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A limit theorem for singular stochastic differential equations

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Abstract We study the weak limits of solutions to SDEs

$$dX_n(t) = a_n\big(X_n(t)\big)\,dt + dW(t),$$

where the sequence $\{a_n\}$ converges in some sense to $(c_-\mathbb{1}_{x<0} + c_+\mathbb{1}_{x>0})/x + \gamma\delta_0$. Here δ_0 is the Dirac delta function concentrated at zero. A limit of $\{X_n\}$ may be a Bessel process, a skew Bessel process, or a mixture of Bessel processes.

Keywords Bessel process, skew Bessel process, limit theorems
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1 Introduction

Consider the stochastic differential equation

$$dX(t) = a\big(X(t)\big)\,dt + dW(t), \quad t \geqslant 0, \tag{1}$$

where a is a locally integrable function.

The aim of this paper is to study convergence in distribution of the sequence of processes $\{X(nt)/\sqrt{n},\ t \geqslant 0\}$ as $n \to \infty$.

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Observe that

$$dX_n(t) = \sqrt{n}a\big(\sqrt{n}X_n(t)\big)\,dt + dW_n(t), \quad t \geqslant 0,$$

where $W_n(t) = W(nt)/\sqrt{n}$, $t \geqslant 0$ is a Wiener process, and $X_n(t) = X(nt)/\sqrt{n}$, $t \geqslant 0$.

Hence, to study the sequence $\{X(nt)/\sqrt{n}\}$, it suffices to investigate the SDEs

$$dX_n(t) = a_n\big(X_n(t)\big)\,dt + dW(t), \quad t \geqslant 0,$$

where $a_n(x) = na(nx)$.

If $a \in L_1(\mathbb{R})$, then a_n converges in generalized sense to $\alpha\delta_0$, where δ_0 is the Dirac delta function at zero, where $\alpha = \int_{\mathbb{R}} a(x)\,dx$. It is well known that in this case the sequence $\{X_n\}$ converges weakly to a skew Brownian motion with parameter $\gamma = \operatorname{th}(\alpha) = \frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}+e^{-\alpha}}$; see, for example, [14, 10]. Recall that [5, 10] the skew Brownian motion $W_\gamma(t)$ with parameter γ, $|\gamma| \leqslant 1$, is a unique (strong) solution to the SDE

$$dW_\gamma(t) = dW(t) + \gamma\, dL^0_{W_\gamma}(t),$$

where $L^0_{W_\gamma}(t) = \lim_{\varepsilon\to 0+}(2\varepsilon)^{-1}\int_0^t \mathbb{1}_{|W_\gamma(s)|\leqslant\varepsilon}\,ds$ is the local time of the process W_γ at 0. The process W_γ is a continuous Markov process with transition probability density function $p_t(x, y) = \varphi_t(x-y) + \gamma\operatorname{sign}(y)\,\varphi_t(|x|+|y|)$, $x, y \in \mathbb{R}$, where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$ is the density of the normal distribution $N(0, t)$. Note also that W_γ can be obtained from excursions of a Wiener process pointing them (independently of each other) up and down with probabilities $p = (1+\gamma)/2$ and $q = (1-\gamma)/2$, respectively.

Kulinich et al. [8, 7] considered limit theorems in the case where a is nonintegrable function such that

$$\lim_{x\to\pm\infty}\frac{1}{x}\int_0^x \big|va(v) - c_\pm\big|\,dv = 0, \quad \big|xa(x)\big| \leqslant C, \tag{2}$$

where $c_\pm > -1/2$ are constants. In this case, $a_n(x)$ converges in some sense to $c_-\mathbb{1}_{x<0} + c_+\mathbb{1}_{x\geqslant 0}$ as $n \to \infty$.

For instance, if $a(x) = c_\pm/x$ for $\pm x > x_0$, then, for $c_- < 1/2 < c_+$, the sequence X_n converges weakly to a Bessel process. If $c_- = c_+ > -1/2$, then $|X_n|$ also converges weakly to a Bessel process. The problem of weak convergence of X_n for (e.g.) $c_- = c_+ > -1/2$ or $c_- < c_+ \leqslant 1/2$ was not considered.

In this paper, we generalize the results of [14, 8] to the case

$$a(x) = \widetilde{a}(x) + \frac{\bar{c}(x)}{x}, \quad x \in \mathbb{R},$$

where $\widetilde{a}$ is integrable on $(-\infty; \infty)$, and

$$\bar{c}(x) = c_+ \cdot \mathbb{1}_{x>1} + c_- \cdot \mathbb{1}_{x<-1}, \quad x \in \mathbb{R}.$$

We consider all possible limit processes (depending on c_+ and c_-). In particular, we show that, for $c_+ = c_- < 1/2$, the limit process is a skew Bessel process (see Section 2).

2 Bessel process. Skew Bessel process. Definition, properties

We recall the definition and some properties of Bessel processes.

Let $\delta \geqslant 0$ and $x_0 \in \mathbb{R}$. Consider the SDE

$$Z\big(x_0^2, t\big) = x_0^2 + 2\int_0^t \sqrt{\big|Z\big(x_0^2, s\big)\big|}\, dW(s) + \delta t, \quad t \geqslant 0, \tag{3}$$

where W is a Wiener process.

