Modern Stochastics: Theory and Applications 3 (2016) 237–248
DOI: 10.15559/16-VMSTA62

Stochastic wave equation in a plane driven by spatial stable noise

Larysa Pryhara, Georgiy Shevchenko*

Mechanics and Mathematics Faculty,
Taras Shevchenko National University of Kyiv,
Volodymyrska 64/11, 01601 Kyiv, Ukraine

pruhara7@gmail.com (L. Pryhara), zhora@univ.kiev.ua (G. Shevchenko)

Received: 14 October 2016, Revised: 20 October 2016, Accepted: 20 October 2016,
Published online: 8 November 2016

Abstract The main object of this paper is the planar wave equation

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \Delta\right) U(x,t) = f(x,t), \quad t \geq 0,\ x \in \mathbb{R}^2,$$

with random source f. The latter is, in certain sense, a symmetric α-stable spatial white noise multiplied by some regular function σ. We define a candidate solution U to the equation via Poisson's formula and prove that the corresponding expression is well defined at each point almost surely, although the exceptional set may depend on the particular point (x,t). We further show that U is Hölder continuous in time but with probability 1 is unbounded in any neighborhood of each point where σ does not vanish. Finally, we prove that U is a generalized solution to the equation.

Keywords Stochastic partial differential equation, wave equation, LePage series, stable random measure, Hölder continuity, generalized solution

2010 MSC 60H15, 35L05, 35R60, 60G52

*Corresponding author.

Introduction

Stochastic partial differential equations are widely used in modeling different phenomena involving randomness, and the area of their application is constantly increasing. This is reflected by increasing number of works devoted to such equations. Vast majority of these articles is devoted to the case where the underlying noise is Gaussian. In particular, a stochastic wave equation with Gaussian noise was studied in [1–3, 5, 6, 8], to mention only a few authors. However, many phenomena are characterized by heavy tails of the corresponding distributions; often not only variances, but also expectations of underlying random variables are infinite. In such cases, the underlying random noise is better modeled by a stable distribution.

In this paper, we study a wave equation in the plane, where the random source has a stable distribution. We prove that a candidate solution to the equation, constructed by means of Poisson's formula, is a generalized solution. We also show that it is Hölder continuous is time variable, but it is irregular in the spatial variable.

The paper is organized as follows. Section 1 contains the notation and auxiliary information on objects involved. In Section 2, we introduce the main object of the paper, a planar wave equation with stable noise, and establish main results. The existence and spatial properties of a candidate solution to the equation, constructed via Poisson's formula, are studied in Section 2.1. In Section 2.2, we prove that the candidate solution is a generalized solution to the equation. Finally, in Section 2.3, we establish the Hölder regularity of the solution in the time variable.

1 Preliminaries

1.1 Notational conventions

Throughout the article, the symbol C denotes a generic constant, the exact value of which is not important and may change from line to line. Similarly, $C(\omega)$ is be used to denote a generic a.s. finite random variable. We use the notation $|x|$ both for the absolute value of a real number and for the Euclidean norm of a vector; the particular meaning will be always clear from the context. The Euclidean ball $\{y : |y - x| \le r\}$ is denoted by $B(x, r)$. Finally, $\mathbb{R}_+ = [0, +\infty)$.

1.2 Stable random variables and measures

In this section, we give essential information on symmetric α-stable SαS random variables and measures; for details, we refer the reader to [7].

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space. For a number $\alpha \in (0, 2)$, called the stability parameter, a random variable ξ is SαS with the scale parameter σ^α, $\sigma \ge 0$, if its characteristic function is

$$\mathsf{E}\big[e^{i\lambda\xi}\big] = e^{-\sigma^\alpha|\lambda|^\alpha}.$$

We also denote $\|\xi\|_\alpha = \sigma$; note that this is a (quasi-)norm for $\alpha \ge 1$.

SαS random variables and fields are often constructed by means of an independently scattered SαS random measure, which is defined as follows. Denoting by $\mathcal{B}_f(\mathbb{R}^d)$ the family of Borel sets of finite Lebesgue measure, a random set function $M\colon \mathcal{B}_f(\mathbb{R}^d) \times \Omega \to \mathbb{R}$ is called an independently scattered SαS random measure with Lebesgue control measure if

1) for any $A \in \mathcal{B}_f(\mathbb{R}^d)$, the random variable $M(A)$ is SαS with scale parameter equal to $\lambda_d(A)$, the Lebesgue measure of A;

2) for any disjoint $A_1, \dots, A_n \in \mathcal{B}_f(\mathbb{R}^d)$, the values $M(A_1), \dots, M(A_n)$ are independent.

3) for any disjoint $A_n \in \mathcal{B}_f(\mathbb{R}^d)$, $n \geq 1$, such that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_f(\mathbb{R}^d)$,

$$M(A) = \sum_{n=1}^{\infty} M(A_n)$$

almost surely.

