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Tempered Hermite process

Farzad Sabzikar

Department of Statistics, Iowa State University, Ames, IA 50010, USA

sabzikar@iastate.edu (F. Sabzikar)

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Abstract A tempered Hermite process modifies the power law kernel in the time domain representation of a Hermite process by multiplying an exponential tempering factor $\lambda > 0$ such that the process is well defined for Hurst parameter $H > \frac{1}{2}$. A tempered Hermite process is the weak convergence limit of a certain discrete chaos process.

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1 Introduction

The Hermite processes of order $k = 1, 2, \ldots$ are defined as multiple Wiener–Itô integrals

$$Z_H^k(t) := \int_{\mathbb{R}^k}^{\prime} \int_0^t \left(\prod_{i=1}^{k} (s - y_i)_+^{d-1} \right) ds\, B(dy_1) \ldots B(dy_k), \tag{1}$$

where $d = \frac{1}{2} - \frac{1-H}{k} \in (\frac{1}{2} - \frac{1}{2k}, \frac{1}{2})$ and $\frac{1}{2} < H < 1$ (the prime $'$ on the integral sign shows that one does not integrate on the diagonals $x_i = x_j$, $i \neq j$). They are self-similar processes with stationary increments (see [8, 26]).

In this paper, we introduce a new class of stochastic processes, which we call tempered Hermite processes. Tempered Hermite processes modify the kernel of Z_H^k by multiplying an exponential tempering factor $\lambda > 0$ such that they are well defined for Hurst parameter $H > \frac{1}{2}$. Tempered Hermite processes are not self-similar processes, but they have a scaling property, involving both the time scale and the tempering parameter. The scaling property enable us to show that the tempered Hermite processes are the weak convergence limits of certain discrete chaos processes.

The paper is organized as follows. In Section 2, we define tempered Hermite processes and derive some their basic properties. In Section 3, we present our main result on the weak convergence to tempered Hermite processes.

2 Tempered Hermite process

Let $B = \{B(t), t \in \mathbb{R}\}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $|t|$ for all $t \in \mathbb{R}$. Then the Wiener–Itô integrals

$$I_k(f) := \int_{\mathbb{R}^k}' f(x_1, \ldots, x_k) B(dx_1) \ldots B(dx_k)$$

are defined for all functions $f \in L^2(\mathbb{R}^k)$. The prime $'$ on the integral sign shows that one does not integrate on the diagonals $x_i = x_j$, $i \neq j$. See, for example, [12, Chapter 4].

Definition 1. Let $H > \frac{1}{2}$ and $\lambda > 0$. The process

$$Z_{H,\lambda}^k(t) := \int_{\mathbb{R}^k}' \int_0^t \prod_{i=1}^k \left((s - y_i)_+^{d-1} e^{-\lambda (s-y_i)_+} \right) ds\, B(dy_1) \ldots B(dy_k), \tag{2}$$

where $(x)_+ = xI(x > 0)$ and $d = \frac{1}{2} - \frac{1-H}{k} \in (\frac{1}{2} - \frac{1}{2k}, \infty)$, is called a tempered Hermite process of order k.

The next lemma shows that $Z_{H,\lambda}^k(t)$, given by (2), is well defined for any $t \geq 0$.

Lemma 1. *The function*

$$h_t(y_1, \ldots, y_k) := \int_0^t \prod_{i=1}^k (s - y_i)_+^{d-1} e^{-\lambda (s-y_i)_+} ds \tag{3}$$

is well defined in $L^2(\mathbb{R}^k)$ for any $H > \frac{1}{2}$ and $\lambda > 0$.

Proof. To show that $h_t(y_1, \ldots, y_k)$ is in $L^2(\mathbb{R}^k)$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^k} h_t(y_1, \ldots, y_k)^2\, dy_1 \ldots dy_k \\
&= \int_{\mathbb{R}^k} \Bigg[\int_0^t \int_0^t \prod_{i=1}^k (s_1 - y_i)_+^{d-1} e^{-\lambda (s_1 - y_i)_+} (s_2 - y_i)_+^{d-1} \\
&\quad \times e^{-\lambda (s_2 - y_i)_+}\, ds_1\, ds_2 \Bigg] dy_1 \ldots dy_k \\
&= 2 \int_0^t ds_1 \int_{s_1}^t ds_2 \Bigg[\int_{\mathbb{R}^k} \prod_{i=1}^k (s_1 - y_i)_+^{d-1} e^{-\lambda (s_1 - y_i)_+} (s_2 - y_i)_+^{d-1} \\
&\quad \times e^{-\lambda (s_2 - y_i)_+}\, dy_1 \ldots dy_k \Bigg]
\end{aligned}$$

$$= 2\int_0^t ds \int_0^{t-s} du \left[\int_{\mathbb{R}_+^k} \prod_{i=1}^k w_i^{d-1} e^{-\lambda w_i} (w_i + u)^{d-1} e^{-\lambda(w_i+u)}\, dw_1 \dots dw_k\right]$$

