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Ruin probability in the three-seasonal discrete-time risk model

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Abstract This paper deals with the discrete-time risk model with nonidentically distributed claims. We suppose that the claims repeat with time periods of three units, that is, claim distributions coincide at times $\{1, 4, 7, \dots\}$, at times $\{2, 5, 8, \dots\}$, and at times $\{3, 6, 9, \dots\}$. We present the recursive formulas to calculate the finite-time and ultimate ruin probabilities. We illustrate the theoretical results by several numerical examples.

Keywords Inhomogeneous model, three-seasonal model, finite-time ruin probability, ultimate ruin probability

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1 Introduction

The discrete-time risk model is a classical collective risk model for insurance. In the homogeneous version of this model, the insurer's surplus at each time $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is defined by the following equality:

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i, \tag{1}$$

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where $u \in \mathbb{N}_0$ is the initial insurer's surplus, and the claim amounts $Z_1, Z_2, \dots$ are assumed to be independent copies of a nonnegative integer-valued random variable Z. This random variable and the initial surplus u generate the homogeneous discrete-time risk model. A typical path of the surplus process $W_u(n)$ is shown in Fig. 1.

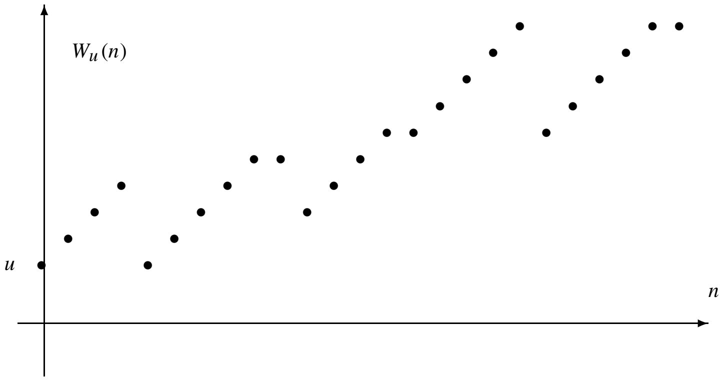

Fig. 1. Behavior of the surplus sequence $W_u(n)$

The claim amount generator Z can be characterized by the probability mass function (p.m.f.)

$$z_k = \mathbb{P}(Z = k), \quad k \in \mathbb{N}_0,$$

or by the cumulative distribution function (c.d.f.)

$$F_Z(x) = \sum_{k=0}^{\lfloor x \rfloor} z_k, \quad x \in \mathbb{R},$$

where $\lfloor x \rfloor$ denotes the integer part of x.

The homogeneous discrete-time risk model has been extensively investigated by De Vylder and Goovaerts [5, 6], Dickson [7, 8], Gerber [10], Seal [17], Shiu [19, 18], Picard and Lefèvre [15, 16], Lefèvre and Loisel [11], Leipus and Šiaulys [12], Tang [20], and other authors. The ruin time, the ultimate ruin probability, and the finite-time ruin probability are the main extremal characteristics of any risk model. The first time T_u when the surplus $W_u(n)$ becomes negative or null is called the ruin time, that is,

$$T_u = \begin{cases} \inf\{n \in \mathbb{N} : W_u(n) \leqslant 0\}, \\ \infty \ \text{ if } W_u(n) > 0 \text{ for all } \ n \in \mathbb{N}. \end{cases}$$

The ruin probability until time $T \in \mathbb{N}$ is called the *finite-time ruin probability* and is defined by

$$\psi(u, T) = \mathbb{P}(T_u \leqslant T).$$

The infinite-time or *ultimate ruin probability* is defined by

$$\psi(u) = \mathbb{P}(T_u < \infty).$$

So, for the *ultimate survival probability*, we have

$$\varphi(u) = 1 - \psi(u) = \mathbb{P}(T_u = \infty).$$

The presented definitions imply that

$$\psi(u,T) = \mathbb{P}\Bigg(\bigcup_{n=1}^{T}\Bigg\{u+n-\sum_{i=1}^{n} Z_i \leqslant 0\Bigg\}\Bigg) = \mathbb{P}\Bigg(\max_{1\leqslant n\leqslant T}\sum_{i=1}^{n}(Z_i-1)\geqslant u\Bigg),$$
$$\psi(u) = \mathbb{P}\Bigg(\bigcup_{n=1}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i \leqslant 0\Bigg\}\Bigg) = \mathbb{P}\Bigg(\sup_{n\geqslant 1}\sum_{i=1}^{n}(Z_i-1)\geqslant u\Bigg),$$
$$\varphi(u) = \mathbb{P}\Bigg(\bigcap_{n=1}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\}\Bigg),$$
$$\lim_{T\nearrow\infty}\psi(u,T) = \psi(u).$$

Several formulas and procedures for computing finite-time ruin probability and ultimate ruin probability have been proposed in the literature. Here we present some of them having the recursive form.

- *For the homogeneous discrete-time risk model, we have* (*see, for instance,* [5, 8, 9]):

$$\psi(u,1) = 1 - F_Z(u), \quad u \in \mathbb{N}_0,$$
$$\psi(u,T) = \psi(u,1) + \sum_{k=0}^{u}\psi(u+1-k, T-1)z_k, \quad u \in \mathbb{N}_0, \quad T \in \{2,3,\dots\}.$$

- *If model* (1) *is generated by the claim generator Z such that $\mathbb{E}Z < 1$, then the ultimate ruin probability can be calculated by the formulas* (*see, for instance,* [8, 9, 18]):

$$\psi(0) = \mathbb{E}Z, \tag{2}$$
$$\psi(u) = \sum_{j=1}^{u-1}\big(1-F_Z(j)\big)\psi(u-j) + \sum_{j=u}^{\infty}\big(1-F_Z(j)\big), \quad u \in \mathbb{N}. \tag{3}$$

If the homogeneous discrete-time risk model is generated by Z satisfying condition $\mathbb{E}Z \geqslant 1$, then we say that the net profit condition does not hold, and, in such a case, we have that $\psi(u) = 1$ for all $u \in \mathbb{N}_0$ according to the general renewal theory (see, e.g., [14] and the references therein).

The formulas presented enable us to calculate $\psi(u)$ and $\psi(u,T)$ for $u \in \mathbb{N}_0$ and $T \in \mathbb{N}$. Nevertheless, there exist many other methods that allow us to calculate or estimate the finite-time and the ultimate ruin probabilities. Some of them can be found in [1, 13, 16].

The assumption for claim amounts $\{Z_1, Z_2, \dots\}$ to be nonidentically distributed random variables is a natural generalization of the homogeneous model. If r.v.s

$\{Z_1, Z_2, \ldots\}$ are independent but not necessarily identically distributed, then the model defined by Eq. (1) is called the *inhomogeneous discrete-time risk model*. For such a model, a recursive procedure for calculation of finite-time ruin probabilities can be found in [2, 3]. For the finite-time ruin probabilities

$$\psi^{(j)}(u,T) = \mathbb{P}\bigg(\bigcup_{n=1}^{T}\bigg\{u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\bigg\}\bigg), \quad j \in \mathbb{N}_0,$$

we have the following theorem.

Theorem 1. *Let us consider the inhomogeneous discrete-time risk model defined by Eq. (1) in which $u \in \mathbb{N}_0$, $z_k^{(j)} = \mathbb{P}(Z_{1+j} = k)$, $k, j \in \mathbb{N}_0$, and $F_Z^{(j)}(x) = \mathbb{P}(Z_{1+j} \leqslant x)$, $x \in \mathbb{R}$. Then*

$$\psi^{(j)}(u,1) = 1 - F_Z^{(j)}(u),$$
$$\psi^{(j)}(u,T) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, T-1) z_k^{(j)}$$

for all $u \in \mathbb{N}_0$ and $T \in \{2, 3, \ldots\}$.