It is known (see [15], XI.1, (1.1)), that there exists a unique strong solution $Z(x_0^2, \cdot)$ of (3). This solution is called the squared δ-dimensional Bessel process. The process $Z(x_0^2, \cdot)$ is nonnegative a.s.

Definition 1. The process $B_c(x_0, t) = \sqrt{Z(x_0^2, t)}$ with $x_0 \geqslant 0$ is called the (nonnegative) *Bessel process* with parameter $c = (\delta - 1)/2$.

We will call the process $B_c^-(x_0, t) = -B_c(x_0, t) = -\sqrt{Z(x_0^2, t)}$ with $x_0 \leqslant 0$ the nonpositive Bessel process.

Recall the following properties of the Bessel process (see [15, Chap. XI]).

The Bessel process $\xi(t) = B_c(x_0, t)$ satisfies the SDE

$$d\xi_t = dW_t + \frac{c}{\xi_t}\, dt, \quad t < T_0,$$

where T_0 is the first hitting time of 0. If $\delta \geqslant 2$ (i.e., $c \geqslant 1/2$), then the Bessel process with probability 1 does not hit 0.

If $0 < \delta < 2$ (i.e., $-1/2 < c < 1/2$), then with probability 1 the Bessel process hits 0 but spends zero time at 0. In particular, if $\delta = 1$ (i.e., $c = 0$), then the Bessel process is a reflecting Brownian motion.

If $\delta = 0$ (i.e., $c = -1/2$), then with probability 1 the process attains 0 and remains there forever.

The scale function of the Bessel process B_c equals

$$\psi_c(x) = \begin{cases} -x^{-2c+1} & \text{if } c > 1/2, \\ \ln x & \text{if } c = 1/2, \\ x^{-2c+1} & \text{if } c < 1/2, \end{cases} \tag{4}$$

that is,

$$P_x(T_a < T_b) = \frac{\psi_c(b) - \psi_c(x)}{\psi_c(b) - \psi_c(a)} \quad \text{for any } 0 < a < x < b,$$

where $T_y = \inf\{t \geqslant 0: \ B_c(t) = y\}$.

The transition density for $c > -1/2$, $x, y > 0$, and $t > 0$ equals

$$p_t^c(x, y) = t^{-1}(y/x)^\nu y \exp\big(-\big(x^2 + y^2\big)/2t\big) I_\nu(xy/t),$$

where I_ν is a Bessel function of index $\nu = c - 1/2$.

Let $c \in (-1/2, 1/2)$, and let $p_t^{0,c}(x, y)$ be the transition density of the Bessel process B_c killed at 0.

Set

$$p_t^{skew}(x,y) = p_t^{0,c}\big(|x|,|y|\big)\cdot \mathbb{1}_{xy>0} + \frac{1+\gamma\,\mathrm{sign}\,y}{2}\big(p_t^c\big(|x|,|y|\big) - p_t^{0,c}\big(|x|,|y|\big)\big),\quad x,y\in\mathbb{R}.$$

It is easy to verify that this function satisfies the Chapman–Kolmogorov equation, is nonnegative, and $\int_{\mathbb{R}} p_t^{skew}(x,y)\,dy = 1,\ x\in\mathbb{R}$.

Definition 2. A time-homogeneous Markov process with the transition density p_t^{skew} is called *the skew Bessel process* $B_{c,\gamma}^{skew}$ with parameters c and $\gamma\in[-1,1]$.

Remark 1. We do not consider the skew Bessel process for $c\geqslant 1/2$ because $B_c(x_0,\cdot)$ does not hit 0 if $x_0\neq 0$.

Remark 2. The skew Bessel process B^{skew} can be obtained from a nonnegative Bessel process by pointing its excursions up with probability $p=\frac{1+\gamma}{2}$ and down with probability $q=\frac{1-\gamma}{2}$, similarly to the case of a skew Brownian motion; see arguments in [1], Section 2.

Thus, the scale function of the skew Bessel process equals

$$\psi_{skew}(x) = (q\mathbb{1}_{x\geqslant 0} - p\mathbb{1}_{x<0})|x|^{-2c+1},\quad x\in\mathbb{R}. \tag{5}$$

For other properties of the skew Bessel process, we refer to [2].

Remark 3. If $x_0>0$ and $p=1$ (i.e., $\gamma=1$), then $B_{c,\gamma}^{skew}(x_0,\cdot)$ is a (nonnegative) Bessel process $B_c(x_0,\cdot)$ with parameter c: $B_{c,1}^{skew}(x_0,\cdot)\stackrel{d}{=}B_c(x_0,\cdot)$.

Also, the absolute value of the skew Bessel process $|B_{c,\gamma}^{skew}|$ is a (nonnegative) Bessel process $B_c(x_0,\cdot)$ with parameter c: $|B_{c,\gamma}^{skew}(x_0,\cdot)|\stackrel{d}{=}B_c(x_0,\cdot)$.

If $c=0$, then $B_{c,\gamma}^{skew}$ is a skew Brownian motion: $B_{0,\gamma}^{skew}(\cdot)\stackrel{d}{=}W_\gamma(\cdot)$.