$(s = s_1, u = s_2 - s_1, w_i = s_1 - y_i)$

$$= 2\int_0^t ds \int_0^{t-s} e^{-\lambda uk}\, du \left[\int_{\mathbb{R}_+} w^{d-1}(w+u)^{d-1} e^{-2\lambda w}\, dw\right]^k$$

$$= 2\int_0^t ds \int_0^{t-s} e^{-\lambda uk} u^{k(2d-1)}\, du \left[\int_{\mathbb{R}_+} x^{d-1}(x+1)^{d-1} e^{-2\lambda ux}\, dx\right]^k$$

$$= 2\int_0^t ds \int_0^{t-s} e^{-\lambda uk} u^{k(2d-1)}\, du \left[\frac{\Gamma(d)}{\sqrt{\pi}}\left(\frac{1}{2\lambda u}\right)^{d-\frac{1}{2}} e^{\lambda u} K_{\frac{1}{2}-d}(\lambda u)\right]^k$$

$$= 2\left[\frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-\frac{1}{2}}}\right]^k \int_0^t ds \int_0^{t-s} \left[u^{d-\frac{1}{2}} K_{\frac{1}{2}-d}(\lambda u)\right]^k du$$

$$= 2\left[\frac{\Gamma(d)}{\sqrt{\pi}2^{d-\frac{1}{2}}\lambda^{2d-1}}\right]^k \int_0^t ds \int_0^{\lambda(t-s)} \left[z^{d-\frac{1}{2}} K_{\frac{1}{2}-d}(z)\right]^k dz, \tag{4}$$

where we applied the standard integral formula [13, p. 344]

$$\int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}} \Gamma(\nu) K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right) \tag{5}$$

for $|\arg\beta| < \pi$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$. Here $K_\nu(x)$ is the modified Bessel function of the second kind (see [1, Chapter 9]). Next, we need to show that

$$\int_0^t ds \int_0^{\lambda(t-s)} \left[z^{d-\frac{1}{2}} K_{\frac{1}{2}-d}(z)\right]^k dz$$

is finite for $d > \frac{1}{2} - \frac{1}{2k}$ (equivalently, for $H > \frac{1}{2}$). First, suppose $\frac{1}{2} - \frac{1}{2k} < d < \frac{1}{2}$ (or $\frac{1}{2} < H < 1$). Since $k_\nu(z) < z^{-\nu}2^{\nu-1}\Gamma(\nu)$ for $z > 0$ (Theorem 3.1 in [11]), we have

$$\begin{aligned}
&\int_0^t ds \int_0^{\lambda(t-s)} \left[z^{d-\frac{1}{2}} K_{\frac{1}{2}-d}(z)\right]^k dz \\
&\quad\le \left[2^{-(\frac{1}{2}+d)}\Gamma\left(\frac{1}{2}-d\right)\right]^k \int_0^t ds \int_0^{\lambda(t-s)} z^{k(2d-1)}\, dz \\
&\quad= \frac{[\lambda^{2d-1}2^{-(\frac{1}{2}+d)}\Gamma(\frac{1}{2}-d)]^k\lambda}{(k(2d-1)+1)(k(2d-1)+2)} t^{2kd-k+2},
\end{aligned} \tag{6}$$

which is finite, and, consequently, from (4) and (6) we get

$$\int_{\mathbb{R}^k} h_t(y_1,\dots,y_k)^2\, dy_1 \dots dy_k < \frac{2\lambda[\frac{\Gamma(d)\Gamma(\frac{1}{2}-d)}{\sqrt{\pi}2^{2d}}]^k}{(k(2d-1)+1)(k(2d-1)+2)} t^{2kd-k+2}$$

for $\frac{1}{2} - \frac{1}{2k} < d < \frac{1}{2}$. Next, suppose $d > \frac{1}{2}$ (equivalently $H > 1$). In this case,

$$\int_0^t ds \int_0^{\lambda(t-s)} \left[z^{d-\frac{1}{2}} K_{\frac{1}{2}-d}(z)\right]^k dz = \int_0^t ds \int_0^{\lambda(t-s)} \left[z^{d-\frac{1}{2}} K_{d-\frac{1}{2}}(z)\right]^k dz$$
$$\leq \int_0^t ds \int_0^{\lambda(t-s)} \left[2^{d-\frac{3}{2}} \Gamma\left(d-\frac{1}{2}\right)\right]^k dz \leq \frac{\lambda[2^{d-\frac{3}{2}}\Gamma(d-\frac{1}{2})]^k}{2} t^2, \tag{7}$$

where we applied the fact that $K_\nu(z) = K_{-\nu}(z)$ and $k_\nu(z) < z^{-\nu}2^{\nu-1}\Gamma(\nu)$ for $z > 0$. Hence, from (4) and (7) it follows that

$$\int_{\mathbb{R}^k} h_t(y_1, \ldots, y_k)^2 \, dy_1 \ldots dy_k < \left[\frac{\Gamma(d)\Gamma(d-\frac{1}{2})}{2\sqrt{\pi}\lambda^{2d-1}}\right]^k \lambda \, t^2$$

for $d > \frac{1}{2}$. Finally, let $d = \frac{1}{2}$ (equivalently, $H = 1$). Consider

$$\int_0^{\lambda(t-s)} \left(K_0(z)\right)^k dz = \int_0^{\eta} \left(K_0(z)\right)^k dz + \int_\eta^{\lambda(t-s)} \left(K_0(z)\right)^k dz$$
$$:= I_1 + I_2. \tag{8}$$

Since $K_0(z) \sim -\log(z)$ as $z \to 0$ (see [1, Eq. 9.6.8], we have

$$I_1 = \int_0^{\eta} \left(\frac{K_0(z)}{\log(z)} \log(z)\right)^k dz \leq (1+\epsilon)^k \int_0^{\eta} \left(-\log(z)\right)^k dz$$
$$= (1+\epsilon)^k \int_{-\log(\eta)}^{+\infty} w^k e^{-w} \, dz \leq (1+\epsilon)^k \int_0^{+\infty} w^k e^{-w} \, dz$$
$$= (1+\epsilon)^k \Gamma(k+1). \tag{9}$$

