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The consistent criteria of hypotheses

L. Aleksidze, M. Mumladze, Z. Zerakidze*

Gori University, Gori, Georgia

lelaaleksidze@gmail.com (L. Aleksidze), mmumladze@mail.ru (M. Mumladze),
Zura.zerakidze@mail.ru (Z. Zerakidze)

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Abstract In this paper we define the consistent criteria of hypotheses such as the probability of any kind of errors is zero for given criteria. We prove necessary and sufficient conditions for the existence of such criteria.

Keywords Consistent criteria, singular, orthogonal, weakly separable, strongly separable probability measures
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1 Basic notions and consistent criterion on hypothesis for countable family of probability measures

Let (E, S) be a measurable space with a given family of probability measures: $\{\mu_i, i \in I\}$.

Definition 1. The family $\{\mu_i,\ i \in I\}$ of probability measures is called orthogonal (singular) if μ_i and μ_j are orthogonal for each $i \neq j$.

Definition 2. The family $\{\mu_i,\ i \in I\}$ of probability measures is called separable if there exists a family of S-measurable sets $\{X_i,\ i \in I\}$ such that the relations are fulfilled:

*Corresponding author.

(1) $\forall i \in I\ \forall j \in I\ \ \mu_i(X_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$

(2) $\forall i \in I\ \forall j \in I\ \ \mathrm{card}(X_i \cap X_j) < c$ if $i \neq j$, where c denotes a power of continuum.

Definition 3. The family $\{\mu_i,\ i \in I\}$ of probability measures is called weakly separable if there exists a family of S-measurable sets $\{X_i,\ i \in I\}$ such that the relations are fulfilled:

$$\forall i \in I\ \forall j \in I \quad \mu_i(X_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Definition 4. The family $\{\mu_i,\ i \in I\}$ of probability measures is called strongly separable if there exists a disjoint family of S-measurable sets $\{X_i,\ i \in I\}$ such that the relations are fulfilled: $\forall i \in I\ \ \mu_i(X_i) = 1$.

Remark 1. A strong separability implies separability, a separability implies a weak separability, and a weak separability implies orthogonality, but not vice versa.

Example 1. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of subsets of E. Take the S-measurable sets $X_i = \{0 \leq x \leq 1,\ y = i,\ i \in [0,1]\}$ and assume that L_i are the linear Lebesgue probability measures on X_i. Then the family $\{L_i,\ i \in [0,1]\}$ is strongly separable.

Example 2. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of subsets of E. Take the S-measurable sets

$$X_i = \{(x,y) \mid 0 \leq x \leq 1,\ y = i,\ \text{if } i \in [0,1];\ x = i - 2,\ 0 \leq y \leq 1,\ \text{if } i \in [2,3]\}.$$

Let L_i be the linear Lebesgue probability measures on X_i. Then the family $\{L_i,\ i \in [0,1] \cup [2,3]\}$ is separable but not strongly separable.

Example 3. Let $E = [0,1] \times [0,1] \times [0,1]$, S be a Borel σ-algebra of subsets of E. Take the S-measurable sets:

$$\begin{aligned} X_i = \{(x,y,z) \mid\ & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ z = i,\ \text{if } i \in [0,1]; \\ & x = i - 2,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1,\ \text{if } i \in [2,3]; \\ & 0 \leq x \leq 1,\ y = i - 4,\ 0 \leq z \leq 1,\ \text{if } i \in [4,5]\}. \end{aligned}$$

Let L_i be the planar Lebesgue probability measures on X_i. Then the family $\{L_i,\ i \in [0,1] \cup [2,3] \cup [4,5]\}$ is weakly separable but not separable.

Example 4. Let $E = [0,1] \times [0,1]$, S be a Borel σ-algebra of subsets of E. Take the S-measurable sets

$$X_i = \{(x,y) \mid 0 \leq x \leq 1,\ y = i,\ i \in (0,1]\}.$$

Let L_i be the linear Lebesgue probability measures on X_i and L_0 be the planar Lebesgue probability measure on $E = [0,1] \times [0,1]$. Then the family $\{L_i,\ i \in [0,1]\}$ is orthogonal, but not weakly separable.

