

Randomly stopped sums with consistently varying distributions

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Abstract Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent random variables, and η be a counting random variable independent of this sequence. We consider conditions for $\{\xi_1, \xi_2, \dots\}$ and η under which the distribution function of the random sum $S_\eta = \xi_1 + \xi_2 + \dots + \xi_\eta$ belongs to the class of consistently varying distributions. In our consideration, the random variables $\{\xi_1, \xi_2, \dots\}$ are not necessarily identically distributed.

Keywords Heavy tail, consistently varying tail, randomly stopped sum, inhomogeneous distributions, convolution closure, random convolution closure

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1 Introduction

Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \dots\}$, and let η be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v. η and r.v.s $\{\xi_1, \xi_2, \dots\}$ are independent. Let $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \dots + \xi_n$ for $n \in \mathbb{N}$,

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and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s $\{\xi_1, \xi_2, \dots\}$.

We are interested in conditions under which the d.f. of S_η ,

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(S_n \leq x), \quad (1)$$

belongs to the class of consistently varying distributions.

Throughout this paper, $f(x) = o(g(x))$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$, and $f(x) \sim g(x)$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ for two vanishing (at infinity) functions f and g . Also, we denote the support of a counting r.v. η by

$$\text{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}.$$

Before discussing the properties of F_{S_η} , we recall the definitions of some classes of heavy-tailed d.f.s, where $\overline{F}(x) = 1 - F(x)$ for all real x and a d.f. F .

- A d.f. F is heavy-tailed ($F \in \mathcal{H}$) if for every fixed $\delta > 0$,

$$\lim_{x \rightarrow \infty} \overline{F}(x) e^{\delta x} = \infty.$$

- A d.f. F is long-tailed ($F \in \mathcal{L}$) if for every y (equivalently, for some $y > 0$),

$$\overline{F}(x + y) \sim \overline{F}(x).$$

- A d.f. F has a dominatedly varying tail ($F \in \mathcal{D}$) if for every fixed $y \in (0, 1)$ (equivalently, for some $y \in (0, 1)$),

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

- A d.f. F has a consistently varying tail ($F \in \mathcal{C}$) if

$$\lim_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

- A d.f. F has a regularly varying tail ($F \in \mathcal{R}$) if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

for some $\alpha \geq 0$ and any fixed $y > 0$.

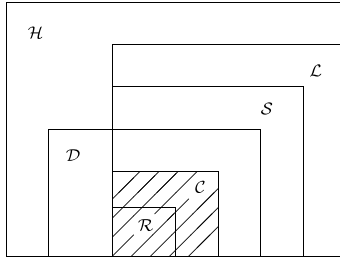


Fig. 1. Classes of heavy-tailed distributions.

- A d.f. F supported on the interval $[0, \infty)$ is subexponential ($F \in \mathcal{S}$) if

$$\lim_{x \rightarrow \infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2. \tag{2}$$

If a d.f. G is supported on \mathbb{R} , then we suppose that G is subexponential ($G \in \mathcal{S}$) if the d.f. $F(x) = G(x)\mathbb{1}_{[0, \infty)}(x)$ satisfies relation (2).

It is known (see, e.g., [4, 11, 13], and Chapters 1.4 and A3 in [8]) that these classes satisfy the following inclusions:

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

These inclusions are depicted in Fig. 1 with the class \mathcal{C} highlighted.

There exist many results on sufficient or necessary and sufficient conditions in order that the d.f. of the randomly stopped sum (1) belongs to some heavy-tailed distribution class. Here we present a few known results concerning the belonging of the d.f. F_{S_η} to some class. The first result on subexponential distributions was proved by Embrechts and Goldie (see Theorem 4.2 in [9]) and Cline (see Theorem 2.13 in [5]).

Theorem 1. Let $\{\xi_1, \xi_2, \dots\}$ be independent copies of a nonnegative r.v. ξ with subexponential d.f. F_ξ . Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. If $\mathbb{E}(1 + \delta)^\eta < \infty$ for some $\delta > 0$, then the d.f. $F_{S_\eta} \in \mathcal{S}$.

Similar results for the class \mathcal{D} can be found in Leipus and Šiaulyš [14]. We present the statement of Theorem 5 from this work.

