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Asymptotics of exponential moments of a weighted local time of a Brownian motion with small variance

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Abstract We prove a large deviation type estimate for the asymptotic behavior of a weighted local time of εW as $\varepsilon \to 0$.

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1 Introduction and the main result

Let $\{W_t, t \geq 0\}$ be a real-valued Wiener process, and μ be a σ-finite measure on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \mu\big([x-1, x+1]\big) < \infty. \tag{1}$$

Recall that the *local time* $L_t^{\mu}(W)$ *of the process* W *with the weight* μ can be defined as the limit of the integral functionals

$$L_t^{\mu_n}(W) := \int_0^t k_n(W_s)\,ds, \quad k_n(x) := \frac{\mu_n(dx)}{dx}, \; n \geq 1, \tag{2}$$

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where μ_n, $n \geq 1$, is a sequence of absolutely continuous measures such that

$$\int_{\mathbb{R}} f(x)\mu_n(dx) \to \int_{\mathbb{R}} f(x)\mu(dx)$$

for all continuous f with compact support, and (1) holds for μ_n, $n \geq 1$, uniformly. The limit $L_t^\mu(W)$ exists in the mean square sense due to the general results from the theory of W-functionals; see [3], Chapter 6. This definition also applies to εW instead of W for any positive ε. In what follows, we will treat εW as a Markov process whose initial value may vary, and with a slight abuse of notation, we denote by $\mathbf{P}_x$ the law of εW with $\varepsilon W_0 = x$ and by $\mathbf{E}_x$ the expectation w.r.t. this law.

In this note, we study the asymptotic behavior as $\varepsilon \to 0$ of the exponential moments of the family of weighted local times $L_t^\mu(\varepsilon W)$. Namely, we prove the following theorem.

Theorem 1. *For arbitrary finite measure μ on $\mathbb{R}$,*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^\mu(\varepsilon W)} = \frac{t}{2} \sup_{y\in\mathbb{R}} \mu\big(\{y\}\big)^2. \tag{3}$$

For arbitrary σ-finite measure μ on $\mathbb{R}$ that satisfies (1),

$$\sup_{x\in\mathbb{R}} \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_x e^{L_t^\mu(\varepsilon W)} = \frac{t}{2} \sup_{y\in\mathbb{R}} \mu\big(\{y\}\big)^2. \tag{4}$$

We note that in this statement the measure μ can be changed to a signed measure; in this case, in the right-hand side, only the atoms of the positive part of μ should appear. We also note that, in the σ-finite case, the uniform statement (3) may fail; one example of such a type is given in Section 3.

Let us briefly discuss the problem that was our initial motivation for the study of such exponential moments. Consider the one-dimensional SDE

$$dX_t^\varepsilon = a\big(X_t^\varepsilon\big)\,dt + \varepsilon\sigma\big(X_t^\varepsilon\big)\,dW_t \tag{5}$$

with discontinuous coefficients a, σ. In [7], a Wentzel–Freidlin-type large deviation principle (LDP) was established in the case $a \equiv 0$ under mild assumptions on the diffusion coefficient σ. In [8], this result was extended to the particular class of SDEs such that the function a/σ^2 has a bounded derivative. This limitation had appeared because of formula (7) in [8] for the *rate transform* of the family X^ε. This formula contains an integral functional with kernel $(a/\sigma^2)'$ of a certain diffusion process obtained from εW by the time change procedure. If a/σ^2 is not smooth but is a function of a bounded variation, this integral function still can be interpreted as a weighted local time with weight $\mu = (a/\sigma^2)'$. Thus, Theorem 1 can be used in order to study the LDP for the SDE (5) with discontinuous coefficients. One of such particular results can be derived immediately. Namely, if μ is a *continuous* measure, then by Theorem 1 the exponential moments of $L_t^\mu(\varepsilon W)$ are negligible at the logarithmic scale with rate function ε^2. This, after simple rearrangements, allows us to neglect the corresponding term in (7) of [8] and to obtain the statement of Theorem 2.1 of [8] under the weaker condition that a/σ^2 is a continuous function of bounded variation. The problem how to describe in a more general situation the influence of the jumps of a/σ^2 on the LDP for the solution to (5) still remains open and is the subject of our ongoing research. We just remark that due to Theorem 1 the respective integral term is no longer negligible, which well corresponds to the LDP results for piecewise smooth coefficients a, σ obtained in [1, 2, 6].

