Modern Stochastics: Theory and Applications 3 (2016) 119–131
DOI: 10.15559/16-VMSTA55

On fractal faithfulness and fine fractal properties of random variables with independent Q^*-digits

Muslem Ibragim[a], **Grygoriy Torbin**[a,b,*]

[a]*National Pedagogical Dragomanov University, Pyrogova str. 9, 01030, Kyiv, Ukraine*
[b]*Institute for Mathematics of National Academy of Sciences, Tereshchenkivska str. 3, 01601, Kyiv, Ukraine*

muslemhussen1978@yahoo.com (M. Ibragim), torbin7@gmail.com (G. Torbin)

Received: 20 May 2016, Accepted: 3 June 2016,
Published online: 9 June 2016

Abstract We develop a new technique to prove the faithfulness of the Hausdorff–Besicovitch dimension calculation of the family $\Phi(Q^*)$ of cylinders generated by Q^*-expansion of real numbers. All known sufficient conditions for the family $\Phi(Q^*)$ to be faithful for the Hausdorff–Besicovitch dimension calculation use different restrictions on entries q_{0k} and $q_{(s-1)k}$. We show that these restrictions are of purely technical nature and can be removed. Based on these new results, we study fine fractal properties of random variables with independent Q^*-digits.

Keywords Hausdorff–Besicovitch dimension, fractals, faithful Vitali coverings, Q^*-expansion, singularly continuous probability measures
2010 MSC 11K55, 26A30, 28A80, 60G30

1 Introduction

Hausdorff measures and the Hausdorff dimension are important tools in the study of fractals and singularly continuous probability measures. The determination or even estimation of the Hausdorff dimension of a set or measure is the crucial problem in fractal analysis, and a lot of research papers were devoted to these problems. Because

*Corresponding author.

of this reason, many interesting methods for the simplification of the procedure of the determination of the Hausdorff dimension were invented and developed during the last 20 years. One approach to such a simplification consists in some restrictions of admissible coverings. This idea came from Besicovitch's works and has been used by Rogers and Taylor to construct comparable net measures [18] as approximations of the Hausdorff measures. In this paper, we develop this approach via construction of net coverings that lead to a special family of net measures, which are more general that comparable ones. We discuss the notion of faithfulness and nonfaithfulness of the family of cylinders generated by different systems of numerations for the Hausdorff dimension calculation.

Let us shortly recall that the *α-dimensional Hausdorff measure* of a set $E \subset [0, 1]$ with respect to a given fine family of coverings Φ is defined by

$$H^{\alpha}(E, \Phi) = \lim_{\epsilon \to 0} \inf_{|E_j| \le \epsilon} \sum_j |E_j|^{\alpha} = \lim_{\epsilon \to 0} H^{\alpha}_{\epsilon}(E, \Phi),$$

where the infimum is taken over all at most countable ϵ-coverings $\{E_j\}$ of E, $E_j \in \Phi$. The nonnegative number

$$\dim_H(E, \Phi) = \inf\left\{\alpha : H^{\alpha}(E, \Phi) = 0\right\}$$

is called the Hausdorff dimension of the set $E \subset [0, 1]$ w.r.t. the family Φ. If Φ is the family of all subsets of $[0, 1]$ or Φ coincides with the family of all closed (open) subintervals of [0,1], then $\dim_H(E, \Phi)$ is equal to the classical Hausdorff dimension $\dim_H(E)$ of a subset $E \subset [0, 1]$.

A fine covering family Φ is said to be a *faithful family of coverings* (*nonfaithful family of coverings*) for the Hausdorff dimension calculation on [0, 1] if

$$\dim_H(E, \Phi) = \dim_H(E), \quad \forall E \subseteq [0, 1]$$
$$(\text{resp. } \exists E \subseteq [0, 1] : \dim_H(E, \Phi) \neq \dim_H(E)).$$

It is clear that any family Φ of comparable net-coverings (i.e., net-coverings that generate comparable net-measures) is faithful. Conditions for Vitali coverings to be faithful were studied by many authors (see, e.g., [2, 6, 7, 10] and the references therein). First steps in this direction have been done by Besicovitch [9], who proved the faithfulness of the family of cylinders of a binary expansion. His result was extended by Billingsley [10] to the family of s-adic cylinders, by Turbin and Pratsiovytyi [21] to the family of Q-S-cylinders, and by Albeverio and Torbin [2] to the family of Q^*-cylinders for the matrices Q^* with elements p_{0k}, $p_{(s-1)k}$ bounded away from zero.