According to this theorem, we can calculate the finite-time ruin probability $\psi^{(0)}(u,T)$ of the initial model for all $u \in \mathbb{N}_0$ and $T \in \mathbb{N}$. Unfortunately, it is impossible to get formulas for $\psi(u)$ similar to formulas (2) and (3) in the general case because in the case of nonidentically distributed claims, the future of model behavior at each time can be completely new. In paper [4], the general discrete-time risk model was restricted to the model with two kinds of claims. In this model, there are two differently distributed claim amounts that are changing periodically. We call such a model the bi-seasonal discrete-time risk model. In [4] (see Theorem 2.3), the following statement is proved for the calculation of the ultimate ruin probability.

Theorem 2. *Let us consider a bi-seasonal discrete-time risk model generated by independent random claim amounts X and Y, that is, $Z_{2k+1} \stackrel{d}{=} X$ for $k \in \{0, 1, \ldots\}$ and $Z_{2k} \stackrel{d}{=} Y$ for $k \in \{1, 2, \ldots\}$. Denote $S = X + Y$ and $x_n = \mathbb{P}(X = n)$, $y_n = \mathbb{P}(Y = n)$, $s_n = \mathbb{P}(S = n)$ for $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$.*

- *If $\mathbb{E}X + \mathbb{E}Y < 2$, then*
$$\lim_{u\to\infty} \psi(u) = 0.$$
- *If $s_0 = x_0 y_0 \neq 0$, then:*
$$\psi(0) = 1 - (2 - \mathbb{E}S) \lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_n - a_{n+1}},$$
$$1 - \psi(u) = \alpha_u\big(1 - \psi(0)\big) + \beta_u(2 - \mathbb{E}S), \quad u \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $n \in \mathbb{N}_0$, are two sequences of real numbers defined recursively by formulas:

$$\alpha_0 = 1, \qquad \alpha_1 = -\frac{1}{y_0},$$

$$\alpha_n = \frac{1}{s_0}\Bigl(\alpha_{n-2} - \sum_{i=1}^{n-1} s_i \alpha_{n-i} - x_{n-1}\Bigr), \quad n \geqslant 2,$$
$$\beta_0 = 0, \qquad \beta_1 = \frac{1}{y_0},$$
$$\beta_n = \frac{1}{s_0}\Bigl(\beta_{n-2} - \sum_{i=1}^{n-1} s_i \beta_{n-i} + x_{n-1}\Bigr), \quad n \geqslant 2.$$

- *If* $s_0 \neq 0$*, then*

$$\psi(1) = 1 - \bigl(1 + \psi(0) - \mathbb{E}S\bigr)/y_0,$$
$$\psi(u) = 1 + \frac{1}{s_0}\Biggl(\psi(u-2) - 1 + \sum_{k=1}^{u-1} s_k\bigl(1 - \psi(u-k)\bigr)\Biggr) - \frac{x_{u-1}(1-\psi(1))}{x_0}, \quad u \in \{2, 3, \ldots\}.$$

- *If* $x_0 = 0$*,* $y_0 \neq 0$*, then* $s_1 \neq 0$ *and* $\psi(0) = 1$*.*
- *If* $x_0 \neq 0$*,* $y_0 = 0$*, then* $s_1 \neq 0$ *and* $\psi(0) = \mathbb{E}S - 1$*.*
- *If* $s_0 = 0$*, then, for* $u \in \mathbb{N}$*,*

$$\psi(u) = 1 - \frac{1}{s_1}\Biggl(1 - \psi(u-1) - \sum_{k=2}^{u} s_k\bigl(1 - \psi(u-k+1)\bigr)\Biggr).$$

In this paper, we consider the discrete-time risk model with three seasons. We obtain a list of formulas similar to those in Theorem 2 to calculate the ultimate ruin probability in such a model. In Section 2, we present a precise definition of the three-seasonal discrete-time risk model and our main statements, whereas in Sections 3 and 4, we give detailed proofs. Finally, Section 5 deals with some numerical examples.

2 Main results

We now present the model under consideration.

Definition 1. We say that the insurer's surplus $W_u(n)$ follows the three-seasonal risk model if $W_u(n)$ is given by Eq. (1) for each $n \in \mathbb{N}_0$ and the following assumptions hold:

- the initial insurer's surplus $u \in \mathbb{N}_0$,
- the random claim amounts $Z_1, Z_2, \ldots$ are nonnegative integer-valued independent r.v.s,
- for all $k \in \mathbb{N}_0$, we have $Z_{3k+1} \stackrel{d}{=} Z_1$, $Z_{3k+2} \stackrel{d}{=} Z_2$, and $Z_{3k+3} \stackrel{d}{=} Z_3$.

Let us define p.m.f.s and p.d.f.s by the following equalities:

$$a_k = \mathbb{P}(Z_1 = k), \qquad b_k = \mathbb{P}(Z_2 = k),$$
$$c_k = \mathbb{P}(Z_3 = k), \qquad s_k = \mathbb{P}(S = k), \quad k \in \mathbb{N}_0,$$

where $S = Z_1 + Z_2 + Z_3$,

$$A(x) = \sum_{k=0}^{\lfloor x \rfloor} a_k, \qquad B(x) = \sum_{k=0}^{\lfloor x \rfloor} b_k,$$
$$C(x) = \sum_{k=0}^{\lfloor x \rfloor} c_k, \qquad D(x) = \sum_{k=0}^{\lfloor x \rfloor} s_k, \quad x \geqslant 0.$$

It is not difficult to see that the definitions of ruin time, finite-time ruin probability, ultimate ruin probability, and ultimate survival probability remain the same. All expressions of these quantities are the same as in the homogeneous discrete-time risk model. However, the procedures to calculate the finite-time or the ultimate probabilities are more complex than in the homogeneous or be-seasonal discrete-time risk models.

Our first result immediately follows from Theorem 1. The obtained formulas allow us to calculate the finite-time ruin probabilities $\psi(u, T) = \psi^{(0)}(u, T)$ in the three-seasonal risk model for all $u \in \mathbb{N}_0$ and all $T \in \mathbb{N}$.

Theorem 3. *In the three-seasonal discrete-time risk model, for each $u \in \mathbb{N}_0$, we have*

$$\psi^{(0)}(u, 1) = \sum_{k>u} a_k, \qquad \psi^{(1)}(u, 1) = \sum_{k>u} b_k, \qquad \psi^{(2)}(u, 1) = \sum_{k>u} c_k,$$

and for all $u \in \mathbb{N}_0$ and $T \in \{2, 3, \ldots\}$, we have the following recursive formulas:

$$\psi^{(0)}(u, T) = \psi^{(0)}(u, 1) + \sum_{k=0}^{u} \psi^{(1)}(u + 1 - k, T - 1)a_k,$$
$$\psi^{(1)}(u, T) = \psi^{(1)}(u, 1) + \sum_{k=0}^{u} \psi^{(2)}(u + 1 - k, T - 1)b_k,$$
$$\psi^{(2)}(u, T) = \psi^{(2)}(u, 1) + \sum_{k=0}^{u} \psi^{(0)}(u + 1 - k, T - 1)c_k.$$

Our second result describes the meaning of the net profit condition in the three-seasonal discrete-time risk model. The proof of the theorem is presented in Section 3.

