, set $Y_n = \mathbb{E}[Y \mid \mathcal{G}_n]$. Then $Y_n \to Y$ in L^2, since $\bigvee_n \mathcal{G}_n$ generates $\mathcal{F}_T$ up to completion. Let (θ^n, ξ^n) represent $Y_n - \mathbb{E}[Y]$, extending ξ^n by zero outside A_n. Orthogonality and the two isometries give

$$\|\theta^n - \theta^m\|^2_{L^2(\mathrm{d}t\otimes\mathbb{P})} + \|\xi^n - \xi^m\|^2_{L^2(\nu\otimes\mathbb{P})} = \|Y_n - Y_m\|^2_{L^2(\mathbb{P})}.$$

The integrands therefore converge in their completed predictable L^2-spaces, and their integrals represent $Y - \mathbb{E}[Y]$. Taking $Y = M_T$ and conditioning at time t proves the assertion; $\mathcal{F}_0$ is trivial up to null sets. The same isometry gives the stated energy identity and uniqueness. □

Lemma 2.15 (Uniqueness of finite-horizon representation integrands)**.** *Let (θ, ξ) and (θ', ξ') be predictable integrands such that the stochastic integrals appearing below are integrable martingales on $[0, T]$. If*

$$\int_0^t \langle \theta_s, \mathrm{d}B_s\rangle + \int_{[0,t]\times\mathbb{R}_0^d} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) = \int_0^t \langle \theta'_s, \mathrm{d}B_s\rangle + \int_{[0,t]\times\mathbb{R}_0^d} \xi'_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z)$$

for every $0 \le t \le T$, up to indistinguishability, then

$$\theta = \theta' \quad \mathrm{d}t \otimes \mathbb{P}\text{-a.e.}, \qquad \xi = \xi' \quad \nu \otimes \mathbb{P}\text{-a.e.}$$

Consequently, if a bounded martingale has a finite-horizon representation in the sense of Definition 2.12(i), its integrands agree with the integrands of the square-integrable representation of Proposition 2.14 on the corresponding null-set classes.

Proof. Subtract the two representations and write the difference as $Z = Z^c + Z^d$, where

$$Z^c_t = \int_0^t \langle \theta_s - \theta'_s, \mathrm{d}B_s\rangle, \qquad Z^d_t = \int_{[0,t]\times\mathbb{R}_0^d} (\xi_{s,z} - \xi'_{s,z})\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

The local martingale Z^c is continuous and Z^d is purely discontinuous. Since $Z \equiv 0$, uniqueness of the decomposition of a local martingale into its continuous and purely discontinuous parts gives $Z^c \equiv 0$ and $Z^d \equiv 0$; see, for instance, Jacod–Shiryaev [8, Theorem I.4.18]. Localizing the continuous stochastic integral and applying the Brownian isometry yield

$$\mathbb{E}\int_0^{T\wedge\tau_n} |\theta_s - \theta'_s|^2\, \mathrm{d}s = 0$$

for a reducing sequence (τ_n), and monotone convergence gives $\theta = \theta'$ $\mathrm{d}t \otimes \mathbb{P}$-a.e.

We first verify that no two Poisson atoms share a time coordinate. On a finite-ν window A with $\nu(A) > 0$, conditional on $N(A) = m$, the points have the law of m independent samples from $\nu|_A/\nu(A)$. Its time marginal is atomless by Assumption 2.1, so any two sampled times agree with probability zero. The assertion is immediate if $\nu(A) = 0$. Applying this argument to increasing finite windows covering E_T gives a single event of probability one on which every pair of distinct atoms has distinct times.

For the jump part, $Z^d \equiv 0$ implies that every jump is zero. The jump formula defining compensated random-measure integrals [8, Definition II.1.27], together with time-diffuseness and the preceding time-simplicity, gives $\Delta Z^d_t = \xi_{t,z} - \xi'_{t,z}$ at each atom (t, z) of N. With

$$H_{t,z} := 1 \wedge |\xi_{t,z} - \xi'_{t,z}|,$$

we therefore have $\int_{E_T} H_{t,z}\, N(\mathrm{d}t, \mathrm{d}z) = 0$ a.s. The compensation formula gives

$$\mathbb{E} \int_{E_T} H_{t,z}\, \nu(\mathrm{d}t, \mathrm{d}z) = \mathbb{E} \int_{E_T} H_{t,z}\, N(\mathrm{d}t, \mathrm{d}z) = 0,$$

and hence $\xi = \xi'$ $\nu \otimes \mathbb{P}$-a.e. The final assertion follows because a bounded terminal martingale is square-integrable, so Proposition 2.14 supplies an L^2 representation and the preceding uniqueness argument compares it with the given integrable representation. □

Lemma 2.16 (Predictable add-one invariance)**.** *Let $K : [0,T] \times \mathbb{R}^d_0 \times \Omega \to \mathbb{R}$ be bounded and $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d_0)$-measurable, where $\mathcal{P}$ is the predictable σ-field of the usual augmented product filtration. Then*

$$K_{t,z}(\varepsilon^+_{t,z}\omega) = K_{t,z}(\omega) \quad \textit{for } \nu \otimes \mathbb{P}\textit{-a.e. } (t, z, \omega).$$

Proof. First work with the unaugmented product filtration, denoted by $(\mathcal{F}^0_t)$. Its predictable sigma-field is generated by sets of the form $(s, u] \times A$, with $A \in \mathcal{F}^0_s$, together with the time-zero sets. For an elementary field

$$K_{t,z}(\omega) = 1_{(s,u]}(t) 1_S(z) 1_A(\omega), \qquad A \in \mathcal{F}^0_s, \quad S \in \mathcal{B}(\mathbb{R}^d_0),$$

the equality is pointwise apart from the harmless endpoint $t = 0$: if $t \in (s, u]$, then the map $\varepsilon^+_{t,z}$ adds a Poisson atom strictly after time s and leaves the Brownian coordinate unchanged, so it does not change the $\mathcal{F}^0_s$-measurable event A. The time-zero exceptional set is ν-null because the deterministic compensator has no fixed time atoms under the standing product realization. A monotone-class argument extends the identity to all bounded unaugmented predictable fields.