Definition 5. We consider the notion of Hypothesis as any assumption that defines the form of the distribution selection.

Let H be set of hypotheses and B(H) be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 6. The family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$ is said to admit a consistent criteria of a hypothesis if there exists at least one measurable map $\delta : (E, S) \to (\mathrm{H}, \mathrm{B}(\mathrm{H}))$, such that $\mu_H(\{x \mid \delta(x) = H\}) = 1$, for all $H \in \mathrm{H}$.

Definition 7. The following probability:

$$\alpha_H(\delta) = \mu_H(\{x \mid \delta(x) \neq H\})$$

is called the probability of error of the H-th kind for a given criterion δ.

Definition 8. The family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$ is said to admit a consistent criterion of any parametric function if for any real bounded measurable function $g : (\mathrm{H}, \mathrm{B}(\mathrm{H})) \to \mathbb{R}$ there exists at least one measurable function $f : (E, S) \to \mathbb{R}$ such that $\mu_H(\{x \mid f(x) = g(H)\}) = 1$, for all $H \in \mathrm{H}$.

Definition 9. The family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$is said to admit an unbiased criterion of any parametric function if for any real bounded measurable function $g : (\mathrm{H}, \mathrm{B}(\mathrm{H})) \to \mathbb{R}$ there exists at least one measurable function $\beta : (E, S) \to \mathbb{R}$, such that $\int_E \beta(x)\mu_H(dx) = g(H)$ for all $H \in \mathrm{H}$.

Remark 2. If M is a family of probability measures admitting a consistent criterion for a hypothesis, then it is clear that M is a family of probability measures which admits a consistent criterion for any parametric function and a family of probability measures which admits an unbiased criterion of any parametric function.

Theorem 1. *The family of probability measures* $\{\mu_H,\ H \in \mathrm{H}\}$ *admits a consistent criterion* δ *of a hypothesis if and only if the probability of error of all kinds is equal to zero for the criterion* δ.

Proof. *Necessity*. As the family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$ admits a consistent criterion of a hypothesis, so there exists such a measurable map $\delta : (E, S) \to (\mathrm{H}, \mathrm{B}(\mathrm{H}))$ that $\mu_H(\{x \mid \delta(x) = H\}) = 1$ for all $H \in \mathrm{H}$. It follows $\alpha_H(\delta) = \mu_H(\{x \mid \delta(x) \neq H\} = 0$.

Sufficiency. As the probability of error of all kinds is equal to zero, so $\alpha_H(\delta) = \mu_H(\{x \mid \delta(x) \neq H\} = 0$ for all $H \in \mathrm{H}$, we have

$$\{x \mid \delta(x) = H\} \cap \{x \mid \delta(x) = H'\} = \emptyset$$

for any $H \neq H'$.

On the other hand $\{x \mid \delta(x) = H\} \cup \{x \mid \delta(x) \neq H\} = E$ and $\mu_H(\{x \mid \delta(x) = H\}) = 1$, for all $H \in \mathrm{H}$.

Therefore δ is a consistent criterion of a hypothesis. The Theorem 1 is proved. □

Theorem 2. *Let* $\mathrm{H} = \{H_1, H_2, \dots, H_n, \dots\}$ *be the set of hypotheses. The family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$, $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ *admits the consistent criterion of hypotheses if and only if the family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *is strongly separable.*

Proof. *Necessity.* Since the family $\{\mu_{H_i},\ i \in \mathbb{N}\}$ admits a consistent criterion of hypotheses, then there exists a measurable map δ of the space (E, S) to $(\mathrm{H}, \mathrm{B}(\mathrm{H}))$ such that $\mu_{H_i}(x \mid \delta(x) = H_i) = 1,\ i \in \mathbb{N}$. Let $X_i = (x : \delta(x) = H_i)$, then it is obvious, that $X_i \cap X_i \neq \emptyset$ for all $i \neq j$ and $\mu_{H_i}(X_i) = 1,\ \forall i \in \mathbb{N}$. Therefore, the family of probability measures $\{\mu_{H_i},\ i \in N\}$ is strongly separable.