Theorem 2. Let $\{\xi_1, \xi_2, \dots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{D}$ and finite mean $\mathbb{E}\xi$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ with d.f. F_η and finite mean $\mathbb{E}\eta$. Then d.f. $F_{S_\eta} \in \mathcal{D}$ iff $\min\{F_\xi, F_\eta\} \in \mathcal{D}$.

We recall only that the d.f. F belongs to the class \mathcal{D} if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = - \lim_{y \rightarrow \infty} \frac{1}{\log y} \log \left(\liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right).$$

The random convolution closure for the class \mathcal{L} was considered, for instance, in [1, 14, 16, 17]. We now present a particular statement of Theorem 1.1 from [17].

Theorem 3. Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ with d.f. F_η . Then the d.f. $F_{S_\eta} \in \mathcal{L}$ if the following five conditions are satisfied:

- (i) $\mathbb{P}(\eta \geq \kappa) > 0$ for some $\kappa \in \mathbb{N}$;
- (ii) for all $k \geq \kappa$, the d.f. F_{S_k} of the sum S_k is long tailed;
- (iii) $\sup_{k \geq 1} \sup_{x \in \mathbb{R}} (F_{S_k}(x) - F_{S_k}(x - 1))\sqrt{k} < \infty$;
- (iv) $\limsup_{z \rightarrow \infty} \sup_{k \geq \kappa} \sup_{x \geq k(z-1)+z} \frac{\overline{F}_{S_k}(x-1)}{\overline{F}_{S_k}(x)} = 1$;
- (v) $\overline{F}_\eta(ax) = o(\sqrt{x} \overline{F}_{S_\kappa}(x))$ for each $a > 0$.

We observe that the case of identically distributed r.v.s is considered in Theorems 1 and 2. In Theorem 3, r.v.s $\{\xi_1, \xi_2, \dots\}$ are independent but not necessarily identically distributed. A similar result for r.v.s having d.f.s with dominatedly varying tails can be found in [6].

Theorem 4 ([6], Theorem 2.1). Let r.v.s $\{\xi_1, \xi_2, \dots\}$ be nonnegative independent, not necessarily identically distributed, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{D} if the following three conditions are satisfied:

- (i) $F_{\xi_\kappa} \in \mathcal{D}$ for some $\kappa \in \text{supp}(\eta)$,
- (ii) $\limsup_{x \rightarrow \infty} \sup_{n \geq \kappa} \frac{1}{n \overline{F}_{\xi_\kappa}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty$,
- (iii) $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_{\xi_\kappa}}^+$.

In this work, we consider randomly stopped sums of independent and not necessarily identically distributed r.v.s. As noted before, we restrict ourselves on the class \mathcal{C} . If r.v.s $\{\xi_1, \xi_2, \dots\}$ are not identically distributed, then different collections of conditions on $\{\xi_1, \xi_2, \dots\}$ and η imply that $F_{S_\eta} \in \mathcal{C}$. We suppose that some r.v.s from $\{\xi_1, \xi_2, \dots\}$ have distributions belonging to the class \mathcal{C} , and we find minimal conditions on $\{\xi_1, \xi_2, \dots\}$ and η for the distribution of the randomly stopped sum S_η to remain in the same class. It should be noted that we use the methods developed in [6] and [7].

The rest of the paper is organized as follows. In Section 2, we present our main results together with two examples of randomly stopped sums S_η with d.f.s having consistently varying tails. Section 3 is a collection of auxiliary lemmas, and the proofs of the main results are presented in Section 4.

2 Main results

In this section, we present three statements in which we describe the belonging of a randomly stopped sum to the class \mathcal{C} . In the conditions of Theorem 5, the counting r.v. η has a finite support. Theorem 6 describes the situation where no moment conditions on the r.v.s $\{\xi_1, \xi_2, \dots\}$ are required, but there is strict requirement for η . Theorem 7 deals with the opposite case: the r.v.s $\{\xi_1, \xi_2, \dots\}$ should have finite means, whereas the requirement for η is weaker. It should be noted that the case of real-valued r.v.s $\{\xi_1, \xi_2, \dots\}$ is considered in Theorems 5 and 6, whereas Theorem 7 deals with non-negative r.v.s.

