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Averaged deviations of Orlicz processes and majorizing measures

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Abstract This paper is devoted to investigation of supremum of averaged deviations $|X(t) - f(t) - \int_{\mathbb{T}}(X(u) - f(u))\,\mathrm{d}\mu(u)/\mu(\mathbb{T})|$ of a stochastic process from Orlicz space of random variables using the method of majorizing measures. An estimate of distribution of supremum of deviations $|X(t) - f(t)|$ is derived. A special case of the L_q space is considered. As an example, the obtained results are applied to stochastic processes from the L_2 space with known covariance functions.

Keywords Orlicz space, Orlicz process, supremum distribution, method of majorizing measures, Ornstein–Uhlenbeck process

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1 Introduction

This paper is devoted to investigation of the supremum of averaged deviations of stochastic processes from Orlicz spaces of random variables using the method of majorizing measures. In particular, we estimate functionals of the following type:

$$\sup_{t\in\mathbb{T}}\left|X(t) - f(t) - \frac{1}{\mu(\mathbb{T})}\int_{\mathbb{T}}\big(X(u) - f(u)\big)\,\mathrm{d}\mu(u)\right|$$

where $(\mathbb{T}, \mathcal{B}, \mu)$ is a measurable space with finite measure $\mu(\mathbb{T}) < \infty$, and $f(u)$ is some function. In particular, using the obtained with probability one estimates for such a functional, we are able to estimate the distribution of $\sup_{t\in\mathbb{T}}|X(t) - f(t)|$. A special attention is devoted to the Orlicz spaces such as the L_q spaces.

The method of majorizing measures is used in the theory of Gaussian stochastic processes to determine conditions of boundedness and sample path continuity with probability one of these processes. Application of the method gives a possibility to obtain estimates for the distributions of stochastic processes. Papers by Fernique [3, 4] are among the first in this direction. In some cases, the method of majorizing measures turns out to be more effective than the entropy method exploited by Dudley [2], Fernique [4], Nanopoulos and Nobelis [14], and Kôno [5]. For example, Talagrand [15] proposed necessary and sufficient conditions in terms of majorizing measures for the sample path continuity with probability one of Gaussian stochastic processes. Such conditions in entropy terms were found by Fernique [4] for stationary Gaussian processes only. More details on the method of majorizing measures can be found in papers by Talagrand [15, 16], Ledoux and Talagrand [13], and Ledoux [12].

Particular cases of problems considered in this paper were investigated by Kozachenko and Moklyachuk [7], Kozachenko and Ryazantseva [8], Kozachenko, Vasylyk, and Yamnenko [10], Kozachenko and Sergiienko [9], Yamnenko [18]. Kozachenko and Ryazantseva [8] obtained conditions of boundedness and sample path continuity with probability one of stochastic processes from the Orlicz space of random variables generated by exponential Orlicz functions. Kozachenko, Vasylyk, and Yamnenko [10] estimated the probability that the supremum of a stochastic process from Orlicz spaces of exponential type exceeds some function. Kozachenko and Moklyachuk [7] obtained conditions of boundedness and estimates of the distribution of the supremum of stochastic processes from the Orlicz space of random variables. Kozachenko and Sergiienko [9] constructed tests for a hypothesis concerning the form of the covariance function of a Gaussian stochastic process. Yamnenko [18] obtained an estimate for distributions of norms of deviations of a stochastic process from the Orlicz space of exponential type from a given function in $L_p(\mathbb{T})$.

As a simple example, we apply the obtained results to a stochastic process with the same covariance function as that of the Ornstein–Uhlenbeck process but with trajectories from the L_2 space. In [17], a similar problem is considered for a generalized Ornstein–Uhlenbeck process from the Orlicz space of exponential type $\mathrm{Sub}_\varphi(\Omega)$.

2 Orlicz spaces. Basic definitions

Definition 1 (Orlicz N-function [1])**.** A continuous even convex function $\{U(x), x \in \mathbb{R}\}$ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $U(0) = 0$, and

$$\frac{U(x)}{x} \to 0 \quad \text{as } x \to 0 \quad \text{and} \quad \frac{U(x)}{x} \to \infty \quad \text{as } x \to \infty.$$

Any Orlicz N-function U has the following properties [11]:

a) $U(\alpha x) \le \alpha U(x)$ for any $0 \le \alpha \le 1$;

b) $U(x) + U(y) \le U(|x| + |y|)$;

c) the function $U(x)/x$ increases for $x > 0$.

Example 1. The following functions are N-functions:

- $U(x) = \alpha|x|^\beta,\ \alpha > 0,\ \beta > 1;$
- $U(x) = \exp\{|x|\} - |x| - 1;$
- $U(x) = \exp\{\alpha|x|^\beta\} - 1,\ \alpha > 0,\ \beta > 1;$
- $U(x) = \begin{cases} (e\alpha/2)^{2/\alpha}x^2, & |x| \le (2/\alpha)^{1/\alpha}, \\ \exp\{|x|^\alpha\}, & |x| > (2/\alpha)^{1/\alpha}, \quad 0 < \alpha < 1. \end{cases}$

Definition 2 (Class Δ_2 [11]). An N-function $U(x)$ belongs to the class Δ_2 if there exist a constant $x_0 \ge 0$ and an increasing function $K(x) > 0$, $x \ge 0$, such that

$$U(zx) \le K(z)U(x) \quad \text{for } z \ge 1,\ x \ge x_0.$$

Example 2. The following functions are from the class Δ_2:

- $U(x) = |x|^\alpha/\alpha,\ \alpha > 1;$
- $U(x) = |x|^\alpha(|\ln|x|| + 1),\ \alpha > 1;$
- $U(x) = (1 + |x|)(|\ln(1 + |x|) + 1) - |x|.$

The function $U(x) = \exp\{|x|\} - |x| - 1$ increases faster than any power function, and therefore it does not belong to the class Δ_2.

Definition 3 (Class E [1, 6]). An N-function $U(x)$ belongs to the class E if there exist constants $z_0 \ge 0$, $B > 0$, and $D > 0$ such that, for all $x \ge z_0$ and $y \ge z_0$,

$$U(x)U(y) \le BU(Dxy).$$

Example 3. (i) Let $U(x) = c|x|^p$, $c > 0$, $p > 1$. Then U belongs to the class E with constants $B = c$, $z_0 = 0$, and $D = 1$.

(ii) The function $U(x) = |x|^\beta/(\log(c + |x|))^\alpha$ belongs to the class E if c is a number large enough such that the function $U(x)$ be convex. In this case, $z_0 = \max\{0, \exp\{2^{-1/\alpha}\} - c\}$.

We will further also consider functions that belong to the intersection of the classes Δ_2 and E.

Example 4. Let $U(x) = |x|^q$, $q > 1$. Then $U \in \Delta_2 \cap E$.

Example 5. There exist functions from the class E that do not belong to the class Δ_2, for example, $U(x) = \exp\{|x|^\alpha\} - 1$, $\alpha > 1$, and $U(x) = \exp\{\phi(x)\} - 1$, where $\phi(x)$ is an N-function.