Sufficiency. As the family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ is strongly separable, then there exist such pairwise disjoint S-measurable sets $X_i,\ i \in \mathbb{N}$ that $\mu_{H_i}(X_i) = 1,\ \forall i \in \mathbb{N}$.

Let's define δ as such a mapping $(E, S) \to (\mathrm{H}, \mathrm{B}(\mathrm{H}))$ that $\delta(X_i) = H_i,\ i \in \mathbb{N}$. We have $\{x : \delta(x) = H_i\} = X_i$ and $\mu_{H_i}\{x : \delta(x) = H_i\} = 1,\ \forall i \in \mathbb{N}$. Therefore δ is a consistent criterion of hypotheses. The Theorem 2 is proved. □

Theorem 3. *Let* $\mathrm{H} = \{H_1, H_2, \dots, H_n, \dots\}$ *and the family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *be separable or weakly separable. Then the family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *admits a consistent criterion of hypotheses.*

Proof. Since the family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ is separable or weakly separable, then there exists a family $X_1, X_2, \dots, X_n, \dots$ of S-measures sets such that

$$\mu_{H_i}(X_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let us consider the sets:

$$\begin{aligned}
\overline{X}_1 &= X_1 - X_1 \cap \Big(\bigcup_{k \neq 1} X_k\Big) \\
\overline{X}_2 &= X_2 - X_2 \cap \Big(\bigcup_{k \neq 2} X_k\Big) \\
&\dots \\
\overline{X}_n &= X_n - X_n \cap \Big(\bigcup_{k \neq n} X_k\Big) \\
&\dots
\end{aligned}$$

It is obvious that $\{\overline{X}_1, \overline{X}_2, \dots, \overline{X}_n, \dots\}$ is a disjoint family of S-measurable sets and $\mu_{H_i}(\overline{X}_i) = 1,\ \forall i \in \mathbb{N}$. Therefore, the family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ is strongly separable and $\{\mu_{H_i},\ i \in \mathbb{N}\}$ admits a consistent criterion of hypotheses by the Theorem 1. The Theorem 3 is proved. □

Theorem 4. *Let* $\mathrm{H} = \{H_1, H_2, \dots, H_n, \dots\}$ *and the family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ *be orthogonal (singular). Then the family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *admits a consistent criterion of hypotheses.*

Proof. The singularity of probability measures implies an existence of the family $\{X_{ik}\}$ of S-measurable sets such that for any $i \neq k$ we have $\mu_{H_k}(X_{ik}) = 0$ and $\mu_{H_i}(E - X_{ik}) = 0$.

Let us consider the sets $X_i = \bigcup_{k \neq i}(E - X_{ik})$, then

$$\mu_{H_i}(X_i) = \mu_{H_i}\left(\bigcup_{k \neq i}(E - X_{ik})\right) \leq \sum_{k \neq i} \mu_{H_i}(E - X_{ik}) = 0.$$

Therefore, $\mu_{H_i}(X_i) = 0$; $\mu_{H_i}(E - X_i) = 1$. On the other hand, for $k \neq i$ we have $\mu_{H_k}(E - X_i) = 0$. This means that the family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ is weakly separable. By the Theorem 3 this family of probability measures admits a consistent criterion of hypotheses. The Theorem 4 is proved. $\square$

It follows from the Theorems 3, 4, that for the countable family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *the notions* of weakly separable, separable, orthogonal and strong separable are equivalent.