3 Main results

Let

$$a(x) = \widetilde{a}(x) + \frac{\bar{c}(x)}{x},\quad x\in\mathbb{R},$$

where $\widetilde{a}\in L_1(\mathbb{R})$ and

$$\bar{c}(x) = c_+\cdot\mathbb{1}_{x>1} + c_-\cdot\mathbb{1}_{x<-1},\quad x\in\mathbb{R}.$$

Let $X_n(t), t\geqslant 0$, be the solution of the SDE

$$\begin{cases} dX_n(t) = na\big(nX_n(t)\big)\,dt + dW(t) \\ \qquad = \left(n\widetilde{a}\big(nX_n(t)\big) + \dfrac{\bar{c}\big(nX_n(t)\big)}{X_n(t)}\right)dt + dW(t),\quad t\geqslant 0, \\ X_n(0) = x_0. \end{cases}$$

The existence and uniqueness of a strong solution of this SDE follows from [3, Thm. 4.53].

Theorem 1. *If c_+ and $c_- > -1/2$, then the sequence of processes $\{X_n\}$ converges weakly to a limit process X_∞. In particular:*

A1. If

(a) $x_0 > 0$ *and* $c_+ \geqslant 1/2$, *or*
(b) $x_0 \geqslant 0$ *and* $c_- < c_+ < 1/2$, *or*
(c) $x_0 = 0$ *and* $c_- < 1/2 \leqslant c_+$,

then

$$X_\infty(t) = B^+_{c_+}(x_0, t), \quad t \geqslant 0.$$

A2. Similarly, if

(a) $x_0 < 0$ *and* $c_- \geqslant 1/2$, *or*
(b) $x_0 \leqslant 0$ *and* $c_+ < c_- < 1/2$, *or*
(c) $x_0 = 0$ *and* $c_+ < 1/2 \leqslant c_-$,

then

$$X_\infty(t) = B^-_{c_-}(x_0, t), \quad t \geqslant 0.$$

A3. If $x_0 < 0$, $c_- < 1/2$, and $c_- < c_+$, then the limiting process evolves as $B^-_{c_-}$ until hitting 0 and then proceeds as $B^+_{c_+}$ indefinitely, that is,

$$X_\infty(t) = B^-_{c_-}(x_0, t) \cdot \mathbb{1}_{t \leqslant \tau} + B^+_{c_+}(0, t-\tau) \cdot \mathbb{1}_{t > \tau}, \quad t \geqslant 0,$$

where $\tau = \inf\{t \colon X_\infty(t) \geqslant 0\}$ and $B^\pm_{c_\pm}$ are independent (positive and negative) Bessel processes.

A4. Similarly, if $x_0 > 0$, $c_+ < 1/2$, and $c_+ < c_-$, then

$$X_\infty(t) = B^+_{c_+}(x_0, t) \cdot \mathbb{1}_{t \leqslant \tau} + B^-_{c_-}(0, t-\tau) \cdot \mathbb{1}_{t > \tau}, \quad t \geqslant 0,$$

where $\tau = \inf\{t \colon X_\infty(t) \leqslant 0\}$.

A5. If $c_+ = c_- =: c < 1/2$, then, for any x_0,

$$X_\infty(t) = B^{skew}_{c,\gamma}(x_0, t), \quad t \geqslant 0,$$

where $\gamma = \operatorname{th}(\int_{-\infty}^{+\infty} \widetilde{a}(z)\,dz) = \frac{1-\exp\{-2\int_{-\infty}^{+\infty} \widetilde{a}(z)\,dz\}}{1+\exp\{-2\int_{-\infty}^{+\infty} \widetilde{a}(z)\,dz\}}$.

A6. Finally, if $x_0 = 0$, $c_+ \geqslant 1/2$, and $c_- \geqslant 1/2$, then the distribution of the limit process X_∞ equals

$$p \cdot \mathbb{P}_{B^+_{c_+}} + (1-p) \cdot \mathbb{P}_{B^-_{c_-}},$$

where

$$p = \frac{\int_0^\infty A(-y)(y \vee 1)^{-2c_-}\,dy}{\int_0^\infty (A(-y)(y \vee 1)^{-2c_-} + A(y)(y \vee 1)^{-2c_+})\,dy}, \tag{6}$$

$A(y) = \exp\{-2\int_0^y \widetilde{a}(z)\,dz\}$, and $\mathbb{P}_{B^\pm_{c_\pm}}$ are the distributions of positive and negative Bessel processes $B^\pm_{c_\pm}(0, \cdot)$ starting from 0.

Remark 4. Some results of the theorem follow from [8]. However, we apply here the general approach applicable to all cases simultaneously. Condition (2) is somewhat weaker than $\tilde{a} \in L_1(\mathbb{R})$. However, we do not assume that $\sup_x |x\tilde{a}(x)| < \infty$, contrary to the paper [8].