Now, we find an upper bound for I_2. It can be shown that $\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x}$ for $0 < x < y$ and an arbitrary real number ν (see [4]). Therefore,

$$I_2 = \int_\eta^{\lambda(t-s)} \left(K_0(z)\right)^k dz < \int_\eta^{\lambda(t-s)} \left(K_0(\eta)e^{\eta-z}\right)^k dz \left[K_0(\eta)e^{\eta}\right]^k \left(\lambda(t-s)-\eta\right). \tag{10}$$

From (8), (9), and (10) we can see that

$$\int_0^{\lambda(t-s)} \left(K_0(z)\right)^k dz < (1+\epsilon)^k \Gamma(k+1) + \left[K_0(\eta)e^{\eta}\right]^k \left(\lambda(t-s)-\eta\right)$$

and hence

$$\int_0^t ds \int_0^{\lambda(t-s)} \left[K_{\frac{1}{2}-d}(z)\right]^k dz < \left((1+\epsilon)^k \Gamma(k+1)\right)t \; + \; \left[K_0(\eta)e^{\eta}\right]^k \left(\frac{\lambda t^2}{2} - \eta t\right) \tag{11}$$

for $\epsilon > 0$, and this shows that

$$\int_{\mathbb{R}^k} h_t(y_1, \ldots, y_k)^2 \, dy_1 \ldots dy_k$$
$$< 2\left[\frac{\Gamma(d)}{\sqrt{\pi}\lambda^{2d-1}} 2^{d-\frac{1}{2}}\right]^k \left[\left((1+\epsilon)^k \Gamma(k+1)\right)t + \left[K_0(\eta)e^{\eta}\right]^k \left(\frac{\lambda t^2}{2} - \eta t\right)\right]$$

for $d = \frac{1}{2}$ ($H = 1$), which completes the proof. □

The next result shows that although a tempered Hermite process is not a self-similar process, it does have a nice scaling property. Here the symbol $\triangleq$ indicates the equivalence of finite-dimensional distributions.

Proposition 1. *The process $Z^k_{H,\lambda}$ given by (2) has stationary increments such that*

$$\big\{Z^k_{H,\lambda}(ct)\big\}_{t\in\mathbb{R}} \triangleq \big\{c^H Z^k_{H,c\lambda}(t)\big\}_{t\in\mathbb{R}} \tag{12}$$

for any scale factor $c > 0$.

Proof. Since $B(dy)$ has the control measure $m(dy) = \sigma^2\,dy$, the random measure $B(c\,dy)$ has the control measure $c^{1/2}\sigma^2\,dy$. Given t_j, $j = 1,\dots,n$, by the change of variables $s = cs'$ and $y_i = cy_i^{'}$ for $i = 1\dots,k$ we have

$$\begin{aligned}
Z^k_{H,\lambda}(ct_j) &= \int_{\mathbb{R}^k}\int_0^{ct_j}\Bigg(\prod_{i=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(s-y_i)_+}\Bigg)\,ds\,B(dy_1)\dots B(dy_k)\\
&= \int_{\mathbb{R}^k}\int_0^{t_j}\Bigg(\prod_{i=1}^k \big(cs'-cy_i^{'}\big)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(cs'-cy_i^{'})_+}\Bigg) c\,ds'\,B\big(cdy_1'\big)\dots B\big(cdy_k'\big)\\
&\triangleq c^H\int_{\mathbb{R}}\int_0^{t_j}\big((s'-y')_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda c(s'-y')_+}\big)\,ds'\,B\big(dy_1'\big)\dots B\big(dy_k'\big)\\
&= c^H Z^k_{H,c\lambda}(t_j),
\end{aligned}$$

so that (12) holds. Suppose now that $s_j < t_j$ and change the variables $x = x' + s_j$, $y_i = s_j + y_i'$ (for $j = 1,\dots,n$) to get

$$\begin{aligned}
&\big(Z^k_{H,\lambda}(t_j) - Z^k_{H,\lambda}(s_j)\big)\\
&\quad= \int_{\mathbb{R}^k}\int_{s_j}^{t_j}\Bigg(\prod_{i=1}^k (x-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(x-y_i)_+}\Bigg)\,dx\,B(dy_1)\dots B(dy_k)\\
&\quad= \int_{\mathbb{R}^k}\int_0^{t_j-s_j}\Bigg(\prod_{i=1}^k \big(x'+s_j-y_i\big)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(x'+s_j-y_i)_+}\Bigg)\,dx'\,B(dy_1)\dots B(dy_k)\\
&\quad\triangleq \int_{\mathbb{R}^k}\int_0^{t_j-s_j}\Bigg(\prod_{i=1}^k \big(x_i'-y_i'\big)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(x_i'-y_i')_+}\Bigg)\,dx'\,B\big(dy_1'\big)\dots B\big(dy_k'\big)\\
&\quad= Z^k_{H,\lambda}(t_j-s_j),
\end{aligned}$$

which shows that a tempered Hermite process of order k has stationary increments. □

As a consequence of Lemma 1 and Proposition 1, we get the following:

Proposition 2. *The stochastic process $Z^k_{H,\lambda}(t)$ has a continuous version.*

