Modern Stochastics: Theory and Applications 3 (2016) 133–144
DOI: 10.15559/16-VMSTA56

Approximations for a solution to stochastic heat equation with stable noise

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Received: 25 April 2016, Revised: 15 June 2016, Accepted: 15 June 2016,
Published online: 30 June 2016

Abstract We consider a Cauchy problem for stochastic heat equation driven by a real harmonizable fractional stable process Z with Hurst parameter $H > 1/2$ and stability index $\alpha > 1$. It is shown that the approximations for its solution, which are defined by truncating the LePage series for Z, converge to the solution.

Keywords Heat equation, real harmonizable fractional stable process, LePage series, stable random measure, general stochastic measure

2010 MSC 60H15, 60G22, 60G52

1 Introduction

Partial differential equations with randomness are widely used to model physical, chemical, biological phenomena, financial asset prices, economical processes, etc. The popularity of such models is due to the combination of deterministic and stochastic features among their characteristics.

The majority of existing literature is devoted to the case where the random noise has some Gaussian or sub-Gaussian distribution. To mention only few papers, a heat equation with Gaussian noise was considered in [1, 2, 8, 14], and a wave equation in

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[3, 9, 14]. Equations with sub-Gaussian measures were intensively studied in [6, 7]. The articles [10, 11] consider equations with general stochastic measures.

The research carried out in the cited articles does not allow one to consider phenomena where the randomness has a heavy-tailed distribution. But heavy tails are ubiquitous when modeling extreme risks, so it is quite important to consider equations with heavy-tailed noise.

The main object of this article is a stochastic heat equation in which the source of randomness is a real harmonizable fractional stable process Z. The solution is understood in the mild sense, with the integral defined pathwise as a fractional integral [15].

We consider approximations for the solution of this equation, which are obtained by truncating the LePage representation series of Z. The main result of this paper is that such representations converge to the true solution.

The paper is organized as follows. Section 2 contains basic facts about stable random variables and related processes. It also establishes an auxiliary analytical lemma. In Section 3, we formulate and prove the main result of this article.

2 Preliminaries

2.1 Stable random variables and related processes

In this paper, we consider only symmetric α-stable ($S\alpha S$) random variables with $\alpha \in (1, 2)$. We further provide basic information about such variables and related objects; for a more detailed exposition, we refer the reader to [13].

A random variable ξ is symmetric α-stable (SαS) with scale parameter σ^α, $\sigma \geq 0$, if it has the characteristic function

$$\mathsf{E}\big[e^{i\lambda\xi}\big] = e^{-|\sigma\lambda|^\alpha}.$$

Given some linear space of SαS random variables, the scale parameter is a quasi-norm on this space, denoted $\|\cdot\|_\alpha$.

To construct families of stable random variables, in particular, stable random processes, one frequently uses some stable random measures. We will be interested in the so-called complex rotationally invariant SαS measure μ on $\mathbb{R}$. By definition this is a complex-valued random measure on $\mathcal{B}(\mathbb{R})$ with the following properties:

1. for any Borel set $A \in \mathcal{B}(\mathbb{R})$, the random variable $\operatorname{Re}\mu(A)$ is SαS with scale parameter equal to $\lambda(A)$, the Lebesgue measure of A;
2. for any $A \in \mathcal{B}(\mathbb{R})$, the random variable $\mu(A)$ is rotationally invariant, that is, for any $\theta \in \mathbb{R}$, the distribution of $e^{i\theta}\mu(A)$ coincides with that of $\mu(A)$;
3. for any disjoint sets $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$, the random variables $\mu(A_1), \dots, \mu(A_n)$ are independent.

For a function $f : \mathbb{R} \to \mathbb{C}$ with

$$\|f\|^\alpha_{L^\alpha(\mathbb{R})} = \int_{\mathbb{R}} \big|f(x)\big|^\alpha dx < \infty,$$

it is possible to define the stochastic integral

$$I(f) = \int_{\mathbb{R}} f(x)\mu(dx)$$

such that $\operatorname{Re} I(f)$ is an SαS random variable with scale parameter $\|f\|^\alpha_{L^\alpha(\mathbb{R})}$. In other words,

$$\big\|\operatorname{Re} I(f)\big\|^\alpha_\alpha = \|f\|^\alpha_{L_\alpha(\mathbb{R})},$$

that is, the real part of the stochastic integral $I(\cdot)$ maps $L^\alpha(\mathbb{R})$ isometrically into some family of SαS random variables.