4 Proof

It follows from [9, Section 3] or [11, Section 3.7] that if A1 is satisfied, then, for any $\alpha > 0$, we have the convergence

$$X_n\big(\cdot \wedge \tau^{n,\alpha}\big) \quad \Rightarrow \quad B^+_{c_+}\big(x_0, \cdot \wedge \tau^{0,\alpha}\big), \quad n \to \infty,$$

where $\tau^{n,\alpha} = \inf\{t \geqslant 0\colon X_n(t) \leqslant \alpha\}$ and $\tau^{0,\alpha} = \inf\{t \geqslant 0\colon B^+_{c_+}(x_0, t) \leqslant \alpha\}$. Since the process $B^+_{c_+}(x_0, \cdot)$ does not hit 0, this yields the proof. Case A2 is considered similarly.

To prove all other items of Theorem 1, we use the method proposed in [13].

Let $\{\xi^{(n)}, n \geqslant 0\}$ be a sequence of continuous homogeneous strong Markov processes. For $\alpha > 0$, set

$$\tau^{n,\alpha} := \inf\big\{t \geqslant 0\colon \big|\xi^{(n)}(t)\big| \leqslant \alpha\big\}, \qquad \sigma^{n,\alpha} := \inf\big\{t \geqslant 0\colon \big|\xi^{(n)}(t)\big| \geqslant \alpha\big\}.$$

We denote by $\xi^{(n)}_{x_0}$ a process that has the distribution of $\xi^{(n)}$ conditioned by $\xi^{(n)}(0) = x_0$.

The next statement is a particular case of Theorem 2 of [13].

Lemma 1. *Assume that the sequence* $\{\xi^{(n)}, n \geqslant 0\}$ *satisfies the following conditions*:

$$\xi^{(n)}(0) \quad \Rightarrow \quad \xi^{(0)}(0); \tag{7}$$

$$\forall T > 0\ \forall \varepsilon > 0\ \exists \delta > 0\ \exists n_0\ \forall n \geqslant n_0$$
$$\mathrm{P}\Big(\sup_{\substack{|s-t|<\delta,\\ s,t\in[0,T]}} \big|\xi^{(n)}(t) - \xi^{(n)}(s)\big| \geqslant \varepsilon\Big) \leqslant \varepsilon; \tag{8}$$

$$\forall T > 0 \qquad \lim_{\varepsilon\to 0+} \sup_n \mathrm{E}\int_0^T \mathbb{1}_{|\xi^{(n)}(t)|\leqslant\varepsilon}\,dt = 0; \tag{9}$$

$$\int_0^\infty \mathbb{1}_{\xi^{(0)}(t)=0}\,dt = 0 \quad a.s. \tag{10}$$

Assume that, for any $\alpha > 0$, $x_0 \in \mathbb{R}$, *and any sequence* $\{x_n\}$ *such that* $\lim_{n\to\infty} x_n = x_0$, *we have*

$$\big(\xi^{(n)}_{x_n}\big(\cdot \wedge \tau^{n,\alpha}\big), \tau^{n,\alpha}\big) \quad \Rightarrow \quad \big(\xi^{(0)}_{x_0}\big(\cdot \wedge \tau^{0,\alpha}\big), \tau^{0,\alpha}\big), \quad n \to \infty; \tag{11}$$

$$\xi^{(n)}_{x_n}\big(\sigma^{n,\alpha}\big) \quad \Rightarrow \quad \xi^{(0)}_{x_0}\big(\sigma^{0,\alpha}\big), \quad n \to \infty. \tag{12}$$

Then $\xi^{(n)} \Rightarrow \xi^{(0)}$ *in* $C([0,\infty))$ *as* $n \to \infty$.

We apply this lemma for $\xi^{(n)} = X_n, n \geqslant 1$, and $\xi^{(0)} = X_\infty$ in cases A1–A5 of the theorem. Case A6 will be considered separately.

Remark 5. Condition (12) is the only condition that is not true in case A6. It fails if $x_0 = 0$. Indeed, for any $x > 0$, the process $B^+_{c_+}(x, \cdot)$ does not hit 0. So, we may select a sequence $\{x_n\} \subset (0, \infty)$ that converges to 0 sufficiently slowly and such that, given $X_n(0) = x_n$, we have $X_n(\cdot) \Rightarrow B_+(0, \cdot)$ and $\lim_{n\to\infty} P(\exists t \geq 0\colon\ X_n(t) = 0) = 0$. The concrete selection of $\{x_k\}$ can be done using formulas (15) and (16). Since $B_+(0, \sigma^{0,\alpha}) = \alpha$ a.s., we get $X_n(\sigma^{n,\alpha}) \Rightarrow \alpha$. However, if all x_n were negative, then the limit might be $-\alpha$.

Conditions (7) and (10) are obvious.

The convergence

$$\forall \alpha > 0 \quad \xi^{(n)}_{x_n}\big(\cdot \wedge \tau^{n,\alpha}\big) \quad \Rightarrow \quad \xi^{(0)}_{x_0}\big(\cdot \wedge \tau^{0,\alpha}\big), \quad n \to \infty, \tag{13}$$

follows from [9, Section 3] or [11, Section 3.7]. Since

$$P\big(\forall \varepsilon > 0\, \exists t \in \big(\tau^{0,\alpha}, \tau^{0,\alpha} + \varepsilon\big)\colon\ \big|\xi^{(0)}_{x_0}(t)\big| < \alpha \mid \tau^{0,\alpha} < \infty\big) = 1,$$

convergence (13) yields the convergence of pairs (11).