Theorem 5. *Let $\{\xi_1, \xi_2, \dots, \xi_D\}$, $D \in \mathbb{N}$, be independent real-valued r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots, \xi_D\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{C} if the following conditions are satisfied:*

- (a) $\mathbb{P}(\eta \leq D) = 1$,
- (b) $F_{\xi_1} \in \mathcal{C}$,
- (c) for each $k = 2, \dots, D$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$.

Theorem 6. *Let $\{\xi_1, \xi_2, \dots\}$ be independent real-valued r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{C} if the following conditions are satisfied:*

- (a) $F_{\xi_1} \in \mathcal{C}$,
- (b) for each $k \geq 2$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$,
- (c) $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty$,
- (d) $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_{\xi_1}}^+$.

When $\{\xi_1, \xi_2, \dots\}$ are identically distributed with common d.f. $F_\xi \in \mathcal{C}$, conditions (a), (b), and (c) of Theorem 6 are satisfied obviously. Hence, we have the following corollary.

Corollary 1 (See also Theorem 3.4 in [3]). *Let $\{\xi_1, \xi_2, \dots\}$ be i.i.d. real-valued r.v.s with d.f. $F_\xi \in \mathcal{C}$, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{C} if $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_\xi}^+$.*

Theorem 7. *Let $\{\xi_1, \xi_2, \dots\}$ be independent nonnegative r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{C} if the following conditions are satisfied:*

- (a) $F_{\xi_1} \in \mathcal{C}$,
- (b) for each $k \geq 2$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$,
- (c) $\mathbb{E}\xi_1 < \infty$,

- (d) $\overline{F}_\eta(x) = o(\overline{F}_{\xi_1}(x)),$
- (e) $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty,$
- (f) $\limsup_{u \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{\substack{k=1 \\ \mathbb{E}\xi_k \geq u}}^n \mathbb{E}\xi_k = 0.$

Similarly to Corollary 1, we can formulate the following statement. We note that, in the i.i.d. case, conditions (a), (b), (e), and (f) of Theorem 7 are satisfied.

Corollary 2. *Let $\{\xi_1, \xi_2, \dots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{C}$, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class \mathcal{C} under the following two conditions: $\mathbb{E}\xi < \infty$ and $\overline{F}_\eta(x) = o(\overline{F}_\xi(x)).$*

Further in this section, we present two examples of r.v.s $\{\xi_1, \xi_2, \dots\}$ and η for which the random sum F_{S_η} has a consistently varying tail.

Example 1. Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s such that ξ_k are exponentially distributed for all even k , that is,

$$\overline{F}_{\xi_k}(x) = e^{-x}, \quad x \geq 0, \quad k \in \{2, 4, 6, \dots\},$$

whereas, for each odd k , ξ_k is a copy of the r.v.

$$(1 + \mathcal{U}) 2^{\mathcal{G}},$$

where \mathcal{U} and \mathcal{G} are independent r.v.s, \mathcal{U} is uniformly distributed on the interval $[0, 1]$, and \mathcal{G} is geometrically distributed with parameter $q \in (0, 1)$, that is,

$$\mathbb{P}(\mathcal{G} = l) = (1 - q) q^l, \quad l = 0, 1, \dots$$

In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ and distributed according to the Poisson law.

Theorem 6 implies that the d.f. of the randomly stopped sum S_η belongs to the class \mathcal{C} because:

- (a) $F_{\xi_1} \in \mathcal{C}$ due to considerations in pp. 122–123 of [2],
- (b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3, 5, \dots\}$, and $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$ for $k \in \{2, 4, 6, \dots\}$,
- (c) $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) \leq 1,$
- (d) all moments of the r.v. η are finite.

Note that ξ_1 does not satisfy condition (c) of Theorem 7 in the case $q \geq 1/2$. Hence, Example 1 describes the situation where Theorem 6 should be used instead of Theorem 7.

Example 2. Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s such that ξ_k are distributed according to the Pareto law (with tail index $\alpha = 2$) for all odd k , and ξ_k are exponentially distributed (with parameter equal to 1) for all even k , that is,

$$\begin{aligned} \bar{F}_{\xi_k}(x) &= \frac{1}{x^2}, \quad x \geq 1, \quad k \in \{1, 3, 5, \dots\}, \\ \bar{F}_{\xi_k}(x) &= e^{-x}, \quad x \geq 0, \quad k \in \{2, 4, 6, \dots\}. \end{aligned}$$

In addition, let η be a counting r.v independent of $\{\xi_1, \xi_2, \dots\}$ that has the Zeta distribution with parameter 4, that is,

$$\mathbb{P}(\eta = m) = \frac{1}{\zeta(4)} \frac{1}{(m + 1)^4}, \quad m \in \mathbb{N}_0,$$

where ζ denotes the Riemann zeta function.