Let now $\mathbf{T}$ be a parametric set. For a measurable function $f\colon \mathbf{T}\times\mathbb{R}\to\mathbb{C}$ such that $f(t,\cdot)\in L_\alpha(\mathbb{R})$ for all $t\in\mathbf{T}$, we may define the random field $\{Z(t), t\in\mathbf{T}\}$ by

$$Z(t) = \operatorname{Re}\int_{\mathbb{R}} f(t,x)\mu(dx). \tag{1}$$

This random field $Z(t)$ has the so-called LePage series representation constructed as follows. Let φ be an arbitrary positive probability density on $\mathbb{R}$, and let the independent families $\Gamma_k,\ k\ge 1,\ \xi_k,\ k\ge 1,\ g_k,\ k\ge 1$, of random variables satisfy:

1. $\Gamma_k,\ k\ge 1$, is a sequence of Poisson arrival times with unit intensity;
2. $\xi_k,\ k\ge 1$, are independent random variables having density φ;
3. $g_k,\ k\ge 1$, are independent complex-valued rotationally invariant Gaussian random variables[1] with $\mathsf{E}[|\operatorname{Re} g_k|^\alpha]=1$.

Then the random field $Z(t),\ t\in\mathbf{T}$, defined by (1) has the same finite-dimensional distributions as

$$Z'(t) = C_\alpha \operatorname{Re}\sum_{k\ge 1}\Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha} f(t,\xi_k)g_k, \tag{2}$$

where

$$C_\alpha = \left(\frac{\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}{1-\alpha}\right)^{1/\alpha};$$

the series converges almost surely for all $t\in\mathbf{T}$ (see [5, Lemma 1] and [13, Theorem 1.4.2]).

In the rest of our paper, C denotes a generic constant whose value may change from line to line; $C_{a,b,\dots}$ denotes a constant depending on $a, b, \dots$.

2.2 Fractional integration

We will use the pathwise fractional integration; for more detail, see [12, 15]. Let functions $f, g\colon [a,b]\to\mathbb{R}$ be such that, for some $\beta\in(0,1)$, the following fractional derivatives are defined:

$$\big(D^\beta_{a+}f\big)(x) = \frac{1}{\Gamma(1-\beta)}\left(\frac{f(x)}{(x-a)^\beta} + \beta\int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\beta}}du\right)1_{(a,b)}(x),$$

[1] Note that the rotational invariance implies $\mathsf{E}[g_k]=0$.

$$\big(D_{b-}^{1-\beta}g\big)(x)=\frac{1}{\Gamma(\beta)}\left(\frac{g(x)}{(b-x)^{1-\beta}}+(1-\beta)\int_x^b\frac{g(x)-g(u)}{(x-u)^{2-\beta}}du\right)1_{(a,b)}(x).$$

Provided that $D_{a+}^{\beta}f\in L^1[a,b]$ and $D_{b-}^{1-\beta}g_{b-}\in L^\infty[a,b]$, where $g_{b-}(x)=g(x)-g(b)$, the fractional integral is defined as

$$\int_a^b f(x)dg(x)=\int_a^b\big(D_{a+}^{\beta}f\big)(x)\big(D_{b-}^{1-\beta}g_{b-}\big)(x)dx.$$

It is worth mentioning that for $f\in C^\nu[a,b]$ and $g\in C^\mu[a,b]$ with $\mu+\nu>1$, the fractional integral $\int_a^b f(x)dg(x)$ is well defined for any $\beta\in(1-\nu,\mu)$ and equals the limit of Riemann sums.

2.3 Estimates of Fourier-type integrals

The following result specifies the rate of convergence in the Riemann–Lebesgue lemma. It may be known that, for example, for periodic functions and integer parameter, this is Zygmund's theorem; however, we failed to find it in the literature. Moreover, a similar reasoning will be used later in the proof of our main results, so we found it suitable to present its proof.

Lemma 1. *Let $f\in C[a,b]$ and $h\colon[0,+\infty)\to[0,+\infty)$ be a nondecreasing function such that $|f(x)-f(y)|\le h(|x-y|)$ for all $x,y\in[a,b]$. Then, for any nonzero $\lambda\in\mathbb{R}$,*

$$\left|\int_a^b f(x)e^{i\lambda x}dx\right|\le 3(b-a)h\big(|\lambda|^{-1}\big)+2|\lambda|^{-1}\sup_{x\in[a,b]}\big|f(x)\big|.$$

Proof. If $|\lambda|\le(b-a)^{-1}$, then

$$\begin{aligned}\left|\int_a^b f(x)e^{i\lambda x}dx\right|&\le\int_a^b\big|f(x)e^{i\lambda x}\big|dx\\&\le\sup_{x\in[a,b]}\big|f(x)\big|(b-a)\le|\lambda|^{-1}\sup_{x\in[a,b]}\big|f(x)\big|.\end{aligned}$$