Let us check condition (8). Set

$$A(y) = \exp\bigg\{-2\int_0^y \widetilde{a}(z)\, dz\bigg\},$$

$$A_n(y) = \exp\bigg\{-2\int_0^y n\widetilde{a}(nz)\, dz\bigg\} = A(ny), \quad y \in \mathbb{R},$$

$$\varPhi_n(x) = \int_0^x A_n(y)\, dy, \quad x \in \mathbb{R}.$$

Observe that $\varPhi_n : \mathbb{R} \to \mathbb{R}$ is a bijection, $\varPhi_n(0) = 0$, and

$$\exists L > 0\ \forall n\ \forall x, y \in \mathbb{R} \qquad L^{-1}|x - y| \leqslant \big|\varPhi_n(x) - \varPhi_n(y)\big| \leqslant L|x - y|.$$

Itô's formula yields

$$d\varPhi_n\big(X_n(t)\big) = A\big(nX_n(t)\big)\bigg(\frac{\bar{c}(nX_n(t))}{X_n(t)}\, dt + dW(t)\bigg).$$

So

$$\begin{aligned}\big|X_n(t) - X_n(s)\big| &\leqslant L\big|\varPhi_n\big(X_n(t)\big) - \varPhi_n\big(X_n(s)\big)\big| \\ &\leqslant L\bigg|\int_s^t A\big(nX_n(z)\big)\frac{\bar{c}(nX_n(z))}{X_n(z)}\, dz\bigg| + L\bigg|\int_s^t A\big(nX_n(z)\big)\, dW(z)\bigg|.\end{aligned}$$

Let $|s - t| < \delta$, and let $\varDelta > 0$ be fixed. Denote $f_n(t) := \int_0^t A(nX_n(z))\, dW(z)$.

a) Assume that $|X_n(z)| > \varDelta$, $z \in [s, t]$. Then

$$\bigg|\int_s^t A\big(nX_n(z)\big)\frac{\bar{c}(nX_n(z))}{X_n(z)}\, dz\bigg| \leqslant C\delta/\varDelta,$$

where $C = \|A\|_\infty \max(|c_-|, |c_+|) < \infty$. Hence, we have the estimate

$$\big|X_n(t) - X_n(s)\big| \leqslant LC\delta/\varDelta + L\omega_{f_n}(\delta),$$

where $\omega_f(\delta) = \sup_{|s-t|<\delta,\ s,t\in[0,T]} |f(t) - f(s)|$ is the modulus of continuity.

b) Assume that $|X_n(z_0)| \leqslant \Delta$ for some $z_0 \in [s,t]$.
Denote $\tau := \inf\{z \geqslant s\colon |X_n(z)| \leqslant \Delta\}$ and $\sigma := \sup\{z \leqslant t\colon |X_n(z)| \leqslant \Delta\}$. Then

$$\begin{aligned}\big|X_n(t) - X_n(s)\big| &\leqslant \big|X_n(s) - X_n(\tau)\big| + \big|X_n(\sigma) - X_n(t)\big| + 2\Delta\\ &\leqslant 2LC\delta/\Delta + 2L\omega_{f_n}(\delta) + 2\Delta.\end{aligned}$$

Thus, in any case, we have the following estimate of the modulus of continuity:

$$\omega_{X_n}(\delta) \leqslant 2LC\delta/\Delta + 2L\omega_{f_n}(\delta) + 2\Delta.$$

Let $\Delta \leqslant \varepsilon/6$. Then, for $\delta \leqslant \frac{\varepsilon\Delta}{6LC}$, we have

$$\begin{aligned}\sup_n \mathrm{P}\big(\omega_{X_n}(\delta) \geqslant \varepsilon\big) &\leqslant \sup_n \mathrm{P}\big(\varepsilon/3 + 2L\omega_{f_n}(\delta) + \varepsilon/3 \geqslant \varepsilon\big)\\ &= \sup_n \mathrm{P}\big(\omega_{f_n}(\delta) \geqslant \varepsilon/6L\big) \to 0, \quad \delta \to 0+.\end{aligned}$$

The last convergence follows from the fact that the sequence of distributions of $\{f_n(\cdot) = \int_0^{\cdot} A(nX_n(z))\,dW(z)\}_{n\geqslant 1}$ in the space of continuous functions is weakly relatively compact because the function A is bounded.

Let us prove (12) in cases A1–A5.

Remark 6. The proof below yields that condition (12) is true if $x_n = 0$ for all $n \geqslant 0$.

Let $|x| < \alpha$. It is easy to see that $P_x(\sigma^{n,\alpha} < \infty) = 1,\ n \in \mathbb{N} \cup \{\infty\}$. Since the process X_n is continuous, we have $|X_n(\sigma^{n,\alpha})| = \alpha$ a.s.

By $p_x^n = P_x(X_n(\sigma^{n,\alpha}) = \alpha),\ n \in \mathbb{N} \cup \{\infty\}$, we denote the probability to reach α before reaching $-\alpha$ when starting from x.