Let $(\mathbb{T}, \mathcal{B}, \mu)$ be a measurable space with finite measure $\mu(\mathbb{T})$.

Definition 4 (Orlicz space). The space $L_U^\mu(\mathbb{T})$ of measurable functions on $(\mathbb{T}, \mathcal{B}, \mu)$ such that, for every $f \in L_U^\mu(\mathbb{T})$, there exists a constant r_f such that

$$\int_{\mathbb{T}} U\left(\frac{f(t)}{r_f}\right) \mathrm{d}\mu(t) < \infty$$

is called the Orlicz space.

The space $L_U^{\mu}(\mathbb{T})$ is a Banach space with the Luxembourg norm

$$\|f\|_{U,\mu}^{\mathbb{T}} = \inf\left\{r > 0\colon \int_T U\left(\frac{f(t)}{r}\right) d\mu(t) \le 1\right\}. \tag{1}$$

We will also consider the Orlicz space $L_U^{\mu\times\mu}(\mathbb{T}\times\mathbb{T})$ of measurable functions on $(\mathbb{T}\times\mathbb{T}, \mathcal{B}\times\mathcal{B}, \mu\times\mu)$, where $\mathcal{B}\times\mathcal{B}$ is the tensor-product sigma-algebra on the product space, and $\mu\times\mu$ is the product measure on the measurable space $(\mathbb{T}\times\mathbb{T}, \mathcal{B}\times\mathcal{B})$, that is, for every $f \in L_U^{\mu\times\mu}(\mathbb{T}\times\mathbb{T})$, there exists a constant r_f such that

$$\int_{\mathbb{T}}\int_{\mathbb{T}} U\left(\frac{f(t,s)}{r_f}\right) d\big(\mu(t)\times\mu(s)\big) < \infty.$$

Definition 5 (Young–Fenchel transform)**.** Let $\{U(x), x \in \mathbb{R}\}$ be an Orlicz N-function. The function $\{U^*(x), x \in \mathbb{R}\}$ for which

$$U^*(x) = \sup_{y\in\mathbb{R}}\big(xy - U(y)\big)$$

is called the Young–Fenchel transform of the function U.

Remark 1. If $x > 0$, then

$$U^*(x) = \sup_{y>0}\big(xy - U(y)\big), \qquad U^*(-x) = U^*(x).$$

Theorem 1 (Fenchel–Moreau [1])**.** *Suppose that U is an N-function. Then*

$$(U^*)^* = U.$$

Let us give two examples of convex conjugate functions.

Example 6. (i) Suppose that $p > 1$ and q is the conjugate exponent of p: $1/p + 1/q = 1$. Let $U(x) = |x|^p/p$. Then $U^*(x) = |x^q|/q$.

(ii) Assume that $U(x) = e^{|x|} - |x| - 1$, $x \in \mathbb{R}$. Then

$$U^*(x) = \big(1+|x|\big)\big(\ln(1+|x|)+1\big) - |x|, \quad x \in \mathbb{R}.$$

Let U be an N-function, and f be a function from the space $L_U^{\mu}(\mathbb{T})$. Consider

$$s(f;U) = \int_{\mathbb{T}} U\big(f(t)\big)\, d\mu(t) < \infty.$$

In the space $L_U^{\mu}(\mathbb{T})$, we can introduce a different norm equivalent to the Luxembourg norm. This is the Orlicz norm

$$\|f\|_{(U),\mu}^{\mathbb{T}} = \sup_{v\colon s(v;U^*)\le 1}\left|\int_{\mathbb{T}} f(t)\, d\mu(t)\right|, \tag{2}$$

where U^* is the Young–Fenchel transform of the function U.

Lemma 1 (Hölder inequality [11]). *Let $\{f(t), t \in \mathbb{T}\}$ be a function from the space $L_U^{\mu}(\mathbb{T})$ endowed with the Luxembourg norm (1), and let $\{\varphi(t), t \in \mathbb{T}\}$ be a function from the space $L_{(U^*)}^{\mu}(\mathbb{T})$ endowed with the Orlicz norm (2). Then*

$$\int_{\mathbb{T}} \left|f(t)\varphi(t)\right| \mathrm{d}\mu(t) \le \|f\|_{U,\mu}^{\mathbb{T}} \times \|\varphi\|_{(U^*),\mu}^{\mathbb{T}}. \tag{3}$$

Lemma 2 (Krasnoselskii and Rutitskii [11]). *Let $U(x)$ be an N-function, let $U^*(x)$ be the Young–Fenchel transform of $U(x)$, and let $\chi_A(t)$ be the indicator function of a set $A \subset \mathcal{B}$. Then*

$$\|\chi_A\|_{(U^*),\mu}^{\mathbb{T}} = \mu(A)U^{(-1)}\left(\frac{1}{\mu(A)}\right). \tag{4}$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.

Definition 6. The space $L_U^{\mathbf{P}}(\Omega) = L_U(\Omega)$ of random variables $\xi = \{\xi(\omega), \omega \in \Omega\}$ is called an Orlicz space of random variables, that is, the Orlicz space $L_U(\Omega)$ is the family of random variables ξ for which that there exists a constant $r_\xi > 0$ such that

$$\mathbf{E}U\left(\frac{\xi}{r_\xi}\right) < \infty.$$

The Luxembourg norm in this space is denoted by $\|\xi\|_U$, that is,

$$\|\xi\|_U = \inf\left\{r > 0\colon \mathbf{E}U\left(\frac{\xi}{r}\right) \le 1\right\}.$$

Example 7. Suppose that $U(x) = |x|^p, x \in \mathbb{R}, p \ge 1$. Then $L_U(\Omega)$ is the space $L_p(\Omega)$, and the Luxembourg norm $\|\xi\|_U$ coincides with the norm $\|\xi\|_p = (\mathbf{E}|\xi|^p)^{1/p}$.

The following lemma follows from the Chebyshev inequality.

Lemma 3 (Buldygin and Kozachenko [1]). *Let ξ be a random variable from $L_U(\Omega)$. Then, for all $x > 0$,*

$$\mathbf{P}\big\{|\xi| > x\big\} \le \left(U\left(\frac{x}{\|\xi\|_U}\right)\right)^{-1}. \tag{5}$$

Definition 7. Let $\{X(t), t \in \mathbb{T}\}$ be a random process. The process X belongs to the Orlicz space $L_U(\Omega)$ if all random variables $X(t), t \in \mathbb{T}$, belong to the space $L_U(\Omega)$ and $\sup_{t\in\mathbb{T}} \|X(t)\|_U < \infty$.

Example 8. Suppose that there exists a nonnegative function $c(t), t \in \mathbb{T}$, such that $\mathbf{P}\{|X(t)| \le c(t)\} = 1, t \in \mathbb{T}$. Then X is an $L_U(\Omega)$-process for any Orlicz space $L_U(\Omega)$.