For the usual augmentation, the raw-version argument in the proof of Lemma 2.19 gives a bounded unaugmented predictable field K^0 and a $\mathbb{P}$-null set D such that $K_{t,z} = K^0_{t,z}$ for all $t > 0, z$ whenever $\omega \notin D$. Enlarge D to a raw measurable null set. Lemma 2.2, applied to $\mathbf{1}_D$, gives $\varepsilon^+_{t,z}\omega \notin D$ for $\nu \otimes \mathbb{P}$-a.e. (t, z, ω). Thus K and K^0 agree both before and after insertion, and the raw identity proves the claim. □

Lemma 2.17 (Mecke duality for random predictable tests)**.** *Let F be bounded and $\mathcal{F}_T$-measurable, and let K be bounded predictable with support contained in a deterministic set $A \subset E_T$ satisfying $\nu(A) < \infty$. Then*

$$\mathbb{E}\left[F \int_{E_T} K_{t,z}\, \widetilde{N}(\mathrm{d}t, \mathrm{d}z) \right] = \mathbb{E} \int_{E_T} \Psi_{t,z} F\, K_{t,z}\, \nu(\mathrm{d}t, \mathrm{d}z).$$

Proof. All terms are integrable because F and K are bounded and K is supported on a finite-ν set. Mecke's identity applied to the N-integral gives

$$\mathbb{E}\left[F \int K\, \mathrm{d}N \right] = \mathbb{E} \int F(\varepsilon^+_{t,z}\omega) K_{t,z}(\varepsilon^+_{t,z}\omega)\, \nu(\mathrm{d}t, \mathrm{d}z).$$

By Lemma 2.16, the last factor $K_{t,z}(\varepsilon^{+}_{t,z}\omega)$ may be replaced by $K_{t,z}(\omega)$. Subtracting the compensator term

$$\mathbb{E}\left[F\int K\ \mathrm{d}\nu\right] = \mathbb{E}\int F(\omega)K_{t,z}(\omega)\,\nu(\mathrm{d}t,\mathrm{d}z)$$

gives the displayed identity. □

Lemma 2.18 (Fiberwise Gaussian integration by parts)**.** *Let $F \in \mathscr{C}_{\mathrm{prod}}$, and let H be a bounded product-filtration predictable simple $\mathbb{R}^k$-valued process. Then*

$$\mathbb{E}\left[F\int_0^T \langle H_t, \mathrm{d}B_t\rangle\right] = \mathbb{E}\int_0^T \langle D^B_t F, H_t\rangle\ \mathrm{d}t.$$

Proof. Write $H_t = \sum_i h_i 1_{(t_i,t_{i+1}]}(t)$, with bounded $\mathcal{F}_{t_i}$-measurable $\mathbb{R}^k$-valued coefficients. Denote the coordinate filtrations by $\mathcal{F}^B_t, \mathcal{F}^N_t$. Up to completion, $\mathcal{F}_{t_i}$ is their product sigma-field. Conditional expectations onto finite product partitions approximate each h_i in L^2 by finite sums of bounded products $a(\omega^B)b(\eta)$, with a $\mathcal{F}^B_{t_i}$-measurable and b $\mathcal{F}^N_{t_i}$-measurable, while preserving its uniform bound. For these coefficients, fixing η gives a bounded Brownian predictable simple process and a C^1_b Brownian cylindrical functional $F(\cdot,\eta)$. Apply Wiener-space integration by parts on each fiber and integrate over the Poisson law. The Brownian isometry and Cauchy–Schwarz inequality permit passage to the L^2 limit. □

Lemma 2.19 (Density of predictable simple tests)**.** *Let $(\mathcal{F}^0_t)$ denote the unaugmented product filtration. Bounded unmarked predictable simple processes of the form*

$$H_t = \sum_{j=1}^{m} h_j \mathbf{1}_{(s_j,u_j]}(t), \qquad h_j \text{ bounded and } \mathcal{F}^0_{s_j}\text{-measurable},$$

are dense in the completed predictable space $L^2([0,T]\times\Omega, \mathrm{d}t\otimes\mathbb{P})$. Bounded marked predictable simple fields of the form

$$K_{t,z} = \sum_{j=1}^{m} c_j \mathbf{1}_{(s_j,u_j]}(t)\mathbf{1}_{S_j}(z)\mathbf{1}_{A_j}(\omega), \qquad A_j \in \mathcal{F}^0_{s_j},$$

where $c_j \in \mathbb{R}$, $S_j \in \mathcal{B}(\mathbb{R}^d_0)$, and each S_j is contained in an annulus $\{n^{-1} < |z| \le n\}$, are dense in the completed predictable space $L^2(E_T\times\Omega, \nu\otimes\mathbb{P})$.

Proof. For the unaugmented filtration, the predictable sigma-field is generated by rectangles $(s,u]\times A$, with $A \in \mathcal{F}^0_s$, together with the time-zero sets. On a finite time–mark window, the marked predictable sigma-field is generated by rectangles $(s,u]\times A\times S$. The class of bounded functions that can be approximated in L^2 by the displayed time-simple tests contains these generators and is closed under bounded pointwise monotone limits by dominated convergence. The functional monotone-class theorem therefore gives the approximation of bounded unaugmented predictable processes and fields.

For an augmented predictable rectangle $(s,u]\times D$, with $D \in \mathcal{F}_s$, choose $s_\ell \downarrow s$ in (s,u) and $D_\ell \in \mathcal{F}^0_{s_\ell}$ such that $\mathbb{P}(D\triangle D_\ell) = 0$. Such sets exist by the definition of

usual augmentation. The raw predictable indicators $\mathbf{1}_{(s_\ell,u]}\mathbf{1}_{D_\ell}$ converge to $\mathbf{1}_{(s,u]}\mathbf{1}_D$ for every time outside one $\mathbb{P}$-null event. Adjoining deterministic mark indicators and applying the monotone-class theorem yields raw versions for all bounded predictable fields, simultaneously in $t > 0, z$ outside a $\mathbb{P}$-null event. Time zero is null for both integrand measures. Value truncation and finite-window exhaustion now give the stated L^2 density, componentwise for vector-valued processes. □

Predictable projection notation. Let $\mathcal{P}$ be the predictable sigma-field of the product filtration. On $E_T \times \Omega$, we regard $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^d)$ as a sigma-field after the coordinate rearrangement $(t, \omega, z) \mapsto (t, z, \omega)$. For $H \in L^2(\nu \otimes \mathbb{P})$, write ${}^pH$ for its orthogonal projection onto the closed subspace of marked predictable fields. The same notation denotes the unmarked orthogonal projection in $L^2(\mathrm{d}t \otimes \mathbb{P})$, componentwise for vector fields. The notation $\mathbb{E}[H_{t,z} \mid \mathcal{F}_{t-}]$ below abbreviates this projection as a jointly predictable field in the corresponding measure class. Lemma 2.21 gives the compatible construction on finite windows in L^1.