Using formulas (4) and (5) for the scale of a Bessel process and a skew Bessel process, it is easy to check that

$$p_x^\infty = \begin{cases} \mathbb{1}_{x\geqslant 0} - \big(1 - \frac{\psi_{c_-}(-x)}{\psi_{c_-}(\alpha)}\big)\mathbb{1}_{x<0} & \text{in cases A1, A3,}\\ \frac{\psi_{c_+}(x)}{\psi_{c_+}(\alpha)} \cdot \mathbb{1}_{x>0} & \text{in cases A2, A4,}\\ \frac{\psi_c(|x|)}{\psi_c(\alpha)}(q\mathbb{1}_{x\geqslant 0} - p\mathbb{1}_{x<0}) + p & \text{in case A5,}\end{cases} \tag{14}$$

where ψ_c is given in (4).

For $n \in \mathbb{N}$, we have (see [4, Section 15] and [15])

$$p_x^n = \frac{\varphi_n(x) - \varphi_n(-\alpha)}{\varphi_n(\alpha) - \varphi_n(-\alpha)}, \tag{15}$$

where

$$\begin{aligned}\varphi_n(x) &= \int_0^x \exp\bigg\{-2\int_0^y a_n(z)\,dz\bigg\}\,dy = \int_0^x \exp\bigg\{-2\int_0^y na(nz)\,dz\bigg\}\,dy\\ &= \frac{1}{n}\int_0^{nx} \exp\bigg\{-2\int_0^y a(z)\,dz\bigg\}\,dy = \frac{1}{n}\varphi(nx),\\ \varphi(x) &:= \int_0^x \exp\bigg\{-2\int_0^y a(z)\,dz\bigg\}\,dy.\end{aligned} \tag{16}$$

The function φ is increasing. It follows from the definition of a that φ is bounded from above (below) iff $c_+ > 1/2$ ($c_- > 1/2$). The function φ has the following asymptotic behavior:

$$\varphi(x) \sim \begin{cases} \pm A(\pm\infty)\frac{|x|^{1-2c_\pm}}{1-2c_\pm} & \text{if } c_\pm < 1/2, \\ \pm A(\pm\infty)\ln|x| & \text{if } c_\pm = 1/2, \end{cases} \quad x \to \pm\infty, \tag{17}$$

where

$$A(y) = \exp\left\{-2\int_0^y \widetilde{a}(z)\,dz\right\}, \quad y \in \mathbb{R},$$

and

$$\lim_{x\to\pm\infty} \varphi(x) = \varphi(\pm\infty) \in \mathbb{R} \quad \text{if } c_\pm > 1/2. \tag{18}$$

Condition (12) follows from (14), (15), (16), (17), (18) in cases A1–A5 (and in case A6 if $x_n = 0,\ n \geqslant 0$).

Set $\tau_n = \inf\{t \geqslant 0\colon |X_n(t)| \geqslant 1\}$.

Lemma 2. *Assume that*

$$\lim_{\varepsilon\to 0+} \sup_{|x|\leqslant 1} \sup_n \mathrm{E}_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\,dt = 0. \tag{19}$$

Then (9) *is satisfied, that is,*

$$\forall T > 0 \quad \lim_{\varepsilon\to 0+} \sup_n \mathrm{E}\int_0^T \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\,dt = 0.$$

Proof. Introduce the notations

$$\begin{aligned}
&S^0_{n,\pm} := 0, \qquad T^k_{n,\pm} := \inf\bigl\{t \geq S^{k-1}_{n,\pm}\colon\ X_n(t) = \pm 1\bigr\},\\
&S^k_{n,\pm} := \inf\bigl\{t \geq T^k_{n,\pm}\colon\ X_n(t) = \pm\varepsilon\bigr\},\\
&\tilde{T}^k_{n,\pm} := \inf\bigl\{t \geq S^k_{n,\pm}\colon\ \bigl|X_n(t)\bigr| = 1\bigr\},\\
&\beta^k_{n,\pm} := \int_{S^k_{n,\pm}}^{\tilde{T}^k_{n,\pm}} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\,dt, \qquad \alpha^k_{n,\pm} := S^k_{n,\pm} - T^k_{n,\pm}, \quad k \geqslant 1.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^T \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\,dt\\
&\quad\leqslant \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\,dt + \sum_k \bigl(\beta^1_{n,+} + \cdots + \beta^k_{n,+}\bigr)\mathbb{1}_{\alpha^1_{n,+}<T,\dots,\alpha^k_{n,+}<T,\alpha^{k+1}_{n,+}\geqslant T}\\
&\qquad + \sum_k \bigl(\beta^1_{n,-} + \cdots + \beta^k_{n,-}\bigr)\mathbb{1}_{\alpha^1_{n,-}<T,\dots,\alpha^k_{n,-}<T,\alpha^{k+1}_{n,-}\geqslant T}\,.
\end{aligned}$$

It follows from the strong Markov property that

$$\sum_k \mathrm{E}\bigl(\beta^1_{n,+} + \cdots + \beta^k_{n,+}\bigr)\mathbb{1}_{\alpha^1_{n,+}<T,\dots,\alpha^k_{n,+}<T,\alpha^{k+1}_{n,+}\geqslant T}$$

$$= \sum_k k\mathrm{E}_\varepsilon \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\, dt\, (1-p_{n,+})^k p_{n,+} = (p_{n,+})^{-1}\mathrm{E}_\varepsilon \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\, dt,$$

where $p_{n,+} = \mathrm{P}_1(S^1_{n,+} \geqslant T)$.