3 Distribution of deviations of stochastic processes from Orlicz spaces

Let $(\mathbb{T}, \rho)$ be a compact separable metric space equipped with the metric ρ, and let $\mathcal{B}$ be the Borel σ-algebra on $(\mathbb{T}, \rho)$.

Consider a separable stochastic process $X = \{X(t), t \in \mathbb{T}\}$ from the Orlicz space $L_U(\Omega)$, that is, $X(t) \in L_U(\Omega), t \in \mathbb{T}$, is continuous in the norm $\|\cdot\|_U$.

Assumption 1. Consider such a function $\sigma = \{\sigma(h), h > 0\}$, $t \in \mathbb{T}$, such that

- $\sigma(h) \ge 0$,
- $\sigma(h)$ increases in $h > 0$,
- $\sigma(h) \to 0$ as $h \to 0$,
- $\sigma(h)$ is continuous, and
- $\sup_{\rho(t,s)\le h} \|X(t) - X(s)\|_U \le \sigma(h)$.

Note that at least one such function exists, for example,

$$\sigma(h) = \sup_{\rho(t,s)\le h} \big\|X(t) - X(s)\big\|_U.$$

Denote by $\sigma^{(-1)}(h)$ the generalized inverse to $\sigma(h)$, that is, $\sigma^{(-1)}(h) = \sup\{s: \sigma(s) \le h\}$. Put

$$d(u, v) = \big\|X(u) - X(v)\big\|_U$$

and

$$d_f(u, v) = \big\|X(u) - X(v) - f(u) + f(v)\big\|_U$$

and let S be a set from $\mathcal{B}$ such that

$$(\mu \times \mu)\big\{(u, v) \in S \times S: \rho(u, v) \ne 0\big\} > 0. \tag{6}$$

Consider a sequence $\epsilon_k(t) > 0$ such that $\epsilon_k(t) > \epsilon_{k+1}(t)$, $\epsilon_k(t) \to 0$ as $k \to \infty$, and $\epsilon_1(t) = \sup_{s\in S} \rho(t, s)$. Put $C_t(u) = \{s: \rho(t, s) \le u\}$, $C_{t,k} = C_t(\epsilon_k(t))$, $\mu_k(t) = \mu(C_{t,k} \cap S)$.

Assumption 2. Assume that, for a continuous function $f = \{f(t), t \in \mathbb{T}\}$, there exists a continuous increasing function $\delta(y) > 0$, $y > 0$, such that $\delta(y) \to 0$ as $y \to 0$ and the following condition is satisfied:

$$\big|f(u) - f(v)\big| \le \delta\big(\big\|X(u) - X(v)\big\|_U\big) \le d(u, v).$$

Throughout the paper, we will assume that, for all $B \in \mathcal{B}$,

$$\int_B \big|X(u) - f(u)\big|\, \mathrm{d}\mu(u) < \infty.$$

Lemma 4. *Suppose that $X = \{X(t), t \in \mathbb{T}\}$ is a separable stochastic process from the Orlicz space $L_U(\Omega)$ that satisfies Assumption 1. Let f be a function satisfying Assumption 2, let $\zeta(y)$, $y > 0$, be an arbitrary continuous increasing function such that $\zeta(y) \to 0$ as $y \to 0$, and let*

$$\frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u, v))} \in L_U^{\mu\times\mu}(\mathbb{T} \times \mathbb{T}).$$

Then, for any $S \in \mathcal{B}$ satisfying (6), we have the following inequality with probability one:

$$\sup_{t\in S}\left|X(t) - f(t) - \frac{1}{\mu(S)}\int_S \big(X(u) - f(u)\big)\,\mathrm{d}\mu(u)\right| \\ \le \left\|\frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))}\right\|_{U,\mu\times\mu}^{S\times S} \sup_{t\in S}\sum_{l=1}^{\infty}\zeta\big(2\sigma(\epsilon_l(t))\big)U^{(-1)}\left(\frac{1}{\mu_{l+1}^2(t)}\right). \tag{7}$$

Proof. Let V be the set of separability of the process X, and let t be an arbitrary point from $S \cap V$. Put

$$\tau_l(u) = \frac{\chi_{C_{t,l}\cap S}(u)}{\mu_l(t)},$$

where $\chi_A(u)$ is the indicator function of A. Then

$$\left\|X(t) - f(t) - \int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u)\right\|_U \\ \le \int_S \big\|\big(X(t) - X(u)\big) - \big(f(t) - f(u)\big)\big\|_U \tau_l(u)\,\mathrm{d}\mu(u) \\ \le \int_S \big\|X(t) - X(u)\big\|_U \tau_l(u)\,\mathrm{d}\mu(u) + \int_S \big|f(t) - f(u)\big|\tau_l(u)\,\mathrm{d}\mu(u) \\ \le \sigma\big(\epsilon_l(t)\big) + \delta\big(\sigma(\epsilon_l(t))\big) \to 0 \tag{8}$$

as $l \to \infty$. If follows from Lemma 3 and (8) that

$$\int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u) \to X(t) - f(t)$$

in probability as $l \to \infty$. Therefore, there exists a sequence l_n such that

$$\int_S \big(X(u) - f(u)\big)\tau_{l_n}(u)\,\mathrm{d}\mu(u) \to X(t) - f(t)$$

with probability one as $l_n \to \infty$. It is easy to see that

$$\left|X(t) - f(t) - \int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u)\right| \\ = \Bigg|X(t) - f(t) - \int_S \big(X(u) - f(u)\big)\tau_{l_n}(u)\,\mathrm{d}\mu(u) \\ + \sum_{l=1}^{l_n-1}\left(\int_S \big(X(u) - f(u)\big)\tau_{l+1}(u)\,\mathrm{d}\mu(u) - \int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u)\right)\Bigg| \\ \le \left|X(t) - f(t) - \int_S \big(X(u) - f(u)\big)\tau_{l_n}(u)\,\mathrm{d}\mu(u)\right| \\ + \sum_{l=1}^{l_n-1}\left|\int_S \big(X(u) - f(u)\big)\tau_{l+1}(u)\,\mathrm{d}\mu(u) - \int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u)\right|. \tag{9}$$

It follows from (9) that the following inequality holds with probability one:

$$
\begin{aligned}
&\left|X(t) - f(t) - \frac{1}{\mu(S)} \int_S \big(X(u) - f(u)\big)\,\mathrm{d}\mu(u)\right| \\
&\quad \le \sum_{l=1}^{\infty} \left| \int_S \big(X(u) - f(u)\big)\tau_{l+1}(u)\,\mathrm{d}\mu(u) - \int_S \big(X(u) - f(u)\big)\tau_l(u)\,\mathrm{d}\mu(u)\right| \\
&\quad = \sum_{l=1}^{\infty} \left| \int_S \int_S \big(X(u) - X(v) - f(u) + f(v)\big)\tau_{l+1}(u)\tau_l(v)\,\mathrm{d}\mu(u)\,\mathrm{d}\mu(v)\right| \\
&\quad \le \int_{S\times S} \left| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))}\right| \\
&\qquad \times \left(\sum_{l=1}^{\infty} \tau_{l+1}(u)\tau_l(v)\zeta(d_f(u,v))\right) \mathrm{d}(\mu(u)\times\mu(v)).
\end{aligned} \tag{10}
$$