Otherwise, set $n=[|\lambda|(b-a)]+1$, so that $|\lambda|(b-a)\le n\le 2|\lambda|(b-a)$. Consider the equipartition of $[a,b]$ by points $x_k=a+(b-a)k/n$, $k=0,\dots,n$. Then, for any $|\lambda|\ge(b-a)^{-1}$, the following relations hold:

$$\begin{aligned}&\left|\int_a^b f(x)e^{i\lambda x}dx\right|\\&\quad\le\left|\sum_{k=1}^n\int_{x_{k-1}}^{x_k}f(x_k)e^{i\lambda x}dx\right|+\left|\sum_{k=1}^n\int_{x_{k-1}}^{x_k}\big(f(x)-f(x_k)\big)e^{i\lambda x}dx\right|\\&\quad\le\left|\sum_{k=1}^n f(x_k)\frac{e^{i\lambda x_k}-e^{i\lambda x_{k-1}}}{i\lambda}\right|+\sum_{k=1}^n\int_{x_{k-1}}^{x_k}\big|f(x)-f(x_k)\big|dx\\&\quad\le\left|\sum_{k=1}^n f(x_k)\frac{e^{i\lambda x_k}}{\lambda}-\sum_{k=0}^{n-1}f(x_{k+1})\frac{e^{i\lambda x_k}}{\lambda}\right|+h\left(\frac{b-a}{n}\right)(b-a)\end{aligned}$$

$$
\begin{aligned}
&\le \left| f(b)\frac{e^{i\lambda b}}{\lambda} - f(a)\frac{e^{i\lambda a}}{\lambda} \right| + \left| \sum_{k=1}^{n-1} \bigl(f(x_k) - f(x_{k+1})\bigr)\frac{e^{i\lambda x_k}}{\lambda} \right| + h\left(\frac{b-a}{n}\right)(b-a) \\
&\le \frac{2\sup_{x\in[a,b]}|f(x)|}{|\lambda|} + \sum_{k=1}^{n-1} \frac{1}{|\lambda|} h\left(\frac{b-a}{n}\right) + h\left(\frac{b-a}{n}\right)(b-a) \\
&\le \frac{2\sup_{x\in[a,b]}|f(x)|}{|\lambda|} + \sum_{k=1}^{n-1} \frac{2(b-a)}{n} h\left(\frac{1}{|\lambda|}\right) + h\left(\frac{1}{|\lambda|}\right)(b-a) \\
&\le \frac{2\sup_{x\in[a,b]}|f(x)|}{|\lambda|} + 3(b-a)h\left(\frac{1}{|\lambda|}\right). \qquad \square
\end{aligned}
$$

3 Stochastic heat equation with stable noise and its approximations

Consider a Cauchy problem for the one-dimensional heat equation

$$
\begin{cases} d_t U(t,x) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} U(t,x)dt + \sigma(t,x)dZ(t), \quad t > 0, x \in \mathbb{R}, \\ U(0,x) = U_0(x). \end{cases} \tag{3}
$$

Here $\sigma(t,x)$ is a bounded function that is jointly Hölder continuous of order $\gamma \in (1/2, 1)$, that is,

$$
\bigl|\sigma(t_1,x_1) - \sigma(t_2,x_2)\bigr| \le C\bigl(|t_1-t_2|^\gamma + |x_1-x_2|^\gamma\bigr), \tag{4}
$$

and U_0 is a bounded measurable function. The random force in this equation is a real harmonizable fractional stable process

$$
Z(t) = \operatorname{Re} \int_{\mathbb{R}} \frac{e^{itx}-1}{|x|^{1/\alpha+H}} M(dx),
$$

where M is a complex rotationally invariant SαS measure μ on $\mathbb{R}$, defined in Section 2.1, and $H \in (1/2, 1)$ is the Hurst parameter of the process. In what follows, we denote

$$
f(t,x) = \frac{e^{itx}-1}{|x|^{1/\alpha+H}}.
$$

It is well known (see, e.g., [13]) that Z is an H-self-similar process having a continuous modification; henceforth, we assume that Z itself is continuous. Moreover, it is almost surely pathwise Hölder continuous with any exponent $\gamma \in (0, H)$ (see [5]).

We consider Eq. (3) in the mild sense. We recall that a mild solution is given by the variation-of-constants formula

$$
U(t,x) = \int_{\mathbb{R}} \rho(t,x-y)U_0(y)dy + \int_0^t dZ(s)\int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,y)dy, \tag{5}
$$

where $\rho(t,x) = (4\pi t)^{-1/2}\exp\{-\frac{|x|^2}{4t}\}$.

