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On spectra of probability measures generated by GLS-expansions

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Abstract We study properties of distributions of random variables with independent identically distributed symbols of generalized Lüroth series (GLS) expansions (the family of GLS-expansions contains Lüroth expansion and Q_∞- and G^2_∞-expansions). To this end, we explore fractal properties of the family of Cantor-like sets $C[GLS, V]$ consisting of real numbers whose GLS-expansions contain only symbols from some countable set $V \subset N \cup \{0\}$, and derive exact formulae for the determination of the Hausdorff–Besicovitch dimension of $C[GLS, V]$. Based on these results, we get general formulae for the Hausdorff–Besicovitch dimension of the spectra of random variables with independent identically distributed GLS-symbols for the case where all but countably many points from the unit interval belong to the basis cylinders of GLS-expansions.

Keywords Random variables with independent GLS-symbols, Q_∞-expansion, N-self-similar sets, Hausdorff–Besicovitch dimension
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1 Introduction

During the last 20 years, many authors studied singularly continuous probability measures generated by different expansions of real numbers (see, e.g., [2, 9, 10, 12–15]). All these measures are the distributions of random variables of the form

$$\xi = \Delta^F_{\xi_1 \xi_2 \dots \xi_k \dots},$$

where $\{\xi_k\}$ are independent or Markovian, and F stands for some expansion of real numbers. For the case of expansions over finite alphabets, fractal properties of the spectra of the corresponding measures are relatively well studied. For the case of infinite alphabets, the situation is essentially more complicated. In [8] and [9], it has been shown that even for self-similar Q_∞-expansion and for i.i.d. case, the Hausdorff–Besicovitch dimension of the corresponding spectra cannot be calculated in a traditional way (as a root of the corresponding equation), and formulae for the Hausdorff dimension of the measure μ_ξ are also unknown.

In this paper, we generalize results from [8] and [9] for the case of distributions of random variables with independent identically distributed GLS digits

$$\xi = \Delta^{GLS}_{\xi_1\xi_2\ldots\xi_k\ldots}$$

and get general formulae for the determination of the Hausdorff–Besicovitch dimension of spectra of ξ for the case where all but countably many points from the unit interval belong to the basis cylinders of GLS-expansion.

2 On GLS-expansion and fractal properties of related probability measures

Let $Q_\infty = (q_0, q_1, \ldots, q_n, \ldots)$ be an infinite stochastic vector with positive coordinates. Let us consider a countable sequence $\Delta_i = [a_i, b_i]$ of intervals such that $Int(\Delta_i) \cap Int(\Delta_j) = \emptyset$ $(i \neq j)$ and $|\Delta_i| = q_i$. The sets Δ_i are said to be cylinders of GLS-expansion (generalized Lüroth series).

Let us remark that the placement of cylinders of rank 1 is completely determined by the preselected procedure.

For every cylinder Δ_{i_1} of rank 1, we consider a sequence of nonoverlapping closed intervals $\Delta_{i_1 i_2} \subset \Delta_{i_1}$ such that

$$\frac{|\Delta_{i_1 i_2}|}{|\Delta_{i_1}|} = q_{i_2}$$

and the placement of $\Delta_{i_1 i_2}$ in Δ_{i_1} is the same as Δ_{i_1} in [0; 1]. The closed intervals $\Delta_{i_1 i_2}$ are said to be cylinders of rank 2 of the GLS-expansion.

Similarly, for every cylinder of rank $(n-1)$ $\Delta_{i_1 i_2 \ldots i_{n-1}}$, we consider the sequence of nonoverlapping closed intervals $\Delta_{i_1 i_2 \ldots i_n} \subset \Delta_{i_1 i_2 \ldots i_{n-1}}$ such that

$$\frac{|\Delta_{i_1 i_2 \ldots i_n}|}{|\Delta_{i_1 i_2 \ldots i_{n-1}}|} = q_{i_n}, \quad i \in N \cup \{0\},$$

and the placement of $\Delta_{i_1 i_2 \ldots i_n}$ in $\Delta_{i_1 i_2 \ldots i_{n-1}}$ is the same as Δ_{i_1} in [0; 1].

The closed intervals $\Delta_{i_1 i_2 \ldots i_n}$ are said to be cylinders of rank n of the GLS-expansion. From the construction it follows that

$$|\Delta_{i_1 i_2 \ldots i_n}| = q_{i_1} \cdot q_{i_2} \cdot \ldots \cdot q_{i_n} \leq (q_{\max})^n \to 0 \quad (n \to \infty),$$

where $q_{\max} := \max_i q_i$.

