

Extreme residuals in regression model. Minimax approach

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Abstract We obtain limit theorems for extreme residuals in linear regression model in the case of minimax estimation of parameters.

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1 Introduction

Consider the model of linear regression

$$y_j = \sum_{i=1}^q \theta_i x_{ji} + \epsilon_j, \quad j = \overline{1, N}, \quad (1)$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_q)$ is an unknown parameter, ϵ_j are independent identically distributed (i.i.d.) random variables (r.v.-s) with distribution function (d.f.) $F(x)$, and $X = (x_{ji})$ is a regression design matrix.

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Let $\widehat{\theta} = (\widehat{\theta}_1, \dots, \widehat{\theta}_q)$ be the least squares estimator (LSE) of θ . Introduce the notation

$$\begin{aligned} \widehat{y}_j &= \sum_{i=1}^q \widehat{\theta}_i x_{ji}, & \widehat{\epsilon}_j &= y_j - \widehat{y}_j, & j &= \overline{1, N}; \\ Z_N &= \max_{1 \leq j \leq N} \epsilon_j, & \widehat{Z}_N &= \max_{1 \leq j \leq N} \widehat{\epsilon}_j, \\ Z_N^* &= \max_{1 \leq j \leq N} |\epsilon_j|, & \widehat{Z}_N^* &= \max_{1 \leq j \leq N} |\widehat{\epsilon}_j|. \end{aligned}$$

Asymptotic behavior of the r.v.-s Z_N, Z_N^* is studied in the theory of extreme values (see classical works by Frechet [10], Fisher and Tippett [3], and Gnedenko [5] and monographs [4, 8]). In the papers [6, 7], it was shown that under mild assumptions asymptotic properties of the r.v.-s $Z_N, \widehat{Z}_N, Z_N^*$, and \widehat{Z}_N^* are similar in the cases of both finite variance and heavy tails of observation errors ϵ_j .

In the present paper, we study asymptotic properties of minimax estimator (MME) of θ and maximal absolute residual. For MME, we keep the same notation $\widehat{\theta}$.

Definition 1. A random variable $\widehat{\theta} = (\widehat{\theta}_1, \dots, \widehat{\theta}_q)$ is called MME for θ by the observations (1)

$$\widehat{\Delta} = \Delta(\widehat{\theta}) = \min_{\tau \in \mathbb{R}^q} \Delta(\tau), \tag{2}$$

where

$$\Delta(\tau) = \max_{1 \leq j \leq N} \left| y_j - \sum_{i=1}^q \tau_i x_{ji} \right|.$$

Denote $W_N = \min_{1 \leq j \leq N} \epsilon_j$ and let $R_N = Z_N - W_N$ and $Q_N = \frac{Z_N + W_N}{2}$ be the range and midrange of the sequence $\epsilon_j, j = \overline{1, N}$.

The following statement shows essential difference in the behavior of MME and LSE.

Statement 1. (i) *If the model (1) contains a constant term, namely, $x_{j1} = 1, j = \overline{1, N}$, then almost surely (a.s.)*

$$\widehat{\Delta} \leq \frac{R_N}{2}. \tag{3}$$

(ii) *If the model (1) has the form*

$$y_j = \theta + \epsilon_j, \quad j = \overline{1, N}, \tag{4}$$

then a.s.

$$\widehat{\Delta} = \frac{R_N}{2}, \quad \widehat{\theta} - \theta = Q_N.$$

Remark 1. From the point (ii) of Statement 1 it follows that MME $\widehat{\theta}$ is not consistent in the model (4) with some ϵ_j having all the moments (see Example 2).

Remark 2. The value $\widehat{\Delta}$ can be represented as a solution of the following linear programming problem (LPP):

$$\widehat{\Delta} = \min_{\Delta \in \mathcal{D}} \Delta, \tag{5}$$

$$\mathcal{D} = \left\{ (\tau, \Delta) \in \mathbb{R}^q \times \mathbb{R}_+ : \left| y_j - \sum_{i=1}^q \tau_i x_{ji} \right| \leq \Delta, j = \overline{1, N} \right\}$$

$$= \left\{ (\tau, \Delta) \in \mathbb{R}^q \times \mathbb{R}_+ : \sum_{i=1}^q \tau_i x_{ji} + \Delta \geq y_j, -\sum_{i=1}^q \tau_i x_{ji} + \Delta \geq -y_j, j = \overline{1, N} \right\}.$$