From Lemma 1 and (10) we have the inequality

$$
\begin{aligned}
&\left|X(t) - f(t) - \frac{1}{\mu(S)} \int_S \big(X(u) - f(u)\big)\,\mathrm{d}\mu(u)\right| \\
&\quad \le \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))}\right\|_{U,\mu\times\mu}^{S\times S} \left\| \sum_{l=1}^{\infty} \tau_{l+1}(u)\tau_l(v)\zeta\big(d_f(u,v)\big)\right\|_{(U^*),\mu\times\mu}^{S\times S}.
\end{aligned} \tag{11}
$$

Also, we have

$$
\begin{aligned}
\tau_{l+1}(u)\tau_l(u)\zeta\big(d_f(u,v)\big) &\le \tau_{l+1}(u)\tau_l(u)\zeta\big(d_f(u,t) + d_f(u,t)\big) \\
&\le \tau_{l+1}(u)\tau_l(u)\zeta\big(\sigma\big(\epsilon_l(t)\big) + \sigma\big(\epsilon_{l+1}(t)\big)\big) \\
&\le \tau_{l+1}(u)\tau_l(u)\zeta\big(2\sigma\big(\epsilon_l(t)\big)\big).
\end{aligned} \tag{12}
$$

From (11) and (12) we have that with probability one the following inequality holds:

$$
\begin{aligned}
&\left|X(t) - f(t) - \frac{1}{\mu(S)} \int_S \big(X(u) - f(u)\big)\,\mathrm{d}\mu(u)\right| \\
&\quad \le \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))}\right\|_{U,\mu\times\mu}^{S\times S} \\
&\qquad \times \sum_{l=1}^{\infty} \zeta\big(2\sigma\big(\epsilon_l(t)\big)\big) \left\|\tau_{l+1}(u)\tau_l(v)\right\|_{(U^*),\mu\times\mu}^{S\times S}.
\end{aligned} \tag{13}
$$

It follows from Lemma 2 that

$$
\begin{aligned}
\left\|\tau_{l+1}(u)\tau_l(v)\right\|_{(U^*),\mu\times\mu}^{S\times S} &= \frac{1}{\mu_l(t)\mu_{l+1}(t)} \left\|\chi_{C_{t,l}\cap S}(u)\chi_{C_{t,l+1}\cap S}(v)\right\|_{(U^*),\mu\times\mu}^{S\times S} \\
&= U^{(-1)}\left(\frac{1}{\mu_l(t)\mu_{l+1}(t)}\right) \le U^{(-1)}\left(\frac{1}{\mu_{l+1}^2(t)}\right).
\end{aligned} \tag{14}
$$

Since $t \in S \cap V$ and $S \cap V$ is a countable set, (14) holds with probability one for all $t \in S \cap V$. The process X is separable, and therefore

$$\sup_{t\in S}\left|X(t)-f(t)-\frac{1}{\mu(S)}\int_S \big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right| = \sup_{t\in S\cap V}\left|X(t)-f(t)-\frac{1}{\mu(S)}\int_S \big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right|$$

with probability one. □

Remark 2. If the right side of (7) is finite, then the measure μ is called a majorizing measure on S for the process X.

Corollary 1. *Let the assumptions of Lemma 4 be satisfied. Put*

$$\zeta_1(t) = \zeta\big(2\sigma\big(\epsilon_1(t)\big)\big) = \zeta\Big(2\sigma\Big(\sup_{s\in S}\rho(t,s)\Big)\Big)$$

and

$$\nu_t(u) = \mu\big(C_t\big(\sigma^{(-1)}\big(\zeta^{(-1)}(u)/2\big)\big)\cap S\big).$$

Then, for any $0 < p < 1$*, we have the inequality*

$$\sup_{t\in S}\left|X(t)-f(t)-\frac{1}{\mu(S)}\int_S \big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right| \le \eta_f C_p \tag{15}$$

with probability one, where

$$\eta_f = \left\|\frac{X(u)-X(v)-f(u)+f(v)}{\zeta(d_f(u,v))}\right\|_{U,\mu\times\mu}^{S\times S} \tag{16}$$

and

$$C_p = \sup_{t\in S}\frac{1}{p(1-p)}\int_0^{p\zeta_1(t)} U^{(-1)}\Big(\big(\nu_t(u)\big)^{-2}\Big)\,\mathrm{d}u. \tag{17}$$

Proof. Let the sequence $\epsilon_k(t)$, $k \ge 1$, be defined as

$$\epsilon_k(t) = \sigma^{(-1)}\big(\zeta^{(-1)}\big(\zeta_1(t)p^{k-1}\big)\big).$$

Then

$$\zeta\big(2\sigma\big(\epsilon_l(t)\big)\big) = \zeta_1(t)p^{l-1}$$

and

$$\mu_{l+1}(t) = \mu\big(C_t\big(\epsilon_{l+1}(t)\big)\cap S\big) = \nu_t\big(\zeta_1(t)p^l\big).$$

Therefore, from (7) and the following inequality we obtain the assertion of the corollary:

$$\sum_{l=1}^{\infty}\zeta\big(2\sigma\big(\epsilon_l(t)\big)\big)U^{(-1)}\bigg(\frac{1}{\mu_{l+1}^2(t)}\bigg)=\sum_{l=1}^{\infty}\zeta_1(t)p^{l-1}U^{(-1)}\big(\big(\nu_t\big(\zeta_1(t)p^l\big)\big)^{-2}\big)$$
$$\le\sum_{l\ge 1}\frac{1}{p(1-p)}\int_{\zeta_1(t)p^{l+1}}^{\zeta_1(t)p^l}U^{(-1)}\big(\nu_t(u)^{-2}\big)\,\mathrm{d}u$$
$$\le\int_0^{\zeta_1(t)p}U^{(-1)}\big(\nu_t(u)^{-2}\big)\,\mathrm{d}u. \qquad \square$$

Remark 3. We will further find additional conditions on η_f and C_p from (16) and (17) such that the constant C_p is finite and the random variable η_f is finite with probability one. In this case, we get that μ is a majorizing measure on S for X. In Theorems 3 and 4, these conditions will be formulated for processes from the class Δ_2 and space $L_q(\Omega)$.

Theorem 2. *Let assumptions of Lemma 4 be satisfied, and let the following conditions hold:*

a)
$$\sup_{t\in S}\int_0^{\zeta_1(t)}U^{(-1)}\big(\big(\nu_t(u)\big)^{-2}\big)\,\mathrm{d}u<\infty,$$

b) there exists a constant $r>0$ such that
$$\int_S\int_S\mathbf{E}U\left(\frac{|X(u)-X(v)|+|f(u)-f(v)|}{\zeta(d_f(u,v))r}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big)<\infty. \qquad (18)$$

Then, for all $x>0$, we have the inequality
$$\mathbf{P}\Big\{\sup_{t\in S}\big|X(t)-f(t)\big|>x\Big\}$$
$$\le\inf_{0\le\alpha\le 1}\inf_{0<p<1}\left[\left(U\left(\alpha x\Big/\left\|\int_S\big(X(u)-f(u)\big)\frac{\mathrm{d}\mu(u)}{\mu(S)}\right\|_U\right)\right)^{-1}\right.$$
$$\left.+\mathbf{P}\left\{\eta_f>\frac{(1-\alpha)x}{C_p}\right\}\right], \qquad (19)$$

where η_f and C_p are defined in (16) and (17), respectively.