So, for any sequence of indices $\{i_k\}$ $(i_k \in N \cup \{0\})$, there exists the sequence of embedded closed intervals

$$\Delta_{i_1} \subset \Delta_{i_1 i_2} \subset \Delta_{i_1 i_2 i_2} \subset \cdots \subset \Delta_{i_1 i_2 \ldots i_k} \subset \cdots$$

with $|\Delta_{i_1 i_2 \dots i_k}| \to 0,\ k \to \infty$. Therefore, there exists a unique point $x \in [0,1]$ that belongs to all these cylinders.

Conversely, if $x \in [0,1]$ belongs to some cylinder of rank k for any $k \in N$ and x is not an end-point for any cylinder, then there exists a unique sequence of the cylinders

$$\Delta_{i_1(x)} \supset \Delta_{i_1(x) i_2(x)} \supset \Delta_{i_1(x) i_2(x) i_3(x)} \supset \cdots \supset \Delta_{i_1(x) i_2(x) \dots i_k(x)} \supset \cdots$$

containing x, and

$$x = \bigcap_{k=1}^{\infty} \Delta_{i_1(x) i_2(x) \dots i_k(x)} = \Delta_{i_1(x) i_2(x) \dots i_k(x) \dots}.$$

The latter expression is called the GLS-expansion of x (see, e.g., [1, 3, 4, 6, 7] for details).

Let us remark that the Lüroth expansion and Q_∞-expansion [8, 9] are particular cases of the GLS-expansion. For the case where the ratio of lengths of two embedded cylinders of successive ranks depends on the last index and it is a power of $\varphi = \frac{1+\sqrt{5}}{2}$, we get the G^2_∞-expansion of x [11].

Let $Q_\infty = (q_0, q_1, \dots, q_n, \dots)$ be a stochastic vector with positive coordinates, and let $x = \Delta^{GLS}_{i_1(x) i_2(x) \dots i_k(x) \dots}$ be the GLS-expansion of $x \in [0,1]$.

Let $\{\xi_k\}$ be a sequence of independent identically distributed random variables:

$$P(\xi_k = i) := p_i \geq 0,$$

where

$$\sum_{i=0}^{\infty} p_i = 1.$$

Using the sequence $\{\xi_k\}$ and a given GLS-expansion, let us consider the random variable

$$\xi = \Delta^{GLS}_{\xi_1 \xi_2 \dots \xi_k \dots},$$

which is said to be the random variable with independent identically distributed GLS-symbols. Let μ_ξ be the corresponding probability measure.

To investigate metric, topological, and fractal properties of the spectrum of the random variable with independent identically distributed GLS-symbols, let us study properties of the following family of sets. Let V be a subset of $N_0 := \{0, 1, 2, \dots\}$, and let

$$C[GLS, V] = \left\{x : x = \Delta^{GLS}_{\alpha_1(x) \dots \alpha_k(x) \dots}, \alpha_k \in V\right\}.$$

If the set V is finite, then $C[GLS, V]$ is a self-similar set satisfying the open set condition (see, e.g., [5]). So, its Hausdorff–Besicovitch dimension coincides with the root of the equation

$$\sum_{i \in V} q_i^x = 1. \tag{1}$$

If the set V is countable, then the situation is essentially more complicated. In particular, there exist stochastic vectors Q_∞ and subsets V such that equation (1) has no roots on the unit interval.

For example, if $q_i = \frac{A}{(i+2)\ln^2(i+2)}$ and $V = N$, then the equation $\sum_{i\in V} q_i^x = 1$ has no roots on [0; 1].

Theorem 1. *If a stochastic vector Q_∞ and a set $V \subset N_0$ are such that the equation $\sum_{i\in V} q_i^x = 1$ has a root α_0 on* [0, 1], *then*

$$\dim_H\bigl(C[GLS, V]\bigr) = \alpha_0.$$

Proof. First, let us show that for any $k \in N$, the set $C[GLS, V]$ can be covered by cylinders of rank k and that the α_0-volume of this covering is equal to 1.