For a function $f(x) \in L^\alpha(\mathbb{R}^d)$, the integral

$$I(f) = \int \cdots \int_{\mathbb{R}^d} f(x) M(dx)$$

is defined as the limit in probability of integrals of simple compactly supported functions; its value is an SαS random variable with

$$\|I(f)\|_\alpha^\alpha = \int \cdots \int_{\mathbb{R}^d} |f(x)|^\alpha \, dx.$$

Our analysis is based on the LePage series representation of M defined as follows. Let φ be an arbitrary continuous positive probability density function on $\mathbb{R}^d$, and $\{\Gamma_k, k \geq 1\}$, $\{\xi_k, k \geq 1\}$, $\{g_k, k \geq 1\}$ be three independent families of random variables satisfying:

- Γ_k, $k \geq 1$, is a sequence of arrivals of a Poisson process with unit intensity;
- ξ_k, $k \geq 1$, are independent random vectors in $\mathbb{R}^d$ with density φ;
- g_k, $k \geq 1$, are independent centered Gaussian variables with $\mathsf{E}[|g_k|^\alpha] = 1$.

Then M (as a random process indexed by finite measure Borel sets) has the same finite-dimensional distributions as

$$M'(A) = C_\alpha \sum_{k \geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \mathbb{1}_A(\xi_k) g_k, \tag{1}$$

where $C_\alpha = \left(\frac{\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}{1-\alpha}\right)^{1/\alpha}$; the corresponding series converges almost surely for any Borel set $A \subset \mathbb{R}^d$ of finite Lebesgue measure. Moreover, for any $f_1, f_2, \dots, f_n \in L^\alpha(\mathbb{R}^d)$, the vector $(I(f_1), I(f_2), \dots, I(f_n))$ has the same distribution as $(I'(f_1), I'(f_2), \dots, I'(f_n))$, where

$$I'(f) = C_\alpha \sum_{k \geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(\xi_k) g_k. \tag{2}$$

Throughout the paper, we work with a planar SαS measure M, that is, we consider the case $d = 2$. We will assume, without loss of generality, that M is given by (1), so that, for any function $f \in L^\alpha(\mathbb{R}^2)$, the integral

$$I(f) = \iint_{\mathbb{R}^2} f(x)M(dx)$$

is given by an almost surely convergent series (2). Moreover, we assume that

$$(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_\Gamma \otimes \Omega_\xi \otimes \Omega_g, \mathcal{F}_\Gamma \otimes \mathcal{F}_\xi \otimes \mathcal{F}_g, \mathsf{P}_\Gamma \otimes \mathsf{P}_\xi \otimes \mathsf{P}_g)$$

and, for all $\omega = (\omega_\Gamma, \omega_\xi, \omega g)$ and $k \geq 1$, $\Gamma_k(\omega) = \Gamma_k(\omega_\Gamma)$, $\xi_k(\omega) = \xi_k(\omega_\xi)$, and $g_k(\omega) = g_k(\omega_g)$. This will not harm the generality but will considerably simplify our exposition.

2 Main results

For a positive constant $a > 0$, consider the planar wave equation

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)U(x,t) = \sigma(x,t)\dot{M}(x) \tag{3}$$

with zero initial conditions. The random source is a product of a continuous function σ and SαS white noise $\dot{M}(x)$, which is a formal derivative of a planar SαS random measure M introduced in the previous section. The precise meaning of this equality is not immediately obvious. Clearly, there can be no classical (belonging to $C^2(\mathbb{R}^2 \times \mathbb{R}_+)$ solution to this equation, so we will look at generalized solutions.

Let $\mathcal{D}(\mathbb{R}^2 \times \mathbb{R}_+)$ denote the class of all compactly supported infinitely continuously differentiable functions on $\mathbb{R}^2 \times \mathbb{R}_+$. By a generalized solution we mean a function satisfying

$$\begin{aligned}\int_0^\infty \iint_{\mathbb{R}^2} U(x,t)\left(\frac{\partial^2}{\partial t^2}\theta(x,t) - a^2\Delta\theta(x,t)\right)dx\,dt \\ = \int_0^\infty \iint_{\mathbb{R}^2} \theta(x,t)\sigma(x,t)M(dx)\,dt\end{aligned} \tag{4}$$

for all $\theta \in \mathcal{D}(\mathbb{R}^2 \times \mathbb{R}_+)$.