Proposition 2.20 (L^2 Clark projection formula on the Brownian–Poisson product space)**.** *Let $F \in \mathbb{D}^{1,2}_{B,N}$, the L^2 Brownian–Poisson graph domain of Definition 2.7. Let (θ, ξ) be the predictable integrands in the Brownian–Poisson representation of*

$$M_t := \mathbb{E}[F \mid \mathcal{F}_t].$$

Then

$$\theta = {}^p(D^B F), \qquad \xi = {}^p(\Psi F).$$

The projections are those just defined, and the equalities hold in $L^2(\mathrm{d}t \otimes \mathbb{P})$ and $L^2(\nu \otimes \mathbb{P})$, respectively.

Proof. For $F \in \mathscr{C}_{\mathrm{prod}}$, the formula is obtained by testing the martingale representation against bounded predictable simple Brownian integrals and bounded predictable simple compensated-Poisson integrals. Lemma 2.18 gives, for every bounded predictable simple H,

$$\mathbb{E}\left[F \int_0^T \langle H_t, \mathrm{d}B_t \rangle\right] = \mathbb{E} \int_0^T \langle D_t^B F, H_t \rangle \, \mathrm{d}t,$$

and Lemma 2.17 gives, for every bounded predictable simple K supported on a finite-ν-measure window,

$$\mathbb{E}\left[F \int_{E_T} K_{t,z} \, \widetilde{N}(\mathrm{d}t, \mathrm{d}z)\right] = \mathbb{E} \int_{E_T} \Psi_{t,z} F \, K_{t,z} \nu(\mathrm{d}t, \mathrm{d}z).$$

By orthogonality and the two stochastic-integral isometries, the left-hand sides equal $\mathbb{E}\int_0^T \langle \theta_t, H_t \rangle \, \mathrm{d}t$ and $\mathbb{E}\int_{E_T} \xi_{t,z} K_{t,z} \nu(\mathrm{d}t, \mathrm{d}z)$, respectively. The density of these tests from Lemma 2.19 therefore identifies $\theta = {}^p(D^B F)$ and $\xi = {}^p(\Psi F)$ on the core.

We then pass to the graph-norm closure. Let $F_n \in \mathscr{C}_{\mathrm{prod}}$ converge to F in $\mathbb{D}^{1,2}_{B,N}$, and let (θ^n, ξ^n) and (θ, ξ) be the L^2 predictable-representation integrands of $\mathbb{E}[F_n \mid \mathcal{F}_t]$ and $\mathbb{E}[F \mid \mathcal{F}_t]$. By the Brownian–Poisson isometry in Proposition 2.14,

$$\|\theta^n - \theta\|^2_{L^2(\mathrm{d}t \otimes \mathbb{P})} + \|\xi^n - \xi\|^2_{L^2(\nu \otimes \mathbb{P})} = \|F_n - F - \mathbb{E}[F_n - F]\|^2_{L^2(\mathbb{P})} \to 0.$$

Graph-norm convergence gives $D^B F_n \to D^B F$ and $\Psi F_n \to \Psi F$ in their respective derivative spaces. Predictable projection is an orthogonal projection in each space and hence a contraction. Therefore

$$ {}^p(D^B F_n) \to {}^p(D^B F) \quad \text{in } L^2(\mathrm{d}t \otimes \mathbb{P}), $$

and

$$ {}^p(\Psi F_n) \to {}^p(\Psi F) \quad \text{in } L^2(\nu \otimes \mathbb{P}). $$

Since the projection identities have already been proved on the core, the uniqueness of limits in the two Hilbert spaces gives the identities for F. □

Lemma 2.21 (Marked predictable projection)**.** *Let $\mathcal{P}$ denote the predictable σ-field of the product filtration on $[0,T]\times\Omega$. In this lemma, $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)$ is understood on $[0,T]\times\Omega\times\mathbb{R}_0^d$, after the coordinate rearrangement from (t,ω,z) to (t,z,ω). Let*

$$ H : E_T \times \Omega \to \mathbb{R} $$

be $\mathcal{B}(E_T)\otimes\mathcal{F}_T$-measurable and locally integrable in the sense that

$$ \mathbb{E}\int_A |H_{t,z}|\,\nu(\mathrm{d}t,\mathrm{d}z) < \infty $$

for every Borel set $A \subset E_T$ with $\nu(A) < \infty$. Then there is a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)$-measurable field, denoted by ${}^pH$, unique up to $\nu\otimes\mathbb{P}$-null sets on finite windows, such that for every bounded $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)$-measurable K supported on a finite-ν window,

$$ \mathbb{E}\int_{E_T} H_{t,z}K_{t,z}\,\nu(\mathrm{d}t,\mathrm{d}z) = \mathbb{E}\int_{E_T} {}^pH_{t,z}K_{t,z}\,\nu(\mathrm{d}t,\mathrm{d}z). \tag{2.2} $$

The map $H \mapsto {}^pH$ is linear and local on finite windows. It is a contraction on L^1 and on L^2 over each finite window. Moreover, if $H \geq a$ $\nu\otimes\mathbb{P}$-a.e. for a constant a, then ${}^pH \geq a$ $\nu\otimes\mathbb{P}$-a.e. In the applications below, whenever a jointly $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)$-measurable representative of $\mathbb{E}[H_{t,z} \mid \mathcal{F}_{t-}]$ is constructed, that representative agrees with ${}^pH$ in the $\nu\otimes\mathbb{P}$-class.

Proof. The construction is local, so fix a finite-ν window A. For nonnegative H supported on A, define a finite measure on $[0,T]\times\Omega\times\mathbb{R}_0^d$ by

$$ \mu_H(C) := \mathbb{E}\int_A \mathbf{1}_C(t,\omega,z)H_{t,z}(\omega)\,\nu(\mathrm{d}t,\mathrm{d}z), \qquad C \in \mathcal{B}([0,T])\otimes\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_0^d). $$

Its restriction to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)$ is absolutely continuous with respect to the reference measure

$$ m_A(C) := \mathbb{E}\int_A \mathbf{1}_C(t,\omega,z)\,\nu(\mathrm{d}t,\mathrm{d}z). $$

Let ${}^pH$ be the Radon–Nikodym derivative $\mathrm{d}\mu_H|_{\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0^d)}/\mathrm{d}m_A$. This gives (2.2) on A. General integrable H follows by positive and negative parts, and a sigma-finite exhaustion gives the local version on E_T. Uniqueness follows from testing against bounded predictable indicators on finite windows.