In all these papers, their authors used essentially the same approach to prove the faithfulness of the corresponding family of coverings: it has been shown that there exist positive constants C and $n_0 \in N$ such that, for any $\varepsilon > 0$ and for any interval $(a; b)$ with $b - a < \varepsilon$, there exist at most n_0 cylinders from fine covering families that cover the interval (a, b) and their lengths do not exceed the value $C(b - a)$. It is rather obvious that such families Φ of cylinders generates comparable Hausdorff measures [18], and, therefore, they are faithful for the Hausdorff dimension calculation. Albeverio et al. [7] correctly mentioned that it was rather paradoxical that initial examples of nonfaithful families of coverings first appeared in the two-dimensional case (as a result of active studies of self-affine sets during the last decade of XX century, see, e.g., [8]). The family of cylinders of the classical continued fraction expansion can probably be considered as the first (and rather unexpected) example of nonfaithful one-dimensional net-family of coverings [16]. By using approach which has been invented by Yuval Peres to prove the nonfaithfulness of the family of continued fraction cylinders, in [7] the nonfaithfulness of the family $\Phi(Q_\infty)$ of cylinders of the Q_∞-expansion with polynomially decreasing elements $\{q_i\}$ has been proven. This shows, in particular, that the family of cylinders of the classical Lüroth expansion is nonfaithful. Rather general sufficient conditions for $\Phi(Q_\infty)$ to be faithful were also obtained in [7, 15].

In 2012, Ibragim and Torbin [13] developed a new method to prove the faithfulness of the family of cylinders of Q^*-expansion for the matrices Q^* with elements $p_{0k}, p_{(s-1)k}$ not tending to zero "too quickly." In particular, they proved the following result.

Theorem. *Let* $q_k^* := \max\{q_{0k}, q_{1k}, \dots, q_{s-1k}\}$. *If*

$$\begin{cases} \lim\limits_{k\to\infty} \frac{\ln q_{0,k}}{\ln(q_1^* q_2^* \dots q_k^*)} = 0, \\ \lim\limits_{k\to\infty} \frac{\ln q_{s-1,k}}{\ln(q_1^* q_2^* \dots q_k^*)} = 0, \end{cases} \tag{1}$$

then

$$\dim_H(E) = \dim_H\big(E, \Phi(Q^*)\big), \quad \forall E \subset [0,1].$$

This theorem, a generalization of [2], extended the family of faithful coverings generated by cylinders of Q^*-expansion. In particular, we can easily apply this theorem to prove the faithfulness of the family of cylinders generated by the matrix

$$Q^* = \begin{pmatrix} \frac{1}{10} & \dots & \frac{1}{10k} & \dots \\ \frac{1}{2} - \frac{1}{10} & \dots & \frac{1}{2} - \frac{1}{10k} & \dots \\ \frac{1}{2} - \frac{1}{10} & \dots & \frac{1}{2} - \frac{1}{10k} & \dots \\ \frac{1}{10} & \dots & \frac{1}{10k} & \dots \end{pmatrix}.$$

On the other hand, if $p_{0k}, p_{(s-1)k}$ tend to zero "too quickly" (e.g., $s = 4$, $q_{0k} = q_{3k} = \frac{1}{10^k}$, $q_{1k} = q_{2k} = \frac{1}{2} - \frac{1}{10^k}$), then the above theorem does not work.

In the next section, we develop a new approach to prove the faithfulness of families of coverings and prove essentially new sufficient conditions for Q^*-cylinders to be faithful (we do not need any information about the boundedness from zero of the elements q_{0k} and $q_{(s-1)k}$ or any information about the rate of their convergence to zero).

2 On new sufficient conditions of fractal faithfulness for the family of cylinders of Q^*-expansions

Theorem 2.1. *Let* $q_k := \max_i q_{ik}$, *let*

$$S(m,\delta) := \sum_{k=1}^{\infty} \left(\prod_{i=m+1}^{m+k} q_i \right)^{\delta},$$

and let

$$S(\delta) := \sup_m S(m, \delta).$$

If

$$S(\delta) < +\infty, \quad \forall \delta > 0, \tag{2}$$

then the family $\Phi(Q^*)$ of cylinders generated by Q^*-expansion of real numbers is faithful for the calculation of the Hausdorff dimension on the unit interval, that is,

$$\dim_H E = \dim_H\big(E, \Phi(Q^*)\big), \quad \forall E \subset [0, 1].$$

Proof. It is clear that for the determination of the Hausdorff dimension of subsets from [0, 1] it suffices to consider coverings by intervals (a_j, b_j), where a_j and b_j belong to a set A that is dense in [0, 1]. Let A be the set of all Q^*-irrational points, that is, the set of points that are not end-points of Q^*-cylinders (the Q^*-expansion of these points does not contain digits 0 or $s-1$ in a period).