Theorem 7 implies that the d.f. of the randomly stopped sum S_η belongs to the class \mathcal{C} because:

- (a) $F_{\xi_1} \in \mathcal{C}$,
- (b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3, 5, \dots\}$, and $\bar{F}_{\xi_k}(x) = o(\bar{F}_{\xi_1}(x))$ for $k \in \{2, 4, 6, \dots\}$,
- (c) $\mathbb{E}\xi_1 = 2$,
- (d) $\bar{F}_\eta(x) = o(\bar{F}_{\xi_1}(x))$,
- (e) $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \bar{F}_{\xi_1}(x)} \sum_{i=1}^n \bar{F}_{\xi_i}(x) \leq 1$,
- (f) $\max_{k \in \mathbb{N}} \mathbb{E}\xi_k = 2$.

Regarding condition (d), it should be noted that the Zeta distribution with parameter 4 is a discrete version of Pareto distribution with tail index 3.

Note that η does not satisfy the condition (d) of Theorem 6 because $J_{F_{\xi_1}}^+ = 2$ and $\mathbb{E}\eta^3 = \infty$. Hence, Example 2 describes the situation where Theorem 7 should be used instead of Theorem 6.

3 Auxiliary lemmas

This section deals with several auxiliary lemmas. The first lemma is Theorem 3.1 in [3] (see also Theorem 2.1 in [15]).

Lemma 1. Let $\{X_1, X_2, \dots, X_n\}$ be independent real-valued r.v.s. If $F_{X_k} \in \mathcal{C}$ for each $k \in \{1, 2, \dots, n\}$, then

$$\mathbb{P}\left(\sum_{i=1}^n X_i > x\right) \sim \sum_{i=1}^n \bar{F}_{X_i}(x).$$

The following statement about nonnegative subexponential distributions was proved in Proposition 1 of [10] and later generalized to a wider distribution class in Corollary 3.19 of [12].

Lemma 2. *Let $\{X_1, X_2, \dots, X_n\}$ be independent real-valued r.v.s. Assume that $\overline{F}_{X_i}/\overline{F}(x) \xrightarrow{x \rightarrow \infty} b_i$ for some subexponential d.f. F and some constants $b_i \geq 0, i \in \{1, 2, \dots, n\}$. Then*

$$\frac{\overline{F_{X_1 * X_2 * \dots * X_n}}(x)}{\overline{F}(x)} \xrightarrow{x \rightarrow \infty} \sum_{i=1}^n b_i.$$

In the next lemma, we show in which cases the convolution $F_{X_1 * X_2 * \dots * X_n}$ belongs to the class \mathcal{C} .

Lemma 3. *Let $\{X_1, X_2, \dots, X_n\}, n \in \mathbb{N}$, be independent real-valued r.v.s. Then the d.f. F_{Σ_n} of the sum $\Sigma_n = X_1 + X_2 + \dots + X_n$ belongs to the class \mathcal{C} if the following conditions are satisfied:*

- (a) $F_{X_1} \in \mathcal{C}$,
- (b) for each $k = 2, \dots, n$, either $F_{X_k} \in \mathcal{C}$ or $\overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x))$.

Proof. Evidently, we can suppose that $n \geq 2$. We split our proof into two parts.

First part. Suppose that $F_{X_k} \in \mathcal{C}$ for all $k \in \{1, 2, \dots, n\}$. In such a case, the lemma follows from Lemma 1 and the inequality

$$\frac{a_1 + a_2 + \dots + a_m}{b_1 + b_2 + \dots + b_m} \leq \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m} \right\} \tag{3}$$

for $a_i \geq 0$ and $b_i > 0, i = 1, 2, \dots, m$.