Theorem 2. *The Cauchy problem* (3) *has a solution given by* (5)*, where the integral with respect to Z is understood as a fractional integral.*

Proof. Take some $\beta \in (1-H, \gamma)$. Fix $t > 0$ and $x \in \mathbb{R}^d$ and denote $f(s) = \int_{\mathbb{R}} \rho(t-s, x-y)\sigma(s, y)dy$, $s \in [0, t]$. In view of the Hölder continuity of Z, the fractional derivative $D_{t-}^{1-\beta} Z_{t-}$ is almost surely bounded on $[0, t]$. So in order to prove the claim, we only need to show that $D_{0+}^{\beta} f$ is integrable on $[0, t]$. To this end, write

$$\int_0^t \left|D_{0+}^{\beta} f(s)\right| ds \le C \int_0^t \left(\frac{|f(s)|}{s^{\beta}} + \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\beta}} du \right) ds.$$

Since σ is bounded, so is f, supplying the finiteness of the first integral. To establish that of the second one, use the change of variable $y = x + z\sqrt{t-s}$ and note that $\rho(t, x) = \rho(1, x/\sqrt{t})/\sqrt{t}$ to represent f as

$$f(s) = \int_{\mathbb{R}} \rho(1, z)\sigma(s, x + z\sqrt{t-s})dz.$$

Therefore, for any $u < s < t$,

$$\begin{aligned} \left|f(s) - f(u)\right| &\le \int_{\mathbb{R}} \rho(1, z) \left|\sigma(s, x + z\sqrt{t-s}) - \sigma(s, x + z\sqrt{t-u})\right| dz \\ &\quad + \int_{\mathbb{R}} \rho(1, z) \left|\sigma(s, x + z\sqrt{t-u}) - \sigma(u, x + z\sqrt{t-u})\right| dz \\ &=: I_1 + I_2. \end{aligned}$$

Thanks to (4), $I_2 \le C(s-u)^{\gamma}$ and

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}} \rho(1, z)|z\sqrt{t-s} - z\sqrt{t-u}|^{\gamma} dz \le C|\sqrt{t-s} - \sqrt{t-u}|^{\gamma} \\ &= C\left(\frac{s-u}{\sqrt{t-s} + \sqrt{t-u}}\right)^{\gamma} \le C(t-u)^{-\gamma/2}(s-u)^{\gamma}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_0^t \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\beta}} du\, ds &\le C \int_0^t \int_0^s (t-u)^{-\gamma/2}(s-u)^{\gamma-1-\beta} du\, ds \\ &\le C \int_0^t (t-s)^{-\gamma/2} s^{\gamma-\beta} ds < \infty, \end{aligned}$$

which concludes the proof. □

The process Z is a particular example of a random field given by (1). In view of this, we assume that Z is given by its LePage series representation (2) corresponding to the density

$$\varphi(x) = K_{\eta}|x|^{-1}\left|\log|x| + 1\right|^{-1-\eta},$$

where η is some positive number, and $K_{\eta} = (\int_{\mathbb{R}} |x|^{-1}|\ln|x| + 1|^{-1-\eta}dx)^{-1}$ is a normalizing constant.

To simplify the following reasoning, we assume that

$$(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_{\Gamma} \otimes \Omega_{\xi} \otimes \Omega_g, \mathcal{F}_{\Gamma} \otimes \mathcal{F}_{\xi} \otimes \mathcal{F}_g, \mathsf{P}_{\Gamma} \otimes \mathsf{P}_{\xi} \otimes \mathsf{P}_g)$$

and

$$\Gamma_k(\omega) = \Gamma_k(\omega_\Gamma),\ \xi_k(\omega) = \xi_k(\omega_\xi),\ g_k(\omega) = g_k(\omega_g)$$

for all $\omega = (\omega_\Gamma, \omega_\xi, \omega_g) \in \Omega,\ k \geq 1$.

Let us consider the approximation of the process Z by partial sums of its LePage series (2). Specifically, define

$$Z_N(t) = C_\alpha \operatorname{Re} \sum_{k=1}^{N} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(t, \xi_k) g_k.$$

The following result establishes a uniform, in N, Hölder continuity of the family $\{Z^N, N \geq 1\}$.

Proposition 3. *For any $\theta \in (0, H)$ and $T > 0$, there is an almost surely finite random variable $C = C_{\theta,T}(\omega)$ such that, for all $N \geq 1$ and $t, s \in [0, T]$, we have the following inequality:*

$$\left|Z_N(t) - Z_N(s)\right| \leq C_\theta(\omega)|t-s|^\theta.$$

Proof. For fixed $\omega_\Gamma \in \Omega_\Gamma$, $\omega_\xi \in \Omega_\xi$, and $t > 0$, the sequence

$$\left\{Z_N(t) = Z_N\big(t, (\omega_\Gamma, \omega_\xi, \omega g)\big), N \geq 1\right\}$$

is a martingale on $(\Omega_g, F_g, \mathsf{P}_g)$. Then, for any $N_0 \geq 1$ and $t, s > 0$, the Doob inequality yields

$$\mathsf{E}_g\Big[\sup_{1\leq N\leq N_0} \big(Z_N(t) - Z_N(s)\big)^2\Big] \leq C\mathsf{E}_g\big[\big(Z_{N_0}(t) - Z_{N_0}(s)\big)^2\big].$$