2 Proof of Theorem 1

2.1 Preliminaries

For a measure ν satisfying (1), denote by

$$f_t^{\nu,\varepsilon}(x) = \mathbf{E}_x L_t^\nu(\varepsilon W) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s\varepsilon^2}} e^{-\frac{(y-x)^2}{2s\varepsilon^2}} \nu(dy)\, ds, \quad t \geq 0,\ x \in \mathbb{R}, \tag{6}$$

the characteristic of the local time $L^\nu(\varepsilon W)$ considered as a W-functional of εW; see [3], Chapter 6.

The following statement is a version of *Khas'minskii's lemma*; see [9], Section 1.2.

Lemma 1. *Suppose that*

$$\sup_{x\in\mathbb{R}} f_s^{\nu,\varepsilon}(x) \leq \frac{1}{2}. \tag{7}$$

Then

$$\sup_{x\in\mathbb{R}} \mathbf{E}_x e^{L_s^\nu(\varepsilon W)} \leq 2.$$

Using the Markov property, as a simple corollary, we obtain, for arbitrary $t > 0$,

$$\sup_{x\in\mathbb{R}} \mathbf{E}_x e^{L_t^\nu(\varepsilon W)} \leq 2^{1+t/s} = 2e^{(\log 2)(t/s)}, \tag{8}$$

where $s > 0$ is such that (7) holds. This inequality, combined with (6), leads to the following estimate.

Lemma 2. *For a nonzero measure ν satisfying (1), denote*

$$N(\nu,\gamma) = \sup_{x\in\mathbb{R}} \nu\big([x-\gamma, x+\gamma]\big), \quad \gamma > 0.$$

For any $\lambda \geq 1$ and $\gamma > 0$, there exists $\varepsilon_{\lambda,\gamma} > 0$ such that

$$\sup_{x\in\mathbb{R}} \mathbf{E}_x e^{\lambda L_t^\nu(\varepsilon W)} \leq 2e^{(4\log 2)c_0 N(\nu,\gamma)^2 t\lambda^2\varepsilon^{-2}}, \quad \varepsilon \in (0, \varepsilon_{\lambda,\gamma}), \tag{9}$$

with

$$c_0 = \frac{2}{\pi}\left(1 + 2\sum_{k=1}^\infty e^{-\frac{(2k-1)^2}{2}}\right)^2.$$

Proof. If $\varepsilon\sqrt{s} \leq \gamma$, then we have

$$\begin{aligned} f_s^{\nu,\varepsilon}(x) &= \sum_{k\in\mathbb{Z}} \int_0^s \int_{|y-x-2k\gamma|\leq\gamma} \frac{1}{\sqrt{2\pi v\varepsilon^2}} e^{-\frac{(y-x)^2}{2v\varepsilon^2}} \nu(dy)\, dv \\ &\leq \sqrt{c_0} N(\nu,\gamma) \sqrt{\frac{s}{\varepsilon^2}}. \end{aligned}$$

Take

$$s = \big(2N(\nu,\gamma)\big)^{-2} (c_0)^{-1} \lambda^{-2} \varepsilon^2.$$

Then the inequality $\varepsilon\sqrt{s} \le \gamma$ holds, provided that

$$\varepsilon \le \left(\gamma\big(2N(\nu,\gamma)\big)^2 c_0\lambda^2\right)^{1/3} =: \varepsilon_{\lambda,\gamma}.$$

Under this condition,

$$f_s^{\lambda\nu,\varepsilon}(x) = \lambda f_s^{\nu,\varepsilon}(x) \le \frac{1}{2}.$$