Our approach is to consider a candidate solution given by Poisson's formula

$$\begin{aligned}U(x,t) &= U(x_1,x_2,t) \\ &= \frac{1}{2\pi a}\int_0^t \iint_{B(x,a(t-\tau))} \frac{\sigma(y_1,y_2,\tau)M(dy_1,dy_2)\,d\tau}{\sqrt{a^2(t-\tau)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}} \\ &= \frac{1}{2\pi a}\int_0^t \iint_{B(x,a(t-\tau))} \frac{\sigma(y,\tau)M(dy)\,d\tau}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}}\end{aligned} \tag{5}$$

and later, in Section 2.2, to show that it solves Eq. (3) in a generalized sense.

The integral in (5) is understood in the following sense: we define

$$G(x, y, t) = \frac{1}{2\pi a} \int_0^{t - \frac{|x-y|}{a}} \frac{\sigma(y, \tau)}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}} \, d\tau \, \mathbb{1}_{|x-y|<at}$$

and set

$$U(x, t) = \iint_{\mathbb{R}^2} G(x, y, t) M(dy). \tag{6}$$

In what follows, we need some assumptions about the coefficient σ.

(S1) Boundedness: $|\sigma(x, t)| \le C$ for all $t \ge 0$ and $x \in \mathbb{R}^2$.

(S2) Continuity: $\sigma \in C(\mathbb{R}^2 \times \mathbb{R}_+)$.

(S3) Hölder continuity in time: there exists $\gamma \in (0, 1]$ such that, for all $t, s \ge 0$ and $x \in \mathbb{R}^2$,

$$\left|\sigma(x, t) - \sigma(x, s)\right| \le |t - s|^\gamma.$$

2.1 Existence and spatial properties of a candidate solution

First, we establish a result on the existence of the integral defining the candidate solution $U(x, t)$.

Theorem 1. *Under* (S1), *for any* $t \ge 0$ *and* $x \in \mathbb{R}^2$, *the integral in* (6) *is well defined.*

Proof. According to the definition of the integral with respect to M, the integral is well defined, provided that

$$\iint_{\mathbb{R}^2} \left|G(x, y, t)\right|^\alpha dy < \infty. \tag{7}$$

Taking into account (S1), we have the estimate

$$\begin{aligned}
\left|G(x, y, t)\right| &= \frac{1}{2\pi a} \left| \int_0^{t - \frac{|x-y|}{a}} \frac{\sigma(y, \tau)}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}} \, d\tau \right| \mathbb{1}_{|x-y|<at} \\
&\le C \int_0^{t - \frac{|x-y|}{a}} \frac{d\tau}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}} \, \mathbb{1}_{|x-y|<at} \\
&= \frac{C}{a} \ln \left(\frac{at}{|x-y|} + \sqrt{\frac{a^2t^2}{|x-y|^2} - 1} \right) \mathbb{1}_{|x-y|<at}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \left|G(x, y, t)\right|^\alpha dy \le C \iint_{B(x,at)} \left| \ln \left(\frac{at}{|x-y|} + \sqrt{\frac{a^2t^2}{|x-y|^2} - 1} \right) \right|^\alpha dy \\
&\quad = \left| \begin{matrix} y_1 = x_1 + at\, r \cos\phi \\ y_2 = x_2 + at\, r \sin\phi \end{matrix} \right| = C \int_0^{2\pi} d\phi \int_0^1 r \big(\ln \left| r^{-1} + \sqrt{r^{-2} - 1} \right| \big)^\alpha dr \\
&\quad \le C \int_0^1 r \left| \ln \left(2r^{-1}\right) \right|^\alpha dr \le C \int_0^1 r^{1-\varepsilon} \, dr < \infty,
\end{aligned}$$

where ε is a small positive number. This proves the statement. ☐

Recall that M is assumed to coincide with its LePage series, so we have that, for all $t \geq 0$ and $x \in \mathbb{R}^2$, $U(x, t)$ is given by the almost surely convergent series

$$U(x, t) = \sum_{k \geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} G(t, x, \xi_k) g_k.$$

We will further see that the exceptional event of zero probability generally depends on x and t. Moreover, if σ is continuous, then U is unbounded in any neighborhood of any point where σ does not vanish. In order to prove this, we first note that $G(x, x, t)$ is infinite for any $t \geq 0$ and $x \in \mathbb{R}^2$ such that $\sigma(x, t) \neq 0$. Indeed, let $\sigma(x, t) > 0$ for some $t \geq 0$ and $x \in \mathbb{R}^2$. Then there is $\varepsilon > 0$ such that $\sigma(x, s) > \varepsilon$ for all $s \in [t - \varepsilon, t]$. Write

$$G(x, x, t) = \int_0^{t-\varepsilon} \frac{\sigma(x, \tau)}{a(t - \tau)} \, d\tau + \int_{t-\varepsilon}^{t} \frac{\sigma(x, \tau)}{a(t - \tau)} \, d\tau.$$

The first integral is finite, whereas

$$\int_{t-\varepsilon}^{t} \frac{\sigma(x, \tau)}{a(t - \tau)} \, d\tau \geq \varepsilon \int_{t-\varepsilon}^{t} \frac{d\tau}{a(t - \tau)} = +\infty.$$

This observation leads to the following statement.