Proof. Using Fubini's theorem and (18), we obtain that with probability one
$$\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{\zeta(d_f(u,v))r}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big)$$
$$\le\int_S\int_S U\left(\frac{|X(u)-X(v)|+|f(u)-f(v)|}{\zeta(d_f(u,v))r}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big)<\infty,$$

that is, the process
$$\frac{X(u)-X(v)-f(u)+f(v)}{\zeta(d_f(u,v))}$$

with probability one belongs to the space $L_U^{\mu\times\mu}(S\times S)$. Therefore, with probability one
$$\eta_f=\left\|\frac{X(u)-X(v)-f(u)+f(v)}{\zeta(d_f(u,v))}\right\|_{U,\mu\times\mu}^{S\times S}$$

is a finite random variable. It follows from (15) that

$$\sup_{t\in S}\left|X(t)-f(t)\right|\le\frac{1}{\mu(S)}\left|\int_S\big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right|+\eta_f C_p \tag{20}$$

with probability one. Since $X(u)\in L_U(\Omega)$ for $u\in S$, we have $X(u)-f(u)\in L_U(\Omega)$ for $u\in S$ and

$$\frac{1}{\mu(S)}\int_S\big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\in L_U(\Omega).$$

Moreover,

$$\begin{aligned}\left\|\frac{1}{\mu(S)}\int_S\big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right\|_U&\le\frac{1}{\mu(S)}\int_S\left\|X(u)-f(u)\right\|_U\mathrm{d}\mu(u)\\&\le\sup_{u\in S}\left\|X(u)-f(u)\right\|_U<\infty.\end{aligned}$$

It follows from Lemma 3 that, for any $y>0$,

$$\mathbf{P}\left\{\left|\int_S\big(X(u)-f(u)\big)\frac{\mathrm{d}\mu(u)}{\mu(S)}\right|>y\right\}\le 1/U\left(\frac{y}{\|\frac{1}{\mu(S)}\int_S(X(u)-f(u))\,\mathrm{d}\mu(u)\|_U}\right). \tag{21}$$

It follows from (20) that, for any $0\le\alpha\le1$ and $x>0$,

$$\begin{aligned}&\mathbf{P}\left\{\sup_{t\in S}\left|X(t)-f(t)\right|>x\right\}\\&\quad\le\mathbf{P}\left\{\frac{1}{\mu(S)}\left|\int_S\big(X(u)-f(u)\big)\,\mathrm{d}\mu(u)\right|>\alpha x\right\}+\mathbf{P}\big\{\eta_f C_p>(1-\alpha)x\big\}.\end{aligned} \tag{22}$$

The statement of the theorem follows from (21) and (22). ☐

4 Distribution of deviations of stochastic processes from classes $\boldsymbol{\Delta_2}$ and $\boldsymbol{\Delta_2\cap E}$

Definition 8. A stochastic process $X=\{X(t),t\in\mathbb{T}\}$ belongs to the class Δ_2 if $X\in L_U(\Omega)$, where U is an Orlicz function from the class Δ_2.

Theorem 3. *Suppose that $X=\{X(t),t\in\mathbb{T}\}$ is a separable stochastic process from the class Δ_2 that satisfies Assumption 1. Let f be a function satisfying Assumption 2, where U is the Orlicz N-function from the class Δ_2, let $\zeta(y)$, $y>0$, be an arbitrary continuous increasing function such that $\zeta(y)\to0$ as $y\to0$, and let*

$$\frac{X(u)-X(v)-f(u)+f(v)}{\zeta(d_f(u,v))}\in L_U^{\mu\times\mu}(\mathbb{T}\times\mathbb{T}).$$

Suppose that the following conditions are satisfied:

a) there exists a constant $r > 0$ such that

$$\int_S \int_S K\left(\frac{\gamma(d_f(u,v))}{r}\right) \mathrm{d}\big(\mu(u) \times \mu(v)\big) < \infty, \tag{23}$$

where K and x_0 are introduced in Definition 2 of the class Δ_2 and $\gamma(u) = u/\zeta(u)$;

b)
$$\sup_{t \in S} \int_0^{\zeta_1(t)} U^{(-1)}\Big(\big(\nu_t(u)\big)^{-2}\Big)\,\mathrm{d}u < \infty, \tag{24}$$

where $\zeta_1(t)$ and $\nu_t(u)$ are defined in Corollary 1.

Then, for any $0 < p < 1$, the following inequality holds with probability one:

$$\sup_{t \in S} \left| X(t) - f(t) - \int_S \big(X(u) - f(u)\big) \frac{\mathrm{d}\mu(u)}{\mu(S)} \right| \\ \le \frac{\eta_f}{p(1-p)} \sup_{t \in S} \int_0^{\zeta_1(t)p} U^{(-1)}\Big(\big(\nu_t(u)\big)^{-2}\Big)\,\mathrm{d}u, \tag{25}$$

where

$$\eta_f = \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))} \right\|_{U,\mu\times\mu}^{S\times S}$$

is a finite with probability one random variable.

Proof. It is easy to see that the assumptions of Lemma 4 are satisfied. Consider the function η_f. In order to show that it is finite with probability one, it suffices to prove that the random function

$$\frac{(X(u) - X(v) - f(u) + f(v))}{\zeta(d_f(u,v))}$$

belongs to the space $L_U^{\mu\times\mu}(S \times S)$ with probability one. For this, it suffices to show that there exists a number $r > 0$ such that

$$\int_S \int_S U\left(\frac{X(u) - X(v) - f(u) + f(v)}{r\zeta(d_f(u,v))}\right) \mathrm{d}\big(\mu(u) \times \mu(v)\big) < \infty$$

with probability one. It follows from Fubini's theorem that it suffices to prove that

$$\int_S \int_S \mathbf{E}U\left(\frac{(X(u) - X(v) - f(u) + f(v))}{r\zeta(d_f(u,v))}\right) \mathrm{d}\big(\mu(u) \times \mu(v)\big) < \infty. \tag{26}$$