So, the problem (2) of determination of the values $\widehat{\Delta}$ and $\widehat{\theta}$ is reduced to solving LPP (5). The LPP can be efficiently solved numerically by the simplex method; see [2, 12]. Investigation of asymptotic properties of maximal absolute residual $\widehat{\Delta}$ and MME $\widehat{\theta}$ is quite difficult in the case of general model (1). However, under additional assumptions on regression experiment design and observation errors ϵ_j , it is possible to find the limiting distribution of $\widehat{\Delta}$, to prove the consistency of MME $\widehat{\theta}$, and even estimate the rate of convergence $\widehat{\theta} \rightarrow \theta, N \rightarrow \infty$.

2 The main theorems

First, we recall briefly some results of extreme value theory. Let r.v.-s (ϵ_j) have the d.f. $F(x)$. Assume that for some constants $b_n > 0$ and a_n , as $n \rightarrow \infty$,

$$b_n(Z_n - a_n) \xrightarrow{D} \zeta, \tag{6}$$

and ζ has a nondegenerate d.f. $G(x) = \mathbb{P}(\zeta < x)$. If assumption (6) holds, then we say that d.f. F belongs to the domain of maximum attraction of the probability distribution G and write $F \in D(G)$.

If $F \in D(G)$, then G must have just one of the following three types of distributions [5, 8]:

Type I:

$$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-\alpha}), & \alpha > 0, x > 0; \end{cases}$$

Type II:

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & \alpha > 0, x \leq 0, \\ 1, & x > 0; \end{cases}$$

Type III:

$$\Lambda(x) = \exp(-e^{-x}), \infty < x < \infty. \tag{7}$$

Necessary and sufficient conditions for convergence to each of d.f.-s $\Phi_\alpha, \Psi_\alpha, \Lambda$ are also well known.

Suppose in the model (1) that:

(A1) (ϵ_j) are symmetric r.v.-s;

(A2) (ϵ_j) satisfy relation (6), that is, $F \in D(G)$ with normalizing constants a_n and b_n , where G is one of the d.f.-s. $\Phi_\alpha, \Psi_\alpha, \Lambda$ defined in (7).

Assume further that regression experiment design is organized as follows:

$$\begin{aligned} x_j &= (x_{j1}, \dots, x_{jq}) \in \{v_1, v_2, \dots, v_k\}, \quad v_l = (v_{l1}, \dots, v_{lq}) \in \mathbb{R}^q, \\ v_m &\neq v_l, \quad m \neq l; \end{aligned} \tag{8}$$

that is, x_j take some fixed values only. Besides, suppose that

$$x_j = V_l \quad \text{for } j \in I_l, \quad l = \overline{1, k}, \tag{9}$$

$\text{card}(I_l) = n, I_m \cap I_l = \emptyset, m \neq l, N = kn$ is the sample size,

$$V = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1q} \\ v_{21} & v_{22} & \dots & v_{2q} \\ \dots & \dots & \dots & \dots \\ v_{k1} & v_{k2} & \dots & v_{kq} \end{pmatrix}.$$

Theorem 1. Under assumptions (A1), (A2), (8), and (9),

$$\Delta_n = b_n(\widehat{\Delta} - a_n) \xrightarrow{D} \Delta_0, \quad n \rightarrow \infty, \tag{10}$$

where

$$\begin{aligned} \Delta_0 &= \max_{u \in \mathcal{D}^*} L_0^*(u), \\ L_0^*(u) &= \sum_{l=1}^k (u_l \zeta_l + u'_l \zeta'_l), \quad u = (u_1, \dots, u_k, u'_1, \dots, u'_k), \\ \mathcal{D}^* &= \left\{ u \geq 0 : \sum_{l=1}^k (u_l - u'_l) v_{li} = 0, \sum_{l=1}^k (u_l + u'_l) = 1, i = \overline{1, q} \right\}, \end{aligned} \tag{11}$$

$\zeta_l, \zeta'_l, l = \overline{1, k}$, are i.r.v.-s having d.f. $G(x)$.