Let E be an arbitrary subset of [0, 1]. Let us fix $\varepsilon > 0$ and $\alpha > 0$. Let $\{E_j\}$ be an arbitrary ε-covering of the set E, $E_j = (a_j, b_j)$, $a_j \in A$, $b_j \in A$, $|E_j| < \varepsilon$.

For the interval E_j, there exists a unique cylinder $\Delta_{\alpha_1\alpha_2\ldots\alpha_{n_j}}$ containing E_j such that any cylinder of a higher rank does not contain E_j. In the case where a_j and b_j belong to different cylinders of the first rank, we define $\Delta_{\alpha_1\alpha_2\ldots\alpha_{n_j}} := [0, 1]$.

Let us split $\Delta_{\alpha_1\ldots\alpha_{n_j}}$ on the next rank cylinders. From the maximality of the rank of the cylinder $\Delta_{\alpha_1\alpha_2\ldots\alpha_{n_j}}$ it follows that there exists at least one point that is an end-point of a cylinder of rank $n_j + 1$ and belongs to the interval (a_j, b_j). It is clear that the point

$$c_j = \Delta_{\alpha_1(a_j)\ldots\alpha_{n_j}(a_j)(\alpha_{n_j+1}(a_j)+1)0\ldots0\ldots}$$

possesses such properties.

Let $M_0 = M_0(j)$ be a family of cylinders of rank $n_j + 1$ belonging to (a_j, b_j). It is clear that M_0 contains less than s cylinders (if the points a_j and b_j belong to neighboring cylinders of rank $n_j + 1$, then M_0 is empty). Therefore, the α-volume of these cylinders does not exceed $s|E_j|^\alpha$.

Let $d_j := \sup M_0 = \Delta_{\alpha_1(a_j)\ldots\alpha_{n_j}(a_j)\alpha_{n_j+1}(b_j)\,0\ldots0\ldots}$.

To cover the set E_j by cylinders from $\Phi(Q^*)$, let us cover the sets (a_j, c_j) and $[d_j, b_j)$ separately. Let us choose $\delta \in (0, \alpha)$.

First, let us estimate the α-volume of coverings of the set (a_j, c_j).

Let $L_1 = L_1(j)$ be the family of all cylinders of rank $n_j + 2$ belonging to the cylinder $\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)}$ and to the set $(a_j, c_j]$. Let

$$A_1 = A_1(j) := \big\{i : i \in \big\{\alpha_{n_j+2}(a_j) + 1, \ldots, s-1\big\}\big\}.$$

The corresponding α-volume of these cylinders is equal to

$$\sum_{i\in A_1} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)\,i}|^\alpha \le s \cdot \max_{i\in A_1} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)\,i}|^\alpha$$
$$= s \cdot \max_{i\in A_1}\big(|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)\,i}|^{\alpha-\delta}|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)\,i}|^\delta\big)$$

$$\le s|E_j|^{\alpha-\delta} \cdot \max_{i\in A_1} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+1}(a_j)\, i}|^{\delta}$$

$$\le s|E_j|^{\alpha-\delta} \cdot \left(q_{n_j+2}\, |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j}(a_j)}|\right)^{\delta} \le s|E_j|^{\alpha-\delta} \cdot q^{\delta}_{n_j+2}.$$

Let $L_2 = L_2(j)$ be the family of all cylinders of rank $n_j + 3$ belonging to the cylinder $\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)}$ and to the set $(a_j, c_j]$. Let

$$A_2 = A_2(j) := \left\{i : i \in \left\{\alpha_{n_j+3}(a_j) + 1, \ldots, s-1\right\}\right\}.$$

The corresponding α-volume of these cylinders is equal to

$$\sum_{i\in A_2} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)\, i}|^{\alpha}$$

$$\le s \cdot \max_{i\in A_2} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)\, i}|^{\alpha}$$

$$= s \cdot \max_{i\in A_2}\left(|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)\, i}|^{\alpha-\delta}|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)\, i}|^{\delta}\right)$$

$$\le s|E_j|^{\alpha-\delta} \cdot \max_{i\in A_2} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+2}(a_j)\, i}|^{\delta}$$

$$\le s|E_j|^{\alpha-\delta} \cdot \left(q_{n_j+2}q_{n_j+3}\, |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j}(a_j)}|\right)^{\delta} \le s \cdot |E_j|^{\alpha-\delta} \cdot (q_{n_j+2}q_{n_j+3})^{\delta}.$$

Similarly, let $L_k = L_k(j)$ be the family of all cylinders of rank $n_j + k + 1$ belonging to the cylinder $\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)}$ and to the set $(a_j, c_j]$. Let

$$A_k = A_k(j) := \left\{i : i \in \left\{\alpha_{n_j+k+1}(a_j) + 1, \ldots, s-1\right\}\right\}.$$