Since $U \in \Delta_2$, using Assumption 2, we have

$$\mathbf{E}U\left(\frac{(X(u) - X(v) - f(u) + f(v))}{r\zeta(d_f(u,v))}\right) \\ = \mathbf{E}\chi_{\frac{|X(u)-X(v)-f(u)+f(v)|}{d_f(u,v)} > x_0} \chi_{\frac{\gamma(d_f(u,v))}{r} > 1}$$

$$\times U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\right)K\left(\frac{\gamma(d_f(u,v))}{r}\right)$$
$$+ \mathbf{E}\chi_{\frac{|X(u)-X(v)-f(u)+f(v)|}{d_f(u,v)}\le x_0}\chi_{\frac{\gamma(d_f(u,v))}{r}>1} U\left(x_0\frac{\gamma(d_f(u,v))}{r}\right)$$
$$+ \chi_{\frac{\gamma(d_f(u,v))}{r}\le 1}\mathbf{E}U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\right)$$
$$\le \mathbf{E}U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\right)K\left(\frac{\gamma(d_f(u,v))}{r}\right)$$
$$+ U(x_0)K\left(\frac{\gamma(d_f(u,v))}{r}\right)$$
$$+ \chi_{\frac{\gamma(d_f(u,v))}{r}\le 1}\mathbf{E}U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\right)$$
$$\le \left(K\left(\frac{\gamma(d_f(u,v))}{r}\right) + \chi_{\frac{\gamma(d_f(u,v))}{r}\le 1}\right)\mathbf{E}U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\right)$$
$$+ U(x_0)K\left(\frac{\gamma(d_f(u,v))}{r}\right)$$
$$\le K\left(\frac{\gamma(d_f(u,v))}{r}\right)\big(1+U(x_0)\big) + \chi_{\frac{\gamma(d_f(u,v))}{r}\le 1}. \tag{27}$$

Therefore, for all r such that inequality (23) holds, we have the relation

$$\mathbf{E}\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{r\zeta(d_f(u,v))}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big)$$
$$\le \int_S\int_S \chi_{\frac{\gamma(d_f(u,v))}{r}\le 1}\,\mathrm{d}\big(\mu(u)\times\mu(v)\big)$$
$$+ \big(1+U(x_0)\big)\int_S\int_S K\left(\frac{\gamma(d_f(u,v))}{r}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big) < \infty. \tag{28}$$

Inequality (26) and the statement of Theorem 3 follows from the last relation. □

Corollary 2. *Let the assumptions of Theorem 3 be satisfied. Let r be a number such that condition (23) holds. Then, for any $x > r$, we have the inequality*

$$\mathbf{P}\{\eta_f > x\} \le Z(x),$$

where

$$Z(x) = \int_S\int_S\left[\chi_{\frac{\gamma(d_f(u,v))}{x}\le 1} + \big(1+U(x_0)\big)K\left(\frac{\gamma(d_f(u,v))}{x}\right)\right]\mathrm{d}\big(\mu(u)\times\mu(v)\big).$$

Proof. It follows from (25) and Chebyshev's inequality that

$$\mathbf{P}\{\eta_f > x\}$$
$$= \mathbf{P}\left\{\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{x\zeta(d_f(u,v))}\right)\mathrm{d}\big(\mu(u)\times\mu(v)\big) > 1\right\}$$

$$\leq \mathbf{E}\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{x\zeta(d_f(u,v))}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)$$
$$\leq Z(x). \tag{29}$$

□

Corollary 3. *Let the assumptions of Theorem 3 be satisfied. Let $U(x)\in\Delta_2\cap E$ and $z_0=0$ in Definition 3. Then, for any $x>0$, we have the inequality*

$$\mathbf{P}\{\eta_f > x\} \leq \frac{Z(r)B}{U(x/Dr)},$$

where B and D are the constants from Definition 3, and r is a constant such that condition (23) holds, $Z(r)$ is defined in Corollary 2, and

$$Z(r)\leq \mu^2(S)+(1+U(x_0))\int_S\int_S K\left(\frac{\gamma(d_f(u,v))}{r}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)=Z_1(r).$$

Proof. It follows from (28), the definition of class E, and Chebyshev's inequality that

$$\begin{aligned}
&\mathbf{P}\{\eta_f > x\}\\
&\quad=\mathbf{P}\left\{\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{x\zeta(d_f(u,v))}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)>1\right\}\\
&\quad\leq \mathbf{E}\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\frac{\gamma(d_f(u,v))}{x}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)\\
&\quad=\frac{1}{U(\frac{x}{Dr})}\mathbf{E}\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\frac{\gamma(d_f(u,v))}{x}\right)\\
&\qquad\times U\left(\frac{x}{Dr}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)\\
&\quad\leq\frac{B}{U(x/(Dr))}\\
&\qquad\times\mathbf{E}\int_S\int_S U\left(\frac{X(u)-X(v)-f(u)+f(v)}{d_f(u,v)}\frac{\gamma(d_f(u,v))}{r}\right)\mathrm{d}\bigl(\mu(u)\times\mu(v)\bigr)\\
&\quad\leq\frac{Z(r)B}{U(x/(Dr))}.
\end{aligned}\tag{30}$$

□

Corollary 4. *Let the assumptions of Theorem 3 be satisfied. Then*

a) for all $x>r$, we have the inequality

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{t\in S}\bigl|X(t)-f(t)\bigr|>x\right\}\\
&\quad\leq\inf_{0<\alpha<1}\inf_{0<p<1}\left(1/U\left(\frac{x\alpha}{\|\int_S(X(u)-f(u))\,\mathrm{d}\mu(u)/\mu(S)\|_U}\right)+Z\left(\frac{x(1-\alpha)}{C_p}\right)\right),
\end{aligned}\tag{31}$$

where $Z(x)$ is determined in Corollary 2, C_p is determined in Theorem 2, and r is a constant such that condition (23) holds;

b) if $U \in \Delta_2 \cap E$ with $z_0 = 0$, then, for all $x > 0$, we have the inequality

$$\mathbf{P}\Big\{\sup_{t\in S}\big|X(t) - f(t)\big| > x\Big\} \le \inf_{0<\alpha<1}\inf_{0<p<1}\left(1/U\left(\frac{x\alpha}{\| \int_S (X(u) - f(u))\,\mathrm{d}\mu(u)/\mu(S)\|_U}\right) + Z(r)B/U\left(\frac{x(1-\alpha)}{DrC_p}\right)\right), \tag{32}$$

where B and D are the constants determined in Definition 3, r is a constant such that condition (23) holds true and $Z(x)$ is determined in Corollary 2.