Linearity and locality follow from uniqueness. Positivity and lower bounds follow from the same Radon–Nikodym construction applied to $H - a$. The L^1 contraction follows from $|{}^pH| \le {}^p|H|$ and the duality with bounded predictable signs. For L^2, the identity (2.2) says that ${}^pH$ is the orthogonal projection of H onto the closed subspace of $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d_0)$-measurable fields in $L^2(A\times\Omega, \nu\otimes\mathbb{P})$. Hence it is an L^2 contraction. Finally, a jointly predictable representative of the conditional expectation, when constructed, satisfies the same duality first on predictable rectangles and then, by a monotone-class argument, for all bounded predictable tests. Uniqueness then identifies it with ${}^pH$. □

Lemma 2.22 (Marked predictable Jensen)**.** *Fix a finite-ν window $A \subset E_T$ and let*

$$m_A(C) := \mathbb{E}\int_A \mathbf{1}_C(t,\omega,z)\,\nu(\mathrm{d}t,\mathrm{d}z)$$

for $C \in \mathcal{B}([0,T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d_0)$. On $A \times \Omega$, the marked predictable projection ^{p} of Lemma 2.21 is the conditional expectation with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d_0)$ under m_A. Consequently, if $\varphi : \mathbb{R}^m \to (-\infty, \infty]$ is proper, convex and lower semicontinuous, if $H = (H^1, \dots, H^m)$ has integrable components on $A \times \Omega$, and if $\varphi(H)^-$ is integrable, then

$$\varphi({}^pH^1, \dots, {}^pH^m) \le {}^p(\varphi(H)) \qquad m_A\text{-a.e.}$$

Here ${}^p(\varphi(H))$ denotes the extended conditional expectation and may equal $+\infty$. The unmarked version holds with $\mathrm{d}t \otimes \mathbb{P}$ in place of $\nu \otimes \mathbb{P}$.

Proof. The first assertion is exactly the duality in Lemma 2.21, restricted to the finite measure space $(A \times \Omega, m_A)$; uniqueness of conditional expectation identifies ${}^pH$ with $\mathbb{E}_{m_A}[H \mid \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d_0)]$. For Jensen's inequality, write the proper convex lower semicontinuous function as the supremum of a countable family of affine minorants, which is possible because $\mathbb{R}^m$ is separable. Conditional expectation preserves affine functions and order, and taking the countable supremum gives the displayed inequality. The unmarked statement is identical. □

Lemma 2.23 (Fixed-time predictable projection)**.** *Let F be bounded and $\mathcal{F}_T$-measurable, and let $M_t = \mathbb{E}[F \mid \mathcal{F}_t]$ be its càdlàg martingale version. Then, for every deterministic $t \in (0,T]$, $\Delta M_t = 0$ a.s. and $M_{t-} = \mathbb{E}[F \mid \mathcal{F}_{t-}]$ a.s. Moreover, with $M_{0-} := M_0$, the marked predictable projection of the field $(t,z,\omega) \mapsto F(\omega)$ is the field $(t,z,\omega) \mapsto M_{t-}(\omega)$.*

Proof. By Proposition 2.14, the bounded martingale M has a Brownian–Poisson representation. The Brownian part is continuous. The jump of the compensated-Poisson integral at time t is the integral of its integrand over $N(\{t\}, \mathrm{d}z)$; this is the standard jump identity for integrals with respect to integer-valued random measures. Since the compensator is diffuse in time, $N(\{t\} \times \mathbb{R}^d_0) = 0$ a.s. for every deterministic t, and hence $\Delta M_t = 0$ a.s. Therefore $M_{t-} = M_t = \mathbb{E}[F \mid \mathcal{F}_t]$ a.s. at deterministic t, and the tower property gives $M_{t-} = \mathbb{E}[F \mid \mathcal{F}_{t-}]$ a.s.

Let K be a bounded marked predictable test supported on a finite-ν window. By the completion argument in Lemma 2.19, choose a bounded unaugmented predictable version K^0 with the same support and with $K = K^0$ $\nu \otimes \mathbb{P}$-a.e. For each deterministic (t,z) with $t > 0$, the section $K^0_{t,z}$ is $\mathcal{F}^0_{t-}$-measurable: this holds for predictable rectangles and extends by the monotone-class theorem. Apply the fixed-time identity

to these sections and then use Fubini. Since the time-zero set is ν-null, and replacing K^0 by K changes neither integral, this gives

$$\mathbb{E}\int_{E_T} FK_{t,z}\,\nu(\mathrm{d}t,\mathrm{d}z) = \mathbb{E}\int_{E_T} M_{t-}K_{t,z}\,\nu(\mathrm{d}t,\mathrm{d}z).$$

The uniqueness part of Lemma 2.21 therefore identifies ${}^{p}F$ with M_{t-} as a marked field. □

Lemma 2.24 (Predictable add-one conditional representative)**.** *Let F be bounded and $\mathcal{F}_T$-measurable, choose a Borel representative of F, and assume that $\Psi F \in L^1_{\mathrm{loc}}(\nu\otimes\mathbb{P})$, meaning integrability on every finite-ν window. Let*

$$M_t := \mathbb{E}[F \mid \mathcal{F}_t].$$

Suppose that a predictable field ξ is the marked predictable projection ${}^{p}(\Psi F)$. In particular, this holds for the jump integrand in the L^2 Clark representation whenever $F \in \mathbb{D}^{1,2}_{B,N}$. Then ξ has a predictable version such that

$$M_{t-} + \xi_{t,z} = {}^{p}\big(F(\varepsilon^{+}_{t,z}\cdot)\big) \qquad \nu\otimes\mathbb{P}\text{-a.e.},$$

where the right-hand side is the marked predictable projection of the field $(t,z,\omega)\mapsto F(\varepsilon^{+}_{t,z}\omega)$. Consequently, if $F \geq \varepsilon > 0$ a.s. and the representative is chosen as in Remark 2.3, then $M_{t-} + \xi_{t,z} \geq \varepsilon$ for $\nu\otimes\mathbb{P}$-a.e. (t,z,ω).