Proof. According to the proof of Lemma 1,

$$\mathbb{E}\big|Z^k_{H,\lambda}(t) - Z^k_{H,\lambda}(s)\big|^2 \le \begin{cases} c_1|t-s|^{2H} & \frac{1}{2} < H < 1,\\ c_2|t-s|^2 & H > 1,\end{cases} \tag{13}$$

where c_1 and c_2 are some constants. Kolmogorov's continuity criterion states that a stochastic process $X(t)$ has a continuous version if there exist some positive constants p, β, and c such that

$$\mathbb{E}\big|X(t) - X(s)\big|^p \le c|t-s|^{1+\beta}. \tag{14}$$

Apply (14) for tempered Hermite process $Z^k_{H,\lambda}(t)$ by taking $p = 1$, $\beta = \min\{1, 2H-1\}$, and $c = \min\{c_1, c_2\}$ to get the desired result. □

We next compute the covariance function of $Z^k_{H,\lambda}(t)$. Unlike Hermite processes, the covariance function of a tempered Hermite process is different for different $k \ge 1$.

Proposition 3. *The process $Z^k_{H,\lambda}$ given by (2) has the covariance function*

$$R(t,s) = 2\Big[\frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-\frac{1}{2}}}\Big]^k \int_0^t\int_0^s \big[|u-v|^{d-\frac{1}{2}} K_{\frac{1}{2}-d}\big(\lambda|u-v|\big)\big]^k \,dv\,du$$

for $\lambda > 0$ and $d > \frac{1}{2} - \frac{1}{2k}$ (equivalently, $H > \frac{1}{2}$).

Proof. By applying the Fubini theorem and the isometry of multiple Wiener–Itô integrals we have

$$\begin{aligned}
R(t,s) &= 2\int_{\mathbb{R}^k}\bigg(\int_0^t\int_0^s \prod_{i=1}^k (u-y_i)_+^{d-1}(v-y_i)_+^{d-1} \\
&\qquad \times e^{-\lambda(u-y_i)_+}e^{-\lambda(v-y_i)_+}\,dv\,du\bigg)\,dy_1\ldots dy_k \\
&= 2\int_0^t\int_0^s\int_{\mathbb{R}^k}\bigg[\prod_{i=1}^k (u-y_i)_+^{d-1}(v-y_i)_+^{d-1} \\
&\qquad \times e^{-\lambda(u-y_i)_+}e^{-\lambda(v-y_i)_+}\,dy_1\ldots dy_k\bigg]\,dv\,du \\
&= 2\int_0^t\int_0^s\Big[\int_{\mathbb{R}}(u-y)_+^{d-1}(v-y)_+^{d-1}e^{-\lambda(u-y)_+}e^{-\lambda(v-y)_+}\,dy\Big]^k\,dv\,du \\
&= 2\int_0^t\int_0^s\Big[\int_{-\infty}^{\min(u,v)}(u-y)^{d-1}(v-y)^{d-1}e^{-\lambda(u-y)}e^{-\lambda(v-y)}\,dy\Big]^k\,dv\,du \\
&= 2\int_0^t\int_0^s\Big[\int_0^{+\infty} w^{d-1}\big(|u-v|+w\big)^{d-1}e^{-\lambda w}e^{-\lambda(|u-v|+w)}\,dw\Big]^k\,dv\,du \\
&= 2\int_0^t\int_0^s e^{-\lambda k|u-v|}\,|u-v|^{k(2d-1)} \\
&\qquad \times\Big[\int_0^{+\infty} x^{-(\frac{1}{2}+\frac{1-H}{k})}(x+1)^{-(\frac{1}{2}+\frac{1-H}{k})}e^{-2\lambda|u-v|x}\,dx\Big]^k\,dv\,du
\end{aligned}$$

$$= 2\int_0^t \int_0^s e^{-\lambda k|u-v|} \, |u-v|^{k(2d-1)}$$

$$\times \left[\frac{\Gamma(d)}{\sqrt{\pi}} \left(\frac{1}{2\lambda|u-v|} \right)^{d-\frac{1}{2}} e^{\lambda|u-v|} K_{\frac{1}{2}-d}\big(\lambda|u-v|\big) \right]^k dv\, du$$

$$= 2\left[\frac{\Gamma(d)}{\sqrt{\pi}(2\lambda)^{d-\frac{1}{2}}} \right]^k \int_0^t \int_0^s \left[|u-v|^{d-\frac{1}{2}} K_{\frac{1}{2}-d}\big(\lambda|u-v|\big) \right]^k dv\, du$$

for any $H > \frac{1}{2}$ and $\lambda > 0$, and hence we get the desired result. □

Let $\widehat{B}_1$ and $\widehat{B}_2$ be independent Gaussian random measures with $\widehat{B}_1(A) = \widehat{B}_1(-A)$, $\widehat{B}_2(A) = -\widehat{B}_2(-A)$, and $\mathbb{E}[(\widehat{B}_i(A))^2] = m(A)/2$, where $m(dx) = \sigma^2\, dx$, and define the complex-valued Gaussian random measure $\widehat{B} = \widehat{B}_1 + i\widehat{B}_2$.