Theorem 4. *Consider the three-seasonal discrete time risk model generated by independent random claim amounts Z_1, Z_2, and Z_3. If $\mathbb{E}S > 3$, then $\psi(u) = 1$ for each initial surplus $u \in \mathbb{N}_0$. If $\mathbb{E}S = 3$, then we have the following possible cases:*

- $\psi(0) = \psi(1) = \psi(2) = 1$ *and* $\psi(u) = 0$ *for* $u \in \{3, 4, \ldots\}$ *if* $\{a_3 = b_0 = c_0 = 1\}$*;*

- $\psi(0) = \psi(1) = 1$ *and* $\psi(u) = 0$ *for* $u \in \{2, 3, \ldots\}$ *if* $\{a_0 = b_3 = c_0 = 1\}$, $\{a_2 = b_1 = c_0 = 1\}$, $\{a_1 = b_2 = c_0 = 1\}$, *or* $\{a_2 = b_0 = c_1 = 1\}$*;*
- $\psi(0) = 1$ *and* $\psi(u) = 0$ *for all* $u \in \mathbb{N}$ *if* $\{a_0 = b_0 = c_3 = 1\}$, $\{a_0 = b_2 = c_1 = 1\}$, $\{a_0 = b_1 = c_2 = 1\}$, $\{a_1 = b_0 = c_2 = 1\}$, *or* $\{a_1 = b_1 = c_1 = 1\}$*;*
- $\psi(u) = 1$ *for all* $u \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ *if* $s_3 < 1$.

Our last statement proposes a recursive procedure for calculation of the ultimate survival probabilities $\varphi(u) = 1 - \psi(u),\ u \in \mathbb{N}_0$. The proof of the formulas is given in Section 4.

Theorem 5. *Consider the three-seasonal discrete-time risk model generated by independent random claim amounts* Z_1, Z_2, *and* Z_3. *Denote* $S = Z_1 + Z_2 + Z_3$, $s_n = \mathbb{P}(S = n)$ *for* $n \in \mathbb{N}_0$, *and suppose that* $\mathbb{E}S < 3$. *Then the following statements hold.*

- $\lim\limits_{u\to\infty} \varphi(u) = 1.$
- *If* $s_0 \neq 0$, *then*

$$\varphi(n) = \alpha_n\varphi(0) + \beta_n\varphi(1) + \gamma_n(3 - \mathbb{E}S), \quad n \in \mathbb{N}_0, \tag{4}$$

where

$$\begin{cases} \alpha_0 = 1, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{1}{b_0c_0}, \\ \alpha_n = \frac{1}{s_0}(\alpha_{n-3} - \sum_{k=1}^{n-1} s_k\alpha_{n-k} - a_{n-2}), \quad n \geqslant 3; \end{cases}$$

$$\begin{cases} \beta_0 = 0, \qquad \beta_1 = 1, \qquad \beta_2 = -\frac{c_1}{c_0} - \frac{1}{b_0}, \\ \beta_n = \frac{1}{s_0}(\beta_{n-3} - \sum_{k=1}^{n-1} s_k\beta_{n-k} - a_{n-2}c_0 + c_0\sum_{k=0}^{n-1} a_kb_{n-1-k}), \quad n \geqslant 3; \end{cases}$$

$$\begin{cases} \gamma_0 = 0, \qquad \gamma_1 = 0, \qquad \gamma_2 = \frac{1}{b_0c_0}, \\ \gamma_n = \frac{1}{s_0}(\gamma_{n-3} - \sum_{k=1}^{n-1} s_k\gamma_{n-k} + a_{n-2}), \quad n \geqslant 3. \end{cases}$$

- *If* $\{a_0 = 0, b_0 \neq 0, c_0 \neq 0, a_1 \neq 0\}$, *then*

$$\begin{cases} \varphi(0) = 0, \\ \varphi(n) = \hat{\beta}_n\varphi(1) + \hat{\gamma}_n(3 - \mathbb{E}S), \quad n \in \mathbb{N}, \end{cases} \tag{5}$$

where

$$\begin{cases} \hat{\beta}_1 = \beta_1, \qquad \hat{\beta}_2 = \beta_2, \qquad \hat{\gamma}_1 = \gamma_1, \qquad \hat{\gamma}_2 = \gamma_2, \\ \hat{\beta}_n = \frac{1}{s_1}(\hat{\beta}_{n-2} - \sum_{k=2}^{n} s_k\hat{\beta}_{n-k+1} \\ \qquad - a_{n-1}c_0 - c_0\varphi(1)\sum_{k=1}^{n} a_kb_{n-k}), \quad n \geqslant 3, \\ \hat{\gamma}_n = \frac{1}{s_1}(\hat{\gamma}_{n-2} - \sum_{k=2}^{n} s_k\hat{\gamma}_{n-k+1} + a_{n-1}), \quad n \geqslant 3. \end{cases}$$

- *If* $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$ *then*

$$\varphi(n) = \tilde{\alpha}_n \varphi(0) + \tilde{\gamma}_n (3 - \mathbb{E}S), \quad n \in \mathbb{N}, \tag{6}$$

where

$$\begin{cases} \tilde{\alpha}_1 = -1/c_0, & \tilde{\alpha}_2 = c_1/c_0^2 + 1/(a_0 b_1 c_0), \\ \tilde{\gamma}_1 = 1/c_0, & \tilde{\gamma}_2 = -c_1/c_0^2, \\ \tilde{\alpha}_n = \frac{1}{s_1}(\tilde{\alpha}_{n-2} - \sum_{k=2}^{n} s_k \tilde{\alpha}_{n-k+1} - \sum_{k=0}^{n-1} a_k b_{n-k}), & n \geqslant 3, \\ \tilde{\gamma}_n = \frac{1}{s_1}(\tilde{\gamma}_{n-2} - \sum_{k=2}^{n} s_k \tilde{\gamma}_{n-k+1} + \sum_{k=0}^{n-1} a_k b_{n-k}), & n \geqslant 3. \end{cases}$$

- *If* $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$*, then*

$$\varphi(n) = \breve{\alpha}_n \varphi(0) + \breve{\gamma}_n (3 - \mathbb{E}S), \quad n \in \mathbb{N}_0, \tag{7}$$

where

$$\begin{cases} \breve{\alpha}_0 = 1, \quad \breve{\alpha}_1 = -1/(b_0 c_1), \quad \breve{\gamma}_0 = 0, \quad \breve{\gamma}_1 = 1/(b_0 c_1), \\ \breve{\alpha}_n = \frac{1}{s_1}(\breve{\alpha}_{n-2} - \sum_{k=2}^{n} s_k \breve{\alpha}_{n-k+1} - \sum_{k=0}^{n-1} a_k b_{n-k}), \quad n \geqslant 2, \\ \breve{\gamma}_n = \frac{1}{s_1}(\breve{\gamma}_{n-2} - \sum_{k=2}^{n} s_k \breve{\gamma}_{n-k+1} + \sum_{k=0}^{n-1} a_k b_{n-k}), \quad n \geqslant 2. \end{cases}$$

- *If* $\{a_0 = 0, b_0 = 0, c_0 \neq 0\}$*, then* $\varphi(0) = 0$*,* $\varphi(1) = (3 - \mathbb{E}S)/c_0$*, and*

$$\begin{aligned} \varphi(u+1) = \frac{1}{s_2}\Bigg(& (1 - s_3)\varphi(u) - \sum_{k=1}^{u-1} \varphi(k) s_{u+3-k} \\ & + c_0 \varphi(1) \sum_{k=0}^{u+2} a_k b_{u+2-k} \Bigg), \quad u \in \mathbb{N}. \end{aligned} \tag{8}$$

- *If* $\{a_0 = 0, b_0 \neq 0, c_0 = 0\}$*, then* $\varphi(0) = s_2 \varphi(1)$*,* $\varphi(1) = (3 - \mathbb{E}S)/(s_2 + b_0 c_1)$*, and*

$$\begin{aligned} \varphi(u+1) = \frac{1}{s_2}\Bigg(& (1 - s_3)\varphi(u) - \sum_{k=1}^{u-1} \varphi(k) s_{u+3-k} \\ & + a_{u+1} b_0 c_1 \varphi(1) \Bigg), \quad u \in \mathbb{N}. \end{aligned} \tag{9}$$