Namely, using the relation of Lemma 1 and estimate (3), we get that

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\Sigma_n}(xy)}{\overline{F}_{\Sigma_n}(x)} &= \limsup_{x \rightarrow \infty} \frac{\sum_{k=1}^n \overline{F}_{X_k}(xy)}{\sum_{k=1}^n \overline{F}_{X_k}(x)} \\ &\leq \max_{1 \leq k \leq n} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{X_k}(xy)}{\overline{F}_{X_k}(x)} \end{aligned}$$

for arbitrary $y \in (0, 1)$.

Since $F_{X_k} \in \mathcal{C}$ for each k , the last estimate implies that the d.f. F_{Σ_n} has a consistently varying tail, as desired.

Second part. Now suppose that $F_{X_k} \notin \mathcal{C}$ for some of indexes $k \in \{2, 3, \dots, n\}$. By the conditions of the lemma we have that $\overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x))$ for such k . Let $\mathcal{K} \subset \{2, 3, \dots, n\}$ be the subset of indexes k such that

$$F_{X_k} \notin \mathcal{C} \quad \text{and} \quad \overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x)).$$

By Lemma 2,

$$\overline{F}_{\widehat{\Sigma}_n}(x) \sim \overline{F}_{X_1}(x),$$

where

$$\widehat{\Sigma}_n = X_1 + \sum_{k \in \mathcal{K}} X_k.$$

Hence,

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}_{\widehat{\Sigma}_n}(xy)}{\overline{F}_{\widehat{\Sigma}_n}(x)} = \limsup_{x \rightarrow \infty} \frac{\overline{F}_{X_1}(xy)}{\overline{F}_{X_1}(x)} \tag{4}$$

for every $y \in (0, 1)$.

Equality (4) implies immediately that the d.f. $F_{\widehat{\Sigma}_n}$ belongs to the class \mathcal{C} . Therefore, the d.f. F_{Σ_n} also belongs to the class \mathcal{C} according to the first part of the proof because

$$\Sigma_n = \widehat{\Sigma}_n + \sum_{k \notin \mathcal{K}} X_k$$

and $F_{X_k} \in \mathcal{C}$ for each $k \notin \mathcal{K}$. The lemma is proved. □

The following two statements about dominatedly varying distributions are Lemma 3.2 and Lemma 3.3 in [6]. Since any consistently varying distribution is also dominatingly varying, these statements will be useful in the proofs of our main results concerning the class \mathcal{C} .

Lemma 4. *Let $\{X_1, X_2, \dots\}$ be independent real-valued r.v.s, and $F_{X_\nu} \in \mathcal{D}$ for some $\nu \geq 1$. Suppose, in addition, that*

$$\limsup_{x \rightarrow \infty} \sup_{n \geq \nu} \frac{1}{n \overline{F}_{X_\nu}(x)} \sum_{i=1}^n \overline{F}_{X_i}(x) < \infty.$$

Then, for each $p > J_{F_{X_\nu}}^+$, there exists a positive constant c_1 such that

$$\overline{F}_{S_n}(x) \leq c_1 n^{p+1} \overline{F}_{X_\nu}(x) \tag{5}$$

for all $n \geq \nu$ and $x \geq 0$.

In fact, Lemma 4 is proved in [6] for nonnegative r.v.s. However, the lemma remains valid for real-valued r.v.s. To see this, it suffices to observe that $\mathbb{P}(X_1 + X_2 + \dots + X_n > x) \leq \mathbb{P}(X_1^+ + X_2^+ + \dots + X_n^+ > x)$ and $\mathbb{P}(X_k > x) = \mathbb{P}(X_k^+ > x)$, where $n \in \mathbb{N}$, $k \in \{1, 2, \dots, n\}$, $x \geq 0$, and a^+ denotes the positive part of a .

Lemma 5. *Let $\{X_1, X_2, \dots\}$ be independent real-valued r.v.s, and $F_{X_\nu} \in \mathcal{D}$ for some $\nu \geq 1$. Let, in addition,*

$$\begin{aligned} \limsup_{u \rightarrow \infty} \sup_{n \geq \nu} \frac{1}{n} \sum_{k=1}^n \mathbb{E}(|X_k| \mathbb{1}_{\{X_k \leq -u\}}) &= 0, \\ \limsup_{x \rightarrow \infty} \sup_{n \geq \nu} \frac{1}{n \overline{F}_{X_\nu}(x)} \sum_{i=1}^n \overline{F}_{X_i}(x) &< \infty, \end{aligned}$$

and $\mathbb{E}X_k = \mathbb{E}X_k^+ - \mathbb{E}X_k^- = 0$ for $k \in \mathbb{N}$. Then, for each $\gamma > 0$, there exists a positive constant $c_2 = c_2(\gamma)$ such that

$$\mathbb{P}(S_n > x) \leq c_2 n \overline{F}_{X_\nu}(x)$$

for all $x \geq \gamma n$ and all $n \geq \nu$.