For a number sequence $b_n \rightarrow \infty$ and random sequence (ξ_n) , we will write $\xi_n \stackrel{P}{=} O(b_n^{-1})$ if

$$\sup_n \mathbb{P}(b_n |\xi_n| > C) \rightarrow 0 \quad \text{as } C \rightarrow \infty.$$

Assume that $k \geq q$ and there exists square submatrix $\widetilde{V} \subset V$ of order q

$$\widetilde{V} = \begin{pmatrix} v_{l_1 1} & \dots & v_{l_1 q} \\ \dots & \dots & \dots \\ v_{l_q 1} & \dots & v_{l_q q} \end{pmatrix},$$

such that

$$\det \widetilde{V} \neq 0. \tag{12}$$

Theorem 2. Assume that, under conditions of Theorem 1, $k \geq q$, assumption (12) holds and

$$b_n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{13}$$

Then MME $\widehat{\theta}$ is consistent, and

$$\widehat{\theta}_i - \theta_i \stackrel{P}{=} O(b_n^{-1}), \quad i = \overline{1, q}.$$

Example 1. Let in the model of simple linear regression

$$y_j = \theta_0 + \theta_1 x_j + \epsilon_j, \quad j = \overline{1, N}, \tag{14}$$

$x_j = v, j = \overline{1, N}$, that is, $k = 1$ and $q = 2$.

Then such a model can be rewritten in the form (4) with $\theta = \theta_0 + \theta_1 v$. Clearly, the parameters θ_0, θ_1 cannot be defined unambiguously here. So, it does not make sense to speak about the consistency of MME $\widehat{\theta}$ when $k < q$.

Example 2. Consider regression model (4) with errors ϵ_j having the Laplace density $f(x) = \frac{1}{2}e^{-|x|}$. For this distribution, the famous von Mises condition is satisfied ([8], p. 16) for the type III distribution, that is, $F \in D(\Lambda)$. For symmetric $F \in D(\Lambda)$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}\{2b_n Q_n < x\} = \frac{1}{1 + e^{-x}}.$$

The limiting distribution is a logistic one (see [9], p. 62). Using further well-known formulas for the type Λ ([9], p. 49) $a_n = F^{-1}(1 - \frac{1}{n})$ and $b_n = nf(a_n)$, we find $a_n = \ln \frac{n}{2}$ and $b_n = 1$. From Statement 1 it follows now that MME $\widehat{\theta}$ is not consistent. Thus, condition (13) of Theorem 2 cannot be weakened.

The following lemma allows us to check condition (13).

Lemma 1. Let $F \in D(G)$. Then we have:

1. If $G = \Phi_\alpha$, then

$$x_F = \sup\{x : F(x) < 1\} = \infty, \quad \gamma_n = F^{-1}\left(1 - \frac{1}{n}\right) \rightarrow \infty,$$

$$b_n = \gamma_n^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, (13) does not hold.

2. If $G = \Psi_\alpha$, then

$$x_F < \infty, \quad 1 - F(x_F - x) = x^\alpha L(x),$$

where $L(x)$ is a slowly varying (s.v.) function at zero, and there exists s.v. at infinity function $L_1(x)$ such that

$$b_n = (x_F - \gamma_n)^{-1} = n^\alpha L_1(n) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

So (13) is true.

3. If $G = \Lambda$, then

$$b_n = r(\gamma_n), \quad \text{where } r(x) = R'(x), R(x) = -\ln(1 - F(x)).$$

Clearly, (13) holds if

$$x_F = \infty, \quad r(x) \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

Similar results can be found in [9], Corollary 2.7, pp. 44–45; see also [4, 8].
Set

$$\begin{aligned} Z_{nl} &= \max_{j \in I_l} \epsilon_j, & W_{nl} &= \min_{j \in I_l} \epsilon_j \\ R_{nl} &= Z_{nl} - W_{nl}, & Q_{nl} &= \frac{Z_{nl} + W_{nl}}{2}, \quad l = \overline{1, k}. \end{aligned}$$

It turns out that Theorems 1 and 2 can be significantly simplified in the case $k = q$.

Theorem 3. *Let for the model (1) conditions (8) and (9) be satisfied, $k = q$, and a matrix V satisfies condition (12). Then we have:*

$$(i) \quad \hat{\Delta} = \frac{1}{2} \max_{1 \leq l \leq q} R_{nl}, \tag{15}$$

$$\hat{\theta}_i - \theta_i = \frac{\det V Q_{(i)}}{\det V}, \quad i = \overline{1, q}, \tag{16}$$

where the matrix $V Q_{(i)}$ is obtained from V by replacement of the i th column by the column $(Q_{n1}, \dots, Q_{nq})^T$.