Letting $N_0 \to \infty$ and applying the Fatou lemma, we get

$$\mathsf{E}_g\Big[\sup_{N\geq 1}\big(Z_N(t) - Z_N(s)\big)^2\Big] \leq C\mathsf{E}_g\big[\big(Z(t) - Z(s)\big)^2\big].$$

Using further a reasoning similar to that used in [5, Theorem 1], we get, for any $t, s \in [0, T]$,

$$\mathsf{E}_g\Big[\sup_{N\geq 1}\big(Z_N(t) - Z_N(s)\big)^2\Big] \leq C_T(\omega)|t-s|^{2H}\left|\log|t-s|+1\right|^a$$

with some $a > 0$. Consequently, for any $\theta \in (0, H)$,

$$\sup_{N\geq 1}\left|Z_N(t) - Z_N(s)\right| \leq C_{\theta,T}(\omega)|t-s|^\theta,$$

as required. □

Taking into account that the processes Z^N approximate Z, it is natural to consider corresponding approximations of a mild solution to (3):

$$U_N(t,x) = \int_{\mathbb{R}} \rho(t, x-y)U_0(y)dy + \int_0^t dZ_N(s)\int_{\mathbb{R}} \rho(t-s, x-y)\sigma(s,y)dy.$$

The following theorem is the main result of this paper.

Theorem 4. *For any $t \geq 0$ and $x \in \mathbb{R}$, we have the convergence*

$$U_N(t,x) \to U(t,x),\ N \to \infty,$$

almost surely.

Proof. Fix arbitrary numbers $t > 0$ and $x \in \mathbb{R}$. Let us define the functions $v_k(t,x)$ corresponding to the terms of LePage series:

$$v_k(t,x) = C_\alpha \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \operatorname{Re} \int_0^t d_s f(s,\xi_k) \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,y)dy.$$

Then

$$U_N(t,x) = \int_{\mathbb{R}} \rho(t,x-y)U_0(y)dy + \sum_{k=1}^{N} v_k(t,x).$$

As the first step of our proof, we establish the almost sure convergence of the series $\sum_{k=1}^{\infty} v_k(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}$.

Let us transform the differential

$$d_s f(s,\xi_k) = d_s\left(\frac{e^{is\xi_k}-1}{|\xi_k|^{1/\alpha+H}}\right) = \frac{e^{is\xi_k}\cdot i\xi_k}{|\xi_k|^{1/\alpha+H}}dt = \frac{ie^{is\xi_k}\operatorname{sign}\xi_k}{|\xi_k|^{1/\alpha+H-1}}dt.$$

Then we have

$$\begin{aligned} v_k(t,x) = C_\alpha \Gamma_k^{-1/\alpha} \operatorname{Re}\bigg[& i\frac{\varphi(\xi_k)^{-1/\alpha}\operatorname{sign}\xi_k}{|\xi_k|^{1/\alpha+H-1}} g_k \\ & \times \int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,y)dy\, e^{is\xi_k} ds\bigg]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathsf{E}_g\big[\big|v_k(t,x)\big|^2\big] \geq C_\alpha^2 \Gamma_k^{-2/\alpha} & \frac{\varphi(\xi_k)^{-2/\alpha}}{|\xi_k|^{2/\alpha+2H-2}} \\ & \times \left|\int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,y)dy\, e^{is\xi_k} ds\right|^2. \end{aligned}$$

Let us estimate the last integral:

$$\begin{aligned} & \left|\int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,y)dy\, e^{is\xi_k} ds\right| \\ & \quad \leq \left|\int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,x)dy\, e^{is\xi_k} ds\right| \\ & \qquad + \left|\int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\big(\sigma(s,y)-\sigma(s,x)\big)dy\, e^{is\xi_k} ds\right| =: I_1 + I_2. \end{aligned}$$

Since $\int_{\mathbb{R}} \rho(t-s,x-y)dy = 1$, we have

$$I_1 = \left|\int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\sigma(s,x)dy\, e^{is\xi_k} ds\right| = \left|\int_0^t \sigma(s,x)e^{is\xi_k} ds\right|.$$

First, assume that $|\xi_k| \geq t^{-1}$. Using Lemma 1, the last expression admits the following estimate:

$$\left|\int_0^t \sigma(s,x)e^{is\xi_k}ds\right| \leq 3tC|\xi_k|^{-\gamma} + 2|\xi_k|^{-1}\sup_{s\in[0,t]}\left|\sigma(s,x)\right| \leq C|\xi_k|^{-\gamma}.$$

Let us now estimate I_2, taking into account that $\rho(t,x) = \rho(1,x/\sqrt{t})/\sqrt{t}$:

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}\rho(t-s,x-y)\big(\sigma(s,y)-\sigma(s,x)\big)dye^{is\xi_k}ds\right| \\
&\quad= \left|\int_0^t\int_{\mathbb{R}}\rho\left(1,\frac{x-y}{\sqrt{t-s}}\right)\big(\sigma(s,y)-\sigma(s,x)\big)\frac{dy}{\sqrt{t-s}}e^{is\xi_k}ds\right| \\
&\quad= \left|\int_0^t\int_{\mathbb{R}}\rho(1,z)\big(\sigma(s,x+z\sqrt{t-s})-\sigma(s,x)\big)dze^{is\xi_k}ds\right| \\
&\quad= \left|\int_0^t\tau(s)e^{is\xi_k}\right| =: I_3,
\end{aligned}$$

where

$$\tau(s) = \int_{\mathbb{R}}\rho(1,z)\big(\sigma(s,x+z\sqrt{t-s})-\sigma(s,x)\big)dz.$$

We further estimate

$$\begin{aligned}
\left|\tau(s_1)-\tau(s_2)\right| &= \bigg|\int_{\mathbb{R}}\rho(1,z)\big(\sigma(s_1,x+z\sqrt{t-s_1})-\sigma(s_1,x)\big) \\
&\qquad - \big(\sigma(s_2,x+z\sqrt{t-s_2})-\sigma(s_2,x)\big)dz\bigg| \\
&\leq \left|\int_{\mathbb{R}}\rho(1,z)\big(\sigma(s_1,x+z\sqrt{t-s_1})-\sigma(s_1,x+z\sqrt{t-s_2})\big)dz\right| \\
&\quad + \left|\int_{\mathbb{R}}\rho(1,z)\big(\sigma(s_1,x+z\sqrt{t-s_2})-\sigma(s_2,x+z\sqrt{t-s_2})\big)dz\right| \\
&\quad + \int_{\mathbb{R}}\rho(1,z)\left|\sigma(s_1,x)-\sigma(s_2,x)\right|dz =: J_1+J_2+J_3.
\end{aligned}$$

Thanks to the Hölder continuity (4), $J_3 \leq C(s_1-s_2)^\gamma$ for $s_2 < s_1$. Further, similarly to the proof of Theorem 2, $J_1 \leq C(s_1-s_2)^\gamma(t-s_s)^{-\gamma/2}$ and $J_2 \leq C(s_1-s_2)^\gamma$. Consequently, for $s_2 < s_1$,

$$\left|\tau(s_1)-\tau(s_2)\right| \leq C(s_1-s_2)^\gamma(t-s_2)^{-\gamma/2}.$$

Similarly to J_1, $|\tau(s)| \leq C(t-s)^{\gamma/2}$.

Further, recall that $|\xi_k| \geq t^{-1}$. As in the proof of Lemma 1, set $n = [|\xi_k|t]+1$, so that $|\xi_k|t \leq n \leq 2|\xi_k|t$, and define the equidistant partition of $[0,t]$: $t_j = tj/n$, $j = 0,\dots,n$. Then

$$I_3 = \left|\sum_{j=1}^n\int_{t_{j-1}}^{t_j}\tau(s)e^{is\xi_k}ds\right|$$

$$
\begin{aligned}
&\le \left|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tau(t_{j-1})e^{is\xi_k}ds\right| + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left|\tau(s)-\tau(t_{j-1})\right|ds\\
&\le \left|\sum_{j=1}^{n}\tau(t_{j-1})\left(\frac{e^{it_j\xi_k}}{\xi_k}-\frac{e^{it_{j-1}\xi_k}}{\xi_k}\right)\right| + C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(t-t_{j-1})^{-\gamma/2}(s-t_{j-1})^{\gamma}ds\\
&\le \left|\frac{\tau(0)}{\xi_k}\right| + \left|\frac{\tau(t)}{\xi_k}\right| + \left|\sum_{j=1}^{n}\frac{e^{it_j\xi_k}}{\xi_k}\big(\tau(t_j)-\tau(t_{j-1})\big)\right|\\
&\quad + C\sum_{j=1}^{n}(t-t_{j-1})^{-\gamma/2}(t_j-t_{j-1})^{\gamma}\frac{t}{n}\\
&\le C\left(|\xi_k|^{-1}+|\xi_k|^{-1}\sum_{j=1}^{n}\left(\frac{t}{n}\right)^{\gamma}+\left(\frac{t}{n}\right)^{\gamma}\right)\le C|\xi_k|^{-1}\big(1+n^{1-\gamma}\big)\\
&\le C|\xi_k|^{-1}\big(1+|\xi_k|^{1-\gamma}\big)\le C|\xi_k|^{-\gamma}.
\end{aligned}
$$

Otherwise, if $|\xi_k| \le t^{-1}$, then

$$
I_3 \le \left|\int_0^t \tau(s)e^{is\xi_k}ds\right| \le \int_0^t \left|\tau(s)\right|ds \le C.
$$