- *If* $\{a_0 \neq 0, b_0 = 0, c_0 = 0\}$*, then* $\varphi(0) = 3 - \mathbb{E}S$*,* $\varphi(1) = (3 - \mathbb{E}S)/s_2$*, and*

$$\varphi(u+1) = \frac{1}{s_2}\left((1 - s_3)\varphi(u) + \sum_{k=1}^{u-1} \varphi(k) s_{u+3-k} \right), \quad u \in \mathbb{N}. \tag{10}$$

- *If* $\{a_0 = a_1 = 0, b_0 \neq 0, c_0 \neq 0\}$*, then* $\varphi(0) = 0$*,* $\varphi(1) = (3 - \mathbb{E}S)/(1/a_2 + c_0)$*, and the recursion formula* (8) *is satisfied.*

- *If* $\{a_0 \neq 0, b_0 = b_1 = 0, c_0 \neq 0\}$, *then* $\varphi(0) = 0$, $\varphi(1) = (3 - \mathbb{E}S)/c_0$, *and the same recursion formula* (8) *holds.*

- *If* $\{a_0 \neq 0, b_0 \neq 0, c_0 = c_1 = 0\}$, *then* $\varphi(0) = 3 - \mathbb{E}S$, $\varphi(1) = (3 - \mathbb{E}S)/s_2$, *and the recursion formula* (10) *is satisfied.*

We observe that all formulas presented in Theorem 5 can be used to calculate numerical values of survival or ruin probabilities for an arbitrary three-seasonal risk model and for an arbitrary initial surplus value u. The algorithms based on the derived relations work quite quickly and accurately. A few numerical examples for calculating ruin probability in the various versions of the three-seasonal risk model are presented in Section 5.

3 Net profit condition

In this section, we present a proof of Theorem 4. We recall that we denote the ultimate survival probability by $\varphi(u)$.

Proof of Theorem 4. For an arbitrary $u \in \mathbb{N}_0$, we have

$$\begin{aligned}
\varphi(u) &= \mathbb{P}\Bigg(\bigcap_{n=1}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\}\Bigg) \\
&= \mathbb{P}\Bigg(\bigcap_{n=3}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\} \cap \{Z_1 \geqslant u+1\} \cap \{Z_1+Z_2 \geqslant u+2\}\Bigg) \\
&\quad - \mathbb{P}\Bigg(\bigcap_{n=3}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\} \cap \{Z_1 \geqslant u+1\}\Bigg) \\
&\quad - \mathbb{P}\Bigg(\bigcap_{n=3}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\} \cap \{Z_1+Z_2 \geqslant u+2\}\Bigg) \\
&\quad + \mathbb{P}\Bigg(\bigcap_{n=3}^{\infty}\Bigg\{u+n-\sum_{i=1}^{n} Z_i > 0\Bigg\}\Bigg).
\end{aligned} \tag{11}$$

Since the model is three-seasonal, the last probability in (11) can be expressed by the sum

$$\sum_{k=0}^{u+2} s_k\, \mathbb{P}\Bigg(\bigcap_{n=1}^{\infty}\Bigg\{u+n+3-k-\sum_{i=1}^{n} Z_i > 0\Bigg\}\Bigg) = \sum_{k=0}^{u+2} s_k\, \varphi(u+3-k), \tag{12}$$

where, as before, $s_k = \mathbb{P}(Z_1+Z_2+Z_3 = k)$ for $k \in \mathbb{N}_0$.
The second probability in (11) is equals

$$\sum_{k=u+1}^{\infty} a_k\, \mathbb{P}\Bigg(\bigcap_{n\geqslant 3}\Bigg\{u+n-k-Z_2-Z_3-\sum_{i=4}^{n} Z_i > 0\Bigg\}\Bigg)$$

$$= a_{u+2}\,\mathbb{P}(Z_2+Z_3=0)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-2-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$+ a_{u+1}\,\mathbb{P}(Z_2+Z_3=0)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-1-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$+ a_{u+1}\,\mathbb{P}(Z_2+Z_3=1)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-2-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$= a_{u+2}b_0c_0\varphi(1) + a_{u+1}b_0c_0\varphi(2) + a_{u+1}b_0c_1\varphi(1) + a_{u+1}b_1c_0\varphi(1). \tag{13}$$

Similarly, the third probability in (11) is

$$\mathbb{P}(Z_1+Z_2=u+2)\,\mathbb{P}(Z_3=0)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-2-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$= c_0\varphi(1)\sum_{k=0}^{u+2} a_k b_{u+2-k}, \tag{14}$$

and, finally, the first probability in (11) is

$$\mathbb{P}(Z_1=u+1)\,\mathbb{P}(Z_2=1)\,\mathbb{P}(Z_3=0)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-2-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$+ \mathbb{P}(Z_1=u+2)\,\mathbb{P}(Z_2=0)\,\mathbb{P}(Z_3=0)\,\mathbb{P}\Bigg(\bigcap_{n=4}^{\infty}\Big\{n-2-\sum_{i=4}^{n} Z_i > 0\Big\}\Bigg)$$

$$= a_{u+1}b_1c_0\varphi(1) + a_{u+2}b_0c_0\varphi(1). \tag{15}$$

Substituting (12)–(15) into (11), we get that

$$\varphi(u) = \sum_{k=0}^{u+2} s_k\,\varphi(u+3-k) - a_{u+1}b_0c_0\varphi(2)$$

$$- a_{u+1}b_0c_1\varphi(1) - c_0\varphi(1)\sum_{k=0}^{u+2} a_k b_{u+2-k} \tag{16}$$

for all $u \in \mathbb{N}_0$.

Therefore, for $v \in \mathbb{N}_0$, we have

$$\sum_{u=0}^{v}\varphi(u) = \sum_{u=0}^{v}\sum_{k=0}^{u+2} s_k\varphi(u+3-k)$$

$$- b_0c_0\varphi(2)\big(A(v+1)-a_0\big) - b_0c_1\varphi(1)\big(A(v+1)-a_0\big)$$

$$- c_0\varphi(1)\sum_{u=0}^{v}\sum_{k=0}^{u+2} a_k b_{u+2-k}. \tag{17}$$

We observe that the sum

$$\sum_{u=0}^{v}\sum_{k=0}^{u+2} a_k b_{u+2-k}$$

can be rewritten in the form

$$\begin{aligned}
&a_0\sum_{u=0}^{v} b_{u+2} + a_1\sum_{u=0}^{v} b_{u+1} + a_2\sum_{u=0}^{v} b_u + \sum_{k=3}^{v+2} a_k \sum_{u=k-2}^{v} b_{u+2-k}\\
&\quad = a_0\big(B(v+2)-b_0-b_1\big) + a_1\big(B(v+1)-b_0\big) + a_2 B(v)\\
&\qquad + \sum_{k=3}^{v+2} a_k B(v+2-k)\\
&\quad = \sum_{k=0}^{v+2} a_k B(v+2-k) - a_0 b_0 - a_0 b_1 - a_1 b_0. \qquad (18)
\end{aligned}$$

Similarly, the sum

$$\sum_{u=0}^{v}\sum_{k=0}^{u+2} s_k \varphi(u+3-k)$$

equals

$$\sum_{k=1}^{v+3} \varphi(k) D(v+3-k) - s_0\varphi(1) - s_1\varphi(1) - s_0\varphi(2), \qquad (19)$$

where

$$D(x) = \sum_{k=0}^{\lfloor x\rfloor} s_k = \sum_{k=0}^{\lfloor x\rfloor} \mathbb{P}(Z_1+Z_2+Z_3=k).$$