4 Proofs of the main results

Proof of Theorem 5. It suffices to prove that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} \leq 1. \tag{6}$$

According to estimate (3), for $x > 0$ and $y \in (0, 1)$, we have

$$\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} = \frac{\sum_{n \in \text{supp}(\eta)}^D \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\sum_{n \in \text{supp}(\eta)}^D \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)} \leq \max_{\substack{1 \leq n \leq D \\ n \in \text{supp}(\eta)}} \frac{\mathbb{P}(S_n > xy)}{\mathbb{P}(S_n > x)}.$$

Hence, by Lemma 3,

$$\begin{aligned} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} &\leq \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \max_{\substack{1 \leq n \leq D \\ n \in \text{supp}(\eta)}} \frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} \\ &\leq \max_{\substack{1 \leq n \leq D \\ n \in \text{supp}(\eta)}} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} = 1, \end{aligned}$$

which implies the desired estimate (6). The theorem is proved. □

Proof of Theorem 6. As in Theorem 5, it suffices to prove inequality (6). For all $K \in \mathbb{N}$ and $x > 0$, we have

$$\mathbb{P}(S_\eta > x) = \left(\sum_{n=1}^K + \sum_{n=K+1}^\infty \right) \mathbb{P}(S_n > x) \mathbb{P}(\eta = n).$$

Therefore, for $x > 0$ and $y \in (0, 1)$, we have

$$\begin{aligned} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} &= \frac{\sum_{n=1}^K \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)} \\ &\quad + \frac{\sum_{n=K+1}^\infty \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)} \\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{7}$$

The random variable η is not degenerate at zero, so there exists $a \in \mathbb{N}$ such that $\mathbb{P}(\eta = a) > 0$. If $K \geq a$, then using inequality (3), we get

$$\mathcal{J}_1 \leq \frac{\sum_{n \in \text{supp}(\eta)}^K \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\sum_{n \in \text{supp}(\eta)}^K \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)} \leq \max_{\substack{1 \leq n \leq K \\ n \in \text{supp}(\eta)}} \frac{\mathbb{P}(S_n > xy)}{\mathbb{P}(S_n > x)}.$$

Similarly as in the proof of Theorem 5, it follows that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{J}_1 \leq \max_{\substack{1 \leq n \leq K \\ n \in \text{supp}(\eta)}} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} = 1. \tag{8}$$

Since $\mathcal{C} \subset \mathcal{D}$, we can use Lemma 4 for the numerator of \mathcal{J}_2 to obtain

$$\sum_{n=K+1}^{\infty} \mathbb{P}(S_n > xy)\mathbb{P}(\eta = n) \leq c_3 \bar{F}_{\xi_1}(xy) \sum_{n=K+1}^{\infty} n^{p+1}\mathbb{P}(\eta = n)$$

with some positive constant c_3 . For the denominator of \mathcal{J}_2 , we have that

$$\begin{aligned} \mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \\ &\geq \mathbb{P}(S_a > x)\mathbb{P}(\eta = a). \end{aligned}$$

The conditions of the theorem imply that

$$S_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k + \sum_{k \notin \mathcal{K}_a} \xi_k,$$

where $\mathcal{K}_a = \{k \in \{2, \dots, a\} : F_{\xi_k} \notin \mathcal{C}, \bar{F}_{\xi_k}(x) = o(\bar{F}_{\xi_1}(x))\}$.

By Lemma 2

$$\bar{F}_{\widehat{S}_a}(x)/\bar{F}_{\xi_1}(x) \xrightarrow{x \rightarrow \infty} 1,$$

where $F_{\widehat{S}_a}$ is the d.f. of the sum

$$\widehat{S}_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k$$

In addition, by Lemma 3 we have that the d.f. $F_{\widehat{S}_a}$ belongs to the class \mathcal{C} .