Considering the last term similarly, we get the inequality

$$\mathrm{E}\int_0^T \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\, dt \leqslant \big(1 + (p_{n,+})^{-1} + (p_{n,-})^{-1}\big) \sup_{|x|\leqslant 1} \sup_n \mathrm{E}_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\, dt.$$

It is not difficult to see that $\sup_n (p_{n,\pm})^{-1} < \infty$. The lemma is proved. □

Let us verify (19). It is known [6, Chap. 4.3] that

$$u_{n,\varepsilon}(x) := \mathrm{E}_x \int_0^{\tau_n} \mathbb{1}_{|X_n(t)|\leqslant\varepsilon}\, dt$$

is of the form

$$u_{n,\varepsilon}(x) = \int_{-1}^1 G_n(x,y)\mathbb{1}_{|y|\leqslant\varepsilon}\, m_n(dy), \tag{20}$$

where Green's function G_n equals

$$G_n(x,y) = \begin{cases} \frac{(\varphi_n(x)-\varphi_n(-1))(\varphi_n(1)-\varphi_n(y))}{\varphi_n(1)-\varphi_n(-1)}, & x \leqslant y, \\ G_n(y,x), & x \geqslant y, \end{cases}$$

with φ_n given by formula (16), and

$$m_n(dx) = \exp\left\{2\int_0^x a_n(z)\, dz\right\} dx.$$

The function $u_{n,\varepsilon}(x)$ is a generalized solution (because a_n may be discontinuous) of the equation

$$1/2\, u''_{n,\varepsilon}(x) + a_n(x)u'_{n,\varepsilon}(x) = -\mathbb{1}_{|x|\leqslant\varepsilon}(x), \quad |x| \leqslant 1,$$

with boundary conditions $u_{n,\varepsilon}(\pm 1) = 0$.

A direct verification of the condition $\lim_{\varepsilon\to 0+} \sup_{|x|\leqslant 1} \sup_n u_{n,\varepsilon}(x) = 0$ is possible but cumbersome. We prove the corresponding convergence using the comparison theorem. We consider only the case where a_n satisfies the Lipschitz condition. The general case follows by approximation.

It follows from the Itô–Tanaka formula that

$$\begin{aligned} d\big|X_n(t)\big| &= \operatorname{sign}\big(X_n(t)\big)\, a_n\big(X_n(t)\big)\, dt + \operatorname{sign}\big(X_n(t)\big)\, dW(t) + dl_n(t) \\ &= \operatorname{sign}\big(X_n(t)\big)\, a_n\big(X_n(t)\big)\, dt + dW_n(t) + dl_n(t), \end{aligned}$$

where W_n is a new Wiener process, and l_n is the local time of X_n at zero.

Let $-1/2 < c < \min(c_-, c_+, 0)$. It follows from the arguments of [12] on comparison of reflecting SDEs that $|X_n(t)| \geqslant Y_n(t),\ t \geqslant 0$, where Y_n satisfies the following SDE with reflection at zero:

$$dY_n(t) = \bar{a}_n\big(Y_n(t)\big)\, dt + dW_n(t) + d\tilde{l}_n(t).$$

Here $W_n(t) = \int_0^t \operatorname{sign}(X_n(s))\,dW(s)$ is a Wiener process, $\tilde{l}_n$ is the local time of Y_n at 0, $\bar{a}_n(x) = n\bar{a}(nx)$, $\bar{a}(x) = -(|a(x)| + |a(-x)|) - \frac{c}{x}\mathbb{1}_{|x|>1} + r(x)$, and r is any nonpositive function such that $\bar{a}$ satisfies Lipschitz condition. We will also assume that $\int_{\mathbb{R}} |r(x)|\,dx \leqslant \int_{\mathbb{R}} |b(x)|\,dx$. The Lipschitz property is used only for application of comparison theorem.

To prove (19), it suffices to verify that

$$\lim_{\varepsilon\to 0} \sup_{x\in[0,1]} \sup_n \bar{u}_{n,\varepsilon}(x) := \lim_{\varepsilon\to 0} \sup_{x\in[0,1]} \sup_n \mathrm{E}_x \int_0^{\bar{\tau}_n} \mathbb{1}_{Y_n(s)\in[0,\varepsilon]}\,ds = 0,$$

where $\bar{\tau}_n$ is the entry time of Y_n into $[1,\infty)$.