The corresponding α-volume of these cylinders is equal to

$$\sum_{i\in A_k} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)\, i}|^{\alpha}$$

$$\le s \cdot \max_{i\in A_k} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)\, i}|^{\alpha}$$

$$= s \cdot \max_{i\in A_k}\left(|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)\, i}|^{\alpha-\delta}|\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)\, i}|^{\delta}\right)$$

$$\le s|E_j|^{\alpha-\delta} \cdot \max_{i\in A_k} |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j+k}(a_j)\, i}|^{\delta}$$

$$\le s|E_j|^{\alpha-\delta} \cdot \left(\prod_{i=2}^{k+1} q_{n_j+i}\, |\Delta_{\alpha_1(a_j)\alpha_2(a_j)\ldots\alpha_{n_j}(a_j)}|\right)^{\delta} \le s|E_j|^{\alpha-\delta} \cdot \left(\prod_{i=2}^{k+1} q_{n_j+i}\right)^{\delta}.$$

So, the set (a_j, c_j) can be covered by a countable family of cylinders from L_1, $L_2, \ldots, L_k, \ldots$. The total α-volume of all these cylinders does not exceed the value

$$s|E_j|^{\alpha-\delta} \sum_{k=1}^{\infty} \left(\prod_{i=2}^{k+1} q_{n_j+i}\right)^{\delta} \le S(\delta) \cdot s\, |E_j|^{\alpha-\delta}.$$

Now let us estimate the α-volume of the set $[d_j, b_j)$.

Let $R_1 = R_1(j)$ be the family of all cylinders of rank $n_j + 2$ belonging to the cylinder $\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)}$ and to the set $[d_j, b_j)$. Let

$$B_1 = B_1(j) := \left\{i : i \in \left\{0, \ldots, \alpha_{n_j+2}(b_j) - 1\right\}\right\}.$$

The corresponding α-volume of these cylinders is equal to

$$\sum_{i\in B_1} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)\ i}|^\alpha$$
$$\le s\cdot \max_{i\in B_1} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)\ i}|^\alpha$$
$$= s\cdot \max_{i\in B_1}\big(|\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)\ i}|^{\alpha-\delta}|\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)\ i}|^{\delta}\big)$$
$$\le s|E_j|^{\alpha-\delta}\cdot \max_{i\in B_1} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+1}(b_j)\ i}|^\delta$$
$$\le s|E_j|^{\alpha-\delta}\cdot \big(q_{n_j+2}\ |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j}(b_j)}|\big)^\delta \le s\cdot |E_j|^{\alpha-\delta}\cdot q^\delta_{n_j+2}.$$

Similarly, for $k > 1$, let $R_k = R_k(j)$ be the family of all cylinders of rank $n_j + k + 1$ belonging to the cylinder $\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)}$ and to the set $[d_j, b_j)$. Let

$$B_k = B_k(j) := \big\{i : i \in \big\{0, \ldots, \alpha_{n_j+k+1}(b_j) - 1\big\}\big\}.$$

The corresponding α-volume of these cylinders is equal to

$$\sum_{i\in R_k} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)\ i}|^\alpha$$
$$\le s\cdot \max_{i\in B_k} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)\ i}|^\alpha$$
$$= s\cdot \max_{i\in B_k}\big(|\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)\ i}|^{\alpha-\delta}|\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)\ i}|^{\delta}\big)$$
$$\le s|E_j|^{\alpha-\delta}\cdot \max_{i\in B_k} |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j+k}(b_j)\ i}|^\delta$$
$$\le s|E_j|^{\alpha-\delta}\cdot \left(\prod_{i=2}^{k+1} q_{n_j+i}\right)^\delta |\Delta_{\alpha_1(b_j)\alpha_2(b_j)\ldots\alpha_{n_j}(b_j)}|^\delta \le s\cdot |E_j|^{\alpha-\delta}\left(\prod_{i=2}^{k+1} q_{n_j+i}\right)^\delta.$$

So, the set $[d_j, b_j)$ can be covered by a countable family of cylinders from R_1, $R_2, \ldots, R_k, \ldots$. The total α-volume of these cylinders does not exceed the value

$$s|E_j|^{\alpha-\delta}\sum_{k=1}^{\infty}\left(\prod_{i=2}^{k+1} q_{n_j+i}\right)^\delta \le S(\delta)\cdot s\ |E_j|^{\alpha-\delta}.$$