Theorem 2. *Assume* (S1) *and* (S2). *Then, for all* $t \geq 0$ *and* $x \in \mathbb{R}^2$ *such that* $\sigma(t, x) \neq 0$ *and for all* $\delta > 0$,

$$\sup_{y \in B(x,\delta)} \left| U(y, t) \right| = +\infty$$

almost surely.

Proof. Define

$$\widetilde{\Omega}_\xi = \left\{ \omega_\xi \in \Omega_\xi \mid \exists k \geq 1 : \left| \xi_k(\omega_\xi) - x \right| \leq \delta \right\}.$$

Since $\{\xi_k, k \geq 1\}$ are iid with everywhere positive density, it is clear that $\mathsf{P}_\xi(\widetilde{\Omega}_\xi) = 1$. Fix some $\omega_\xi \in \widetilde{\Omega}_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$. Then $U(x, t)$ has a centered Gaussian distribution, so that by the 0–1 law for Gaussian measures

$$\mathsf{P}_g\Big(\sup_{y \in B(x,\delta)} \left| U(y, t) \right| < +\infty \Big) \in \{0, 1\}.$$

Suppose by contradiction that

$$\mathsf{P}_g\Big(\sup_{y \in B(x,\delta)} \left| U(y, t) \right| < +\infty \Big) = 1.$$

Then by Fernique's theorem

$$\mathsf{E}_g\Big[\sup_{y \in B(x,\delta)} \left| U(y, t) \right|^2 \Big] < \infty. \tag{8}$$

On the other hand,

$$\begin{aligned}\mathsf{E}_g\Big[\sup_{y\in B(x,\delta)}\big|U(y,t)\big|^2\Big] &\geq \sup_{y\in B(x,\delta)}\mathsf{E}_g\big[\big|U(y,t)\big|^2\big]\\ &= C_\alpha^2\sup_{y\in B(x,\delta)}\sum_{k\geq 1}\Gamma_k^{-2/\alpha}\varphi(\xi_k)^{-2/\alpha}\big|G(y,\xi_k,t)\big|^2\\ &\geq C_\alpha^2\Gamma_{k(\omega_\xi)}^{-2/\alpha}\varphi(\xi_{k(\omega_\xi)})^{-2/\alpha}\sup_{y\in B(x,\delta)}G(y,\xi_{k(\omega_\xi)},t)^2,\end{aligned}$$

where $k(\omega_\xi)$ is an integer such that $|\xi_{k(\omega_\xi)}-x|<\delta$ (it exists since $\omega_\xi\in\widetilde{\Omega}_\xi$).

Since σ is continuous, there exists $\varepsilon>0$ such that, for all $y\in\mathbb{R}^2$ with $|x-y|<\varepsilon$, $\sigma(y,t)\neq 0$. Without loss of generality, we can assume that $\varepsilon\geq\delta$. Taking into account that $\xi_{k(\omega_\xi)}\in B(x,\delta)$, we get

$$\mathsf{E}_g\Big[\sup_{y\in B(x,\delta)}\big|U(y,t)\big|^2\Big]\geq C_\alpha^2\Gamma_{k(\omega_\xi)}^{-2/\alpha}\varphi(\xi_{k(\omega_\xi)})^{-2/\alpha}G(\xi_{k(\omega_\xi)},\xi_{k(\omega_\xi)},t)^2.$$

However, since $\sigma(\xi_{k(\omega_\xi)},t)\neq 0$, the observation preceding the theorem yields $G(\xi_{k(\omega_\xi)},\xi_{k(\omega_\xi)},t)^2=+\infty$, which contradicts (8).

Consequently, for all $\omega_\xi\in\widetilde{\Omega}_\xi$ and $\omega_\Gamma\in\Omega_\Gamma$,

$$\mathsf{P}_g\Big(\sup_{y\in B(x,\delta)}\big|U(y,t)\big|<+\infty\Big)=0,$$

whence

$$\begin{aligned}&\mathsf{P}\Big(\sup_{y\in B(x,\delta)}\big|U(y,t)\big|<+\infty\Big)\\ &\quad=\int_{\Omega_\Gamma}\int_{\Omega_\xi}\mathsf{P}_g\Big(\sup_{y\in B(x,\delta)}\big|U(y,t)\big|<+\infty\Big)\,d\mathsf{P}_\xi(\omega_\xi)\,d\mathsf{P}_\Gamma(\omega_\Gamma)=0,\end{aligned}$$

as claimed. □

2.2 Generalized solution

Theorem 2 shows that the function $U(x,t)$ cannot be a classical solution to Eq. (3). Our next aim is to show that it solves (3) in a generalized sense.