If $k \notin \mathcal{K}_a$, then $F_{\xi_k} \in \mathcal{C}$ by the conditions of the theorem. This fact and Lemma 1 imply that

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(S_a > x)}{\bar{F}_{\xi_1}(x)} \geq 1 + \sum_{k \notin \mathcal{K}_a} \liminf_{x \rightarrow \infty} \frac{\bar{F}_{\xi_k}(x)}{\bar{F}_{\xi_1}(x)}.$$

Hence,

$$\mathbb{P}(S_\eta > x) \geq \frac{1}{2} \bar{F}_{\xi_1}(x) \mathbb{P}(\eta = a) \tag{9}$$

for x sufficiently large. Therefore,

$$\begin{aligned} &\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{J}_2 \\ &\leq \frac{2c_3}{\mathbb{P}(\eta = a)} \left(\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\bar{F}_{\xi_1}(xy)}{\bar{F}_{\xi_1}(x)} \right) \sum_{n=K+1}^{\infty} n^{p+1} \mathbb{P}(\eta = n). \end{aligned} \tag{10}$$

Estimates (7), (8), and (10) imply that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} \leq 1 + \frac{2c_3}{\mathbb{P}(\eta = a)} \mathbb{E} \eta^{p+1} \mathbb{1}_{\{\eta > K\}}$$

for arbitrary $K \geq a$.

Letting K tend to infinity, we get the desired estimate (6) due to condition (d). The theorem is proved. \square

Proof of Theorem 7. Once again, it suffices to prove inequality (6).

By condition (e) we have that there exist two positive constants c_4 and c_5 such that

$$\sum_{i=1}^n \overline{F}_{\xi_i}(x) \leq c_5 n \overline{F}_{\xi_1}(x), \quad x \geq c_4, \quad n \in \mathbb{N}.$$

Therefore,

$$\mathbb{E}S_n = \sum_{j=1}^n \mathbb{E}\xi_j = \sum_{j=1}^n \left(\int_0^{c_4} + \int_{c_4}^{\infty} \right) \overline{F}_{\xi_j}(u) du \leq c_4 n + c_5 n \mathbb{E}\xi_1 =: c_6 n \quad (11)$$

for a positive constant c_6 and all $n \in \mathbb{N}$.

If $K \in \mathbb{N}$ and $x > 4Kc_6$, then we have

$$\begin{aligned} \mathbb{P}(S_\eta > x) &= \mathbb{P}(S_\eta > x, \eta \leq K) \\ &\quad + \mathbb{P}\left(S_\eta > x, K < \eta \leq \frac{x}{4c_6}\right) \\ &\quad + \mathbb{P}\left(S_\eta > x, \eta > \frac{x}{4c_6}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} &= \frac{\mathbb{P}(S_\eta > xy, \eta \leq K)}{\mathbb{P}(S_\eta > x)} \\ &\quad + \frac{\mathbb{P}(S_\eta > xy, K < \eta \leq \frac{xy}{4c_6})}{\mathbb{P}(S_\eta > x)} \\ &\quad + \frac{\mathbb{P}(S_\eta > xy, \eta > \frac{xy}{4c_6})}{\mathbb{P}(S_\eta > x)} \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \end{aligned} \quad (12)$$

if $xy > 4Kc_6$, $x > 0$, and $y \in (0, 1)$.

The random variable η is not degenerate at zero, so $\mathbb{P}(\eta = a) > 0$ for some $a \in \mathbb{N}$. If $K \geq a$, then

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{I}_1 \leq 1 \quad (13)$$

similarly to estimate (8) in Theorem 6.

For the numerator of \mathcal{I}_2 , we have

$$\begin{aligned} \mathcal{I}_{2,1} &:= \mathbb{P}\left(S_\eta > xy, K < \eta \leq \frac{xy}{4c_6}\right) \\ &= \sum_{K < n \leq \frac{xy}{4c_6}} \mathbb{P}\left(\sum_{i=1}^n (\xi_i - \mathbb{E}\xi_i) > xy - \sum_{j=1}^n \mathbb{E}\xi_j\right) \mathbb{P}(\eta = n) \\ &\leq \sum_{K < n \leq \frac{xy}{4c_6}} \mathbb{P}\left(\sum_{i=1}^n (\xi_i - \mathbb{E}\xi_i) > \frac{3}{4}xy\right) \mathbb{P}(\eta = n) \end{aligned} \quad (14)$$

by inequality (11).