(ii) *If additionally conditions $(A_1), (A_2)$ are satisfied, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}(2b_n(\hat{\Delta} - a_n) < x) = (G \star G(x))^q, \tag{17}$$

where $G \star G(x) = \int_{-\infty}^{\infty} G(x - y)dG(y)$, and for $i = \overline{1, q}$, as $n \rightarrow \infty$,

$$2b_n(\hat{\theta}_i - \theta_i) \xrightarrow{D} \frac{\det V \zeta_{(i)}}{\det V}, \tag{18}$$

the matrix $V \zeta_{(i)}$ is obtained from the V by the replacement of the i th column by the column $(\zeta_1 - \zeta'_1, \dots, \zeta_q - \zeta'_q)^T$, where all the r.v.-s ζ_i, ζ'_i are independent and have d.f. G .

Remark 3. Suppose that in the model (1), under assumptions (8), (9), $k < q$, and there exists a nondegenerate submatrix $\tilde{V} \subset V$ of order k . Then

$$\hat{\Delta} \leq \frac{1}{2} \max_{1 \leq l \leq k} R_{nl} \quad \text{a.s.}$$

Remark 4. For standard LSE,

$$\hat{\theta}_i - \theta_i \stackrel{P}{=} O(n^{-1/2});$$

therefore, if, under the conditions of Theorems 2 and 3,

$$n^{-1/2}b_n \rightarrow \infty \quad \text{as } n \rightarrow \infty, \tag{19}$$

then MME is more efficient than LSE.

In [6] (see also [9]), it is proved that if $F \in D(\Lambda)$, then for any $\delta > 0$, $b_n = O(n^\delta)$. From this relation and Lemma 1 it follows that (19) is not satisfied for domains of maximum attraction $D(\Phi_\alpha)$ and $D(\Lambda_\alpha)$. In the case of domain $D(\Psi_\alpha)$, condition (19) holds for $\alpha \in (0, 2)$. For example, assume that r.v.-s (ϵ_j) are symmetrically distributed on the interval $[-1, 1]$ and

$$1 - F(1 - h) = h^\alpha L(h) \quad \text{as } h \downarrow 0, \alpha \in (0, 2),$$

where $L(h)$ is an s.v. function at zero. Then $b_n = n^{1/\alpha}L_1(n)$, where L_1 is an s.v. at infinity function, and, under the conditions of Theorems 2 and 3, as $n \rightarrow \infty$,

$$|\hat{\theta}_i - \theta_i| \stackrel{P}{=} O((n^{1/\alpha}L_1(n))^{-1}) = o(n^{-1/2}).$$

The next example also appears to be interesting.

Example 3. Let (ϵ_j) be uniformly distributed in $[-1, 1]$, that is, $F(x) = \frac{x+1}{2}$, $x \in [-1, 1]$. It is well known that $F \in D(\Psi_1)$, $a_n = 1$, $b_n = \frac{n}{2}$. Then, under the conditions of Theorem 3, as $n \rightarrow \infty$,

$$\mathbb{P}(n(1 - \hat{\Delta}) < x) \rightarrow 1 - [\mathbb{P}\{\zeta_1 + \zeta_2 > x\}]^q = 1 - (1 + x)^q \exp(-qx),$$

where ζ_1, ζ_2 are i.i.d. r.v.-s, and $\mathbb{P}(\zeta_i < x) = 1 - \exp(-x)$, $x > 0$.

The following corollary is an immediate consequence of the Theorem 3.