Proof. The identity is local on finite compensator windows. On such a window, Lemma 2.2 gives

$$F(\varepsilon^{+}_{t,z}\omega) = F(\omega) + \Psi_{t,z}F(\omega) \qquad \nu\otimes\mathbb{P}\text{-a.e.}$$

Taking the marked predictable projection and using the linearity in Lemma 2.21 gives

$${}^{p}\big(F(\varepsilon^{+}_{t,z}\cdot)\big) = {}^{p}\big(F(\cdot)\big) + {}^{p}(\Psi F)_{t,z}.$$

By Lemma 2.23, the marked predictable projection of $(t,z,\omega)\mapsto F(\omega)$ is M_{t-}, regarded as a field independent of z. By assumption ${}^{p}(\Psi F) = \xi$, and the displayed identity follows. An increasing finite-window exhaustion gives the statement on E_T. If $F \geq \varepsilon$ for the chosen representative, then $F(\varepsilon^{+}_{t,z}\cdot) \geq \varepsilon$ on a $\nu\otimes\mathbb{P}$-conull set. The lower-bound preservation in Lemma 2.21 gives $M_{t-} + \xi_{t,z} \geq \varepsilon$. □

The derivative core $\mathscr{A}^{\varepsilon}_{T,\Phi}$ records the positivity and integrability assumptions in the derivative inequalities. It is paired with the Clark–Ocone–Haussmann domain $\mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$, where predictable integrands are identified with Malliavin derivatives and jump differences. The next proposition verifies the core conditions under a finite first truncated-moment assumption.

Proposition 2.25 (Positive cylindrical functionals: analytic integrability verification)**.** *Fix $0 < t_1 < \cdots < t_n \leq T$, and let*

$$F = \varphi(X_{t_1},\dots,X_{t_n}),$$

where $\varphi \in C_b^1((\mathbb{R}^d)^n)$ *satisfies*

$$0 < \varepsilon \le \varphi \le M^\star$$

for some constants $\varepsilon, M^\star > 0$. *Assume in addition that*

$$\int_{[0,T]\times\mathbb{R}_0^d} (1 \wedge |z|)\, \nu(\mathrm{d}t, \mathrm{d}z) < \infty. \tag{2.3}$$

Then $F \in \mathscr{A}^{\varepsilon}_{T,\Phi}$ *for every* $\Phi \in \mathcal{C}$. *More precisely,*

$$D_t^B F = \sigma_t^\top \sum_{i=1}^n \nabla_i \varphi(X_{t_1}, \dots, X_{t_n})\, \mathbf{1}_{[0,t_i]}(t), \tag{2.4}$$

and

$$\Psi_{t,z} F = \varphi\big(X_{t_1} + z\mathbf{1}_{\{t\le t_1\}}, \dots, X_{t_n} + z\mathbf{1}_{\{t\le t_n\}}\big) - \varphi(X_{t_1}, \dots, X_{t_n}). \tag{2.5}$$

Moreover, there exist constants $C_1, C_2 > 0$, *with* C_1 *depending only on* φ *and* C_2 *depending only on* φ, Φ, ε, *and* $M^\star$, *such that*

$$|D_t^B F| \le C_1 \|\sigma_t\|_{\mathrm{op}}, \qquad |\Psi_{t,z} F| \le C_1(1 \wedge |z|), \qquad A^\Phi(F, \Psi_{t,z} F) \le C_2(1 \wedge |z|^2). \tag{2.6}$$

Proof. Fix the Poisson coordinate. Then each X_{t_i} is a deterministic translation of $\int_0^{t_i} \sigma_s \, \mathrm{d}B_s$ on the Brownian fiber. Applying the finite-dimensional chain rule to the bounded C^1 function φ gives (2.4). The resulting derivative is jointly measurable, and the bound below permits integration over the Poisson coordinate. This is the fiberwise interpretation of the Gaussian derivative in the additive-process calculus [6, Section 6]. Formula (2.5) follows from Lemma 2.5, applied at the finitely many times t_i.

Since $\varphi \in C_b^1((\mathbb{R}^d)^n)$, one has

$$|D_t^B F| \le \sqrt{n}\, \|\nabla\varphi\|_\infty \, \|\sigma_t\|_{\mathrm{op}}.$$

The Brownian energy is finite because $\int_0^T \|\sigma_t\|_{\mathrm{HS}}^2 \, \mathrm{d}t < \infty$. For the jump difference, the mean value theorem gives

$$|\Psi_{t,z} F| \le \sqrt{n}\, \|\nabla\varphi\|_\infty |z|, \qquad |z| \le 1,$$

whereas boundedness of φ yields

$$|\Psi_{t,z} F| \le 2\|\varphi\|_\infty, \qquad |z| > 1.$$

Hence

$$|\Psi_{t,z} F| \le C_1(1 \wedge |z|)$$

for some constant $C_1 > 0$. The L^1-integrability of ΨF follows from (2.3).

Finally, because $\varepsilon \le F(\omega) \le M^\star$ and $\varepsilon \le F(\varepsilon^+_{t,z}\omega) = F(\omega) + \Psi_{t,z} F(\omega) \le M^\star$ (both bounds following from the global bounds on φ), the convex combination

$F+r\Psi_{t,z}F$ lies in $[\varepsilon, M^\star]$ for every $r \in [0,1]$. Taylor's formula with integral remainder therefore gives

$$\begin{aligned} A^\Phi(F, \Psi_{t,z}F) &= |\Psi_{t,z}F|^2 \int_0^1 (1-r)\Phi''(F + r\Psi_{t,z}F)\ \mathrm{d}r \\ &\le \frac{1}{2} \sup_{u \in [\varepsilon, M^\star]} \Phi''(u)\, |\Psi_{t,z}F|^2 \\ &\le C_2(1 \wedge |z|^2), \end{aligned}$$

for some constant $C_2 > 0$. This is integrable against $\nu(\mathrm{d}t, \mathrm{d}z)$, so all terms in the definition of $\mathscr{A}^{\varepsilon}_{T,\Phi}$ are finite. □

Remark 2.26 (Analytic core for cylindrical functionals). Proposition 2.25 verifies the analytic positivity and integrability conditions under the finite first truncated-moment assumption. The COH identification is a separate hypothesis in Definition 2.27; it is verified for the finite-first-moment terminal class in Part II. The derivative inequalities are applied on $\mathscr{A}^{\varepsilon}_{T,\Phi} \cap \mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$. Infinite-variation small jumps are treated by the terminal L^2 theory developed in Part II; cancellation-based derivative cores are naturally aligned with the raw-difference normalization of the canonical Lévy-space inequality [16, Section 3].