The random variables $\xi_1 - \mathbb{E}\xi_1, \xi_2 - \mathbb{E}\xi_2, \dots$ satisfy the conditions of Lemma 5. Namely, $\mathbb{E}(\xi_k - \mathbb{E}\xi_k) = 0$ for $k \in \mathbb{N}$ and $F_{\xi_1 - \mathbb{E}\xi_1} \in \mathcal{C} \subset \mathcal{D}$ obviously. In addition,

$$\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \mathbb{P}(\xi_1 - \mathbb{E}\xi_1 > x)} \sum_{k=1}^n \mathbb{P}(\xi_i - \mathbb{E}\xi_i > x) < \infty$$

by conditions (a), (c) and (e). Finally,

$$\begin{aligned} & \limsup_{u \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^n \mathbb{E}(|\xi_k - \mathbb{E}\xi_k| \mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leq -u\}}) \\ &= \limsup_{u \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^n \mathbb{E}((\mathbb{E}\xi_k - \xi_k) \mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leq -u\}}) \\ &\leq \limsup_{u \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{\substack{1 \leq k \leq n \\ \mathbb{E}\xi_k \geq u}} \mathbb{E}\xi_k = 0 \end{aligned}$$

because of condition (f). So, applying the estimate of Lemma 5 to (14), we get

$$\begin{aligned} \mathcal{I}_{2,1} &\leq c_7 \sum_{K < n \leq \frac{xy}{4c_6}} n \bar{F}_{\xi_1} \left(\frac{3}{4}xy + \mathbb{E}\xi_1 \right) \mathbb{P}(\eta = n) \\ &\leq c_7 \bar{F}_{\xi_1} \left(\frac{3}{4}xy \right) \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \end{aligned}$$

with a positive constant c_7 . For the denominator of \mathcal{I}_2 , we can use the inequality

$$\begin{aligned} \mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n) \\ &\geq \sum_{n=1}^{\infty} \mathbb{P}(\xi_1 > x) \mathbb{P}(\eta = n) \\ &\geq \bar{F}_{\xi_1}(x) \mathbb{P}(\eta = a) \end{aligned} \tag{15}$$

since the r.v.s $\{\xi_1, \xi_2, \dots\}$ are nonnegative by assumption. Hence,

$$\mathcal{I}_2 \leq \frac{c_7}{\mathbb{P}(\eta = a)} \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \frac{\bar{F}_{\xi_1}(\frac{3}{4}xy)}{\bar{F}_{\xi_1}(x)}.$$

If $y \in (1/2, 1)$, then the last estimate implies that

$$\limsup_{x \rightarrow \infty} \mathcal{I}_2 \leq \frac{c_7}{\mathbb{P}(\eta = a)} \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \limsup_{x \rightarrow \infty} \frac{\bar{F}_{\xi_1}(\frac{3}{8}x)}{\bar{F}_{\xi_1}(x)} \leq c_8 \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \tag{16}$$

with some positive constant c_8 because $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$.

Using inequality (15) again, we obtain

$$\mathcal{I}_3 \leq \frac{\mathbb{P}(\eta > \frac{xy}{4c_6})}{\mathbb{P}(S_\eta > x)} \leq \frac{1}{\mathbb{P}(\eta = a)} \frac{\overline{F}_\eta(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(\frac{xy}{4c_6})} \frac{\overline{F}_{\xi_1}(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(x)}.$$

Therefore, for $y \in (1/2, 1)$, we get

$$\limsup_{x \rightarrow \infty} \mathcal{I}_3 \leq \frac{1}{\mathbb{P}(\eta = a)} \limsup_{x \rightarrow \infty} \frac{\overline{F}_\eta(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(\frac{xy}{4c_6})} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_1}(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(x)} = 0 \quad (17)$$

by condition (d).

Estimates (12), (13), (16), and (17) imply that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} \leq 1 + c_8 \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}}$$

for $K \geq a$.

Letting K tend to infinity, we get the desired estimate (6) because $\mathbb{E}\eta < \infty$ by conditions (c) and (d). The theorem is proved. \square

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