Now the required inequality follows immediately from (8). □

In what follows, we will repeatedly decompose μ into sums of two components and analyze separately the exponential moments of the local times that correspond to these components. We will combine these estimates and obtain an estimate for $L_t^\mu(\varepsilon W)$ itself using the following simple inequality. Let $\mu = \nu + \kappa$ and $p, q > 1$ be such that $1/p + 1/q = 1$. Then

$$L_t^\mu(\varepsilon W) = L_t^\nu(\varepsilon W) + L_t^\kappa(\varepsilon W) = (1/p)L_t^{p\nu}(\varepsilon W) + (1/q)L_t^{q\kappa}(\varepsilon W),$$

and therefore by the Hölder inequality we get

$$\mathbf{E}e^{L_t^\mu(\varepsilon W)} \le \left(\mathbf{E}e^{L_t^{p\nu}(\varepsilon W)}\right)^{1/p}\left(\mathbf{E}e^{L_t^{q\kappa}(\varepsilon W)}\right)^{1/q}. \tag{10}$$

We will also use another version of this upper bound, which has the form

$$\mathbf{E}e^{L_t^\mu(\varepsilon W)}1_A \le \left(\mathbf{E}e^{L_t^{p\mu}(\varepsilon W)}\right)^{1/p}\big(\mathbf{P}(A)\big)^{1/q}, \quad A \in \mathcal{F}. \tag{11}$$

We denote

$$\varDelta = \sup_{x\in\mathbb{R}} \mu\big(\{x\}\big).$$

We will prove Theorem 1 in several steps, in each of them extending the class of measures μ for which the required statement holds.

2.2 *Step I: μ is a finite mixture of δ-measures*

If $\mu = a\delta_z$ is a weighted δ-measure at the point z, then we have

$$L_t^\mu(\varepsilon W) = a\varepsilon^{-1}L_t^{(z)}(W),$$

where

$$L_t^{(z)}(W) = \lim_{\eta\to 0}\frac{1}{2\eta}\int_0^t 1_{|W_s - z|\le\eta}\,ds$$

is the *local time of a Wiener process* at the point z. The distribution of $L_t^{(z)}(W)$ is well known; see, e.g., [5], Chapter 2.2 and expression (6) in Chapter 2.1. Hence, the required statement in the particular case $\mu = a\delta_z$ is straightforward, and we have the following:

$$\lim_{\varepsilon\to 0}\varepsilon^2 \sup_x \log \mathbf{E}_x e^{a\varepsilon^{-1}L_t^{(z)}(W)} = \frac{ta^2}{2}. \tag{12}$$

Note that in this formula the supremum is attained at the point $x = z$.

In this section, we will extend this result to the case where μ is a finite mixture of δ-measures, that is,

$$\mu = \sum_{j=1}^{k} a_j \delta_{z_j}.$$

Let j_* be the number of the maximal value in $\{a_j\}$, that is, $\Delta = a_{j_*}$. Then $L_t^\mu(\varepsilon W) \geq \Delta\varepsilon^{-1} L_t^{(z_{j_*})}(W)$, and it follows directly from (12) that

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^\mu(\varepsilon W)} \geq \frac{t\Delta^2}{2}. \tag{13}$$

In what follows, we prove the corresponding upper bound

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^\mu(\varepsilon W)} \leq \frac{t\Delta^2}{2}, \tag{14}$$

which, combined with this lower bound, proves (3).

Observe that, for $\gamma > 0$ small enough,

$$N(\mu, \gamma) = \Delta.$$

Then by Lemma 2, for any $\lambda \geq 1$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{\lambda L_t^\mu(\varepsilon W)} \leq c_1 \lambda^2 t \Delta^2 \tag{15}$$

with

$$c_1 = (4\log 2)c_0 = \frac{8\log 2}{\pi}\left(1 + 2\sum_{k=1}^{\infty} e^{-\frac{(2k-1)^2}{2}}\right)^2.$$

In particular, taking $\lambda = 1$, we obtain an upper bound of the form (14), but with a worse constant c_1 instead of required $1/2$. We will improve this bound by using the large deviations estimates for εW, the Markov property, and the "individual" identities (12).