Proof. Statement a) follows from Theorem 2 and Corollary 2. Statement b) follows from Theorem 2 and Corollary 3. □

Theorem 4. *Suppose that $X = \{X(t), t \in \mathbb{T}\}$ is a separable stochastic process from the space $L_q(\Omega)$, $q > 1$, satisfying Assumption 1. Let $f \in L_q^{\mu}(S)$ be a function satisfying Assumption 2, let $\zeta(y)$, $y > 0$, be an arbitrary continuous increasing function such that $\zeta(y) \to 0$ as $y \to 0$, and let the following conditions hold:*

$$\Delta_q = \int_S\int_S \big(\gamma\big(d_f(u,v)\big)\big)^q\ \mathrm{d}\big(\mu(u)\times\mu(v)\big) < \infty,$$
$$\sup_{t\in S}\int_0^{\zeta_1(t)}\big(\nu_t(u)\big)^{-2/q}\,\mathrm{d}u < \infty,$$

where $\gamma(y) = y/\zeta(y)$, $\zeta_1(t)$ and $\nu_t(u)$ are defined in Corollary 1. Then, for any $0 < p < 1$ and $x > 0$, we have the inequality

$$\mathbf{P}\Big\{\sup_{t\in S}\big|X(t) - f(t)\big| > x\Big\} \le x^{-q}\big(\Gamma_q^{\frac{1}{q+1}} + \big(D_{p,q}^q\Delta_q\big)^{\frac{1}{q+1}}\big)^{q+1}, \tag{33}$$

where

$$\Gamma_q = \mathbf{E}\left(\int_S\big(X(u) - f(u)\big)\frac{\mathrm{d}\mu(u)}{\mu(S)}\right)^q,$$
$$D_{p,q} = \sup_{t\in S}\frac{1}{p(1-p)}\int_0^{\zeta_1(t)p}\big(\nu_t(u)\big)^{-2/q}\,\mathrm{d}u. \tag{34}$$

Proof. Consider inequality (31). In this case,

$$\left\|\int_S\big(X(u) - f(u)\big)\frac{\mathrm{d}\mu(u)}{\mu(S)}\right\|_U = \Gamma_q^{1/q},$$

$B = D = 1$, $x_0 = 0$, $K(y) = y^q$, $r > 0$,

$$C_p = \sup_{t\in S}\frac{1}{p(1-p)}\int_0^{p\zeta_1(t)}\big(\nu_t(u)\big)^{-\frac{2}{q}}\,\mathrm{d}u,$$

and $Z(r)r^q \to \Delta_q$ as $r \to 0$, where

$$Z(r)r^q = r^q \int_S \int_S \chi_{\frac{\gamma(d_f(u,v))}{r} \le 1} \, \mathrm{d}\big(\mu(u) \times \mu(v)\big) + \int_S \int_S \big(\gamma\big(d_f(u,v)\big)\big)^q \, \mathrm{d}\big(\mu(u) \times \mu(v)\big).$$

It follows from (31) that, for any $0 < p < 1$,

$$\mathbf{P}\Big\{\sup_{t\in S} \big|X(t) - f(t)\big| > x\Big\} \le \inf_{0\le\alpha\le1} \left(\frac{\Gamma_q}{\alpha^q x^q} + \frac{C_p^q \Delta_q}{(1-\alpha)^q x^q}\right).$$

Inequality (33) follows from the last inequality after taking the infimum with respect to α. □

5 Example of existence of majorizing measure for $L_2(\Omega)$-process

In this section, we show that the Lebesgue measure is majorizing on S for some process X from the space $L_2(\Omega)$.

Let $S = \mathbb{T} = [0, T]$. Assume that $\rho(u, v) = d_f(u, v) = |u - v|$ and let μ be the Lebesgue measure, that is, $\mu(S) = T$. Then

$$C_t(u) = \big\{s : |t-s| \le u\big\} = [t-u, t+u]$$

and

$$C_t \cap S = \min\{T, t+u\} - \max\{0, t-u\}.$$

The function $\zeta(u) = u^\alpha$, $\alpha > 0$, satisfies the condition of Lemma 4; therefore, $\gamma(u) = u^{1-\alpha}$ and the expressions in Theorem 4 take the following form:

$$\nu_t(u) = \min\left\{T, t + \sigma^{(-1)}\left(\frac{1}{2}u^{1/\alpha}\right)\right\} - \max\left\{0, t - \sigma^{(-1)}\left(\frac{1}{2}u^{1/\alpha}\right)\right\}, \tag{35}$$

$$\zeta_1(t) = \zeta\Big(2\sigma\Big(\sup_{s\in S}|t-s|\Big)\Big) = \big(2\sigma\big(\max\{t, T-t\}\big)\big)^\alpha,$$

and

$$\Delta_q = \int_0^T \int_0^T \big(d_f(u,v)\big)^{(1-\alpha)q} \, \mathrm{d}u \, \mathrm{d}v.$$

Let $q = 2$, that is, $X(t)$ is a stochastic process from $L_2(\Omega)$. Assume that X is a centered process with covariance function $R_X(u, v) = \mathbf{E}X(u)X(v)$. Then using Fubini's theorem, we obtain the following representation of Γ_q from Theorem 4:

$$\begin{aligned}\Gamma_q &= \mathbf{E}\left(\int_0^T \big(X(u) - f(u)\big)\frac{\mathrm{d}u}{T}\right)^2 \\ &= \frac{1}{T^2}\int_0^T \int_0^T \mathbf{E}\big(X(u) - f(u)\big)\big(X(v) - f(v)\big) \, \mathrm{d}v \, \mathrm{d}u \\ &= \frac{1}{T^2}\int_0^T \int_0^T R_X(u,v) \, \mathrm{d}u \, \mathrm{d}v + \frac{1}{T^2}\left(\int_0^T f(v) \, \mathrm{d}v\right)^2.\end{aligned}$$

Consider the following stochastic process.

Definition 9. A stochastic process $X = \{X(t), t \in \mathbb{T}\}$ is called the generalized Ornstein–Uhlenbeck process from the space $L_2(\Omega)$ if X is an $L_2(\Omega)$-process with the covariance function

$$R_X(t,s)=e^{-\tau|t-s|},\quad \tau>0.$$

Then from Theorem 4 we can state conditions for a majorizing measure on $[0,T]$ for the process X.

Theorem 5. *Let $X=\{X(t),t\in[0,T]\}$ be a centered separable generalized Ornstein–Uhlenbeck stochastic process from the space $L_2(\Omega)$ satisfying Assumption 1, and let a function f satisfy Assumption 2 with the function $\delta(t)$, $t>0$, such that*

$$\int_0^T\int_0^T\big(\delta\big((u-v)^{\beta_1/2}\big)\big)^{2-2\alpha}\,\mathrm{d}u\,\mathrm{d}v<\infty, \tag{36}$$

where $\alpha\in(2/\beta_2,1/\beta_1+1)$ with $\beta_1,\beta_2\in(0,1)$ such that $2/\beta_2<1/\beta_1+1$. Then the Lebesgue measure is majorizing on $[0,T]$ for the process X, and, for any $0<p<1$ and $x>0$, we have the inequality

$$\mathbf{P}\Big\{\sup_{t\in[0,T]}\big|X(t)-f(t)\big|>x\Big\}\le x^{-2}\Big(\Gamma_2^{\frac{1}{3}}+\inf_{\alpha\in(0,1)}(D_{p,2}^2\Delta_2)^{\frac{1}{3}}\Big)^3, \tag{37}$$

where

$$\begin{aligned}
\Gamma_2&=\frac{2(T\tau+e^{-\tau T}-1)}{\tau^2T^2}+\frac{1}{T^2}\bigg(\int_0^T f(v)\,\mathrm{d}v\bigg)^2,\\
\Delta_2&=\int_0^T\int_0^T\big(2\big(\tau|u-v|\big)^{\beta_1}+\big(\delta\big(\big(2\tau(u-v)\big)^{\beta_1/2}\big)\big)^2\big)^{1-\alpha}\,\mathrm{d}u\,\mathrm{d}v,\\
D_{p,2}&=\frac{1}{p(1-p)}\sup_{t\in[0,T]}\bigg(\frac{2\tau'(\tau'\min\{t,T-t\})^{\alpha\beta_2/2-1}}{1-2/\alpha}\\
&\quad+\frac{p2^{3\alpha/2}\,(\tau\max\{t,T-t\})^{\alpha\beta_2/2}-(\tau'\min\{t,T-t\})^{\alpha\beta_2/2}}{T}\bigg),
\end{aligned}$$

where $\tau'=\tau2^{3/\beta_2}$.