Theorem 3. *Assume* (S1) *and* (S2). *1. If $\alpha\in(0,1)$, then there is $\Omega_0\in\mathcal{F}$, $\mathsf{P}(\Omega_0)=1$, such that, for all $\omega\in\Omega_0$ and $\theta\in\mathcal{D}(\mathbb{R}^2\times\mathbb{R}_+)$, Eq.* (4) *holds. 2. If $\alpha\in[1,2)$, then, for any $\theta\in\mathcal{D}(\mathbb{R}^2\times\mathbb{R}_+)$, Eq.* (4) *holds almost surely.*

Remark 4. In the second part of this theorem, the exceptional event of probability zero may depend on θ.

Proof. Write the LePage representation for the left-hand side of Eq. (4):

$$L(\theta)=C_\alpha\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha}K(x,\xi_k,t)g_k,$$

$$K(x, y, t) = \varphi(\xi_k)^{-1/\alpha} \int_0^\infty \iint_{\mathbb{R}^2} G(x, y, t)\Big(\frac{\partial^2}{\partial t^2}\theta(x, t) - a^2\Delta\theta(x, t)\Big)\, dx\, dt$$

and its right-hand side

$$R(\theta) = C_\alpha \int_0^\infty \sum_{k=1}^\infty \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha}\theta(\xi_k, t)\sigma(\xi_k, t) g_k\, dt.$$

The proof consists of two steps: showing the convergence of the LePage series and then proving that (4) holds for partial sums of the LePage series.

Let us estimate the terms in the series for $L(\theta)$. Assume that $\operatorname{supp}\theta \subset [0, R] \times B(0, R)$. Then, denoting $\psi(x, t) = \frac{\partial^2}{\partial t^2}\theta(x, t) - a^2\Delta\theta(x, t)$, we have

$$\begin{aligned} \big|K(x, \xi_k, t)\big| &\le \int_0^R \iint_{B(0,R)} \big|\varphi(\xi_k)\big|^{-1/\alpha}\big|G(x, \xi_k, t)\big|\big|\psi(x, t)\big|\, dx\, dt \\ &\le \Big(\inf_{B(0,R)} \big|\varphi(x)\big|\Big)^{-1/\alpha} \\ &\quad \times \sup_{x\in\mathbb{R}^2, t\ge 0} \big|\psi(x, t)\big| \int_0^R \iint_{B(0,R)} \big|G(x, \xi_k, t)\big|\, dx\, dt \\ &\le C_{R,\theta}. \end{aligned} \tag{9}$$

Consider first the case $\alpha \in (0, 1)$. By the strong law of large numbers and well-known properties of Gaussian random variables there exists $\Omega_0 \in \mathcal{F}$, $\mathsf{P}(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$ and $k \ge 1$,

$$\Gamma_k \ge C_1(\omega)k, \qquad |g_k| \le C_2(\omega)\left(\log k + 1\right),$$

where C_1, C_2 are some positive random variables. Therefore, the kth term in the series for $L(\theta)$ is bounded by

$$C_\alpha \Gamma_k^{-1/\alpha}\big|K(t, x, \xi_k)g_k\big| \le C(\omega)k^{-1/\alpha}\left|\log k + 1\right|.$$

Consequently, the series for $L(\theta)$ is convergent for all $\omega \in \Omega_0$ and $\theta(x, t) \in \mathcal{D}(\mathbb{R}^2 \times \mathbb{R}_+)$. Similarly, we can show the convergence of $R(\theta)$.

For $\alpha \in [1, 2)$, the argument is changed slightly. Specifically, we show the almost sure convergence of the series for $L(\theta)$ and $R(\theta)$ for all $\theta(x, t) \in \mathcal{D}(\mathbb{R}^2 \times \mathbb{R}_+)$. Indeed, for fixed $\omega_\xi \in \Omega_\xi$, $\omega_\Gamma \in \Omega_\Gamma$, in view of (9), we have

$$\mathsf{E}_g\big[L(\theta)^2\big] = C_\alpha^2 \sum_{k\ge 1} \Gamma_k^{-2/\alpha} K(x, \xi_k, t)^2 \le C(\omega) \sum_{k\ge 1} k^{-2/\alpha}$$

almost surely. Therefore, by the Kolmogorov theorem the series for $L(\theta)$ converges P_g-almost surely for almost all $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$ and, therefore, P-almost surely. The almost sure convergence of $R(\theta)$ is shown in a similar way.