It is known [6] that

$$\bar{u}_{n,\varepsilon}(x) = 2\int_x^1 \exp\left\{-2\int_1^y \bar{a}_n(z)\,dz\right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z) \exp\left\{2\int_1^y \bar{a}_n(z)\,dz\right\} dy$$

is a (generalized) solution of the equation

$$1/2\,\bar{u}''_{n,\varepsilon}(x) + \bar{a}_n(x)\bar{u}'_{n,\varepsilon}(x) = -\mathbb{1}_{[0,\varepsilon]}, \quad x \in [0,1],$$

with boundary conditions $u'_{n,\varepsilon}(0) = 0$, $u_{n,\varepsilon}(1) = 0$. So

$$\begin{aligned}
&\sup_{x\in[0,1]} \sup_n \bar{u}_{n,\varepsilon}(x) \\
&\quad = \bar{u}_{n,\varepsilon}(0) \\
&\quad = 2\int_0^1 \exp\left\{-2\int_1^y \bar{a}_n(z)\,dz\right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z) \exp\left\{2\int_1^y \bar{a}_n(z)\,dz\right\} dy \\
&\quad \leqslant K\int_0^1 \exp\left\{\int_0^y \bar{y}^{-2c}\,dz\right\} \int_0^y \mathbb{1}_{[0,\varepsilon]}(z)\, y^{2c}\,dy, \qquad (21)
\end{aligned}$$

where K is a constant that depends only on $\int_{\mathbb{R}} |b(x)|\,dx$ and c (and is independent of n). By our choice, $c \in (-1/2, 0)$, so the right-hand side of (21) tends to 0 as $\varepsilon \to 0+$ by the Lebesgue dominated convergence theorem.

The theorem is proved in cases A1–A5.

Consider case A6. Note that conditions (7)–(11) are satisfied for $\xi^{(n)} = X_n$, $n \geqslant 1$, and $\xi^{(0)} = X_\infty$, where X_∞ is given in the theorem. In particular, this implies that the sequence of distributions of stochastic processes $\{X_n\}$ in the space of continuous functions is weakly relatively compact. Choosing an arbitrary convergent subsequence, without loss of generality, we may assume that $\{X_n\}$ itself converges weakly to a continuous process X. Let $\delta > 0$, and let $\sigma^{n,\delta} = \inf\{t \geqslant 0\colon X_n(t) = \delta\}$, $\sigma^\delta = \inf\{t \geqslant 0\colon X(t) = \delta\}$. It follows from formulas for the scale function of the processes $\{X_n\}$ that $\lim_{n\to\infty} P(X_n(\sigma^{n,\delta}) = \delta) = p$, $\lim_{n\to\infty} P(X_n(\sigma^{n,\delta}) = -\delta) = 1 - p$, where p is given by (6). Formulas (9) and (11) imply that the limit process exits from the interval $[-\delta, \delta]$ with probability 1.

Observe that, for almost all $\delta > 0$, with respect to the Lebesgue measure, the distribution of $X_n(\sigma^{n,\delta} + \cdot)$ converges weakly as $n \to \infty$ to the distribution of $X(\delta + \cdot)$. Indeed, by the Skorokhod theorem on a single probability space (see [16]), without loss of generality, we may assume that the sequence $\{X_n\}$ converges to X uniformly on compact sets with probability 1. For simplicity, we will assume that the convergence holds for all ω and that also $\sigma^{n,\delta}, \sigma^{\delta} < \infty$ for all $\omega, n, \delta > 0$. So we show convergence

$$X_n\big(\sigma^{n,\delta} + \cdot\big) \to X\big(\sigma^{\delta} + \cdot\big) \tag{22}$$

if we prove that

$$\sigma^{n,\delta} \to \sigma^{\delta}, \quad n \to \infty. \tag{23}$$

Convergence (23) may fail only if σ^{δ} is a point of a local maximum of X. It follows from the definition that σ^{δ} is a point of a strict local maximum of X from the left. The set of points of local maximums that are strict maximums from the left is at most countable. This yields that, for almost all ω and almost all $\delta > 0$ with respect to the Lebesgue measure, we have convergence (23) and hence (22).

On the other hand, the distribution of $X_n(\sigma^{n,\delta} + \cdot)$ converges weakly as $n \to \infty$ to the distribution of the process $\mathbb{1}_{\Omega_-} B^-_{c_-}(-\delta, \cdot) + \mathbb{1}_{\Omega_+} B^+_{c_+}(\delta, \cdot)$, where $P(\Omega_-) = 1 - p$, $P(\Omega_+) = p$, and the σ-algebra $\{\emptyset, \Omega_-, \Omega_+, \Omega\}$ is independent of $\sigma(B^{\pm}_{c_\pm}(\pm\delta, t), t \geqslant 0)$.

Recall that assumptions of the theorem yield

$$P\big(\exists t \geqslant 0\colon B^{\pm}_{c_\pm}(\pm\delta, t) = 0\big) = 0.$$

It follows from (9) that

$$P\bigg(\int_0^{\infty} \mathbb{1}_{X(s)=0}\, ds = 0\bigg) = 1.$$

Thus, we have the almost sure convergence in $C([0, \infty))$

$$X\big(\sigma^{\delta} + \cdot\big) \to X(\cdot), \quad \delta \to 0.$$

The processes $\mathbb{1}_{\Omega_-} B^-_{c_-}(-\delta, \cdot) + \mathbb{1}_{\Omega_+} B^+_{c_+}(\delta, \cdot)$ converge in distribution to

$$\mathbb{1}_{\Omega_-} B^-_{c_-}(0, \cdot) + \mathbb{1}_{\Omega_+} B^+_{c_+}(0, \cdot),$$

where the σ-algebras $\{\emptyset, \Omega_-, \Omega_+, \Omega\}$ and $\sigma(B^{\pm}_{c_\pm}(0, t), t \geqslant 0)$ are independent.

This completes the proof of Theorem 1.

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