2 Consistent criteria in Banach space

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 10. A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures if:

(1) a norm can be defined on M_B so that M_B will be a Banach space with respect to this norm, and for any orthogonal measures $\mu, \nu \in M_B$ and any real number $\lambda \neq 0$ the inequality $\|\mu + \lambda\nu\| \geq \|\mu\|$ is fulfilled;

(2) if $\mu \in M_B$, $|f(x)| \leq 1$, than $\nu_f(A) = \int_A f(x)\,\mu(dx) \in M_B$, and $\|\nu_f\| \leq \|\nu\|$, where $f(x)$ is a real measurable function, $A \in S$;

(3) if $\nu_n \in M_B$, $\nu_n > 0$, $\nu_n(E) < \infty$, $n = 1, 2, \ldots$ and $\nu_n \downarrow 0$, then for any linear functional $l^* \in M_B^*$
$$\lim_{n\to\infty} l^*(\nu_n) = 0.$$

The construction of the Banach space of measures is studied in paper [8]. The following theorem have also been proved in this paper:

Theorem 5. *Let M_B be a Banach space of measures, then in M_B there exists a family of pairwise orthogonal probability measures $\{\mu_i,\ i \in I\}$ such that*

$$M_B = \bigoplus_{i \in I} M_B(\mu_i),$$

where $M_B(\mu_i)$ is the Banach space of elements ν of the form:

$$\nu(B) = \int_B f(x)\mu_i\,(dx), \quad B \in S, \qquad \int_E |f(x)|\,\mu_i(dx) < \infty,$$

with the norm
$$\|\nu\|_{M_B(\mu_i)} = \int_E |f(x)|\,\mu_i(dx).$$

Let $\{H_i\}$ be a countable family of hypotheses. Denote by $F = F(M_B)$ the set of real functions f for which $\int_E f(x)\,\mu_{H_i}(dx)$ is defined for all $\mu_{H_i} \in M_B$, where $M_B = \bigoplus_{i\in\mathbb{N}} M_B(\mu_{H_i})$.

Theorem 6. *Let* $M_B = \bigoplus_{i\in\mathbb{N}} M_B(\mu_{H_i})$ *be a Banach space of measures. The family of probability measures* $\{\mu_{H_i},\ i \in \mathbb{N}\}$ *admits a consistent criteria of hypotheses if and only if the correspondence* $f \to l_f$ *defined by the equality*

$$\int_E f(x)\ \mu_H(dx) = l_f(\mu_H), \quad \forall \mu_H \in M_B,$$

is one-to-one.

Here l_f *is the linear functional on* M_B*,* $f \in F(M_B)$*.*

Proof. *Sufficiency*. For $f \in F(M_B)$ we define the linear continuous functional l_f by the equality $\int_E f(x)\ \mu_H dx) = l_f(\mu_H)$. Denote as I_f a countable subset in $\mathbb{N}$, for which $\int_E f(x)\ \mu_{H_i}(dx) = 0$ for $i \notin I_f$. Let us consider the functional $l_{f_{H_i}}$ on $M_B(\mu_{H_i})$ to which it corresponds. Then for $\mu_{H_1}, \mu_{H_2} \in M_B(\mu_{H_i})$ we have:

$$\int_E f_{H_1}(x)\ \mu_H(dx) = l_{f_{H_1}}(\mu_{H_2}) = \int_E f_1(x) f_2(x)\ \mu_{H_i}(dx) = \int f_{H_1}(x)\ \mu_{H_i}(dx).$$

Therefore $f_{H_1} = f_1$ a.e. with respect to the measure μ_{H_i}. Let $f_{H_i} > 0$ a.e. with respect to the measure μ_{H_i} and

$$\int_E f_{H_i}(x)\ \mu_{H_i}(dx) < \infty, \qquad \mu_{H_i}(C) = \int_C f_{H_i}(x)\ \mu_{H_i}(dx),$$

then

$$\int_E f_{H_i}(x)\ \mu_{H_j}(dx) = l_{f_{H_i}}(\mu_{H_j}) = 0, \quad \forall j \in \mathbb{N}.$$