Corollary 1. *If for simple linear regression (14), conditions (8) and (9) are satisfied, $k = q = 2$, and*

$$V = \begin{pmatrix} 1 & v_1 \\ 1 & v_2 \end{pmatrix}, \quad v_1 \neq v_2,$$

then we have:

$$(i) \quad \hat{\Delta} = \frac{1}{2} \max(R_{n1}, R_{n2}),$$

$$\hat{\theta}_1 - \theta_1 = \frac{Q_{n2} - Q_{n1}}{v_2 - v_1}, \quad \hat{\theta}_0 - \theta_0 = \frac{Q_{n1}v_2 - Q_{n2}v_1}{v_2 - v_1};$$

(ii) *under assumptions (A₁) and (A₂), relation (17) holds for $q = 2$, and, as $n \rightarrow \infty$,*

$$2b_n(\hat{\theta}_1 - \theta_1) \xrightarrow{D} \frac{\zeta_2 - \zeta_2' - \zeta_1 + \zeta_1'}{v_2 - v_1},$$

$$2b_n(\hat{\theta}_0 - \theta_0) \xrightarrow{D} \frac{(\zeta_1 - \zeta_1')v_2 - (\zeta_2 - \zeta_2')v_1}{v_2 - v_1},$$

where the r.v.-s $\zeta_1, \zeta_1', \zeta_2, \zeta_2'$ are independent and have d.f. G .

Remark 5. The conditions of Theorem 3 do not require (13). So it describes the asymptotic distribution of $\hat{\theta}$ even for nonconsistent MME.

3 Proofs of the main results

Let us start with the following elementary lemma, where $Z_n(t)$, $W_n(t)$, $R_n(t)$, and $Q_n(t)$ are determined by a sequence $t = \{t_1, \dots, t_n\}$ and are respectively the maximum, minimum, range, and midrange of the sequence t .

Lemma 2. *Let t_1, \dots, t_n be any real numbers, and*

$$\alpha_n = \min_{s \in \mathbb{R}} \max_{1 \leq j \leq n} |t_j - s|. \quad (20)$$

Then $\alpha_n = R_n(t)/2$; moreover, the minimum in (20) is attained at the point $s = Q_n(t)$.

Proof. Choose $s = Q_n(t)$. Then

$$\max_{1 \leq i \leq n} |t_i - s| = Z_n(t) - Q_n(t) = Q_n(t) - W_n(t) = \frac{1}{2} R_n(t).$$

If $s = Q_n(t) + \delta$, then, for $\delta > 0$,

$$\max_{1 \leq i \leq n} |t_i - s| = s - W_n(t) = \frac{1}{2} R_n(t) + \delta,$$

and, for $\delta < 0$,

$$\max_{1 \leq i \leq n} |t_i - s| = Z_n(t) - s = \frac{1}{2} R_n(t) - \delta,$$

that is, $s = Q_n(t)$ is the point of minimum. □

Proof of Statement 1. We will use Lemma 2:

$$\hat{\Delta} = \min_{\tau \in \mathbb{R}^q} \max_{1 \leq j \leq N} \left| \epsilon_j - \sum_{i=1}^q (\tau_i - \theta_i) x_{ji} \right| \leq \min_{\tau_1 \in \mathbb{R}^q} \max_{1 \leq j \leq N} |\epsilon_j - (\tau_1 - \theta_1)| = \frac{1}{2} R_N$$

(we put $\tau_i = 0$, $i \geq 2$). The point (ii) of Statement 2 follows directly from Lemma 2. □

Proof of Theorem 1. Using the notation

$$d = (d_1, \dots, d_q), \quad d_i = \tau_i - \theta_i, \quad i = \overline{1, q},$$

and taking into account Eq. (1), conditions (8) and (9), we rewrite LPP (5) in the following form:

$$\begin{aligned} \hat{\Delta} &= \min_{\Delta \in \mathcal{D}_1} \Delta, \quad (21) \\ \mathcal{D}_1 &= \left\{ (d, \Delta) \in \mathbb{R}^q \times \mathbb{R}_+ : \sum_{i=1}^q d_i x_{ji} + \Delta \geq \epsilon_j, -\sum_{i=1}^q d_i x_{ji} + \Delta \geq -\epsilon_j, j = \overline{1, N} \right\} \\ &= \left\{ (d, \Delta) \in \mathbb{R}^q \times \mathbb{R}_+ : \sum_{i=1}^q d_i v_{li} + \Delta \geq Z_{nl}, -\sum_{i=1}^q d_i v_{li} + \Delta \geq -W_{nl}, l = \overline{1, k} \right\}. \end{aligned}$$

LPP dual to (21) has the form

$$\max_{u \in \mathcal{D}^*} L_n^*(u), \tag{22}$$

where $L_n^*(u) = \sum_{l=1}^k (u_l Z_{nl} - u'_l W_{nl})$, and the domain \mathcal{D}^* is given by (11).