Then, for $|\xi_k| \ge t^{-1}$,

$$
\begin{aligned}
\mathsf{E}_g\big[\left|v_k(t,x)\right|^2\big] &\le C_\alpha^2\Gamma_k^{-2/\alpha}\frac{\varphi(\xi_k)^{-2/\alpha}}{|\xi_k|^{2/\alpha+2H-2}}\left|\int_0^t\int_{\mathbb{R}}\rho(t-s,x-y)\sigma(s,y)dy\,e^{is\xi_k}ds\right|^2\\
&\le C\Gamma_k^{-2/\alpha}|\xi_k|^{2-2H-2\gamma}\left|\ln|\xi_k|+1\right|^{2(1+\eta)/\alpha},
\end{aligned}
$$

whereas, for $|\xi_k| < t^{-1}$,

$$
\mathsf{E}_g\big[\left|v_k(t,x)\right|^2\big] \le C\Gamma_k^{-2/\alpha}|\xi_k|^{2-2H}\left|\ln|\xi_k|+1\right|^{2(1+\eta)/\alpha}.
$$

Hence,

$$
\begin{aligned}
\mathsf{E}_{\xi,g}\big[\left|v_k(t,x)\right|^2\big] &\le C\Gamma_k^{-2/\alpha}\bigg[\int_{|y|\ge t^{-1}}|y|^{2-2H-2\gamma}\big(\left|\ln|y|\right|+1\big)^{2(1+\eta)/\alpha}\varphi(y)dy\\
&\quad + \int_{|y|<t^{-1}}|y|^{2-2H}\big(\left|\ln|y|\right|+1\big)^{2(1+\eta)/\alpha}\varphi(y)dy\bigg]\\
&= C\Gamma_k^{-2/\alpha}\bigg[\int_{|y|\ge t^{-1}}|y|^{1-2H-2\gamma}\big(\left|\ln|y|\right|+1\big)^{(-1+2/\alpha)(1+\eta)}dy\\
&\quad + \int_{|y|<t^{-1}}|y|^{1-2H}\big(\left|\ln|y|\right|+1\big)^{(-1+2/\alpha)(1+\eta)}dy\bigg].
\end{aligned}
$$

The first integral converges since $1-2H-2\gamma < -1$, whereas the second one converges since $1-2H > -1$. Therefore,

$$
\sum_{k=1}^{\infty}\mathsf{E}_{\xi,g}\big[\left|v_k(t,x)\right|^2\big] \le \sum_{k=1}^{\infty}C_k\Gamma_k^{-2/\alpha}.
$$

By the strong law of large numbers, $\Gamma_k \sim \frac{1}{k}$, $k \to +\infty$, P_Γ-almost surely. Therefore, $\sum_{k=1}^{\infty} \mathsf{E}_{\xi,g}[|v_k(t,x)|^2] < \infty$ P_Γ-almost surely.

In particular, $\sum_{k=1}^{\infty} \mathsf{E}_g[|v_k(t,x)|^2]$ converges $\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$-almost surely. For fixed $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$, the random variables $\{v_k(t,x), k \geq 1\}$ are independent centered Gaussian random variables; moreover,

$$\sum_{k=1}^{\infty} \mathsf{E}\big[\big|v_k(t,x)\big|^2\big] < \infty.$$

Then, by the Kolmogorov theorem, $\sum_{k=1}^{\infty} v_k(t,x)$ converges $\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma \otimes \mathsf{P}_g$-almost surely, as claimed.

It remains to prove that the sum $U_0(t,x) + \sum_{k=1}^{\infty} v_k(t,x)$ is equal to $U(t,x)$. We first show that $Z_N(t) \to Z(t)$, $N \to \infty$, almost surely in $C^\theta[0,T]$ for any $T > 0$, $\theta \in (0,H)$. Taking into account that, for any $t \in [0,T]$, $Z_N(t) \to Z(t)$, $N \to \infty$, almost surely, we get that there is a set $\Omega_0 \subset \Omega$ such that $\mathsf{P}(\Omega_0) = 1$ and $Z_N(t) \to Z(t)$, $N \to \infty$, for any $t \in [0,T] \cap \mathbb{Q}$, $\omega \in \Omega_0$. Thanks to Proposition 3, the sequence $\{Z_N, N \geq 1\}$ is almost surely bounded in $C^\theta[0,T]$ for any $\theta \in (0,H)$ and $T > 0$. Therefore, it is precompact in each of these spaces. Take arbitrary $\theta \in (0,H)$. Without loss of generality, the sequence $\{Z_N, N \geq 1\}$ is precompact in $C^\theta[0,T]$ for any $\omega \in \Omega_0$. Fix $\omega \in \Omega_0$ and let $\{Z_{N_k}, k \geq 1\}$ be any subsequence of $\{Z_N, N \geq 1\}$. In view of precompactness, it must contain a subsequence convergent in C^θ; to avoid cumbersome notation, we assume that $Z_{N_k} \to Y$, $k \to \infty$. In particular, $Z_{N_k}(t) \to Y(t)$, $k \to \infty$, $t \in [0,T] \cap \mathbb{Q}$. In view of continuity, $Z(t) = Y(t)$ for any $t \in [0,T]$. Since any subsequence of $\{Z_N, N \geq 1\}$ contains a subsequence convergent to Z in $C^\theta[0,T]$, the sequence itself converges to Z.