Relations (17), (18), and (19) imply that

$$\begin{aligned}
\sum_{k=0}^{v}\varphi(k) = {}&\sum_{k=1}^{v+3}\varphi(k) D(v+3-k)\\
&- b_0 c_0 \varphi(2) A(v+1) - b_0 c_1 \varphi(1) A(v+1)\\
&- c_0\varphi(1)\sum_{k=0}^{v+2} a_k B(v+2-k)
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\sum_{k=0}^{v+3}\varphi(k)\big(1-D(v+3-k)\big)\\
&\quad = \varphi(v+1)+\varphi(v+2)+\varphi(v+3)-\varphi(0)D(v+3) - b_0c_0\varphi(2)A(v+1)\\
&\qquad - b_0 c_1\varphi(1)A(v+1) + c_0\varphi(1)\Bigg(\sum_{k=0}^{v+2} a_k \overline{B}(v+2-k) + A(v+2)\Bigg). \qquad (20)
\end{aligned}$$

For each $K \in [1, v+2)$, we have

$$\sum_{k=0}^{v+2} a_k \overline{B}(v+2-k) = \sum_{k=0}^{K} a_k \overline{B}(v+2-k) + \sum_{k=K+1}^{v+2} a_k \overline{B}(v+2-k).$$

Therefore,

$$\limsup_{v\to\infty} \sum_{k=0}^{v+2} a_k \overline{B}(v+2-k) \leqslant \sum_{k=K+1}^{\infty} a_k$$

for each $K \geqslant 1$, and so

$$\lim_{v\to\infty} \sum_{k=0}^{v+2} a_k \overline{B}(v+2-k) = 0. \tag{21}$$

The sequence $\varphi(u)$, $u \in \mathbb{N}_0$, is bounded and nondecreasing. So, the limit $\varphi(\infty) := \lim_{u\to\infty} \varphi(u)$. Similarly to the derivation of (21), we can get that

$$\limsup_{v\to\infty} \sum_{k=0}^{v+3} \big(\varphi(\infty) - \varphi(k)\big)\big(1 - D(v+3-k)\big) \leqslant \sup_{k\geqslant K+1} \big(\varphi(\infty) - \varphi(k)\big)\mathbb{E}S$$

for each fixed $K \geqslant 1$. Therefore,

$$\begin{aligned}\lim_{v\to\infty} \sum_{k=0}^{v+3} \varphi(k)\big(1 - D(v+3-k)\big) &= \varphi(\infty) \lim_{v\to\infty} \sum_{k=0}^{v+3} \big(1 - D(k)\big)\\ &= \varphi(\infty)\mathbb{E}S.\end{aligned} \tag{22}$$

Relations (20), (21), and (22) imply that

$$\varphi(\infty)(3 - \mathbb{E}S) = \varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1). \tag{23}$$

Now we consider the last equality and examine all possible cases.

(I) If $\mathbb{E}S > 3$, then (23) implies that $\varphi(\infty) = 0$ because the left side of (23) is nonnegative in all cases. So, in this case, $\psi(u) = 1$ for all $u \in \{0, 1, 2, \ldots\}$.

(II) If $\mathbb{E}S = 3$, then (23) implies that

$$\varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1) = 0.$$

Additionally, in this situation we have that $s_3 = 1$ or $s_3 < 1$.

(II-A) If $s_3 = 1$, then we have

$$\begin{cases} a_0b_0c_0 = 0,\\ a_1b_0c_0 + a_0b_1c_0 + a_0b_0c_1 = 0,\\ a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_1b_1c_0 + a_0b_1c_1 + a_1b_0c_1 = 0,\\ a_0b_0c_3 + a_0b_3c_0 + a_3b_0c_0 + a_1b_2c_0 + a_2b_1c_0\\ \qquad +a_0b_1c_2 + a_0b_2c_1 + a_1b_0c_2 + a_2b_0c_1 + a_1b_1c_1 = 1,\\ \varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1) = 0.\end{cases}$$

Taking into account that all numbers a_k, b_k, and c_k are local probabilities for all $k \in \mathbb{N}_0$, the last system implies the following possible cases.

(a) $\{a_3 = b_0 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = \varphi(2) = 0$. In this case, $\psi(0) = \psi(1) = \psi(2) = 1$ and $\psi(u) = 0$, $u \in \{3, 4, \dots\}$ because

$$W_u(n) = \begin{cases} u-2 & \text{if } n \equiv 1 \bmod 3, \\ u-1 & \text{if } n \equiv 2 \bmod 3, \\ u & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

(b) $\{a_0 = b_3 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \dots\}$ because

$$W_u(n) = \begin{cases} u+1 & \text{if } n \equiv 1 \bmod 3, \\ u-1 & \text{if } n \equiv 2 \bmod 3, \\ u & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

(c) $\{a_0 = b_0 = c_3 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \dots\}$ because

$$W_u(n) = \begin{cases} u+1 & \text{if } n \equiv 1 \bmod 3, \\ u+2 & \text{if } n \equiv 2 \bmod 3, \\ u & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

(d) $\{a_2 = b_1 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \dots\}$.

(e) $\{a_1 = b_2 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \dots\}$.

(f) $\{a_0 = b_2 = c_1 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \dots\}$.

(g) $\{a_0 = b_1 = c_2 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \dots\}$.

(h) $\{a_2 = b_0 = c_1 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \dots\}$.

(i) $\{a_1 = b_0 = c_2 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \dots\}$.

(j) $\{a_1 = b_1 = c_1 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \dots\}$.

(II-B) If $s_3 < 1$ and $\mathbb{E}S = 3$, then it is necessary that $s_0 \neq 0$ or $s_1 \neq 0$ or $s_2 \neq 0$ because, on the contrary, $\mathbb{E}S = 3s_3 + 4s_4 + 5s_5 + \cdots > 3(s_3 + s_4 + \cdots) = 3$. In this situation, it suffices to consider the following cases:

$$\{s_0 \neq 0\}, \qquad \{s_0 = 0, s_1 \neq 0\}, \qquad \{s_0 = 0, s_1 = 0, s_2 \neq 0\}.$$

(k) If $s_0 = a_0b_0c_0 \neq 0$, then (23) implies that $\varphi(0) = \varphi(1) = \varphi(2) = 0$, and from (16) we obtain $\varphi(u) = 0$ for all $u \in \{3, 4, \dots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \dots\}$ in this case.

(l) If $s_0 = a_0b_0c_0 = 0$ and $s_1 = a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 \neq 0$, then we have the following possible cases.

(l-1) $\{a_0 = 0, a_1 \neq 0, b_0 \neq 0, c_0 \neq 0\}$. In this situation, Eq. (23) implies that $\varphi(0) = \varphi(1) = \varphi(2) = 0$, whereas (16) implies that $\varphi(u) = 0$ for all $u \in \{3, 4, \dots\}$. So, $\psi(u) = 1$ for all $u \in \{0, 1, \dots\}$ in this case.

(l-2) $\{a_0 \neq 0, b_0 = 0, b_1 \neq 0, c_0 \neq 0\}$. In this situation, Eq. (23) implies that $\varphi(0) = \varphi(1) = 0$, and (16) implies that $\varphi(u) = 0$ for all $u \in \{2, 3, \dots\}$. Therefore, $\psi(u) = 1$ for all $u \in \{0, 1, \dots\}$ again.

(l-3) $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$.

Equality (23) implies that $\varphi(0) = \varphi(1) = 0$, and (16) implies that $\varphi(u) = 0$ for all $u \in \{2, 3, \dots\}$. So, in this case, $\psi(u) = 1$ for all $u \in \{0, 1, \dots\}$.

(m) If $s_0 = a_0b_0c_0 = 0$, $s_1 = a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 = 0$ and $s_2 = a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_1b_1c_0 + a_1b_0c_1 + a_0b_1c_1 \neq 0$, then there exist the following possible cases.