Now we prove that Eq. (4) holds for partial sums of the LePage series; the argument does not depend on the value of α. The counterpart of Eq. (4) for the partial sums reads as

$$C_\alpha \sum_{k=1}^{N} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha}$$
$$\times \int_0^\infty \iint_{\mathbb{R}^2} \int_0^t \frac{\sigma(\xi_k, \tau)}{\sqrt{a^2(t-\tau)^2 - |y-\xi_k|^2}} \mathbb{1}_{|y-\xi_k|<a(t-\tau)} \psi(y,t)\, d\tau\, dy\, dt$$
$$= C_\alpha \sum_{k=1}^{N} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \int_0^\infty \theta(\xi_k, \tau)\sigma(\xi_k, \tau)\, d\tau.$$

It suffices to show the equality of the corresponding terms, that is, to prove that, for any $x \in \mathbb{R}^2$,

$$\int_0^\infty \int_\tau^\infty \iint_{B(x,a(t-\tau))} \frac{\sigma(y,\tau)\psi(y,t)}{\sqrt{a^2(t-\tau)^2 - |x-y|^2}}\, dy\, dt\, d\tau = \int_0^\infty \sigma(x,\tau)\theta(x,\tau)\, d\tau.$$

This equality, in turn, would follow if we show that

$$\int_\tau^\infty \iint_{B(x,a(t-\tau))} \frac{\psi(y,t)}{\sqrt{a^2(t-\tau)^2 - |x-y|^2}}\, dy\, dt = \theta(x,\tau) \tag{10}$$

for all $\tau \geq 0$, $y \in \mathbb{R}^2$.

As before, assume that $\operatorname{supp}\theta \subset [0,R] \times B(0,R)$. Define

$$\widetilde{\theta}(x,u) = \theta(x, R-u), \quad u \leq R.$$

Then

$$\frac{\partial^2}{\partial t^2}\theta(x, R-u) = \frac{\partial^2}{\partial t^2}\widetilde{\theta}(x,u); \qquad \widetilde{\theta}(x,0) = 0; \qquad \frac{\partial}{\partial t}\widetilde{\theta}(x,0) = 0;$$
$$\Delta\theta(x, R-u) = \Delta\widetilde{\theta}(x,u).$$

Consider the following Cauchy problem:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - a^2\Delta\right) V(x,t) = \dfrac{\partial^2}{\partial t^2}\widetilde{\theta}(x,t) - a^2\Delta\widetilde{\theta}(x,t), \\ V(x,0) = 0, \\ \dfrac{\partial V(x,t)}{\partial t}\Big|_{t=0} = 0, \end{cases}$$

Clearly, the function $\widetilde{\theta}$ is a solution. On the other hand, by Poisson's formula, for all $x \in \mathbb{R}^2$ and $r \geq 0$,

$$\widetilde{\theta}(x,r) = \frac{1}{2\pi a} \int_0^r \iint_{B(x,a(r-s))} \frac{\frac{\partial^2}{\partial t^2}\widetilde{\theta}(y,s) - a^2\Delta\widetilde{\theta}(y,s)}{\sqrt{a^2(r-s)^2 - |x-y|^2}}\, dy\, ds.$$

Changing the variables $r \to R-\tau$, $s \to R-t$ and noticing that $\psi(t,x)$ vanishes for $\tau \geq R$, we get (10) for all $t \in [0,R]$. For $\tau \geq R$, the both sides of the equality are zero, whence the proof follows. □

2.3 Regularity of solution in time variable

In this section, adapting the argument of [4], we show that the solution U constructed by means of Poisson's formula (6) is Hölder continuous in the time variable. Since we have already shown that U is highly irregular in the spatial variable, our findings for the planar wave equation are in a sharp contrast with the scalar case, where the time and space regularity are the same.

Theorem 5. *Assume* (S1)–(S3). *Then for any* $T > 0$ *and* $x \in \mathbb{R}^2$ *the function* $U(x, \cdot)$ *is Hölder continuous on* $[0, T]$ *almost surely with any exponent less than* $\gamma \wedge \frac{1}{2}$. *This implies the required statement.*

Proof. Take some $h > 0$ and $t \in [0, T - h]$. For fixed $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$, we have the estimate

$$\begin{aligned}
&\mathsf{E}_g\Big[\big(U_1(x,t) - U_1(x,t+h)\big)^2\Big] \\
&\quad = C_\alpha^2 \sum_{k\ge 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \big|G(x,\xi_k,t_2) - G(x,\xi_k,t_2)\big|^2 \le g(h),
\end{aligned}$$

where

$$g(h) = C_\alpha^2 \sum_{k\ge 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \sup_{\substack{t_1,t_2\in[0,T]\\ 0<t_1-t_2<h}} \big|G(x,\xi_k,t_1) - G(x,\xi_k,t_2)\big|^2.$$