Denote $C_{H_i} = \{x \mid f_{H_i}(x) > 0\}$, then $\int_E f_{H_i}(x)\ \mu_{H_j}(dx) = l_{f_{H_i}}(\mu_{H_j}) = 0,\ \forall j \in \mathbb{N}$. Hence it follows, that $\mu_{H_j}(C_{H_i}) = 0,\ \forall j \in \mathbb{N}$. On the other hand $\mu_{H_i}(E - C_{H_i}) = 0$. Therefore the family $\{\mu_{H_i},\ i \in \mathbb{N}\}$ is weekly separable and

$$\mu_{H_i}(C_{H_j}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let us consider the sets $\overline{C}_{H_i} = C_{H_i} \setminus C_{H_i} \cap \bigcup_{k\neq i} C_{H_k}$. It is obvious that $\{\overline{C}_{H_i},\ i \in \mathbb{N}\}$ is a disjunctive family of S-measurable sets and $\mu_{H_i}(C_{H_i}) = 1,\ \forall i \in \mathbb{N}$. Let us define a mapping $\delta : (E, S) \to (\mathrm{H}, \mathrm{B}(H))$ like that $\delta(\overline{C}_{H_i}) = H_i,\ \forall i \in \mathbb{N}$. We have $\mu_{H_i}(\{x \mid \delta(x) = H_i\}) = 1,\ \forall i \in \mathbb{N}$. Therefore δ is a consistent criterion of hypotheses.

Necessity. Since the family of probability measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$ admits a consistent criterion of hypotheses and this family is strongly separable, so there exist S-measurable sets X_i, $i \in \mathbb{N}$, such that

$$\mu_{H_i}(X_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We put the linear continuous functional l_{X_i} into the correspondence to a function $I_{X_i} \in F(M_\beta)$ by the formula:

$$\int_E I_{X_i}(x)\ \mu_{H_i}(dx) = l_{I_{X_i}}(\mu_{H_i}) = \|\mu_{H_i}\|_{M_\beta(\mu_{H_i})}.$$

We put the linear continuous functional $l_{f_{H_1}}$ into the correspondence to the function

$$f_{H_1}(x) = f_1(x)I_{X_i}(x) \in F(M_\beta).$$

Then for any $\mu_{H_2} \in M_B(\mu_i)$

$$\int_E f_{H_1}(x)\ \mu_{H_2}(dx) = \int_E f_1(x)I_{X_i}(x)\ \mu_{H_2}(dx) = \int_E f(x)f_1(x)I_{X_i}(x)\ \mu_{H_i}(dx)$$
$$= l_{f_{H_1}}(\mu_{H_2}) = \|\mu_{H_2}\|_{M_B(\mu_{H_i})}.$$

Let $\Sigma = \{l_f\}$ be the collection of extensions of the functional $l_{f_{H_1}} : M_\beta(\mu_{H_1}) \to R$ satisfying the condition $l_f \le p(x)$ on those subspaces where they are defined.

Let us introduce a partial ordering on Σ having assumed $l_{f_1} < l_{f_2}$ if l_{f_2} is defined on a larger set than l_{f_1} and $l_{f_1}(\mu) = l_{f_2}(\mu)$ where both of them are defined.

Let $\{l_{f_i}\}_{i\in I}$ be a linear ordered subset in Σ. Let $M_B(\mu_{H_i})$ be the subspace on which $l_{f_{H_i}}$ is defined. Define l_f on $\bigoplus_{i\in I} M_B(\mu_{H_i})$, having assumed $l_f(\mu) = l_{f_{H_i}}(\mu)$ if $\mu \in M_B(\mu_{H_i})$.

It is obvious, that $l_{f_{H_i}} < l_f$. Since any linearly ordered subset in Σ has an upper bound, by virtue of Zorn lemma Σ contains a maximal element λ defined on some set X' satisfying the condition $\lambda(x) \le p(x)$ for $x \in X'$. But X' must coincide with the entire space M_B because otherwise we could extend λ to a wider space by adding, as above, one more dimension.

This contradicts with the maximality of λ and hence $X' = M_B$. Therefore the extension of the functional is defined everywhere.