Hence, the interval (a_j, b_j) can be covered by using at most s cylinders from $M_0 = M_0(j)$, a countable family of cylinders from $L_1, L_2, \ldots, L_k, \ldots$ and a countable family of cylinders from $R_1, R_2, \ldots, R_k, \ldots$. We emphasize that all these cylinders are subsets of (a_j, b_j) and their total α-volume does not exceed the value

$$\big(1 + 2S(\delta)\big)\cdot s\ |E_j|^{\alpha-\delta}, \quad \forall\delta \in (0, \alpha).$$

Therefore, given a subset E, $\alpha \in (0, 1]$, $\delta \in (0, \alpha)$, $\varepsilon > 0$, and an ε-covering of the set E by intervals (a_j, b_j), $a_j \in A, b_j \in A$, there exists an ε-covering of E by cylinders from $\Phi(Q^*)$ such that its α-volume does not exceed the value

$$\big(1 + 2S(\delta)\big)\cdot s\ \sum_j |E_j|^{\alpha-\delta}.$$

Hence, for any $\alpha \in (0, 1]$, $\delta \in (0, \alpha)$, and $E \subset [0, 1]$, we have

$$H^\alpha(E) \le H^\alpha\big(E, \Phi\big(Q^*\big)\big) \le \big(1 + 2S(\delta)\big) \cdot s\ H^{\alpha-\delta}(E).$$

So,

$$\dim_H\big(E, \Phi\big(Q^*\big)\big) \le \dim_H(E) + \delta, \quad \forall \delta \in (0, \alpha),$$

which proves the inequality

$$\dim_H\big(E, \Phi\big(Q^*\big)\big) \le \dim_H(E), \quad \forall E \subset [0, 1].$$

Therefore,

$$\dim_H\big(E, \Phi\big(Q^*\big)\big) = \dim_H(E)$$

for any $E \subset [0, 1]$, which proves the theorem. □

Corollary 2.1. *If*

$$\sup_{ik} q_{ik} < 1, \tag{3}$$

then the family $\Phi(Q^*)$ of cylinders generated by Q^*-expansion of real numbers is faithful for the calculation of the Hausdorff dimension on the unit interval.

Proof. If $\sup_{ik} q_{ik} < 1$, then there exists a positive constant $q < 1$ such that $q_k < q$ for all $k \in N$. In such a case,

$$S(m, \delta) := \sum_{k=1}^{\infty} \left(\prod_{i=m+1}^{m+k} q_i \right)^{\delta} \le \sum_{k=1}^{\infty} q^{k\delta} = \frac{q^\delta}{1 - q^\delta}, \quad \forall m \in N,$$

so that

$$S(\delta) := \sup_m S(m, \delta) \le \frac{q^\delta}{1 - q^\delta} < +\infty, \quad \forall \delta > 0.$$

Therefore, the family $\Phi(Q^*)$ is faithful. □

From this corollary it follows, in particular, that the family $\Phi(Q^*)$ of cylinders generated by the matrix

$$Q^* = \begin{pmatrix} \frac{1}{10} & \cdots & \frac{1}{10^k} & \cdots \\ \frac{1}{2} - \frac{1}{10} & \cdots & \frac{1}{2} - \frac{1}{10^k} & \cdots \\ \frac{1}{2} - \frac{1}{10} & \cdots & \frac{1}{2} - \frac{1}{10^k} & \cdots \\ \frac{1}{10} & \cdots & \frac{1}{10^k} & \cdots \end{pmatrix}$$

is faithful.

Let us show how sufficient conditions for the faithfulness obtained in [2] can be easily derived from our results.

Corollary 2.2. *If* $\inf_k\{q_{0k}, q_{(s-1)k}\} > 0$*, then the family $\Phi(Q^*)$ of cylinders generated by Q^*-expansion of real numbers is faithful for the calculation of the Hausdorff–Besicovitch dimension on the unit interval.*

Proof. If $\inf_k\{q_{0k}, q_{(s-1)k}\} > 0$, then there exists a positive constant q_* such that $q_{0k} > q_*, q_{(s-1)k} > q_*$, $\forall k \in N$. Therefore, $\sup_{ik} q_{ik} \le 1 - 2q_* < 1$. So, the faithfulness of $\Phi(Q^*)$ follows from the previous corollary. □

From the proof of the corollary it follows that it is easy to extend the results of [2] in the following way.

Corollary 2.3. *If* $\inf_k\{q_{0k}\} > 0$*, then the family* $\Phi(Q^*)$ *of cylinders generated by* Q^**-expansion of real numbers is faithful for the calculation of the Hausdorff dimension on the unit interval.*

Proof. If $\inf_k\{q_{0k}\} > 0$, then there exists a positive constant q_* such that $q_{0k} > q_*$, $\forall k \in N$. Therefore, $\sup_{ik} q_{ik} \le 1 - q_* < 1$. So, the faithfulness of $\Phi(Q^*)$ follows from Corollary 2.1. □

So, for the faithfulness of the family $\Phi(Q^*)$, it suffices to control only elements of the first raw of the matrix Q^*.