Let $t_1, t_2 \in [0, T]$ be such that $0 < t_2 - t_1 < h$. Write

$$\begin{aligned}
&\big|G(x,y,t_1) - G(x,y,t_2)\big|^2 \\
&\quad = \bigg| \int_0^{t_1 - \frac{|x-y|}{a}} \frac{\sigma(y,\tau)}{\sqrt{a^2(t_1-\tau)^2 - |x-y|^2}}\, d\tau\, \mathbb{1}_{|x-y|<at_1} \\
&\qquad - \int_0^{t_2 - \frac{|x-y|}{a}} \frac{\sigma(y,\tau)}{\sqrt{a^2(t_2-\tau)^2 - |x-y|^2}}\, d\tau\, \mathbb{1}_{|x-y|<at_2} \bigg|^2.
\end{aligned}$$

Consider first the case $|x - y| < at_1$. Changes of variable $\tau = t_1 - s$ and $\tau = t_2 - s$ in the first and second integrals, respectively, give

$$\begin{aligned}
&\bigg| \int_{\frac{|x-y|}{a}}^{t_1} \frac{\sigma(y,t_1-s)\,ds}{\sqrt{a^2s^2 - |x-y|^2}} - \int_{\frac{|x-y|}{a}}^{t_2} \frac{\sigma(y,t_2-s)\,ds}{\sqrt{a^2s^2 - |x-y|^2}} \bigg| \\
&\quad \le \bigg| \int_{\frac{|x-y|}{a}}^{t_1} \frac{\sigma(y,t_1-s) - \sigma(y,t_2-s)}{\sqrt{a^2s^2 - |x-y|^2}}\, ds \bigg| + \bigg| \int_{t_1}^{t_2} \frac{\sigma(y,t_1-s)\,ds}{\sqrt{a^2s^2 - |x-y|^2}} \bigg| =: I_1 + I_2.
\end{aligned}$$

Taking into account (S3), we have

$$\begin{aligned}
I_1 &\le \int_{\frac{|x-y|}{a}}^{t_1} \frac{|\sigma(y,t_1-s) - \sigma(y,t_2-s)|}{\sqrt{a^2s^2 - |x-y|^2}}\, ds \\
&\le |t_1 - t_2|^\gamma \int_{\frac{|x-y|}{a}}^{t_1} \frac{ds}{\sqrt{a^2s^2 - |x-y|^2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\,|t_1-t_2|^{\gamma}\int_{\frac{|x-y|}{a}}^{t_1}\frac{ds}{\sqrt{(as-|x-y|)(as+|x-y|)}}\\
&\le \frac{C\,|t_1-t_2|^{\gamma}\sqrt{at_1-|x-y|}}{\sqrt{|x-y|}}\le \frac{C\,|t_1-t_2|^{\gamma}}{\sqrt{|x-y|}};\\
I_2 &\le C\int_{t_1}^{t_2}\frac{ds}{\sqrt{a^2s^2-|x-y|^2}}\le C\int_{t_1}^{t_2}\frac{ds}{\sqrt{(as-|x-y|)\,(as+|x-y|)}}\\
&\le \frac{2C}{a\sqrt{|x-y|}}\big(\sqrt{at_2-|x-y|}-\sqrt{at_1-|x-y|}\big)\le \frac{C\sqrt{t_2-t_1}}{\sqrt{|x-y|}}.
\end{aligned}
$$

Now let $at_1<|x-y|<at_2$. In this case,

$$
\begin{aligned}
&\big|G(t_1,x,y)-G(t_2,x,y)\big|\\
&\quad=\left|\int_0^{t_2-\frac{|x-y|}{a}}\frac{\sigma(y,\tau)}{\sqrt{a^2(t_2-\tau)^2-|x-y|^2}}\,d\tau\,\mathbb{1}_{|x-y|<at_2}\right|\\
&\quad\le C\int_0^{t_2-\frac{|x-y|}{a}}\frac{\mathbb{1}_{|x-y|<aT}\,d\tau}{\sqrt{a^2(t_2-\tau)^2-|x-y|^2}}\\
&\quad\le C\int_0^{t_2-\frac{|x-y|}{a}}\frac{\mathbb{1}_{|x-y|<aT}\,d\tau}{\sqrt{(a(t_2-\tau)-|x-y|)(a(t_2-\tau)+|x-y|)}}\\
&\quad\le \frac{C}{\sqrt{|x-y|}}\int_0^{t_2-\frac{|x-y|}{a}}\frac{\mathbb{1}_{|x-y|<aT}\,d\tau}{\sqrt{(a(t_2-\tau)-|x-y|)}}\\
&\quad\le \frac{C\sqrt{at_2-|x-y|}}{\sqrt{|x-y|}}\mathbb{1}_{|x-y|<aT}\le \frac{C\sqrt{h}}{\sqrt{|x-y|}}\mathbb{1}_{|x-y|<aT}.
\end{aligned}
$$