(m-1) $\{a_0 = 0, a_1 = 0, a_2 \neq 0, b_0 \neq 0, c_0 \neq 0\}$;

(m-2) $\{a_0 \neq 0, b_0 = 0, b_1 = 0, b_2 \neq 0, c_0 \neq 0\}$;

(m-3) $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 = 0, c_2 \neq 0\}$;

(m-4) $\{a_0 = 0, a_1 \neq 0, b_0 = 0, b_1 \neq 0, c_0 \neq 0\}$;

(m-5) $\{a_0 = 0, a_1 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$;

(m-6) $\{a_0 \neq 0, b_0 = 0, b_1 \neq 0, c_0 = 0, c_1 \neq 0\}$.

In all cases, Eqs. (23) and (16) imply that $\varphi(u) = 0$, and so $\psi(u) = 1$ for all $u \in \mathbb{N}_0$. Theorem 4 is proved. □

4 Recursive formulas

In this section, we prove Theorem 5. Equality (23) from the previous section plays a crucial role.

Proof of Theorem 5. Let we consider the case $\mathbb{E}S < 3$. First, we prove that $\varphi(\infty) = 1$. According to the definition

$$\varphi(\infty) = \lim_{u\to\infty} \mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{\sum_{i=1}^{n}(Z_i - 1) < u\right\}\right) = \lim_{u\to\infty} \mathbb{P}\left(\sup_{u\geqslant 1} \eta_n < u\right),$$

where

$$\eta_n = \sum_{i=1}^{n} (Z_i - 1), \quad n \in \mathbb{N}.$$

If $n = 3N$, $N \in \mathbb{N}$, then

$$\frac{\eta_n}{n} = \frac{\eta_{3N}}{3N} = \frac{1}{3}\left(\frac{1}{N}\sum_{i=1}^{N}(Z_{3i-2} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i-1} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1)\right).$$

If $n = 3N + 1$, $N \in \mathbb{N}$, then

$$\begin{aligned}\frac{\eta_n}{n} = \frac{\eta_{3N+1}}{3N+1} &= \frac{N+1}{3N+1}\,\frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-2} - 1)\\ &\quad + \frac{N}{3N+1}\left(\frac{1}{N}\sum_{i=1}^{N}(Z_{3i-1} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1)\right).\end{aligned}$$

If $n = 3N + 2$, $N \in \mathbb{N}$, then

$$\begin{aligned}\frac{\eta_n}{n} = \frac{\eta_{3N+2}}{3N+2} &= \frac{N}{3N+2}\,\frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1)\\ &\quad + \frac{N+1}{3N+2}\left(\frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-2} - 1) + \frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-1} - 1)\right).\end{aligned}$$

Hence, the strong law of large numbers implies that

$$\frac{\eta_n}{n} \underset{n\to\infty}{\to} \frac{1}{3}(\mathbb{E}Z_1 - 1 + \mathbb{E}Z_2 - 1 + \mathbb{E}Z_3 - 1) = \frac{\mathbb{E}S - 3}{3}$$

almost surely.

From this it follows that

$$\mathbb{P}\left(\sup_{m\geqslant n}\left|\frac{\eta_m}{m} + \mu\right| < \frac{\mu}{2}\right) \underset{n\to\infty}{\to} 1 \tag{24}$$

with $\mu := (\mathbb{E}S - 3)/3 > 0$.

For arbitrary positive u and $N \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{n\geqslant 1}\eta_n < u\right) &\geqslant \mathbb{P}\left(\left(\bigcap_{n=1}^{N}\left\{\eta_n \leqslant \frac{u}{2}\right\}\right) \cap \left(\bigcap_{n=N+1}^{\infty}\left\{\eta_n \leqslant \frac{u}{2}\right\}\right)\right)\\ &\geqslant \mathbb{P}\left(\bigcap_{n=1}^{N}\left\{\eta_n \leqslant \frac{u}{2}\right\}\right) + \mathbb{P}\left(\bigcap_{n=N+1}^{\infty}\{\eta_n \leqslant 0\}\right) - 1\\ &\geqslant \mathbb{P}\left(\bigcap_{n=1}^{N}\left\{\eta_n \leqslant \frac{u}{2}\right\}\right) + \mathbb{P}\left(\sup_{m\geqslant N+1}\left|\frac{\eta_m}{m} + \mu\right| < \frac{\mu}{2}\right) - 1.\end{aligned}$$

The last inequality implies that

$$\lim_{u\to\infty}\mathbb{P}\Big(\sup_{n\geqslant 1}\eta_n<u\Big)\geqslant \mathbb{P}\Big(\sup_{m\geqslant N+1}\Big|\frac{\eta_m}{m}+\mu\Big|<\frac{\mu}{2}\Big)$$

for arbitrary $N\in\mathbb{N}$.

Hence, according to (24), we have that $\varphi(\infty)=1$.

Substituting this into (23), we get

$$3-\mathbb{E}S=\varphi(0)+b_0c_0\varphi(2)+b_0c_1\varphi(1)+c_0\varphi(1). \tag{25}$$

In addition, Eq. (16) can be rewritten as follows:

$$\begin{aligned}\varphi(u)=&\sum_{k=0}^{u+2}s_{u+2-k}\,\varphi(k+1)-a_{u+1}b_0c_0\varphi(2)-a_{u+1}b_0c_1\varphi(1)\\ &-c_0\varphi(1)\sum_{k=0}^{u+2}a_kb_{u+2-k},\quad u\in\mathbb{N}_0.\end{aligned} \tag{26}$$

Now we consider the last two formulas to get a suitable recursion procedure described in Theorem 5.

- First, let $s_0=a_0b_0c_0\neq 0$, and let the sequences $\alpha_n,\beta_n,\gamma_n$ be defined in the statement of Theorem 5.

We prove (4) by induction. We observe that relation (25) implies immediately:

$$\begin{aligned}\varphi(0)&=\alpha_0\varphi(0)+\beta_0\varphi(1)+\gamma_0(3-\mathbb{E}S),\\ \varphi(1)&=\alpha_1\varphi(0)+\beta_1\varphi(1)+\gamma_1(3-\mathbb{E}S),\\ \varphi(2)&=\alpha_2\varphi(0)+\beta_2\varphi(1)+\gamma_2(3-\mathbb{E}S).\end{aligned}$$

Now suppose that Eq. (4) holds for all $n=0,1,\ldots,N-1$, and we will prove that (4) holds for $n=N$. By (26) we have

$$\begin{aligned}\varphi(N-3)=&\sum_{k=0}^{N-1}s_{N-1-k}\,\varphi(k+1)-a_{N-2}b_0c_0\varphi(2)-a_{N-2}b_0c_1\varphi(1)\\ &-c_0\varphi(1)\sum_{k=0}^{N-1}a_kb_{N-1-k}.\end{aligned}$$

Therefore, using the induction hypothesis, we get

$$\begin{aligned}s_0\varphi(N)=&\,\varphi(N-3)-\sum_{k=1}^{N-1}s_k\varphi(N-k)+a_{N-2}b_0c_0\varphi(2)\\ &+a_{N-2}b_0c_1\varphi(1)+c_0\varphi(1)\sum_{k=0}^{N-1}a_kb_{N-1-k}\end{aligned}$$

$$
\begin{aligned}
&= \alpha_{N-3}\varphi(0) + \beta_{N-3}\varphi(1) + \gamma_{N-3}(3 - \mathbb{E}S) \\
&\quad - \sum_{k=1}^{N-1} s_k\big(\alpha_{N-k}\varphi(0) + \beta_{N-k}\varphi(1) \\
&\quad + \gamma_{N-k}(3 - \mathbb{E}S)\big) + a_{N-2}b_0c_0\varphi(2) + a_{N-2}b_0c_1\varphi(1) \\
&\quad + c_0\varphi(1)\sum_{k=0}^{N-1} a_k b_{N-1-k}.
\end{aligned} \tag{27}
$$