Proposition 4. *Let $H > \frac{1}{2}$ and $\lambda > 0$. The process $Z^k_{H,\lambda}$ given by (2) has the spectral domain representation*

$$Z^k_{H,\lambda}(t) = C_{H,k} \int''_{\mathbb{R}^k} \frac{e^{it(\omega_1+\cdots+\omega_k)} - 1}{i(\omega_1 + \cdots + \omega_k)} \times \prod_{j=1}^{k} (\lambda + i\omega_j)^{-(\frac{1}{2}-\frac{1-H}{k})} \widehat{B}(d\omega_1) \ldots \widehat{B}(d\omega_k), \tag{15}$$

where $\widehat{B}(\cdot)$ is a complex-valued Gaussian random measure, and $C_{H,k} = (\frac{\Gamma(\frac{1}{2}-\frac{1-H}{k})}{\sqrt{2\pi}})^k$ is a constant depending on H and k. The double prime $''$ on the integral indicates that one does not integrate on the diagonals $\omega_i = \omega_j$, $i \neq j$.

Proof. We first observe that

$$h_t(y_1, \ldots, y_k) = \int_0^t \prod_{j=1}^{k} (s - y_j)_+^{d-1} e^{-\lambda(s-y_j)_+}\, ds \tag{16}$$

has the Fourier transform

$$\widehat{h_t}(\omega_1, \ldots, \omega_k)$$

$$= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{\mathbb{R}^k} e^{i\sum_{j=1}^k \omega_j y_j} \int_0^t \prod_{j=1}^{k} (s - y_j)_+^{d-1} e^{-\lambda(s-y_j)_+}\, ds\, dy_1 \ldots dy_k$$

$$= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{\mathbb{R}^k} \int_0^t e^{i\sum_{j=1}^k \omega_j(s-u_j)} \prod_{j=1}^{k} (u_j)_+^{d-1} e^{-\lambda(u_j)_+}\, ds\, du_1 \ldots du_k$$

$$= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_0^t \int_{\mathbb{R}^k} e^{is\sum_{j=1}^k \omega_j} \prod_{j=1}^{k} (u_j)_+^{d-1} e^{-(\lambda+i\omega_j)u_j}\, du_1 \ldots du_k\, ds$$

$$= \left[\frac{\Gamma(d)}{\sqrt{2\pi}} \right]^k \frac{e^{it(\omega_1+\cdots+\omega_k)} - 1}{i(\omega_1 + \cdots + \omega_k)} \prod_{j=1}^{k} (\lambda + i\omega_j)^{-d}$$

$$= \left[\frac{\Gamma(\frac{1}{2} - \frac{1-H}{k})}{\sqrt{2\pi}}\right]^k \frac{e^{it(\omega_1+\cdots+\omega_k)} - 1}{i(\omega_1 + \cdots + \omega_k)} \prod_{j=1}^{k} (\lambda + i\omega_j)^{-(\frac{1}{2}-\frac{1-H}{k})},$$

using the well-known formula for the characteristic function of the gamma density. Then (2), together with Proposition 9.3.1 in [21], implies that

$$\begin{aligned} Z^k_{H,\lambda}(t) &= \int'_{\mathbb{R}^k} h_t(y_1, \ldots, y_k) B(dy_1) \ldots B(dy_k) \\ &\triangleq \int''_{\mathbb{R}^k} \widehat{h_t}(\omega_1, \ldots, \omega_k) \widehat{B}(d\omega_1) \ldots \widehat{B}(d\omega_k) \\ &= C_{H,k} \int''_{\mathbb{R}^k} \frac{e^{it(\omega_1+\cdots+\omega_k)} - 1}{i(\omega_1 + \cdots + \omega_k)} \prod_{j=1}^{k} (\lambda + i\omega_j)^{-(\frac{1}{2}-\frac{1-H}{k})} \widehat{B}(d\omega_1) \ldots \widehat{B}(d\omega_k), \end{aligned}$$

which is equivalent to (15). □

3 Limit theorem

In this section, we show that the process $Z^k_{H,\lambda}(t)$ is the weak convergence limit of a certain discrete chaos process. Our approach follows the seminal work of Bai and Taqqu [3]. When $k = 1$ and $\lambda > 0$, the discrete process $Y^{\lambda,k}(n)$, (18), is a time series that is useful to model turbulence [20, 24]. When $k = 1$ and $\lambda = 0$, Davydov [7] (see also Giraitis et al. [12, p. 276] and Whitt [27, Theorem 4.6.1]) established the corresponding invariance principle for $Y^{\lambda,k}(n)$, where the limit involves a fractional Brownian motion. When $k > 1$ and $\lambda = 0$, Taqqu [26] showed that the weak convergence limit of $Y^{\lambda,k}(n)$ is the Hermite process (1).

The following proposition gives a powerful tool for proving the result of this section.

Proposition 5. *Let*

$$Q_k(g_N) := \sum'_{(j_1,\ldots,j_k)\in\mathbb{Z}^k} g_N(j_1, \ldots, j_k)\varepsilon_{j_1} \ldots \varepsilon_{j_k} \tag{17}$$

for $N = 1, 2, \ldots$, where $g_N \in L^2(\mathbb{Z}^k)$ for $k \geq 1$, and $\{\varepsilon_n\}$ is an i.i.d. sequence with mean zero and variance 1. Assume that, for some $f \in L^2(\mathbb{R}^k)$,

$$\int_{\mathbb{R}^k} \left|\tilde{g}_N(u_1, \ldots, u_k) - f(u_1, \ldots, u_k)\right|^2 du_1 \ldots du_k \to 0, \quad \text{as } N \to \infty,$$

where

$$\tilde{g}_N(u_1, \ldots, u_k) := N^{\frac{k}{2}} g_N\big([u_1 N] + c_1, \ldots, [u_k N] + c_k\big), \quad (c_1, \ldots, c_k) \in \mathbb{Z}^k.$$

Then

$$Q_k(g_N) \xrightarrow{f.d.d.} \int_{\mathbb{R}^k} f(u_1, \ldots, u_k) B(du_1) \ldots B(du_k)$$

as $N \to \infty$.