If we put the linear continuous functional l_f into the correspondence to the function

$$f(x) = \sum_{i\in\mathbb{N}} g_{H_i}(x)I_{X_i}(x) \in F(M_B),$$

then we obtain

$$\int_E f(x)\,\mu_H(dx) = \|\mu_H\| = \sum_{i\in\mathbb{N}} \|\mu_{H_i}\|_{M_B(\mu_{H_i})},$$

where $\mu_H = \sum_{i\in N}\int_E g_{H_i}\ \mu_{H_i}(dx)$. The Theorem 6 is proved. $\square$

Remark 3. It follows from the proven theorem that the indicated above correspondence puts some functions $f \in F(M_B)$ into the correspondence to each linear continuous functional l_f. If in $F(M_\beta)$ we identify functions coinciding with respect to the measures $\{\mu_{H_i},\ i \in \mathbb{N}\}$, then the correspondence will be bijective.

In what follows $\mathrm{B}(E,S)$ will always denote a vector space formed by all real bounded measurable functions on (E,S) having the natural order. It is an (AN)-space with identity according to which a function is identically equal to one on E (see [5]). Let $\mathrm{B}'(E,S)$ denote the topological conjugate space of $\mathrm{B}(E,S)$, which is an order-complete Banach lattice. The elements of $\mathrm{B}'(E,S)$ are called finitely-additive measures on (E,S) and the canonical bilinear form which puts $\mathrm{B}(E,S)$ and $\mathrm{B}'(E,S)$ in duality is denoted by

$$\langle f,\mu\rangle = \mu(f) = \int_E f(x)\ \mu(dx), \quad f \in \mathrm{B}(E,S),\ \mu \in \mathrm{B}'(E,S)$$

and called the integral of f with respect to μ. In what follows $\mathrm{B}(\mathrm{H}, B(\mathrm{H}))$ is the space of measurable bounded functions and $\mathrm{B}'(\mathrm{H}, B(\mathrm{H}))$ is the conjugate space of all finitely-additive measures on $(\mathrm{H}, B(\mathrm{H}))$.

Equal units of $\mathrm{B}(\mathrm{H}, B(\mathrm{H}))$ space are denoted by e_{H} and elements $\mathrm{B}(\mathrm{H}, B(\mathrm{H}))$ and $\mathrm{B}'(\mathrm{H}, B(\mathrm{H}))$ are denoted by g and ν respectively. Intersection of $\{\nu \in \mathrm{B}'(\mathrm{H}, \mathrm{B}(\mathrm{H})) \mid \langle e_{\mathrm{H}}, \nu\rangle = 1\}$ and positive cone is denoted by S_{H}. It is clear that S_{H} is a compact subset of the simple share, so a set of extreme points of this cone is not empty.

It is also well known that in the (*ZFC*), (*CH*), (*MA*) theory there exists a continual weekly separable family of probability measures which is not strongly separable. Here and in the sequel we denote by (*MA*) the Martin's axiom (see [3]).

Theorem 7. *Let* $M_B = \bigoplus_{H \in \mathrm{H}} M_B(\mu_H)$ *be the Banach space of measures, E be the complete separable metric space, S be the Borel σ-algebra in E and* $\operatorname{card} \mathrm{H} \leq c$. *Then in the theory (ZFC) and (MA) the family of probability measures* $\{\mu_H, H \in \mathrm{H}\}$ *admits a consistent criteria of hypotheses if and only if the family of probability measures* $\{\mu_H,\ H \in \mathrm{H}\}$ *admits an unbiased criterion of any parametric function and the correspondence* $f \to l_f$ *by the equality*

$$\int_E f(x)\mu_H(dx) = l_f(\mu_H), \quad \forall \mu_H \in M_B$$

is one-to-one. Here l_f *is a linear continuous functional on* M_B, $f \in F(M_B)$.

Proof. *Necessity*. As the family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$ admits a consistent criterion of hypotheses, so the family $\{\mu_H,\ H \in \mathrm{H}\}$ admits an unbiased criterion of any parametric function and it is strongly separable. So, the family $\{\mu_H,\ H \in \mathrm{H}\}$ is weekly separable. The necessity is proved in the same manner as the necessity of the Theorem 6.