Combining the estimates, we get

$$
g(h)\le Ch^{2\beta}\sum_{k\ge 1}\Gamma_k^{-2/\alpha}\varphi(\xi_k)^{-2/\alpha}\frac{1}{|x-\xi_k|}\mathbb{1}_{|x-\xi_k|<aT},
$$

where $\beta=\gamma\wedge\frac{1}{2}$. Taking the expectation w.r.t. ξ, we obtain

$$
\begin{aligned}
\mathsf{E}_{\xi}\big[g(h)\big]&\le Ch^{2\beta}\sum_{k\ge 1}\Gamma_k^{-2/\alpha}\mathsf{E}_{\xi}\left[\frac{\varphi(\xi_k)^{-2/\alpha}}{|x-\xi_k|}\mathbb{1}_{|x-\xi_k|<aT}\right]\\
&\le Ch^{2\beta}\sum_{k\ge 1}\Gamma_k^{-2/\alpha}\inf_{B(x,aT)}\varphi(y)^{-2/\alpha}\iint_{B(x,aT)}\frac{1}{|x-y|}\,dy\le C(\omega_\Gamma)h^{2\beta}.
\end{aligned}
$$

Hence, for any $\eta>0$,

$$
\mathsf{E}_{\xi}\left[\sum_{n=1}^{\infty}\frac{2^{n\beta}}{n^{1+\eta}}g\big(2^{-n}\big)\right]\le C(\omega_\Gamma)\sum_{n=1}^{\infty}n^{-1-\eta}<\infty.
$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2^{n\beta} g(2^{-n})}{n^{1+\eta}} < \infty$$

$\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$-almost surely. In particular, $\frac{2^{n\beta} g(2^{-n})}{n^{1+\eta}} \to 0$, $n \to \infty$, $\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$-almost surely. Since the function g is nondecreasing and the function $f(x) = x^{-2\beta} |\ln x|^{1+\eta}$ satisfies $f(2x) \le Cf(x)$ for x small enough, the latter convergence implies

$$\frac{g(h)}{h^{2\beta} |\ln h|^{1+\eta}} \to 0, \quad h \to 0,$$

$\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$-almost surely. Therefore, the inequality

$$\mathsf{E}_g\big[\big(U(x,t_1) - U(x,t_2)\big)^2\big] \le (t_2 - t_1)^{2\beta} \big|\ln(t_2 - t_1)\big|^{1+\eta},$$

holds $\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$-almost surely for all $t_1, t_2 \in [0, T]$ close enough and such that $t_1 < t_2$. Since U has a centered Gaussian distribution for fixed $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$, the last observation yields that

$$\big|U(x,t_1) - U(x,t_2)\big| \le C(\omega) \, |t_1 - t_2|^\beta \, \big|\ln|t_2 - t_1| + 1\big|^{1+\eta/2}$$

P-almost surely for all $t_1, t_2 \in [0, T]$ close enough. □

References

[1] Balan, R.M., Tudor, C.A.: The stochastic wave equation with fractional noise: A random field approach. Stoch. Process. Appl. **120**(12), 2468–2494 (2010) MR2728174. doi:10.1016/j.spa.2010.08.006

[2] Dalang, R.C., Frangos, N.E.: The stochastic wave equation in two spatial dimensions. Ann. Probab. **26**(1), 187–212 (1998) MR1617046. doi:10.1214/aop/1022855416

[3] Dalang, R.C., Sanz-Solé, M.: Hölder–Sobolev regularity of the solution to the stochastic wave equation in dimension three. Mem. Am. Math. Soc. **199**(931), 70 (2009) MR2512755. doi:10.1090/memo/0931

[4] Kôno, N., Maejima, M.: Hölder continuity of sample paths of some self-similar stable processes. Tokyo J. Math. **14**(1), 93–100 (1991) MR1108158. doi:10.3836/tjm/1270130491

[5] Millet, A., Morien, P.-L.: On a stochastic wave equation in two space dimensions: Regularity of the solution and its density. Stoch. Process. Appl. **86**(1), 141–162 (2000) MR1741200. doi:10.1016/S0304-4149(99)00090-3

[6] Quer-Sardanyons, L., Tindel, S.: The 1-d stochastic wave equation driven by a fractional Brownian sheet. Stoch. Process. Appl. **117**(10), 1448–1472 (2007) MR2353035. doi:10.1016/j.spa.2007.01.009

[7] Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Chapman & Hall, New York, NY (1994) MR1280932

[8] Walsh, J.B.: An introduction to stochastic partial differential equations. In: École d'Été de Probabilités de Saint-Flour, XIV–1984. Lect. Notes Math., vol. 1180, pp. 265–439. Springer, Berlin (1986) MR0876085. doi:10.1007/BFb0074920