According to the basic duality theorem ([11], Chap. 4),

$$\hat{\Delta} = \max_{u \in \mathcal{D}^*} L_n^*(u).$$

Hence, we obtain

$$b_n(\hat{\Delta} - a_n) = \max_{u \in \mathcal{D}^*} b_n(L_n^*(u) - a_n) = \max_{u \in \mathcal{D}^*} g_n(u),$$

$$g_n(u) = \sum_{l=1}^k [u_l b_n(Z_{nl} - a_n) + u'_l b_n(-W_{nl} - a_n)].$$

Denote by Γ^* the set of vertices of the domain \mathcal{D}^* and

$$g_0(u) = \sum_{l=1}^k (u_l \zeta_l + u'_l \zeta'_l).$$

Since the maximum in LPP (22) is attained at one of the vertices Γ^* ,

$$\max_{u \in \mathcal{D}^*} g_n(u) = \max_{u \in \Gamma^*} g_n(u), \quad n \geq 1.$$

Obviously, $\text{card}(\Gamma^*) < \infty$. Thus, to prove (10), it suffices to prove that, as $n \rightarrow \infty$

$$\max_{u \in \mathcal{D}^*} g_n(u) \xrightarrow{D} \max_{u \in \Gamma^*} g_0(u)$$

or

$$(g_n(u), u \in \Gamma^*) \xrightarrow{D} (g_0(u), u \in \Gamma^*). \tag{23}$$

The Cramer–Wold argument (see, e.g., §7 of the book [1]) reduces (23) to the following relation: for any $t_m \in R$, as $n \rightarrow \infty$,

$$\sum_{u^{(m)} \in \Gamma^*} g_n(u^{(m)}) t_m \xrightarrow{D} \sum_{u^{(m)} \in \Gamma^*} g_0(u^{(m)}) t_m.$$

The last convergence holds if for any c_l, c'_l , as $n \rightarrow \infty$,

$$\sum_{l=1}^k [c_l(Z_{nl} - a_n) + c'_l(-W_{nl} - a_n)] \xrightarrow{D} \sum_{l=1}^k (c_l \zeta_l + c'_l \zeta'_l). \tag{24}$$

Under the conditions of Theorem 1,

$$\zeta_{nl} = b_n(Z_{nl} - a_n) \xrightarrow{D} \zeta_l,$$

$$\zeta'_{nl} = b_n(-W_{nl} - a_n) \xrightarrow{D} \zeta'_l, \quad l = \overline{1, k}. \tag{25}$$

The vectors $(Z_{nl}, W_{nl}), l = \overline{1, k}$, are independent, and, on the other hand, Z_{nl} and W_{nl} are asymptotically independent as $n \rightarrow \infty$ ([8], p. 28). To obtain (24), it remains to apply once more the Cramer–Wold argument. \square

Proof of Theorem 2. Let $\hat{d} = (\hat{d}_1, \dots, \hat{d}_q)$, $\hat{\Delta}$ be the solution of LPP (21), and $\gamma_l = \sum_{i=1}^q \hat{d}_i v_{li}$. Then, for any $l = \overline{1, k}$,

$$\begin{aligned} \gamma_l + \hat{\Delta} &\geq Z_{nl}, \\ -\gamma_l + \hat{\Delta} &\geq -W_{nl}. \end{aligned} \tag{26}$$

Rewrite the asymptotic relation (25) and (10) in the form

$$\begin{aligned} Z_{nl} &= a_n + \frac{\zeta_{nl}}{b_n}, & -W_{nl} &= a_n + \frac{\zeta'_{nl}}{b_n}, \\ \zeta_{nl} &\xrightarrow{D} \zeta_l, & \zeta'_{nl} &\xrightarrow{D} \zeta'_l, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \hat{\Delta} &= a_n + \frac{\Delta_n}{b_n}, \\ \Delta_n &\xrightarrow{D} \Delta_0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{28}$$

Combining (26)–(28), we obtain, for $l = \overline{1, k}$,

$$\begin{aligned} \gamma_l &\geq Z_{nl} - \hat{\Delta} = \frac{\zeta_{nl} - \Delta_n}{b_n} = O(b_n^{-1}), \\ \gamma_l &\leq W_{nl} + \hat{\Delta} = \frac{-\zeta'_{nl} + \Delta_n}{b_n} = O(b_n^{-1}). \end{aligned}$$