Sufficiency. According to the Theorem 6 a Borel orthogonal family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$, $\operatorname{card} \mathrm{H} \leq c$ is weakly separable. We represent $\{\mu_H,\ H \in \mathrm{H}\}$ as an inductive sequence $\mu_H < \omega_\alpha$, where ω_α denotes the first ordinal number of the power of the set H. Since the family $\{\mu_H,\ H \in \mathrm{H}\}$ is weakly separable, there exists a family of measurable parts $\{X_H\}_{H<\omega_\alpha}$ of the space E, such that the following relations are fulfilled:

$$\mu_H(X_{H'}) = \begin{cases} 1, & \text{if } H = H', \\ 0, & \text{if } H \neq H' \end{cases}$$

for all $H \in [0, \omega_\alpha)$ and $H' \in [0, \omega_\alpha)$.

We define ω_α-sequence of parts B_H of the space E so that the following relations are fulfilled:

(1) B_H is a Borel subset in E for all $H < \omega_\alpha$.
(2) $B_H \subset X_H$ for all $H < \omega_\alpha$.
(3) $B_H \cap B_{H'} = \emptyset$ for all $H < \omega_\alpha$, $H' < \omega_\alpha$, $H \neq H'$.
(4) $\mu_H(B_H) = 1$ for all $H < \omega_\alpha$.

Assume that $B_0 = X_0$. Let further the partial sequence $\{B_{H'}\}_{H'<H}$ be already defined for $H < \omega_\alpha$.

It is clear, that $\mu^*(\bigcup_{H'<H} B_{H'}) = 0$. Thus there exists a Borel subset Y_H of the space E, such that the following relations are valid: $\bigcup_{H'<H} B_{H'} \subset Y_H$ and $\mu(Y_H) = 0$.

Assume $B_H = X_H \setminus Y_H$, thereby the ω_α sequence of $\{B_H\}_{H<\omega_\alpha}$ disjunctive measurable subsets of the space E is constructed. Therefore $\mu_H(B_H) = 1$ for all $H < \omega_\alpha$. As a family of probability measures $\{\mu_H,\ H \in \mathrm{H}\}$ admits an unbiased criterion for any parametric function, so there exists a subspace $G \subset \mathrm{B}(E,S)$, containing e_E unit and $\mathrm{B}(E,S)$ can be imagine as a topological sum of G and $H_0 = \mu^{-1}(0)$, where the functional

$$\mu(f) = \int_E f(x)\,\mu(dx), \quad f \in \mathrm{B}(E,S),\ \mu \in \mathrm{B}'(E,S)$$

and a family $\{\mu_H,\ H \in \mathrm{H}\}$ is strongly separable, subspace G is a grid towards canonical order on G (see [5]). We assume that S_0 is a minimal σ-algebra of subalgebra S, all function on G are measurable towards S_0. Then $G \subset B(E,S_0) \subset B(E,S)$.

Since a subspace G contains e_E and represents a grid, then $G \supset B(E,S)$ and that's why $G = B(E,S)$.

As family $\{\mu_H,\ H \in \mathrm{H}\}$ represents a dense subspace of exS_H (exS_H are extreme points of S_H), so I_μ is an ideal in the set S_0 which contains zero measured sets for all $\mu \in \{\mu_H,\ H \in \mathrm{H}\}$ and consists only of an empty set.

Hence there exist such sets $\{A_H,\ H \in \mathrm{H}\}$ that $\mu_H(A_H) = 1$ and $A_H \cap A_{H'} = \emptyset$ for a $H \neq H_0$ and $E = \bigcup_{H\in\mathrm{H}} A_H$ is a set S_0. It follows from the condition of this theorem that for every $T \in \mathrm{B}(\mathrm{H})$ in G there exists f_T function, which is a consistent criterion of g_T parametric function. If $A = \{x \mid f_t(x) \neq 0\}$, then $\bigcup_{H\in T} A_H \subset A$, $A \cap A_H = \emptyset$ for all $H \notin T$ and hence $\bigcup_{H\in T} A_H = A$ implying that $\bigcup_{H\in T} A_H \subset S_0$.

Then, the mapping $\delta(x) = H$ if $x \in A_H$ for all $H \in \mathrm{H}$ is a consistent criterion of hypotheses. The theorem is proved. □

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