Let us also mention that Theorem 2.1 can give a positive answer on the faithfulness of $\Phi(Q^*)$ even for the case where $\inf_k\{q_{0k}, q_{(s-1)k}\} = 0$ and $\sup_{ik} q_{ik} = 1$ simultaneously. To illustrate this, let us consider the matrix

$$Q^* = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{2^{n+1}} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{2^n-1}{2^n} & \frac{1}{3} & \cdots \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{2^{n+1}} & \frac{1}{3} & \cdots \end{pmatrix},$$

that is,

$$q_{0k} = q_{1k} = q_{2k} = \frac{1}{3}, \quad k = 2n,\ n \in N,$$

and

$$q_{0k} = q_{2k} = \frac{1}{2^{n+1}}, \qquad q_{1k} = \frac{2^n - 1}{2^n}, \quad k = 2n-1,\ n \in N.$$

In such a case, $\inf_k\{q_{0k}, q_{(s-1)k}\} = 0$ and $\sup_{ik} q_{ik} = 1$, but it is clear that $q_k q_{k+1} < \frac{1}{3}$ for all $k \in N$, and, therefore,

$$S(m,\delta) := \sum_{k=1}^{\infty} \left(\prod_{i=m+1}^{m+k} q_i \right)^{\delta} \le q_{m+1}^{\delta} + 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n\delta} = 1 + \frac{2}{3^{\delta} - 1}, \quad \forall m \in N.$$

So, $S(\delta) < +\infty$ for all $\delta > 0$, which proves the faithfulness of the family $\Phi(Q^*)$.

3 On fine fractal properties of random variables with independent Q^*-symbols

Let $\{\xi_k\}$ be a sequence of independent random variables taking values $0, 1, \dots, s-1$ with probabilities $p_{0k}, p_{1k}, \dots, p_{s-1k}$, respectively. The random variable

$$\xi = \Delta^{Q^*}_{\xi_1 \xi_2 \dots \xi_k \dots} \tag{4}$$

is said to be the random variable with independent Q^*-symbols. Let ν_ξ be the corresponding probability measure.

The Lebesgue structure of ν_ξ is well studied (see, e.g., [2, 4]). It is known, in particular, that the distribution of ξ is of pure type. It is of pure discrete type if and only if

$$\prod_{k=1}^{\infty} \max_i p_{ik} > 0, \tag{5}$$

of pure absolutely continuous type if and only if

$$\prod_{k=1}^{\infty} \left(\sqrt{p_{0k}q_{0k}} + \sqrt{p_{1k}q_{1k}} + \cdots + \sqrt{p_{(s-1)k}q_{(s-1)k}}\right) > 0, \tag{6}$$

and of pure singularly continuous type if and only if infinite products (5) and (6) are equal to zero.

Let us recall that the Hausdorff dimension of the distribution of a random variable τ is defined as follows:

$$\dim_H(\tau) = \inf\{\dim_H(E),\ E \in \mathcal{B}_\tau\},$$

where $\mathcal{B}_\tau$ is the family of all possible (not necessarily closed) supports of the random variable τ, that is,

$$\mathcal{B}_\tau = \{E : E \in \mathcal{B},\ P_\tau(E) = 1\}.$$

Let us also recall the notion of the Hausdorff–Billingsley dimension of a set w.r.t. a probability measure and w.r.t. a system of partitions. Let υ be a continuous probability measure on [0, 1], and let Φ be the family of cylinders generated by some expansion. Then the $(\upsilon - \alpha)$-Hausdorff measure of a set $E \subset [0, 1]$ w.r.t. the family Φ and measure υ is defined as follows:

$$H^\alpha(E, \upsilon, \Phi) = \lim_{\varepsilon \to 0}\left[\inf_{\upsilon(E_j) \le \varepsilon}\left\{\sum_j \upsilon^\alpha(E_j)\right\}\right] = \lim_{\varepsilon \to 0} H^\alpha_\varepsilon(E, \upsilon, \Phi),$$

where $E_j \in \Phi$, $\bigcup_j E_j \supset E$.

The number

$$\dim_\upsilon(E, \Phi) = \inf\{\alpha : H^\alpha(E, \upsilon, \Phi) = 0\}$$

is called *the Hausdorff–Billingsley dimension* of a set E w.r.t. υ and Φ.