For $k = 1$, the set $C[GLS, V]$ can be covered by cylinders of rank 1. It easy to see that the α_0-volume is equal to 1:

$$\sum_{i_1\in V} |\Delta_{i_1}|^{\alpha_0} = \sum_{i_1\in V} q_{i_1}^{\alpha_0} = 1.$$

Suppose that for $k = n - 1$, the α_0-volume of the covering of $C[GLS, V]$ by cylinders of rank $n - 1$ is equal to 1. Let us show that for $k = n$, the α_0-volume of the covering of $C[GLS, V]$ by cylinders of rank n will not change. We have

$$\begin{aligned}\sum_{i_j\in V} |\Delta_{i_1 i_2 \ldots i_{n-1} i_n}|^{\alpha_0} &= \sum_{i_j\in V} (q_{i_1} q_{i_2} \ldots q_{n-1} q_n)^{\alpha_0}\\ &= \sum_{i_1\in V} q_{i_1}^{\alpha_0} \cdot \sum_{i_j\in V} (q_{i_1} q_{i_2} \ldots q_{n-1})^{\alpha_0} = 1.\end{aligned}$$

So, for any $\varepsilon > 0$, we get

$$H_\varepsilon^{\alpha_0}\bigl(C[GLS, V]\bigr) \le 1.$$

Hence,

$$H^{\alpha_0}\bigl(C[GLS, V]\bigr) \le 1.$$

By the definition of the Hausdorff–Besicovitch dimension we get

$$\dim_H\bigl(C[GLS, V]\bigr) \le \alpha_0.$$

Let us show that $\dim_H(C[GLS, V]) \ge \alpha_0$. To this end, let us consider sets $V = \{i_1, \ldots, i_k, \ldots\}$, $V_k = \{i_1, \ldots, i_k\}$, $k \ge 2$, $k \in N$, and the sequence $C[GLS, V_k]$ of subsets of $C[GLS, V]$. For all $k \ge 2$, $k \in N$, we have

$$C[GLS, V_k] \subset C[GLS, V_{k+1}],$$

and, therefore,

$$\dim_H\bigl(C[GLS, V_k]\bigr) \le \dim_H\bigl(C[GLS, V_{k+1}]\bigr).$$

Let $\dim_H(C[GLS, V_k]) = \alpha_k$. The sets $C[GLS, V_k]$ are self-similar and satisfy the open set condition (OSC). Hence, the Hausdorff–Besicovitch dimension α_k of $C[GLS, V_k]$ coincides with the solution of the equation

$$\sum_{i\in V_k} q_i^x = 1.$$

It is clear that $\alpha_2 < \alpha_3 < \cdots < \alpha_k < \cdots$ and $\alpha_k < \alpha_0$. So, the sequence $\{\alpha_k\}$ is increasing and bounded. Therefore, there exists a limit $\lim_{k\to\infty} \alpha_k = \alpha^*$.

It is clear that $\alpha^* \le \alpha_0$ because $\alpha_k < \alpha_0$ $(\forall k \in N)$. Let us prove that $\alpha^* = \alpha_0$.

Assume the opposite: let $\alpha^* < \alpha_0$. Then there exists α' such that $\alpha^* < \alpha' < \alpha_0$. Then $\sum_{i\in V_k} q_i^{\alpha'} < 1$ for all $k \in N$. Since $\sum_{i\in V_k} q_i^{\alpha_k} = 1$, we get $\sum_{i\in V_k} q_i^{\alpha'} < 1$ for all $k \in N$. Let us consider the series $\sum_{k=1}^{\infty} q_{i_k}^{\alpha'}$. It is clear that $\sum_{k=1}^{n} q_{i_k}^{\alpha'} < 1$ for all $n \in N$. So $\lim_{n\to\infty} \sum_{k=1}^{n} q_{i_k}^{\alpha'} \le 1$. Therefore, $\sum_{i\in V} q_i^{\alpha'} \le 1$.

Since $q_i^{\alpha'} > q_i^{\alpha_0}$ for all $i \in V$ and $\sum_{i\in V} q_i^{\alpha_0} = 1$, we get $\sum_{i\in V} q_i^{\alpha'} > \sum_{i\in V} q_i^{\alpha_0} = 1$, which contradicts the already proven inequality $\sum_{i\in V} q_i^{\alpha'} \le 1$. This proves that $\alpha^* = \alpha_0$.