Since

$$
\varphi(2) = -\frac{1}{b_0c_0}\varphi(0) - \frac{c_1}{c_0}\varphi(1) - \frac{1}{b_0}\varphi(1) + \frac{3-\mathbb{E}S}{b_0c_0}
$$

due to (25), we obtain from (27) that

$$
\begin{aligned}
\varphi(N) &= \varphi(0)\,\frac{1}{s_0}\Bigg(\alpha_{N-3} - \sum_{k=1}^{N-1} s_k\alpha_{N-k} - a_{N-2}\Bigg) \\
&\quad + \varphi(1)\,\frac{1}{s_0}\Bigg(\beta_{N-3} - \sum_{k=1}^{N-1} s_k\beta_{N-k} - a_{N-2}c_0 + c_0\sum_{k=0}^{N-1} a_k b_{N-1-k}\Bigg) \\
&\quad + (3 - \mathbb{E}S)\,\frac{1}{s_0}\Bigg(\gamma_{N-3} - \sum_{k=1}^{N-1} s_k\gamma_{N-k} + a_{N-2}\Bigg) \\
&= \alpha_N\varphi(0) + \beta_N\varphi(1) + \gamma_N(3 - \mathbb{E}S).
\end{aligned}
$$

Hence, the desired relation (4) holds for all $n \in \mathbb{N}_0$ by induction.

- If $\{a_0 = 0, b_0 \neq 0, c_0 \neq 0, a_1 \neq 0\}$, then $s_0 = 0$ and $s_1 \neq 0$. Equality (26) with $u = 0$ implies that $\varphi(0) = 0$. The recursive relation (5) can be derived from the basic equalities (25) and (26) in the same manner as relation (4).
- If $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$, then it follows from Eq. (25) that $3 - \mathbb{E}S = \varphi(0) + c_0\varphi(1)$. Hence, $\varphi(1) = \tilde{\alpha}_1\varphi(0) + \tilde{\gamma}_1(3 - \mathbb{E}S)$. This is Eq. (6) for $n = 1$. The validity of (6) for the other n can be derived from (26) using the induction arguments.
- In the case $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$, formula (7) follows from (25) if $n = 1$. For the other n, formula (7) follows from (26) again by using the induction arguments.
- In the case $\{a_0 = 0, b_0 = 0, c_0 \neq 0\}$, we have that $s_0 = s_1 = 0$ and $s_2 \neq 0$ because of $\mathbb{E}S < 3$. It follows immediately from (26) that $\varphi(0) = 0$, whereas from (25) it follows that $\varphi(1) = (3 - \mathbb{E}S)/s_2$. Finally, we can get the recursive formula (8) from (26) using the same induction procedure.
- In the case $\{a_0 = 0, b_0 \neq 0, c_0 = 0\}$, similarly as in the previous one, we derive that $\varphi(0) = s_2\varphi(1)$ from (26), we derive that $3 - \mathbb{E}S = \varphi(0) + b_0c_1\varphi(1)$ from (25), and we derive the desired formula (9) again from (26).

- The case $\{a_0 \neq 0, b_0 = 0, c_0 = 0\}$ is considered completely analogously as both previous cases. Here we omit details.

We have that $\mathbb{E}S < 3$. So, it remains to study the following possible cases:

$$\{a_0 = a_1 = 0, b_0 \neq 0, c_0 \neq 0\}, \qquad \{a_0 \neq 0, b_0 = b_1 = 0, c_0 \neq 0\},$$
$$\{a_0 \neq 0, b_0 \neq 0, c_0 = c_1 = 0\}.$$

In all these cases, the presented recursion relations follow from Eq. (26), and the initial values of survival probability $\varphi(0)$ and $\varphi(1)$ can be obtained using Eq. (25) together with Eq. (26) with $u = 0$ or $u = 1$. Theorem 5 is proved. □

5 Numerical examples

In this section, we present three examples of computing numerical values of the finite-time ruin probability and the ultimate ruin probability. All calculations are carried out using software MATHEMATICA. In all presented tables, the numbers are rounded up to three decimal places.

First example. Suppose that the three-seasonal discrete-time risk model is generated by the r.v.s

Z_1	0	1	2
$\mathbb{P}$	0.5	0.25	0.25

,

Z_2	0	1	2
$\mathbb{P}$	0.4	0.3	0.3

,

Z_3	0	1	2
$\mathbb{P}$	0.3	0.35	0.35

.

In Table 1, we give the finite-time ruin probabilities for initial surpluses $u \in \{0, 1, \ldots 10, 20\}$ and times $T \in \{1, 2, \ldots, 10, 20\}$ together with the ultimate ruin probabilities for the same u.

Numerical values of the finite-time ruin probabilities are calculated using the algorithm presented in Theorem 3, whereas the values of the ultimate ruin probabilities are obtained using the formulas of Theorem 5. Namely, first, we observe that $\mathbb{E}S = 2.7$ and $s_0 \neq 0$ in this case. So, Eq. (4) holds for an arbitrary $n \in \mathbb{N}_0$. In

Table 1. Ruin probabilities for the first model

T	u											
	0	1	2	3	4	5	6	7	8	9	10	20
1	0.5	0.25	0	0	0	0	0	0	0	0	0	0
2	0.65	0.325	0.075	0	0	0	0	0	0	0	0	0
3	0.703	0.404	0.128	0.026	0	0	0	0	0	0	0	0
4	0.733	0.445	0.169	0.046	0.007	0	0	0	0	0	0	0
5	0.751	0.475	0.2	0.066	0.014	0.002	0	0	0	0	0	0
6	0.768	0.503	0.233	0.089	0.026	0.005	0.001	0	0	0	0	0
7	0.779	0.523	0.256	0.106	0.035	0.009	0.002	0	0	0	0	0
8	0.788	0.538	0.275	0.122	0.045	0.014	0.003	0.001	0	0	0	0
9	0.796	0.554	0.295	0.139	0.056	0.019	0.006	0.001	0	0	0	0
10	0.802	0.566	0.310	0.152	0.065	0.024	0.008	0.002	0	0	0	0
20	0.836	0.632	0.402	0.243	0.138	0.075	0.038	0.018	0.008	0.003	0.001	0
∞	0.877	0.722	0.541	0.404	0.301	0.224	0.167	0.125	0.093	0.069	0.052	0.003

Table 2. Ruin probabilities for the Poison model

T	u											
	0	1	2	3	4	5	6	7	8	9	10	20
1	0.393	0.09	0.014	0.002	0	0	0	0	0	0	0	0
2	0.481	0.152	0.037	0.008	0.001	0	0	0	0	0	0	0
3	0.535	0.205	0.066	0.019	0.005	0.001	0	0	0	0	0	0
4	0.549	0.221	0.075	0.023	0.006	0.002	0	0	0	0	0	0
5	0.562	0.236	0.086	0.028	0.009	0.002	0.001	0	0	0	0	0
6	0.576	0.254	0.099	0.036	0.012	0.004	0.001	0	0	0	0	0
7	0.581	0.26	0.103	0.038	0.013	0.004	0.001	0	0	0	0	0
8	0.585	0.266	0.109	0.042	0.015	0.005	0.002	0.001	0	0	0	0
9	0.591	0.274	0.115	0.046	0.017	0.006	0.002	0.001	0	0	0	0
10	0.593	0.277	0.118	0.048	0.018	0.007	0.002	0.001	0	0	0	0
20	0.605	0.295	0.134	0.059	0.026	0.011	0.005	0.002	0.001	0.0003	0.0001	0
∞	0.609	0.3	0.139	0.064	0.029	0.013	0.006	0.003	0.001	0.001	0.0002	0

particular,

$$\begin{cases} \varphi(250) = \alpha_{250}\varphi(0) + \beta_{250}\varphi(1) + 0.3\gamma_{250}, \\ \varphi(251) = \alpha_{251}\varphi(0) + \beta_{251}\varphi(1) + 0.3\gamma_{251}, \end{cases}$$

According to the first statement of Theorem 5, we can suppose that $\varphi(250) = \varphi(251) = 1$. So, we get $\varphi(0)$ and $\varphi(1)$ from this system after calculating the values of $\{\alpha_0, \alpha_1, \dots \alpha_{251}\}$, $\{\beta_0, \beta_1, \dots, \beta_{251}\}$, and $\{\gamma_0, \gamma_1, \dots, \gamma_{251}\}$. Now it remains to use Eq. (4) again to obtain the values $\varphi(u) = 1 - \psi(u)$ for initial surplus values $u \in \{2, 3, \dots, 10, 20\}$.