Proof. See Proposition 4.1 in [3] and also Corollary 4.7.1 in [12]. □

Define the discrete chaos process

$$Y^{\lambda,k}(n) := \sum_{(i_1,i_2,\dots,i_k)\in\mathbb{Z}^k}^{\prime} C^{\lambda}(i_1,i_2,\dots,i_k)\varepsilon_{n-i_1}\dots\varepsilon_{n-i_k}, \tag{18}$$

where the prime $'$ indicates exclusion of the diagonals $i_p = i_q$, $p \neq q$, $\{\varepsilon_n\}$ is as before, and

$$C^{\lambda}(i_1,i_2,\dots,i_k) = \prod_{j=1}^{k}(i_j)_+^{d-1}e^{-\lambda(i_j)_+} \tag{19}$$

for $d \in (\frac{1}{2} - \frac{1}{2k}, \infty)$ and $\lambda > 0$. Now, consider

$$S_N^{\lambda}(t) = \sum_{n=1}^{[Nt]} Y^{\lambda,k}(n), \quad 0 \le t \le 1.$$

Theorem 1. *Let $Y^{\lambda,k}(n)$ be the discrete chaos process given by* (18)*. Then*

$$\frac{1}{N^H} S_N^{\frac{\lambda}{N}}(t) \Rightarrow Z_{H,\lambda}^{k}(t), \tag{20}$$

where $\Rightarrow$ means weak convergence in the Skorokhod space $D[0,1]$ with uniform metric, $Z_{H,\lambda}^k(t)$ is the tempered Hermite process in (2)*, and $H = 1 + kd - \frac{k}{2}$.*

Remark 1. The Lamperti's theorem [15] states that if

$$\frac{1}{d(N)}\sum_{k=1}^{[Nt]} Y_k \stackrel{f.d.d.}{\longrightarrow} Z(t)$$

and $d(N) \to \infty$ as $N \to \infty$, where $\{Y_k\}$ is stationary, then $\{Z(t)\}_{t\ge0}$ is self-similar with stationary increments ($\stackrel{f.d.d.}{\longrightarrow}$ means the convergence of finite-dimensional distributions). In our case, since the stationary processes $\{Y_k^{\frac{\lambda}{N}}\}$ depend on N through the parameter λ, the limit process $\{Z_{H,\lambda}^k(t)\}_{t\ge0}$ need not be a self-similar process. Therefore, the result of Theorem 1 does not contradict the Lamperti theorem.

Proof of Theorem 1. First, we show that

$$\begin{aligned}\frac{1}{N^H} S_N^{\frac{\lambda}{N}}(t) &= \frac{1}{N^H}\sum_{n=1}^{[Nt]} Y^{\frac{\lambda}{N},k}(n)\\ &= \sum_{(i_1,\dots,i_k)\in\mathbb{Z}^k} \frac{1}{N^H}\sum_{n=1}^{[Nt]} C^{\frac{\lambda}{N}}(n-i_1,\dots,n-i_k)\varepsilon_{i_1}\dots\varepsilon_{i_k}\\ &= Q_k(h_{t,N}) \stackrel{f.d.d.}{\longrightarrow} Z_{H,\lambda}^k(t) \quad \text{as } N\to\infty,\end{aligned} \tag{21}$$

where

$$h_{t,N}(i_1,\dots,i_k) := \frac{1}{N^H}\sum_{n=1}^{[Nt]} C^{\frac{\lambda}{N}}(n-i_1,\dots,n-i_k),$$

and $Q_k(\cdot)$ is defined by (17). In order to show (21), we just need to check that

$$\big\|\tilde{h}_{t,N}(y_1,\ldots,y_k) - h_t(y_1,\ldots,y_k)\big\|_{L^2(\mathbb{R}^k)} \to 0 \tag{22}$$

as $N \to \infty$, where

$$\begin{aligned}\tilde{h}_{t,N}(y_1,\ldots,y_k) &:= N^{\frac{k}{2}} h_{t,N}\big([Ny_1]+1,\ldots,[Ny_k]+1\big)\\ &= \frac{N^{\frac{k}{2}}}{N^H}\sum_{n=1}^{[Nt]} C^{\frac{\lambda}{N}}\big(n-[Ny_1]-1,\ldots,n-[Ny_k]-1\big),\end{aligned}$$

and $h_t(y_1,\ldots,y_k)$ is given by (3). Write

$$\begin{aligned}\tilde{h}_{t,N}(y_1,\ldots,y_k) &= \frac{N^{\frac{k}{2}}}{N^H}\sum_{n=1}^{[Nt]} C^{\frac{\lambda}{N}}\big(n-[Ny_1]-1,\ldots,n-[Ny_k]-1\big)\\ &= \frac{1}{N^{1+kd-k}}\sum_{n=1}^{[Nt]}\prod_{i=1}^{k}\big(n-[Ny_i]-1\big)_+^{d-1} e^{-\frac{\lambda}{N}(n-[Ny_i]-1)_+}\\ &= \frac{1}{N}\sum_{n=1}^{[Nt]}\prod_{i=1}^{k}\left(\frac{n-[Ny_i]-1}{N}\right)_+^{d-1} e^{-\lambda(\frac{n-[Ny_i]-1}{N})_+}\\ &= \int_0^t \prod_{i=1}^{k}\left(\frac{[Ns]-[Ny_i]}{N}\right)_+^{d-1} e^{-\lambda(\frac{[Ns]-[Ny_i]}{N})_+}\,ds.\end{aligned}$$