Since for any $k \ge 2,\ k \in N$,

$$\alpha_k = \dim_H\big(C[GLS, V_k]\big) \le \dim_H\big(C[GLS, V]\big),$$

we get

$$\alpha_0 \le \dim_H\big(C[GLS, V]\big).$$

Thus,

$$\alpha_0 = \dim_H\big([GLS, V]\big). \qquad \square$$

Theorem 2. *If the matrix* Q_∞ *and the set* $V = \{i_1, i_2, \ldots, i_k, i_{k+1}, \ldots\}$ *are such that equation* $\sum_{i\in V} q_i^x = 1$ *has no roots on* $[0, 1]$, *then*

$$\dim_H\big(C[GLS, V]\big) = \lim_{k\to\infty} \dim_H\big([GLS, V_k]\big),$$

where $V_k = \{i_1, i_2, \ldots, i_k\},\ k \in N,\ k \ge 2.$

Proof. The sets $C[GLS, V_k]$ are self-similar and satisfy the OSC. Thus the dimension α_k can be obtained as a solution of the equation $\sum_{i\in V_k} q_i^x = 1$. It is easy to see that

$$\alpha_2 < \alpha_3 < \cdots < \alpha_{k-1} < \alpha_k < 1.$$

Therefore, there exists the limit

$$\lim_{k\to\infty} \alpha_k = \alpha^*.$$

It is clear that $\alpha^* \le \dim_H(C[GLS, V])$ because $C[GLS, V_k] \subset C[GLS, V]$ for all $k \ge 2$. Suppose that $\alpha^* < \dim_H(C[GLS, V])$. Then there exists a number α' such that $\alpha^* < \alpha' < \dim_H(C[GLS, V])$. It is clear that

$$\sum_{i\in V_2} q_i^{\alpha'} < \sum_{i\in V_3} q_i^{\alpha'} < \cdots < \sum_{i\in V_k} q_i^{\alpha'} < 1.$$

So, $\sum_{i\in V} q_i^{\alpha'} \le 1$.

On the other hand, $H^{\alpha'}(C[GLS, V]) = \infty$ by the definition of the Hausdorff–Besicovitch dimension (because $\alpha' < \dim_H(C[GLS, V])$). Then the α'-dimensional Hausdorff measure of the set $C[GLS, V]$ with respect to the family Φ of coverings that are generated by the GLS-expansion of the unit segment is equal to $H^{\alpha'}(C[GLS,$

$\Phi]) = \infty$ (where the family Φ is a locally fine system of the coverings of the unit segment, i.e., for any $\varepsilon > 0$, there exists such a covering of $[0, 1]$ by the subsets $E_j \in \Phi$ such that $|E_j| < \varepsilon$ and $[0, 1] = \bigcup_j E_j$). Since the set $C[GLS, V]$ can be covered by cylindrical segments of the GLS-expansion with indices from V, we deduce that for any $M > 0$, there exists $k(M)$ such that for all $k > k(M)$, we have the inequality

$$\sum_{i_q \in V, q \in \{1,\dots,k\}} |\Delta_{i_1 i_2 \dots i_k}|^{\alpha'} > M,$$

$$\begin{aligned}\sum_{i_q \in V, q \in \{1,\dots,k\}} |\Delta_{i_1 i_2 \dots i_k}|^{\alpha'} &= \sum_{i_q \in V, q \in \{1,\dots,k-1\}} |\Delta_{i_1 i_2 \dots i_{k-1}}|^{\alpha'} \cdot \sum_{i_k \in V} \Delta_{i_k}^{\alpha'} \\ &< \sum_{i_q \in V, q \in \{1,\dots,k-1\}} |\Delta_{i_1 i_2 \dots i_{k-1}}|^{\alpha'} \\ &= \sum_{i_q \in V, q \in \{1,\dots,k-2\}} |\Delta_{i_1 i_2 \dots i_{k-2}}|^{\alpha'} \cdot \sum_{i_{k-1} \in V} q_{i_{k-1}}^{\alpha'} \\ &< \sum_{i_q \in V, q \in \{1,\dots,k-2\}} |\Delta_{i_1 i_2 \dots i_{k-2}}|^{\alpha'} < \cdots \\ &< \sum_{i_1 \in V} |\Delta_{i_1}|^{\alpha'} = \sum_{i_1 \in V} q_{i_1}^{\alpha'} < 1.\end{aligned}$$

From the obtained contradiction it follows that

$$\lim_{k \to \infty} \alpha_k = \dim_H \bigl(C[GLS, V]\bigr),$$

where $\alpha_k = \dim_H (C[GLS, V_k])$. □

Remark 1. Theorems 1 and 2 can be considered as natural generalizations of results from [8].

Theorem 3. *The Hausdorff–Besicovitch dimension can be calculated as follows*:

$$\dim_H \bigl(C[GLS, V]\bigr) = \sup \left\{ x : \sum_{i \in V} q_i^x \geq 1 \right\}$$

for any Q_∞ *and* $V = \{i_1, i_2, \dots, i_k, i_{k+1}, \dots\}$.