The raw difference normalization $\Psi_{t,z}F = F(\varepsilon^+_{t,z}\omega) - F(\omega)$ is used throughout. In the one-dimensional stationary Lévy reduction, the canonical Lévy notation is recovered by writing $D^J_{t,z}F := \Psi_{t,z}F/z$ for $z \neq 0$.

2.3 Representation formulas and convexity input

Two ingredients are used below. The first is the predictable representation of the conditional-expectation martingale, together with the Clark–Ocone–Haussmann identification of the integrands through Malliavin derivatives and jump differences. The second is the convexity input for conditional Jensen estimates of the nonlinear entropy integrands. All predictable projections in this subsection are taken with respect to the product filtration fixed in Section 2.

Definition 2.27 (Finite-horizon COH hypotheses)**.** Let $\mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$ be the class of all $\mathcal{F}_T$-measurable functionals F for which the following four hypotheses hold on the product filtration of Assumption 2.1.

(COH1) $F \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ and there are raw derivative representatives $D^B F$ and ΨF, with ΨF understood through Lemma 2.2, such that

$$D^B F \in L^1([0,T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}), \qquad \Psi F \in L^1(E_T \times \Omega, \nu \otimes \mathbb{P}).$$

(COH2) There exist predictable integrands θ and ξ such that

$$F - \mathbb{E}[F] = \int_0^T \langle \theta_t, \mathrm{d}B_t \rangle + \int_{E_T} \xi_{t,z}\, \widetilde{N}(\mathrm{d}t, \mathrm{d}z),$$

where both stochastic integrals are integrable martingales on $[0,T]$.

(COH3) The Brownian and jump integrands are identified by the conditional derivative projections

$$\theta_t = \mathbb{E}[D_t^B F \mid \mathcal{F}_{t-}] \quad \mathrm{d}t\otimes\mathbb{P}\text{-a.e.}, \qquad \xi_{t,z} = \mathbb{E}[\Psi_{t,z} F \mid \mathcal{F}_{t-}] \quad \nu\otimes\mathbb{P}\text{-a.e.}$$

(COH4) The predictable projections in (COH3) are taken after fixing the canonical add-one representatives of Lemma 2.2; in particular, identities involving $F(\varepsilon^+_{t,z}\cdot)$ are understood on finite-ν windows and then extended by exhaustion.

In the standard scalar additive setting, (COH2)–(COH3) follow from the mixed Clark–Ocone–Haussmann formula of [6, Section 6] under its domain assumptions. In the present $\mathbb{R}^d$-valued raw-difference notation, the formulas in this definition specify the normalization used in this paper. Part II gives terminal examples in both finite-first-moment and infinite-activity regimes.

Remark 2.28 (COH hypotheses and analytic cores). The COH domain contains only representation and derivative-identification hypotheses. Positivity and integrability assumptions for entropy, Poincaré, or exponential estimates are stated in the corresponding analytic cores.

Lemma 2.29 (Finite-horizon COH representation under $\mathbb{P}$)**.** *Let $F \in \mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$. Then the martingale*

$$M_t := \mathbb{E}[F \mid \mathcal{F}_t], \qquad 0 \le t \le T,$$

admits the predictable representation

$$M_t = \mathbb{E}[F] + \int_0^t \langle \theta_s, \mathrm{d}B_s\rangle + \int_{[0,t]\times\mathbb{R}^d_0} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \tag{2.7}$$

where the predictable integrands satisfy

$$\theta_s = \mathbb{E}[D_s^B F \mid \mathcal{F}_{s-}], \qquad \xi_{s,z} = \mathbb{E}[\Psi_{s,z} F \mid \mathcal{F}_{s-}]$$

up to the corresponding null sets. The stochastic integrals in (2.7) *are understood in the finite-horizon L^1 representation sense specified by the representation hypotheses. The stochastic integrals in this representation are integrable martingales on $[0,T]$, hence their future increments have zero conditional expectation. When an Itô isometry or square-integrability is used below, it is obtained separately from boundedness of the martingale or imposed in the relevant representation domain. In that case the L^1 Clark–Ocone–Haussmann integrands and the L^2 martingale-representation integrands agree on the corresponding null-set classes by Lemma 2.15.*

Proof. Assumptions (COH2)–(COH3) give a finite-horizon Clark–Ocone–Haussmann representation on the product filtration, with stochastic integrals that are integrable martingales on $[0,T]$. Taking conditional expectations with respect to $\mathcal{F}_t$ therefore eliminates the increments over $(t,T]$ and yields (2.7). The Gaussian identification is written with respect to the Brownian driver: $\theta_t = \mathbb{E}[D_t^B F \mid \mathcal{F}_{t-}]$. □

Proposition 2.30 (Convexity of the entropy integrands)**.** *Let $\Phi \in \mathcal{C}$. For every integer $m \geq 1$ and every symmetric positive semidefinite matrix $A \in \mathbb{R}^{m\times m}$, the map*

$$\Theta_{1,A}(u,v) := \Phi''(u)\langle v, Av\rangle, \qquad (u,v) \in (0,\infty)\times\mathbb{R}^m,$$

is convex. Moreover, with

$$D_\Phi := \{(u,v) \in \mathbb{R}^2 : u > 0,\ u+v > 0\},$$

the map

$$\Theta_2(u,v) := A^\Phi(u,v)$$

is convex on D_Φ. In particular, taking A to be the identity matrix gives the conditional Jensen argument for the Brownian Gaussian energy term, and the second assertion gives the corresponding argument for the jump remainder term.

Proof. If Φ is affine, then both integrands vanish, so there is nothing to prove. Assume that Φ is non-affine, and set

$$\psi(u) := \frac{1}{\Phi''(u)}, \qquad u > 0.$$

By assumption, ψ is positive and concave on $(0,\infty)$.