Let $d=1$. In this case,

$$\left(\frac{[Ns]-[Ny]}{N}\right)_+^{d-1} e^{-\lambda(\frac{[Ns]-[Ny]}{N})_+} = e^{-\lambda(\frac{[Ns]-[Ny]}{N})_+} \le e^{-\lambda(s-y)_+}e^{\frac{\lambda}{N}}$$

for all $N \ge 1$, and hence

$$\begin{aligned}\left|\prod_{i=1}^{k} e^{-\lambda(\frac{[Ns]-[Ny_i]}{N})_+}\right| &\le e^{\lambda k}\prod_{i=1}^{k} e^{-\lambda(s-y_i)_+}\\ &=: g_1(s-y_1,\ldots,s-y_k).\end{aligned}$$

Next, consider $0<d<1$. Since $[Ns]-[Ny] > Ns-Ny-1$, we get

$$\left(\frac{[Ns]-[Ny]}{N}\right)_+^{d-1} < \left(\frac{Ns-Ny-1}{N}\right)_+^{d-1} \le (s-y-1)_+^{d-1}$$

for all $N \ge 1$, and hence

$$\begin{aligned}\left|\prod_{i=1}^{k}\left(\frac{[Ns]-[Ny_i]}{N}\right)_+^{d-1} e^{-\lambda(\frac{[Ns]-[Ny_i]}{N})_+}\right| &< \prod_{i=1}^{k}(s-y_i-1)_+^{d-1} e^{-\lambda(s-y_i-1)_+}\\ &=: g_2(s-y_1,\ldots,s-y_k).\end{aligned}$$

Finally, suppose that $d > 1$. Since $[Ns] - [Ny] < Ns - Ny + 1$, we get

$$\left(\frac{[Ns]-[Ny]}{N}\right)_+^{d-1} < \left(\frac{Ns - Ny + 1}{N}\right)_+^{d-1} \le (s - y + 1)_+^{d-1}$$

for all $N \ge 1$, and hence

$$\left|\prod_{i=1}^{k}\left(\frac{[Ns]-[Ny_i]}{N}\right)_+^{d-1} e^{-\lambda(\frac{[Ns]-[Ny_i]}{N})_+}\right| < \prod_{i=1}^{k}(s - y_i + 1)_+^{d-1} e^{-\lambda(s-y_i-1)_+}$$
$$=: g_3(s - y_1, \dots, s - y_k).$$

By the similar argument of Lemma 1, we can verify that

$$\int_{\mathbb{R}^k}' \left(\int_0^t g_i(s - y_1, \dots, s - y_k)\, ds\right)^2 dy_1, \dots, dy_k < \infty$$

for $i = 1, 2, 3$. On the other hand, since $C^\lambda(i_1, \dots, i_k)$ is continuous a.e., $C^\lambda(\frac{[Ns]-[Ny_1]}{N}, \dots, \frac{[Ns]-[Ny_k]}{N})$ converges a.e. to $C^\lambda(s - y_1, \dots, s - y_k)$ as $N \to \infty$. Now apply the dominated convergence theorem to get the desired result (22).

In order to show the tightness, we need to verify that

$$\mathbb{E}\left|N^{-H}\left(S_N^{\frac{\lambda}{N}}(t) - S_N^{\frac{\lambda}{N}}(s)\right)\right|^{2\gamma} \le C\left|F_n(t) - F_n(s)\right|^{2\alpha}, \quad 0 \le s < t \le 1, \tag{23}$$

where $\gamma > 0$ and $\alpha > \frac{1}{2}$ (here $\{F_n\}_{n\ge1}$ is a sequence of nondecreasing continuous functions on [0, 1] that are uniformly bounded and satisfy

$$\lim_{\delta\to0}\limsup_{n\to\infty}\omega_\delta(F_n) = 0,$$

where $\omega_\delta(F) := \sup_{|t-s|<\delta}|F(t) - F(s)|$ for $\delta > 0$). See Lemma 4.4.1 in [12] for more details. Observe that

$$\begin{aligned}
S_N^{\frac{\lambda}{N}}(t) &= \sum_{n=1}^{[Nt]} Y^{\frac{\lambda}{N},k}(n) = \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' \sum_{n=1}^{[Nt]} C^{\frac{\lambda}{N},k}(n - i_1, \dots, n - i_k)\varepsilon_{i_1}\dots\varepsilon_{i_k} \\
&= \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' \sum_{n=1}^{[Nt]}\prod_{j=1}^{k}(n - i_j)_+^{d-1} e^{-\frac{\lambda}{N}(n-i_j)_+}\varepsilon_{i_1}\dots\varepsilon_{i_k} \\
&= \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' N^{kd-k+1}\left[\frac{1}{N}\sum_{n=1}^{[Nt]}\prod_{j=1}^{k}\left(\frac{n - i_j}{N}\right)_+^{d-1} e^{-\frac{\lambda}{N}(n-i_j)_+}\right]\varepsilon_{i_1}\dots\varepsilon_{i_k} \\
&= \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' N^{kd-k+1}\left[\int_0^t \prod_{j=1}^{k}\left(\frac{[Ny] + 1 - i_j}{N}\right)_+^{d-1}\right. \\
&\quad \left.\times\, e^{-\frac{\lambda}{N}([Ny]+1-i_j)_+}\, dy\right]\varepsilon_{i_1}\dots\varepsilon_{i_k}
\end{aligned}$$

$$= N \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' \Big[\int_0^t \prod_{j=1}^k \big([Ny]+1-i_j\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-i_j)_+}\,dy\Big]\varepsilon_{i_1}\dots\varepsilon_{i_k}.$$