Let

$$h_j := -\sum_{i=0}^{s-1} p_{ij} \ln p_{ij}, \qquad 0 \ln 0 := 0, \qquad H_n := \sum_{j=1}^{n} h_j,$$

$$b_j := -\sum_{i=0}^{s-1} p_{ij} \ln q_{ij}, \qquad B_n := \sum_{j=1}^{n} b_j,$$

and

$$d_j := -b_j^2 + \sum_{i=0}^{s-1} p_{ij} \ln^2 q_{ij}.$$

Theorem 3.1. *Assume that the following conditions hold:*

$$S(\delta) < +\infty, \quad \forall \delta > 0; \tag{7}$$

$$\sum_{j=1}^{\infty} \frac{d_j}{j^2} < +\infty; \tag{8}$$

$$\varliminf_{n\to\infty} \frac{B_n}{n} > 0. \tag{9}$$

Then the Hausdorff dimension of the distribution of a random variable with independent Q^*-symbols is equal to

$$\dim_H \nu_\xi = \varliminf_{n\to\infty} \frac{H_n}{B_n}. \tag{10}$$

Proof. Let $\Delta_n(x) = \Delta^{Q^*}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)}$ be the cylinder of rank n of the Q^*-expansion of x. Let $\nu = \nu_\xi$, and let μ be the Lebesgue measure on [0, 1].

Then

$$\nu\big(\Delta_n(x)\big) = p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot \cdots \cdot p_{\alpha_n(x)n},$$
$$\mu\big(\Delta_n(x)\big) = q_{\alpha_1(x)1} \cdot q_{\alpha_2(x)2} \cdot \cdots \cdot q_{\alpha_n(x)n}.$$

Let us consider

$$\frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} = \frac{\sum_{j=1}^{n} \ln p_{\alpha_j(x)j}}{\sum_{j=1}^{n} \ln q_{\alpha_j(x)j}}.$$

If a real number $x = \Delta^{Q^*}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}$ is chosen randomly so that $P(\alpha_j(x) = i) = p_{ij}$ (i.e., the distribution of the random variable x coincides with the initial probability measure ν), then $\{\eta_j\} = \{\eta_j(x)\} = \{\ln p_{\alpha_j(x)j}\}$ and $\{\psi_j\} = \{\psi_j(x)\} = \{\ln q_{\alpha_j(x)j}\}$ are sequences of independent random variables with the following distributions:

η_j	$\ln p_{0j}$	$\ln p_{1j}$	$\ldots$	$\ln p_{(s-1)j}$
	p_{0j}	p_{1j}	$\ldots$	$p_{(s-1)j}$

ψ_j	$\ln q_{0j}$	$\ln q_{1j}$	$\ldots$	$\ln q_{(s-1)j}$
	p_{0j}	p_{1j}	$\ldots$	$p_{(s-1)j}$

It is clear that $M\eta_j = -h_j$ and $|h_j| \le \ln s$.

It is not hard to check that $M\eta_j^2 = \sum_{i=0}^{s-1} p_{ij} \ln^2 p_{ij} \le \frac{4}{e^2}$ [2].

From the strong law of large numbers [19] it follows that, for ν-almost all $x \in [0, 1]$, the following equality holds:

$$\lim_{n\to\infty} \frac{(\eta_1 + \eta_2 + \cdots + \eta_n) - M(\eta_1 + \eta_2 + \cdots + \eta_n)}{n} = 0. \tag{11}$$

It is clear that $M(\eta_1 + \eta_2 + \cdots + \eta_n) = -H_n$.

To show that the strong law of large numbers can also be applied to the sequence $\{\psi_j\}$, let us consider

$$M\psi_j = \sum_{i=0}^{s-1} p_{ij} \ln q_{ij}, \qquad M\psi_j^2 = \sum_{i=0}^{s-1} p_{ij} \ln^2 q_{ij}.$$

Since $d_j = D(\psi_j)$ and the series $\sum_{j=1}^{\infty} \frac{d_j}{j^2}$ converges (see (8)), by Kolmogorov's theorem (strong law of large numbers [19]) it follows that, for ν-almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{(\psi_1 + \psi_2 + \cdots + \psi_n) - M(\psi_1 + \psi_2 + \cdots + \psi_n)}{n} = 0. \tag{12}$$

Let us remark that $M(\psi_1 + \psi_2 + \cdots + \psi_n) = -B_n$.

Now let us consider the set

$$\begin{aligned} A &= \left\{x : \lim_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{\psi_1(x) + \psi_2(x) + \cdots + \psi_n(x)} - \frac{H_n}{B_n}\right) = 0\right\} \\ &= \left\{x : \lim_{n\to\infty} \frac{(\frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)+H_n}{n}) - \frac{H_n}{B_n}(\frac{\psi_1(x)+\psi_2(x)+\cdots+\psi_n(x)+B_n}{n})}{(\frac{\psi_1(x)+\psi_2(x)+\cdots+\psi_n(x)+B_n}{n}) - \frac{B_n}{n}} = 0\right\}. \end{aligned}$$

By the Gibbs inequality it follows that $h_j \le b_j$. Hence, $0 \le \frac{H_n}{B_n} = \frac{\sum_{j=1}^n h_j}{\sum_{j=1}^n b_j} \le 1$.