Second example. Suppose now that the three-seasonal discrete-time risk model is generated by three Poison distributions: Z_1 with parameter $1/2$, Z_2 with parameter $2/3$, and Z_3 with parameter $4/5$. In Table 2, we present the finite-time ruin probabilities for $u \in \{0, 1, \dots 10, 20\}$, $T \in \{1, 2, \dots, 10, 20\}$ and the ultimate ruin probabilities for $u \in \{0, 1, \dots 10, 20\}$. All calculations are made similarly as in the first example.

Third example. We write $\xi \sim \mathcal{G}(p)$ if ξ is a r.v. having the geometric distribution with parameter $p \in (0, 1)$, that is, $\mathbb{P}(\xi = k) = p(1-p)^k, k \in \mathbb{N}_0$. Suppose that the

Table 3. Ruin probabilities for the geometric model

T	u											
	0	1	2	3	4	5	6	7	8	9	10	20
1	0.25	0.063	0.016	0.004	0.001	0	0	0	0	0	0	0
2	0.333	0.111	0.037	0.012	0.004	0.001	0	0	0	0	0	0
3	0.556	0.34	0.218	0.143	0.094	0.063	0.042	0.028	0.019	0.012	0.008	0
4	0.566	0.35	0.226	0.149	0.099	0.066	0.044	0.029	0.019	0.013	0.009	0
5	0.576	0.362	0.236	0.156	0.104	0.069	0.046	0.031	0.021	0.014	0.009	0
6	0.653	0.461	0.334	0.243	0.176	0.127	0.091	0.065	0.046	0.033	0.023	0.001
7	0.657	0.466	0.338	0.247	0.18	0.13	0.093	0.067	0.048	0.034	0.024	0.001
8	0.661	0.471	0.344	0.252	0.184	0.133	0.096	0.069	0.049	0.035	0.025	0.001
9	0.703	0.529	0.406	0.312	0.239	0.181	0.137	0.102	0.076	0.056	0.042	0.002
10	0.705	0.532	0.409	0.315	0.241	0.184	0.139	0.104	0.078	0.057	0.042	0.002
20	0.774	0.635	0.528	0.438	0.363	0.298	0.243	0.197	0.159	0.127	0.101	0.008
∞	0.927	0.879	0.84	0.803	0.769	0.736	0.705	0.675	0.647	0.619	0.593	0.385

three-seasonal risk model is generated by r.v.s $Z_1 \sim \mathcal{G}(3/4)$, $Z_2 \sim \mathcal{G}(2/3)$, and $Z_3 \sim \mathcal{G}(1/3)$. In Table 3, we present the finite-time and infinite-time ruin probabilities for this geometric model. The values of initial surpluses u and times T are the same as in the previous examples.

The presented tables show that the behavior of ruin probabilities is closely related to the structure of generating r.v.s and not only to the global model characteristic $\mathbb{E}S$.

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References

[1] Asmussen, S., Albrecher, H.: Ruin Probabilities. World Scientific, Singapore (2010). MR2766220. doi:10.1142/9789814282536

[2] Bieliauskienė, E., Šiaulys, J.: Gerber–Shiu function for the discrete inhomogeneous claim case. Int. J. Comput. Math. **89**, 1617–1630 (2012). MR2949469. doi:10.1080/00207160.2012.693607

[3] Blaževičius, K., Bieliauskienė, E., Šiaulys, J.: Finite-time ruin probability in the inhomogeneous claim case. Lith. Math. J. **50**, 260–270 (2010). MR2719562. doi:10.1007/s10986-010-9084-2

[4] Damarackas, J., Šiaulys, J.: Bi-seasonal discrete-time risk model. Appl. Math. Comput. **247**, 930–940 (2014). MR3270895. doi:10.1016/j.amc.2014.09.040

[5] De Vylder, F.E., Goovaerts, M.J.: Recursive calculation of finite-time ruin probabilities. Insur. Math. Econ. **7**, 1–8 (1988). MR0971858. doi:10.1016/0167-6687(88)90089-3

[6] De Vylder, F.E., Goovaerts, M.J.: Explicit finite-time and infinite-time ruin probabilities in the continuous case. Insur. Math. Econ. **24**, 155–172 (1999). MR1704808. doi:10.1016/S0167-6687(98)00049-3

[7] Dickson, D.C.M.: On numerical evaluation of finite time survival probabilities. Br. Actuar. J **5**, 575–584 (1999). doi:10.1017/S135732170000057X

[8] Dickson, D.C.M.: Insurance Risk and Ruin. Cambridge University Press, Cambridge (2005). MR2160707. doi:10.1017/CBO9780511624155

[9] Dickson, D.C.M., Waters, R.: Recursive calculation of survival probabilities. ASTIN Bull. **21**, 199–221 (1991). doi:10.2143/AST.21.2.2005364

[10] Gerber, H.U.: Mathematical fun with the compound binomial process. ASTIN Bull. **18**, 161–168 (1988). doi:10.2143/AST.18.2.2014949

[11] Lefèvre, Cl.: Loisel St.: On finite-time ruin probabilities for classical risk models. Scand. Actuar. J. **2008**, 41–60 (2008). MR2414622. doi:10.1080/03461230701766882

[12] Leipus, R., Šiaulys, J.: Finite-horizon ruin probability asymptotics in the compound discrete-time risk model. Lith. Math. J. **51**(2), 207–219 (2011). MR2805739. doi:10.1007/s10986-011-9120-x

[13] Li, S., Lu, Y., Garrido, J.: A review of discrete-time risk models. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **103**, 321–337 (2009). MR2582636. doi:10.1007/BF03191910

[14] Mikosch, T.: Non-Life Insurance Mathematics. Springer, Berlin (2009). MR2503328. doi:10.1007/978-3-540-88233-6

[15] Picard, Ph.: Lefèvre Cl.: The probability of ruin in finite time with discrete claim size distribution. Scandinavian Actuarial journal, 58–69 (1997). 1997. MR1440825. doi:10.1080/03461238.1997.10413978

[16] Picard, Ph.: Lefèvre Cl.: Probabilité de ruine éventuelle dans un modèle de risque à temps discret. J. Appl. Probab. **40**, 543–556 (2003). MR1993252. doi:10.1239/jap/1059060887

[17] Seal, H.L.: The Stochastic Theory of a Risk Business. Wiley, New York (1969)

[18] Shiu, E.S.W.: Calculation of the probability of eventual ruin by Beckman's convolution series. Insur. Math. Econ. **7**, 41–47 (1989). MR0971864. doi:10.1016/0167-6687(88)90095-9

[19] Shiu, E.S.W.: The probability of eventual ruin in the compound binomial model. ASTIN Bull. **19**, 179–190 (1989). doi:10.2143/AST.19.2.2014907

[20] Tang, Q.: The ruin probability of a discrete-time risk model under constant interest with heavy tails. Scand. Actuar. J. **2004**, 229–240 (2004).