Now, thanks to the integrability of the fractional derivative of $f(s) = \int_{\mathbb{R}} \rho(t-s, x-y)\sigma(s,y)dy$, which was shown in the proof of Theorem 2, the convergence established in the previous paragraph yields

$$\int_0^t dZ_N(s) \int_{\mathbb{R}} \rho(t-s, x-y)\sigma(s,y)dy \to \int_0^t dZ(s) \int_{\mathbb{R}} \rho(t-s, x-y)\sigma(s,y)dy$$

as $n \to \infty$, concluding the proof. □

Remark 5. It is possible to consider (3) with Z being a real harmonizable multifractional stable motion considered in [4]. Making some minor changes, one can show that Theorem 4 is valid in this case as well.

Acknowledgements

The authors thank an anonymous referee for his careful reading of the manuscript and valuable remarks, which led to a significant improvement of the article.

References

[1] Balan, R., Kim, D.: The stochastic heat equation driven by a Gaussian noise: Germ Markov property. Commun. Stoch. Anal. **2**(2), 229–249 (2008). MR2446691

[2] Balan, R.M., Tudor, C.A.: Stochastic heat equation with multiplicative fractional-colored noise. J. Theor. Probab. **23**(3), 834–870 (2010). MR2679959. doi:10.1007/s10959-009-0237-3

[3] Balan, R.M., Tudor, C.A.: The stochastic wave equation with fractional noise: A random field approach. Stoch. Process. Appl. **120**(12), 2468–2494 (2010). MR2728174. doi:10.1016/j.spa.2010.08.006

[4] Dozzi, M., Shevchenko, G.: Real harmonizable multifractional stable process and its local properties. Stoch. Process. Appl. **121**(7), 1509–1523 (2011). MR2802463. doi:10.1016/j.spa.2011.03.012

[5] Kôno, N., Maejima, M.: Hölder continuity of sample paths of some self-similar stable processes. Tokyo J. Math. **14**(1), 93–100 (1991). MR1108158. doi:10.3836/tjm/1270130491

[6] Kozachenko, Y.V., Slyvka-Tylyshchak, A.I.: The Cauchy problem for the heat equation with a random right side. Random Oper. Stoch. Equ. **22**(1), 53–64 (2014). MR3245299. doi:10.1515/rose-2014-0006

[7] Kozachenko, Y.V., Veresh, K.I.: The heat equation with random initial conditions from Orlicz spaces. Theory Probab. Math. Stat. **80**, 71–84 (2010). MR2541953. doi:10.1090/S0094-9000-2010-00795-2

[8] Nualart, E., Quer-Sardanyons, L.: Gaussian estimates for the density of the non-linear stochastic heat equation in any space dimension. Stoch. Process. Appl. **122**(1), 418–447 (2012). MR2860455. doi:10.1016/j.spa.2011.08.013

[9] Quer-Sardanyons, L., Tindel, S.: The 1-d stochastic wave equation driven by a fractional Brownian sheet. Stoch. Process. Appl. **117**(10), 1448–1472 (2007). MR2353035. doi:10.1016/j.spa.2007.01.009

[10] Radchenko, V.: Mild solution of the heat equation with a general stochastic measure. Stud. Math. **194**(3), 231–251 (2009). MR2539554. doi:10.4064/sm194-3-2

[11] Radchenko, V.M.: Properties of integrals with respect to a general stochastic measure in a stochastic heat equation. Theory Probab. Math. Stat. **82**, 103–114 (2011). MR2790486. doi:10.1090/S0094-9000-2011-00830-7

[12] Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Science Publishers, Yverdon (1993), 976 pp. MR1347689

[13] Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Chapman & Hall, New York, NY (1994). MR1280932

[14] Walsh, J.B.: An introduction to stochastic partial differential equations. In: École D'été de Probabilités de Saint-Flour, XIV, 1984. Lecture Notes in Math., vol. 1180, pp. 265–439. Springer, Berlin (1986). MR0876085. doi:10.1007/BFb0074920

[15] Zähle, M.: On the link between fractional and stochastic calculus. In: Stochastic Dynamics (Bremen, 1997), pp. 305–325. Springer, New York (1999). MR1678495. doi:10.1007/0-387-22655-9_13