Therefore, we get

$$\begin{aligned}
&\mathbb{E}\big|N^{-H}\big(S_N^{\frac{\lambda}{N}}(t)-S_N^{\frac{\lambda}{N}}(s)\big)\big|^2\\
&\quad = N^{2-2H}\\
&\qquad \times \mathbb{E}\Big|\sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' \Big[\int_0^{t-s} \prod_{j=1}^k \big([Ny]+1-i_j\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-i_j)_+}\,dy\Big]\varepsilon_{i_1}\dots\varepsilon_{i_k}\Big|^2\\
&\quad \le k!N^{2-2H} \sum_{(i_1,\dots,i_k)\in\mathbb{Z}}' \Big[\int_0^{t-s} \prod_{j=1}^k \big([Ny]+1-i_j\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-i_j)_+}\,dy\Big]^2\\
&\quad = k!N^{2-2H+k}\\
&\qquad \times \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k \big([Ny]+1-[Nx_j]\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-[Nx_j])_+}\,dy\Big]^2 dx_1\dots dx_k.
\end{aligned}$$

Now, we consider two different cases corresponding with the range of d. First, assume that $\frac{1}{2}-\frac{1}{2k} < d \le 1$ (equivalently, $\frac{1}{2} < H \le 1+\frac{k}{2}$): Since $[Ny]-[Nx_j]+1 > Ny - Nx_j$, we can write

$$\begin{aligned}
&\mathbb{E}\big|N^{-H}\big(S_N^{\frac{\lambda}{N}}(t)-S_N^{\frac{\lambda}{N}}(s)\big)\big|^2\\
&\quad \le k!N^{2-2H+k}\\
&\qquad \times \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k \big([Ny]+1-[Nx_j]\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-[Nx_j])_+}\,dy\Big]^2 dx_1\dots dx_k\\
&\quad \le k!N^{2-2H+2kd-k} \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k (y-x_j)_+^{d-1} e^{-\lambda(y-x_j)_+}\,dy\Big]^2 dx_1\dots dx_k\\
&\quad = k! \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k (y-x_j)_+^{d-1} e^{-\lambda(y-x_j)_+}\,dy\Big]^2 dx_1\dots dx_k.
\end{aligned}$$

Now, let $d > 1$. Since $Ny - Nx_j < [Ny]-[Nx_j]+1 < Ny - Nx_j + N$, we have

$$\begin{aligned}
&\mathbb{E}\big|N^{-H}\big(S_N^{\frac{\lambda}{N}}(t)-S_N^{\frac{\lambda}{N}}(s)\big)\big|^2\\
&\quad \le k!N^{2-2H+k}\\
&\qquad \times \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k \big([Ny]+1-[Nx_j]\big)_+^{d-1} e^{-\frac{\lambda}{N}([Ny]+1-[Nx_j])_+}\,dy\Big]^2 dx_1\dots dx_k\\
&\quad \le k! \int_{\mathbb{R}^k}' \Big[\int_0^{t-s} \prod_{j=1}^k (y-x_j+1)_+^{d-1} e^{-\lambda(y-x_j)_+}\,dy\Big]^2 dx_1\dots dx_k
\end{aligned}$$

$$= k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - x_j + 1)^{d-1} e^{-\lambda(y-x_j+1)} e^{\lambda} \mathbf{1}_{\{y > x_j\}} \, dy \right]^2 dx_1 \dots dx_k$$

$$= e^{2\lambda k} k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - z_j)^{d-1} e^{-\lambda(y-z_j)} \mathbf{1}_{\{y > z_j + 1\}} \, dy \right]^2 dz_1 \dots dz_k$$

$$\le e^{2\lambda k} k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - z_j)_+^{d-1} e^{-\lambda(y-z_j)_+} \, dy \right]^2 dz_1 \dots dz_k.$$

Therefore,

$$\mathbb{E}\big| N^{-H} \big(S_N^{\frac{\lambda}{N}}(t) - S_N^{\frac{\lambda}{N}}(s) \big) \big|^2$$
$$\le e^{2\lambda k} k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - z_j)_+^{d-1} e^{-\lambda(y-z_j)_+} \, dy \right]^2 dz_1 \dots dz_k$$

for any $d > \frac{1}{2} - \frac{1}{2k}$ (equivalently, $H > \frac{1}{2}$). According to the proof of Lemma 1,

$$e^{2\lambda k} k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - z_j)_+^{d-1} e^{-\lambda(y-z_j)_+} \, dy \right]^2 dz_1 \dots dz_k \le C|t-s|^{2H}$$

for $\frac{1}{2} < H < 1$ and

$$e^{2\lambda k} k! \int_{\mathbb{R}^k}' \left[\int_0^{t-s} \prod_{j=1}^{k} (y - z_j)_+^{d-1} e^{-\lambda(y-z_j)_+} \, dy \right]^2 dz_1 \dots dz_k \le C|t-s|^{2}$$

for $H > 1$. Now, it remains to apply (23) by selecting $\gamma = 1$, $\alpha = \min\{H, 1\}$, and $F_n(t) = t$ to get the desired result. □

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