Fix $m \geq 1$ and a positive semidefinite matrix $A \in \mathbb{R}^{m\times m}$, and write $A^{1/2}$ for its symmetric square root. For $(u_i, v_i) \in (0,\infty)\times\mathbb{R}^m$ and $\lambda \in [0,1]$, concavity of ψ gives

$$\psi(\lambda u_1 + (1-\lambda)u_2) \geq \lambda\psi(u_1) + (1-\lambda)\psi(u_2).$$

Hence

$$\begin{aligned}&\Theta_{1,A}(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2)\\ &\quad\leq \frac{|\lambda A^{1/2}v_1 + (1-\lambda)A^{1/2}v_2|^2}{\lambda\psi(u_1) + (1-\lambda)\psi(u_2)}.\end{aligned}$$

The vector Cauchy–Schwarz inequality in the form

$$|\lambda a_1 + (1-\lambda)a_2|^2 \leq (\lambda b_1 + (1-\lambda)b_2)\left(\lambda\frac{|a_1|^2}{b_1} + (1-\lambda)\frac{|a_2|^2}{b_2}\right), \qquad b_1, b_2 > 0,$$

with $a_i = A^{1/2}v_i$ and $b_i = \psi(u_i)$ yields

$$\Theta_{1,A}(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2) \leq \lambda\Theta_{1,A}(u_1,v_1) + (1-\lambda)\Theta_{1,A}(u_2,v_2).$$

Thus $\Theta_{1,A}$ is convex.

For the jump term, Taylor's formula with integral remainder gives

$$A^\Phi(u,v) = v^2\int_0^1 (1-r)\Phi''(u+rv)\,\mathrm{d}r.$$

This formula is valid because $\Phi \in C^2((0,\infty))$ and the segment $\{u+rv : 0 \leq r \leq 1\}$ is contained in $(0,\infty)$ whenever $u > 0$ and $u+v > 0$. For each fixed $r \in [0,1]$,

the map $(u, v) \mapsto \Phi''(u + rv)v^2$ is the scalar case of the already proved convexity with $A = 1$, composed with the linear transformation $(u, v) \mapsto (u + rv, v)$. Hence A^Φ is a nonnegative weighted average of convex functions. This is the same convexity mechanism used in the Φ-entropy framework of [4] and, in the canonical Lévy setting, in [16]. □

Corollary 2.31 (Conditional Jensen form)**.** *Let $\Phi \in \mathcal{C}$ and let $\mathcal{G} \subset \mathcal{F}_T$ be a sub-σ-field. For the Gaussian inequality, let U be positive and let V be $\mathbb{R}^m$-valued. Assume that U, V, and the displayed conditional expectations are integrable, and that $\mathbb{E}[U \mid \mathcal{G}] > 0$ a.s. Then*

$$\Phi''(\mathbb{E}[U \mid \mathcal{G}])|\mathbb{E}[V \mid \mathcal{G}]|^2 \le \mathbb{E}[\Phi''(U)|V|^2 \mid \mathcal{G}].$$

For the jump inequality, let V be real-valued and assume in addition that $U + V > 0$ and $\mathbb{E}[U \mid \mathcal{G}] + \mathbb{E}[V \mid \mathcal{G}] > 0$ a.s.; then

$$A^\Phi(\mathbb{E}[U \mid \mathcal{G}], \mathbb{E}[V \mid \mathcal{G}]) \le \mathbb{E}[A^\Phi(U, V) \mid \mathcal{G}].$$

Both inequalities follow from conditional Jensen applied to the convex maps in Proposition 2.30.

2.4 Role of the representation domains

The terminal L^2-Clark argument in Part II uses the product graph domain $\mathbb{D}^{1,2}_{B,N}$ and the L^2 Clark projection formula.

The derivative-form inequalities are formulated on $\mathscr{A}^{\varepsilon}_{T,\Phi} \cap \mathscr{D}^{\mathrm{COH}}_{\mathbb{P}}$. In concrete cylindrical examples, Proposition 2.25 verifies the analytic entropy core under the finite first truncated-moment condition, while Definition 2.27 supplies the representation component. This separation distinguishes the terminal L^2 identity from the derivative-form inequalities.

3 Martingale entropy balance

The representation results above now give the entropy balance used in Part II.

Lemma 3.1 (Jumps of compensated Poisson integrals)**.** *Let ξ be predictable and suppose that*

$$J_t := \int_{[0,t]\times\mathbb{R}^d_0} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \qquad 0 \le t \le T,$$

is defined as a finite-horizon L^1 or L^2 martingale. Under Assumption 2.1, the compensator has no fixed-time atoms, and

$$\Delta J_t = \int_{\mathbb{R}^d_0} \xi_{t,z} N(\{t\}, \mathrm{d}z) \qquad \textit{for all } 0 \le t \le T$$

up to indistinguishability. In particular, since the time projection of N is simple a.s., this jump is ξ_{t,Z_t} on the event that N has a jump with mark Z_t at time t, and is zero otherwise.

Proof. The general jump formula for a compensated random-measure integral gives its jump as the integral over $N(\{t\}, \mathrm{d}z)$ minus the compensator atom; see [8, Definition II.1.27]. The latter vanishes under Assumption 2.1. For bounded simple integrands on finite windows this is also immediate from the definition. In the L^2 case, the extension follows by the stochastic-integral isometry and Doob's inequality: choose approximants with summable squared errors, so that their integrals converge uniformly in time almost surely and the integrands converge at the atoms of N by the compensation formula. The general compensated-integral jump formula applies to the finite-horizon integrable representations as well. The time-simplicity argument in the proof of Lemma 2.15 gives the final assertion. □

Lemma 3.2 (Martingale Φ-entropy balance on the product representation domain)**.** *Let $\Phi \in \mathcal{C}$, let $\varepsilon > 0$, and let $F \in \mathscr{R}^{\varepsilon}_{T,\Phi}(\mathbb{P})$, with the representation taken with respect to the product filtration. Write*

$$M_t := \mathbb{E}[F \mid \mathcal{F}_t] = \mathbb{E}[F] + \int_0^t \langle \theta_s, \mathrm{d}B_s \rangle + \int_{[0,t]\times\mathbb{R}_0^d} \xi_{s,z}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Then

$$\begin{aligned} \mathrm{Ent}_\Phi(F) = \frac{1}{2}\,\mathbb{E}\left[\int_0^T \Phi''(M_{t-})|\theta_t|^2 \,\mathrm{d}t\right] \\ + \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0^d} A^\Phi(M_{t-}, \xi_{t,z})\, \nu(\mathrm{d}t, \mathrm{d}z)\right]. \end{aligned} \tag{3.1}$$

Proof. Choose $M^\star < \infty$ such that $\varepsilon \le F \le M^\star$. The càdlàg version of $M_t = \mathbb{E}[F \mid \mathcal{F}_t]$ takes values in $[\varepsilon, M^\star]$ simultaneously for all t, by the conditional bounds on a countable dense set containing T and right-continuity. It is therefore square-integrable. Proposition 2.14 and Lemma 2.15 identify the given integrands with their L^2 representation classes. In particular,

$$\mathbb{E}\int_0^T |\theta_t|^2 \,\mathrm{d}t < \infty, \qquad \mathbb{E}\int_{E_T} |\xi_{t,z}|^2\, \nu(\mathrm{d}t, \mathrm{d}z) < \infty.$$

Since Φ' is bounded on $[\varepsilon, M^\star]$, the Brownian integral with integrand $\Phi'(M_{t-})\theta_t$ and the compensated-Poisson integral with integrand $\Phi'(M_{t-})\xi_{t,z}$ are square-integrable martingales.