Denote $\mu_j = a_j \delta_{z_j}$, $j = 1, \dots, k$. Then

$$L_t^\mu(\varepsilon W) = \sum_{j=1}^{k} L_t^{\mu_j}(\varepsilon W).$$

Fix some family of neighborhoods O_j of z_j, $j = 1, \dots, k$, such that the minimal distance between them equals $\rho > 0$, and denote

$$O^j = \mathbb{R} \setminus \bigcup_{i\neq j} O_i.$$

For some $N \geq 1$ whose particular value will be specified later, consider the partition $t_n = t(n/N)$, $n = 0, \dots, N$, of the segment $[0, t]$ and denote

$$B_{n,j} = \left\{ f \in C(0,t) : f_s \in O^j, s \in [t_{n-1}, t_n] \right\}, \quad j \in \{1, \dots, k\},\ n \in \{1, \dots, N\},$$

$$C_{j_1,\dots,j_N}=\bigcap_{n=1}^N B_{n,j_n},\qquad j_1,\dots,j_N\in\{1,\dots,k\}.$$

Observe that if the process εW does not visit O_j on the time segment $[u,v]$, then $L^{\mu_j}(\varepsilon W)$ on this segment stays constant. This means that, on the set $\{\varepsilon W\in C_{j_1,\dots,j_N}\}$, we have

$$L_t^\mu(\varepsilon W)=\sum_{n=1}^N\big(L_{t_n}^{\mu_{j_n}}(\varepsilon W)-L_{t_{n-1}}^{\mu_{j_n}}(\varepsilon W)\big).$$

Because $L^{\mu_j}(\varepsilon W)$ is a time-homogeneous additive functional of the Markov process εW, we have

$$E_x\big[e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W)-L_{t_{n-1}}^{\mu_{j_n}}(\varepsilon W)}\big|\mathcal{F}_{t_{n-1}}\big]=E_y e^{L_{t/N}^{\mu_{j_n}}(\varepsilon W)}\Big|_{y=\varepsilon W_{t_{n-1}}}.$$

Then by (12), for any $j_1,\dots,j_N\in\{1,\dots,k\}$,

$$\limsup_{\varepsilon\to0}\varepsilon^2\sup_{x\in\mathbb{R}}\log\mathbf{E}_x e^{L_t^\mu(\varepsilon W)}1_{\varepsilon W\in C_{j_1,\dots,j_N}}\le\frac{t}{2N}\sum_{n=1}^N(a_{j_n})^2\le\frac{t\Delta^2}{2}.$$

Because we have a fixed number of sets $C_{j_1,\dots,j_N}$, this immediately yields

$$\limsup_{\varepsilon\to0}\varepsilon^2\sup_{x\in\mathbb{R}}\log\mathbf{E}_x e^{L_t^\mu(\varepsilon W)}1_{\varepsilon W\in C}\le\frac{t\Delta^2}{2}\tag{16}$$

with

$$C=\bigcup_{j_1,\dots,j_N\in\{1,\dots,k\}}C_{j_1,\dots,j_N}.$$

Hence, to get the required upper bound (14), it suffices to prove an analogue of (16) with the set C replaced by its complement $D=C(0,t)\setminus C$. Using (11) with $p=2$, $A=\{\varepsilon W\in D\}$, and (15) with $\lambda=2$, we get

$$\limsup_{\varepsilon\to0}\varepsilon^2\sup_{x\in\mathbb{R}}\log\mathbf{E}_x e^{L_t^\mu(\varepsilon W)}1_{\varepsilon W\in D}\le 2c_1t\Delta^2+\frac12\limsup_{\varepsilon\to0}\varepsilon^2\sup_{x\in\mathbb{R}}\log\mathbf{P}_x(\varepsilon W\in D).$$