Proof. Let $\alpha_0 = \dim_H (C[GLS, V])$. Show that $\sup\{x : \sum_{i \in V} q_i^x \geq 1\} \geq \alpha_0$. Let us consider the function

$$\varphi(x) = \sum_{i \in V} q_i^x$$

and denote the set

$$A_+ = \bigl\{x : \varphi(x) \geq 1\bigr\}.$$

Let α_k and α_{k+1} be the solutions of the equations

$$\sum_{i \in V_k} q_i^x = 1$$

and

$$\sum_{i \in V_{k+1}} q_i^x = 1,$$

respectively.

Let us show that $\alpha_k < \alpha_{k+1} < \alpha_0$. If α_0 is the solution of $\sum_{i \in V} q_i^x = 1$, then it is easy to see that $\alpha_k < \alpha_{k+1} < \alpha_0$. If $\sum_{i \in V} q_i^x = 1$ has no roots on [0, 1], then

$$\alpha_0 = \lim_{k \to \infty} \alpha_k$$

and $\alpha_k < \alpha_{k+1}$, so that $\alpha_k < \alpha_{k+1} < \alpha_0$.

Express the function $\varphi(x)$ as follows:

$$\varphi(x) = \underbrace{q_{i_1}^x + \cdots + q_{i_k}^x + q_{i_{k+1}}^x}_{\geq 1} + \sum_{j=k+1}^{\infty} q_{i_j}^x.$$

It is easy to see that for all $x \in [\alpha_k, \alpha_{k+1}]$ $(x < \alpha_0,\ k \in N)$, $\varphi(x) \geq 1$. Then $A_+ \supset (-\infty; \alpha_0)$ and $\sup A_+ \geq \alpha_0$.

Let us show that $\sup A_+ \leq \alpha_0$. Suppose the opposite. If $\sup A_+ > \alpha_0$, then there exists x_1 such that $x_1 \in (\alpha_0; \sup A_+]$, $x_1 \in A_+$, and $\varphi(x_1) \geq 1$. So

$$\sum_{i \in V} q_i^{x_1} \geq 1.$$

Since α_k is a solution of $\sum_{i \in V_k} q_i^x = 1$ and $\alpha_k < \alpha_0$, we get

$$\sum_{i \in V_k} q_i^{\alpha_k} = 1$$

and

$$\sum_{i \in V_k} q_i^{\alpha_0} \leq 1.$$

It is clear that

$$\sum_{i \in V} q_i^{\alpha_0} < 1$$

and

$$\sum_{i \in V} q_i^{x_1} \leq 1.$$

So, from the obtained contradiction it follows that $\sup A_+ = \alpha_0$. $\square$

Let Δ_{∞}^{GLS} be the set of those $x \in [0; 1]$ that do not belong to any cylinder of the first rank of the GLS-expansion. The set Δ_{∞}^{GLS} can be empty, countable, or of continuum cardinality.

Let us recall that the nonempty and bounded set E is called N-self-similar if it can be represented as a union of a countably many sets E_j $(\dim_H(E_i \cap E_j) < \dim_H E, i \neq j)$ such that the set E is similar to the sets E_j with coefficient k_j.

Since the spectrum S_ξ of the distribution of a random variable ξ with independent identically distributed GLS-symbols is a self-similar or N-self-similar set and $S_\xi = (C[GLS, V])^{cl}$, we can apply the above results to calculate the Hausdorff–Besicovitch dimension of the spectrum S_ξ for the case where Δ_{∞}^{GLS} is an at most countable set.

So, we get the following theorem, which can be considered as a corollary of Theorems 1 and 2.

Theorem 4. *Let* $V := \{i : p_i > 0\}$. *If* Δ_{∞}^{GLS} *is at most countable, then the Hausdorff–Besicovitch dimension of the spectrum of the distribution of a random variable* ξ *with independent identically distributed GLS-symbols can be calculated in the following way.*

1) If the equation $\sum_{i\in V} q_i^x = 1$ *has one root* α_0 *on* $[0, 1]$, *then*

$$\dim_H S_\xi = \alpha_0.$$

2) If the equation $\sum_{i\in V} q_i^x = 1$ *has no roots on* $[0, 1]$, *then*

$$\dim_H S_\xi = \lim_{k\to\infty} \alpha_k,$$

where α_k *are the roots of the equations* $\sum_{i\in V_k} q_i^x = 1,\ V_k = \{i_1, i_2, \dots, i_k\} \subset V$.

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