Put $D = \{(u, v) : u > 0,\ u+v > 0\}$, and extend A^Φ by zero off D to a nonnegative Borel function $\widetilde{A}^\Phi$. The field

$$H_{t,z} := \widetilde{A}^\Phi(M_{t-}, \xi_{t,z})$$

is predictable and satisfies $\mathbb{E}\int_{E_T} H \,\mathrm{d}\nu < \infty$ by Definition 2.12(i). The pair $(M_{t-}, \xi_{t,z})$ lies in D outside a predictable $\nu \otimes \mathbb{P}$-null set. By the compensation formula, this exceptional set also has zero N-measure almost surely. Thus the extension changes neither the jump sum nor the compensator integral, and $\mathbb{E}\int H \,\mathrm{d}N = \mathbb{E}\int H \,\mathrm{d}\nu < \infty$.

To justify the L^1 compensated integral, take finite windows $A_n \uparrow E_T$ and set $H_n = (H \wedge n)\mathbf{1}_{A_n}$. Each H_n is square-integrable. The processes

$$Z_t^n = \int_{[0,t]\times\mathbb{R}_0^d} H_n \,\mathrm{d}\widetilde{N}$$

are martingales of mean zero. Define $Z_t = \int_{[0,t]\times\mathbb{R}_0^d} H\,\mathrm{d}N - \int_{[0,t]\times\mathbb{R}_0^d} H\,\mathrm{d}\nu$, whose two terms are finite almost surely. Then

$$\mathbb{E}\sup_{0\le t\le T}|Z_t - Z_t^n| \le 2\mathbb{E}\int_{E_T}(H - H_n)\,\mathrm{d}\nu \longrightarrow 0.$$

Consequently Z is an integrable martingale of mean zero, and this construction defines $\int H\,\mathrm{d}\widetilde{N}$.

Apply Itô's formula using a $C^2(\mathbb{R})$ extension of Φ that agrees with it near $[\varepsilon, M^\star]$. Lemma 3.1 identifies the jump correction with the N-integral of H. Hence

$$\Phi(M_T) - \Phi(M_0) = \int_0^T \Phi'(M_{t-})\langle\theta_t, \mathrm{d}B_t\rangle + \int_{E_T}\Phi'(M_{t-})\xi_{t,z}\,\mathrm{d}\widetilde{N} + \frac{1}{2}\int_0^T \Phi''(M_{t-})|\theta_t|^2\,\mathrm{d}t + \int_{E_T} H_{t,z}\,\mathrm{d}N.$$

The jump terms can equivalently be combined using the explicit identity

$$\int_{E_T}\big(\Phi(M_{t-}+\xi_{t,z}) - \Phi(M_{t-})\big)\,\mathrm{d}\widetilde{N} = \int_{E_T}\Phi'(M_{t-})\xi_{t,z}\,\mathrm{d}\widetilde{N} + \int_{E_T} H_{t,z}\,\mathrm{d}\widetilde{N}.$$

Here the left-hand side is interpreted as the sum on the right: its linear term is an L^2 stochastic integral and its remainder is the L^1 integral just constructed. Values on the exceptional set are immaterial. Both terms have zero expectation. Taking expectations in Itô's formula, and using $M_0 = \mathbb{E}[F]$, $M_T = F$ and the compensation formula for H, proves (3.1). □

Remark 3.3 (Automatic domain conditions and the entropy class). Under Assumption 2.1, let F be $\mathcal{F}_T$-measurable with $0 < \varepsilon \le F \le M^\star < \infty$ a.s., and let (θ, ξ) be the L^2 representation integrands of $M_t = \mathbb{E}[F \mid \mathcal{F}_t]$ supplied by Proposition 2.14. The càdlàg version of M takes values in $[\varepsilon, M^\star]$ simultaneously in time. At every atom (t, z) of N, Lemma 3.1 gives $M_{t-} + \xi_{t,z} = M_t$. The predictable field $K_{t,z} = \mathbf{1}_{\{M_{t-}+\xi_{t,z}\notin[\varepsilon,M^\star]\}}$ therefore satisfies $\int_{E_T} K\,\mathrm{d}N = 0$ a.s. Compensation yields $\mathbb{E}[\int_{E_T} K\,\mathrm{d}\nu] = 0$, so

$$M_{t-} + \xi_{t,z} \in [\varepsilon, M^\star], \qquad |\xi_{t,z}| \le M^\star - \varepsilon \quad \nu\otimes\mathbb{P}\text{-a.e.}$$

For $\Phi \in \mathcal{C}$, put $C_\Phi = \sup_{u\in[\varepsilon,M^\star]}\Phi''(u) < \infty$ and $H_{t,z} = A^\Phi(M_{t-}, \xi_{t,z})$. Taylor's formula gives

$$0 \le H_{t,z} \le \frac{C_\Phi}{2}|\xi_{t,z}|^2, \qquad H_{t,z}^2 \le \frac{C_\Phi^2}{4}(M^\star - \varepsilon)^2|\xi_{t,z}|^2.$$

The L^2 representation thus gives both energy conditions in Definition 2.12(i), so $F \in \mathscr{R}^{\varepsilon}_{T,\Phi}(\mathbb{P})$; moreover, $H \in L^2(\nu\otimes\mathbb{P})$. These conclusions require no Clark identification; the derivative hypotheses remain separate.

The same bounded-range estimates and the proof of Lemma 3.2 show that (3.1) holds for every convex $\Phi \in C^2((0,\infty))$ under the same bounds on F. The additional Chafaï condition on $1/\Phi''$ is used in Proposition 2.30 and the conditional Jensen inequalities, not in the balance identity itself.

4 Concluding remarks

The product representation and Clark formulas identify the predictable Brownian and jump integrands. Marked projections make the associated nonlinear entropy energies well defined. Part II evaluates these energies for terminal additive observables through $P_{t,T}$, using the martingale entropy balance proved here.

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