By the LDP for the Wiener process ([4], Chapter 3, §2),

$$\limsup_{\varepsilon\to0}\varepsilon^2\sup_{x\in\mathbb{R}}\log\mathbf{P}_x(\varepsilon W\in D)=-\inf_{f\in\mathrm{closure}(D)}I(f),$$

where

$$I(f)=\begin{cases}(1/2)\int_0^t(f_s')^2\,ds, & f \text{ is absolutely continuous on } [0,t];\\ +\infty & \text{otherwise.}\end{cases}$$

For any trajectory $f\in D$, there exists n such that f visits at least two sets O_j on the time segment $[t_{n-1},t_n]$. Therefore, any trajectory $f\in\mathrm{closure}(D)$ exhibits an oscillation $\ge\rho$ on this time segment. On the other hand, for an absolutely continuous f,

$$|f_u-f_v|=\left|\int_u^v f_s'\,ds\right|\le|u-v|^{1/2}\bigg(\int_0^t\big(f_s'\big)^2\,ds\bigg)^{1/2}.$$

This means that, for any $f \in \text{closure}(D)$,

$$I(f) \geq \frac{\rho^2 N}{2t},$$

which yields

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^{\mu}(\varepsilon W)} 1_{\varepsilon W\in D} \leq 2c_1 t \Delta^2 - \frac{\rho^2 N}{2t}.$$

If in this construction, N was chosen such that

$$N \geq (4c_1 - 1)\rho^{-2} t^2 \Delta^2,$$

then the latter inequality guarantees the analogue of (16) with D instead of C. This completes the proof of (14).

2.3 Step II: μ is finite

Exactly the same argument as that used in Section 2.2 provides the lower bound (13). In this section, we prove the upper bound (14) for a finite measure μ and thus complete the proof of the first assertion of the theorem. For finite μ and any $\chi > 0$, we can find $\gamma > 0$ and decompose $\mu = \mu_0 + \nu$ in such a way that μ_0 is a finite mixture of δ-measures and $N(\nu, \gamma) < \chi$. Let $p, q > 1$ be such that $1/p + 1/q = 1$. The measure $p\mu_0$ has the maximal weight of an atom equal to $p\Delta$. Since we have already proved the required statement for finite mixtures of δ-measures, we have

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log\big(\mathbf{E}_x e^{L_t^{p\mu_0}(\varepsilon W)}\big)^{1/p} \leq \frac{t}{2} p\Delta^2. \tag{17}$$

On the other hand, we have $N(\nu, \gamma) < \chi$ and then by Lemma 2

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log\big(\mathbf{E}_x e^{L_t^{q\nu}(\varepsilon W)}\big)^{1/q} \leq c_1 q t \chi^2.$$

Hence, by (10),

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^{\mu}(\varepsilon W)} \leq \frac{t}{2} p\Delta^2 + c_1 q t \chi^2.$$

Now we can finalize the argument. Fix $\Delta_1 > \Delta := \max_{x\in\mathbb{R}} \mu(\{x\})^2$ and choose $p, q > 1$ such that $1/p + 1/q = 1$ and $p\Delta^2 < \Delta_1^2$. Then there exists $\chi > 0$ small enough such that

$$p\Delta^2 + 2c_1 q t \chi^2 < \Delta_1^2.$$

Taking the decomposition $\mu = \mu_0 + \nu$ that corresponds to this value of χ and applying the previous calculations, we obtain an analogue of the upper bound (14) with Δ replaced by Δ_1. Since $\Delta_1 > \Delta$ is arbitrary, the same inequality holds for Δ.

2.4 Step III: μ is σ-finite

In this section, we prove the second assertion of the theorem. As before, the lower bound can be obtained directly from the case $\mu = a\delta_z$, and hence we concentrate ourselves on the proof of the upper bound

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_x e^{L_t^{\mu}(\varepsilon W)} \leq \frac{t\Delta^2}{2}, \quad x \in \mathbb{R}. \tag{18}$$

We will use an argument similar to that from the previous section and decompose μ into a sum $\mu = \mu_0 + \nu$ with finite μ_0 and ν, which is negligible in a sense. However, such a decomposition relies on the initial value x, and this is the reason why we obtain an individual upper bound (18) instead of the uniform one (14).