Choose l_1, \dots, l_q satisfying (12). Then

$$\sum_{i=1}^q \hat{d}_i v_{l_j i} = \gamma_{l_j} = O(b_n^{-1}), \quad j = \overline{1, q},$$

and by Cramer’s rule,

$$\hat{\theta}_i - \theta_i = \hat{d}_i = \frac{\det \tilde{V} \gamma_{(i)}}{\det \tilde{V}} = O(b_n^{-1}),$$

where the matrix $\tilde{V} \gamma_{(i)}$ is obtained from \tilde{V} by replacement of the i th column by the column $(\gamma_{l_1}, \dots, \gamma_{l_q})^T$. □

Proof of Theorem 3. (i) We have

$$\begin{aligned} \Delta &= \min_{\tau \in R^q} \max_{1 \leq l \leq q} \max_{j \in l_l} \left| y_j - \sum_{i=1}^q \tau_i v_{li} \right| \\ &= \min_{d \in R^q} \max_{1 \leq l \leq q} \max_{j \in l_l} \left| \epsilon_j - \sum_{i=1}^q d_i v_{li} \right|. \end{aligned} \tag{29}$$

By Lemma 2,

$$\min_{s \in R} \max_{j \in I_l} |\epsilon_j - s| = \frac{1}{2} R_{nl} \quad \text{as } s = Q_{nl}, \quad l = \overline{1, q}.$$

Therefore, the minimum in d is attained in (29) at the point \hat{d} being the solution of the system of linear equations

$$\sum_{i=1}^q d_i v_{li} = Q_{nl}, \quad l = \overline{1, q}.$$

Since the matrix V is nonsingular, by Cramer’s rule

$$\hat{d}_i = \hat{\theta}_i - \theta_i = \frac{\det V_{Q(i)}}{\det V}, \quad i = \overline{1, q}.$$

Obviously, for such a choice of \hat{d} , $\Delta = \frac{1}{2} \max_{1 \leq l \leq q} R_{nl}$, that is, we have obtained formulae (15) and (16).

(ii) Using the asymptotic independence of r.v.-s Z_n and W_n , we derive the following statement.

Lemma 3. *If r.v.-s (ϵ_j) satisfy conditions (A_1) , (A_2) , then, as $n \rightarrow \infty$,*

$$b_n(R_n - 2a_n) \xrightarrow{D} \zeta + \zeta', \tag{30}$$

$$2b_n Q_n \xrightarrow{D} \zeta - \zeta', \tag{31}$$

where ζ and ζ' are independent r.v.-s and have d.f. G .

In fact, this lemma is contained in Theorem 2.9.2 of the book [4] (see also Theorem 2.10 in [9]).

Equality (17) of Theorem 3 follows immediately from relation (30) of Lemma 3.

Similarly, from the asymptotic relation (31) and Eq. (16) we obtain (18) applying once more the Cramer–Wold argument. \square

Remark3 follows directly from Theorem 3. Indeed, let $k < q$, and let there exist a nonsingular submatrix $\tilde{V} \subset V$,

$$\tilde{V} = \begin{pmatrix} v_{1i_1} & \dots & v_{1i_k} \\ \dots & \dots & \dots \\ v_{ki_1} & \dots & v_{ki_k} \end{pmatrix}.$$

Choosing in LPP (21) from Theorem 1, $d_i = 0$ for all $i \neq i_1, i_2, \dots, i_k$ (i.e., taking $\tau_i = \theta_i$ for such indices i), we pass to the problem (29). It remains to apply Eq. (15) of Theorem 3.

Remark 6. Using the notation $\bar{\zeta} - \bar{\zeta}' = (\zeta_1 - \zeta'_1, \dots, \zeta_q - \zeta'_q)^T$, the coordinatewise relation (18) of Theorem 3 can be rewritten in the equivalent vector form

$$2b_n(\hat{\theta} - \theta) \xrightarrow{D} V^{-1}(\bar{\zeta} - \bar{\zeta}') \quad \text{as } n \rightarrow \infty. \tag{32}$$

If $\text{Var } \zeta = \sigma_G^2$ of r.v. ζ having d.f.G exists, then the covariance matrix of the limiting distribution in (32) is $C_G = 2\sigma_G^2(V^T V)^{-1}$.

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