Since $\underline{\lim}_{n\to\infty} \frac{B_n}{n} > 0$ (see (9)), we deduce the existence of a constant $c_1 > 0$ such that $|\frac{B_n}{n}| \ge c_1$ for all $n \in N$.

Therefore, for ν-almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{(\frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)+H_n}{n}) - \frac{H_n}{B_n}(\frac{\psi_1(x)+\psi_2(x)+\cdots+\psi_n(x)+B_n}{n})}{(\frac{\psi_1(x)+\psi_2(x)+\cdots+\psi_n(x)+B_n}{n}) - \frac{B_n}{n}} = 0.$$

So, $\nu(A) = 1$ and $\dim_\nu(A, \Phi) = 1$.

Let us consider the sets

$$\begin{aligned} A_1 &= \left\{x : \quad \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{\psi_1(x) + \psi_2(x) + \cdots + \psi_n(x)} - \frac{H_n}{B_n}\right) = 0\right\}, \\ A_2 &= \left\{x : \quad \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{\psi_1(x) + \psi_2(x) + \cdots + \psi_n(x)}\right) \le \underline{\lim}_{n\to\infty} \frac{H_n}{B_n}\right\} \\ &= \left\{x : \quad \underline{\lim}_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} \le \underline{\lim}_{n\to\infty} \frac{H_n}{B_n}\right\}, \end{aligned}$$

and

$$\begin{aligned} A_3 &= \left\{x : \quad \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{\psi_1(x) + \psi_2(x) + \cdots + \psi_n(x)}\right) \ge \underline{\lim}_{n\to\infty} \frac{H_n}{B_n}\right\} \\ &= \left\{x : \quad \underline{\lim}_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} \ge \underline{\lim}_{n\to\infty} \frac{H_n}{B_n}\right\}. \end{aligned}$$

It is obvious that $A \subset A_1$. By the same arguments as in [2] we can easily check that $A_1 \subset A_3$ and $A \subset A_2$.

Let $D = \underline{\lim}_{n\to\infty} \frac{H_n}{B_n}$.

From $A \subset A_2$ it follows that $\dim_\mu(A, \Phi) \le \dim_\mu(A_2, \Phi)$. From Theorem 2.1 of [10] it follows that $\dim_\mu(A_2, \Phi) \le D$. Therefore, $\dim_\mu(A, \Phi) \le D$.

From the condition $A \subset A_3$ and Theorem 2.2 of [10] it follows that $\dim_\mu(A, \Phi) \ge D \cdot \dim_\nu(A, \Phi) = D \cdot 1 = D$. Hence, $\dim_\mu(A, \Phi) = D$.

Since μ is the Lebesgue measure on $[0,1]$, we get $\dim_H(A,\Phi) = \dim_\mu(A,\Phi) = D$. From (7) and Theorem 1 it follows that the family Φ of cylinders of the Q^*-expansion is faithful for the determination of the Hausdorff–Besicovitch dimension on the unit interval, and, therefore, $\dim_H(A,\Phi) = \dim_H(A)$. Hence, $\dim_H(A) = D$.

Finally, let us prove that the constructed set A is the minimal dimensional support of the measure ν. To this end, let us consider an arbitrary support C of the measure ν. It is clear that the set $C_1 = C \cap A$ is also a support of the measure ν and that $C_1 \subset C$. Then $\dim_H(C_1) \le \dim_H(C)$ and $C_1 \subset A$.

Let us prove that $\dim_H(C_1) = \dim_H(A)$.

From $C_1 \subset A$ it follows that $\dim_H(C_1) \le \dim_H(A) = D$. On the other hand,

$$C_1 \subset A \subset A_3 = \left\{x : \varliminf_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} \ge D\right\}.$$

Therefore, from Theorem 2.2 of [10] it follows that

$$\dim_H(C_1) = \dim_\mu(C_1,\Phi) \ge D \cdot \dim_\nu(C_1,\Phi) = D \cdot 1 = D.$$

So, $\dim_H(C_1) = D = \dim_H(A)$. □

Acknowledgments

This work was partly supported by research projects "Spectral Structures and Topological Methods in Mathematics" (SFB-701, Bielefeld University), STREVCOM FP-7-IRSES 612669 (EU), "Multilevel analysis of singularly continuous probability measures and its applications" (Ministry of Education and Science of Ukraine), and by Alexander von Humboldt Stiftung.

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