Namely, for a given x, we define μ_0, ν by restricting μ to $[x-R, x+R]$ and its complement, respectively. Without loss of generality, we assume that for each R, the corresponding ν is nonzero. Since we have already proved the required statement for finite measures, we get (17).

Next, denote $M = \sup_{x\in\mathbb{R}} \mu([x-1, x+1])$ and observe that $N(\nu, 1) \le M$. Then by Lemma 2 with $\gamma = 1$ and the strong Markov property, for any stopping time τ, the exponential moment of $L_t^{q\nu}(\varepsilon W)$ conditioned by $\mathcal{F}_\tau$ is dominated by $2e^{c_1 M^2 t q^2 \varepsilon^{-2}}$. This holds for $\varepsilon \le \varepsilon_{q,1}^{x,R}$, where we put the indices x, R in order to emphasize that this constant depends on ν, which, in turn, depends on x, R. Since we have assumed that, for any x, R, the respective ν is nonzero, the constants $\varepsilon_{q,1}^{x,R}$ are strictly positive.

Now we take by τ the first time moment when $|\varepsilon W_\tau - x| = R$. Observe that $L_t^{\nu}(\varepsilon W)$ equals 0 on the set $\{\tau > t\}$ and it is well known that

$$\mathbf{P}_x(\tau < t) \le 4\mathbf{P}_x(\varepsilon W_t > R) \le Ce^{-tR^2\varepsilon^{-2}/2}.$$

Summarizing the previous statements, we get

$$\mathbf{E}_x e^{L_t^{q\nu}(\varepsilon W)} \le 1 + 2Ce^{t\varepsilon^{-2}(c_1 M^2 q^2 - R^2/2)}, \quad \varepsilon \le \varepsilon_{\lambda,1}^{x,R},$$

which implies

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log\big(\mathbf{E}_x e^{L_t^{q\nu}(\varepsilon W)}\big)^{1/q} \le t\big(c_1 M^2 q - R^2/(2q)\big)_+, \tag{19}$$

where we denote $a_+ = \max(a, 0)$. By (10) inequalities (17) and (19) yield

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_x e^{L_t^{\mu_0}(\varepsilon W)} \le \frac{t}{2} p\Delta^2 + t\big(c_1 M^2 q - R^2/(2q)\big)_+.$$

Now we finalize the argument in the same way as we did in the previous section. Fix $\Delta_1 > \Delta$ and take $p > 1$ such that $p\Delta^2 \le \Delta_1^2$. Then take R large enough so that, for the corresponding q,

$$c_1 M^2 q - R^2/(2q) \le 0.$$

Under such a choice, the calculations made before yield (18) with Δ replaced by Δ_1. Since $\Delta_1 > \Delta$ is arbitrary, the same inequality holds for Δ.

3 Example

Let

$$\mu = \sum_{k=1}^{\infty} (\delta_{k^2} + \delta_{k^2+2^{-k}}).$$

Then μ satisfies (1) and $\Delta = 1$. However, it is an easy observation that when the initial value x is taken in the form $x_k = k^2$, the respective exponential moments satisfy

$$\mathbf{E}_{x_k} e^{L_t^{\mu}(\varepsilon W)} \to \mathbf{E}_0 e^{L_t^{\nu}(\varepsilon W)}, \quad k \to \infty,$$

with $\nu = 2\delta_0$. Then

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \sup_{x\in\mathbb{R}} \log \mathbf{E}_x e^{L_t^{\mu}(\varepsilon W)} \geq \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}_0 e^{L_t^{\nu}(\varepsilon W)} = 2t > \frac{t}{2},$$

and therefore (3) fails.

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