Proof. Let us apply the inequality

$$1-\exp\{-x\}\le x^{\beta},\quad 0<\beta\le1,\ x\ge0. \tag{38}$$

It is easy to see that, for all $0\le x<1$, we have $1-\exp\{-x\}\le x\le x^{\beta}$. Also, $1-\exp\{-x\}\le1\le x^{\beta}$ for all $x\ge1$.

Then, using (38), we have that

$$\begin{aligned}
d(t,s)&=\big\|X(t)-X(s)\big\|_{L_2}=\big(\mathbf{E}\big(X(t)-X(s)\big)^2\big)^{1/2}\\
&=\big(\mathbf{E}X(t)^2+\mathbf{E}X(s)^2-2R_X(t,s)\big)^{1/2}=\big(2-2\exp\big\{-\tau|t-s|\big\}\big)^{1/2}\\
&\le2^{1/2}\big(\tau|t-s|\big)^{\beta/2},
\end{aligned}$$

that is, the function $\sigma(h)=2^{1/2}(\tau h)^{\beta/2}\ge\sup_{|t-s|\le h}d(t,s)$, $h>0$, satisfies Assumption 1. Then

$$\sigma^{(-1)}(h) = \frac{h^{2/\beta}}{2^{1/\beta}\tau}, \quad h > 0. \tag{39}$$

Also, it is easy to see that, for the centered process X,

$$d_f(t,s) = \left(d^2(t,s) + \left(f(t) - f(s)\right)^2\right)^{1/2} \le \left(2\left(\tau|t-s|\right)^{\beta_1} + \delta^2\left(d(t,s)\right)\right)^{1/2}$$

for any $\beta_1 \in (0, 1]$ and

$$\begin{aligned}\int_0^T \int_0^T R_X(s,t)\,\mathrm{d}s\,\mathrm{d}t &= \int_0^T \int_0^t e^{-\tau(t-s)}\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_t^T e^{-\tau(s-t)}\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{\tau}\int_0^T \left(1 - e^{-\tau t} - e^{-\tau(T-t)} + 1\right)\mathrm{d}t = \frac{2(T\tau + e^{-\tau T} - 1)}{\tau^2}.\end{aligned} \tag{40}$$

From (34) it follows that

$$\Delta_2 = \int_0^T \int_0^T \left(2\left(\tau|t-s|\right)^{\beta_1} + \left(\delta\left(\left(2\tau(u-v)\right)^{1/2}\right)\right)^2\right)^{1-\alpha}\mathrm{d}u\,\mathrm{d}v < \infty$$

if $\beta_1(1-\alpha) + 1 > 0$, that is, if $\alpha < 1/\beta_1 + 1$. Then

$$\int_0^T \int_0^T \left(\left(\delta(u-v)\right)^{\beta_1/2}\right)^{2-2\alpha}\mathrm{d}u\,\mathrm{d}v < \infty.$$

Applying (39) to (35) for some $\beta_2 \in (0, 1]$, we have that

$$\nu_t(u) = \min\left\{T, t + \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau 2^{3/\beta_2}}\right\} - \max\left\{0, t - \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau 2^{3/\beta_2}}\right\}.$$

Put $\tau' = \tau 2^{3/\beta_2}$. It is easy to see that $\nu_t(u) = T$ if $T < t + \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau'}$ and $0 > t - \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau'}$, that is, if $u > (\tau'\max\{t, T-t\})^{\alpha\beta_2/2}$; $\nu_t(u) = t + \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau'} - (t - \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau'}) = \frac{u^{\frac{2}{\alpha\beta_2}}}{2\tau'}$ if $u \le (\tau'\min\{t, T-t\})^{\alpha\beta_2/2}$; and $\nu_t(u) = \max\{t, T-t\} + \frac{u^{\frac{2}{\alpha\beta_2}}}{\tau'}$ if $(\tau'\min\{t, T-t\})^{\alpha\beta_2/2} \le u < (\tau'\max\{t, T-t\})^{\alpha\beta_2/2}$.

Consider

$$D_{p,2} = \sup_{t\in[0,T]} \frac{1}{p(1-p)} \int_0^{p2^{3\alpha/2}(\tau\max\{t,T-t\})^{\alpha\beta_2/2}} \frac{1}{\nu_t(u)}\,\mathrm{d}u.$$

For $\alpha > 2/\beta_2$, we have

$$\begin{aligned}&\int_0^{p2^{3\alpha/2}(\tau\max\{t,T-t\})^{\alpha\beta_2/2}} \frac{\mathrm{d}u}{\nu_t(u)} \\ &\quad = \int_0^{(4\tau\min\{t,T-t\})^{\alpha/2}} 2\tau' u^{-2/(\alpha\beta_2)}\,\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&+\int_{(\tau'\min\{t,T-t\})^{\alpha\beta_2/2}}^{(\tau'\max\{t,T-t\})^{\alpha\beta_2/2}} \frac{\mathrm{d}u}{\max\{t,T-t\}+\frac{u^{2/(\alpha\beta_2)}}{\tau'}}\\
&+\int_{(\tau'\max\{t,T-t\})^{\alpha\beta_2/2}}^{p2^{3\alpha/2}(\tau\max\{t,T-t\})^{\alpha\beta_2/2}} \frac{1}{T}\,\mathrm{d}u\\
\le{}& \frac{2\tau'}{1-2/(\alpha\beta_2)}\big(\tau'\min\{t,T-t\}\big)^{\alpha\beta_2/2-1}\\
&+\frac{(\tau'\max\{t,T-t\})^{\alpha\beta_2/2}-(\tau'\min\{t,T-t\})^{\alpha\beta_2/2}}{\max\{t,T-t\}+\min\{t,T-t\}}\\
&+\frac{p2^{3\alpha/2}\,(\tau\max\{t,T-t\})^{\alpha\beta_2/2}-(\tau'\max\{t,T-t\})^{\alpha\beta_2/2}}{T}\\
={}&\frac{2\tau'(\tau'\min\{t,T-t\})^{\alpha\beta_2/2-1}}{1-2/\alpha}\\
&+\frac{p2^{3\alpha/2}\,(\tau\max\{t,T-t\})^{\alpha\beta_2/2}-(\tau'\min\{t,T-t\})^{\alpha\beta_2/2}}{T}. \qquad (41)
\end{aligned}
$$

□

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